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On Invariants with the Novikov Additive Property

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1. Introduction

As observed by S. P. Novikov, the signature of manifolds behaves additively in the following sense: If M_1 and M_2 are oriented compact bounded differentiable manifolds and φ: $\partial M_1 \to \partial M_2$ is an orientation preserving diffeomorphism, then

$$\tau(M_1 \cup_\varphi - M_2) = \tau(M_1) - \tau(M_2).$$

The present paper is the third in a series of notes ([6, 7]) concerning this type of additivity. In [6] I proved, that any real valued invariant with this additive property coincides (up to a factor in each dimension, of course) with the signature on all closed manifolds. In Section 2 of the present note, we will generalize this result in three ways:

(i) We admit the invariant to take values in any abelian group G (in [6] I made much use of the fact that $\mathbb{R}$ has no elements of order two),

(ii) we also consider the unoriented case, and

(iii) we formulate the additive property in terms of closed manifolds only, we will not assume that the invariant is defined for bounded manifolds at all.

In Section 3 we will prove a similar theorem for real valued invariants of the equivariant oriented diffeomorphism type of orientation preserving involutions. The proof is based on Section 2 and on [7].

2. A Characterization of Signature and Euler Characteristic on Closed Manifolds

Definition. An invariant σ of the oriented diffeomorphism type of closed oriented differentiable manifolds with values in an abelian group G has the property (A), if for any three oriented compact bounded differentiable manifolds M_i (indexed modulo 3) and orientation preserving diffeomorphisms $\varphi_i: \partial M_i \to \partial M_{i+1}$, we have

$$\sigma(M_0 \cup_{\varphi_0} - M_1) + \sigma(M_1 \cup_{\varphi_1} - M_2) + \sigma(M_2 \cup_{\varphi_2} - M_0) = 0.$$

Obviously, property (A) follows from the additive property as formulated for the signature above.

Similarly, we define for unoriented manifolds:

Definition. An invariant ε of the (unoriented) diffeomorphism type of closed differentiable manifolds with values in an abelian group G has the property (B), if ε is additive with respect to disjoint union of closed manifolds and for any two compact bounded differentiable manifolds M_0 and M_1 and diffeomorphisms $\varphi: \partial M_0 \to \partial M_1$ and $\psi: \partial M_0 \to \partial M_1$, we have

$$\varepsilon(M_0 \cup_\varphi M_1) = \varepsilon(M_0 \cup_\psi M_1).$$

In particular, if ε is defined for all bounded manifolds and $\varepsilon(M_0 \cup_\varphi M_1) = \varepsilon(M_0) + \varepsilon(M_1)$ for all diffeomorphisms $\varphi: \partial M_0 \to \partial M_1$, then the restriction of ε to closed manifolds has property (B). The Euler characteristic e in even dimensions provides an example of such an invariant, because $e(M_0^{2n} \cup_\varphi M_1^{2n}) = e(M_0) + e(M_1) - e(\partial M_0) = e(M_0) + e(M_1)$, since ∂M_0 is a closed odd dimensional manifold.

Our main result is:

Theorem 1. (a) *Let σ have property* (A) *and define $a_k = \sigma(P_{2k}(\mathbb{C}))$. Then for any closed oriented manifold M^n we have*

$$\sigma(M^n) = \begin{cases} 0 & \text{if} \quad n \not\equiv 0 \bmod 4 \\ a_k \tau(M^{4k}) & \text{if} \quad n = 4k. \end{cases}$$

(b) *Let ε have property* (B) *and define $b_k = \varepsilon(P_{2k}(\mathbb{R}))$. Then for any closed manifold M^n we have*

$$\varepsilon(M^n) = \begin{cases} 0 & \text{if} \quad n \quad \text{is} \quad \text{odd} \\ b_k e(M^{2k}) & \text{if} \quad n = 2k. \end{cases}$$

Proof. Consider the oriented case. If X is closed and we apply (A) to $M_0 = X$, $M_1 = M_2 = \emptyset$, we obtain $\sigma(X) = -\sigma(-X)$. If X and Y are closed, we get $\sigma(X+Y) + \sigma(-X) + \sigma(-Y) = 0$, thus σ is additive with respect to disjoint union. (The "+" sign between manifolds indicates disjoint union.) Now let M be bounded and $X = M \cup_{\partial M} -M$ the usual "double" of M. Putting $M_0 = M_1 = M_2 = M$, we obtain from (A): $\sigma(X) + \sigma(-X) + \sigma(X) = \sigma(X) = 0$, and since $\sigma(M_0 \cup_\varphi -M_1)$ does not depend on the choice of φ, we see that σ vanishes for all closed manifolds of the form $M \cup_\varphi -M$. As a special case, we have

(1 a) *If the closed oriented manifold X is fibred over a positive dimensional sphere, then $\sigma(X) = 0$.*

Here and in the following we use the word "fibred" in the sense of "being the total space of a locally trivial differentiable fibration."

Now consider the unoriented case. We cut S^1 into two pairs of intervals. Then there are two essentially different ways φ and ψ to re-attach these interval pairs to a closed manifold, giving S^1 in one case and the disjoint union of two copies of S^1 in the other. Thus (B) implies: $\varepsilon(S^1) = \varepsilon(S^1 + S^1) = \varepsilon(S^1) + \varepsilon(S^1)$, hence $\varepsilon(S^1) = 0$. The same argument shows that in fact ε vanishes for any closed manifold which is fibred over S^1. We now generalize this to

(1 b) *If the closed manifold X is fibred over an odd dimensional sphere, then $\varepsilon(X) = 0$.*

Proof of (1b): Let X be fibred over S^{2k+1} with fibre F. Then, clearly, $\varepsilon(X)=\varepsilon(S^{2k+1}\times F)$. Now, $S^{2k+1}\times F$ is the total space E_k of a fibration over $P_k(\mathbb{C})$ with fibre $F_k=S^1\times F$. We cut $P_k(\mathbb{C})$ into D^{2k} and the normal disc bundle DN_{k-1}^k of $P_{k-1}(\mathbb{C})$ in $P_k(\mathbb{C})$ and define

$$M_0=E_k|DN_{k-1}^k+D^{2k}\times F_k \quad \text{and} \quad M_1=E_k|D^{2k}+DN_{k-1}^k\times F_k.$$

Then we can glue M_0 and M_1 together in two ways to get either $E_k+P_k(\mathbb{C})\times F_k$ or the disjoint union of a fibration over the double of D^{2k} and a fibration over the double of DN_{k-1}^k. The double of DN_{k-1}^k, however, is fibred over $P_{k-1}(\mathbb{C})$ with fibre S^2, and hence we get

$$\varepsilon(E_k)+\varepsilon(P_k(\mathbb{C})\times F_k)=\varepsilon(E_{k-1})+\varepsilon(S^{2k}\times F_k),$$

where E_{k-1} is fibred over $P_{k-1}(\mathbb{C})$ with fibre F_{k-1}, which in turn is fibred over S^2 with fibre F_k.

Let us call a manifold Y an ε-annihilator, if for any X we have $\varepsilon(X\times Y)=0$. S^1 is such an ε-annihilator, and also, of course, any product of S^1 with a manifold. Moreover, if Y is fibred over a sphere S^n such that the fibre Y' is an ε-annihilator, then Y is one, because $\varepsilon(X\times Y)=\varepsilon(X\times S^n\times Y')=0$. In particular, F_k is an ε-annihilator and hence we get $\varepsilon(E_k)=\varepsilon(E_{k-1})$.

Now we apply the same "cut and glue" procedure to E_{k-1} and $P_{k-1}(\mathbb{C})\times F_{k-1}$, and since F_{k-1} is fibred over S^2 with the ε-annihilator F_k as fibre, F_{k-1} is an ε-annihilator itself and we get $\varepsilon(E_{k-1})=\varepsilon(E_{k-2})$, where $E_{k-2}\to P_{k-2}(\mathbb{C})$ has an ε-annihilator F_{k-2} as fibre. By induction, we get $\varepsilon(E_k)=\varepsilon(E_{k-1})=\cdots=\varepsilon(E_0)=\varepsilon(F_0)=0$, thus (1b) is proved.

The same trick is used to prove the following two statements on fibrations over projective spaces

(2a) *If the closed oriented manifold E is fibred over $P_k(\mathbb{C})$ with fibre F then* $\sigma(E)=\sigma(P_k(\mathbb{C})\times F)$.

(2b) *If the closed manifold E is fibred over $P_k(\mathbb{R})$ with fibre F, then* $\varepsilon(E)+\varepsilon(P_k(\mathbb{R})\times F)=\varepsilon(S^k\times F)$.

Proof of (2a). As the three bounded manifolds involved in (A) we choose $M_0=E|DN_{k-1}^k$, $M_1=E|D^{2k}\cong D^{2k}\times F$ and $M_2=DN_{k-1}^k\times F$, suitably oriented. Then by (A) we have

$$\sigma(E)-\sigma(P_k(\mathbb{C})\times F)-\sigma(E_{k-1})=0, \tag{$*$}$$

where E_{k-1} is fibred over $P_{k-1}(\mathbb{C})$ with a fibre F_{k-1} which is fibred itself over S^2 with fibre F. By (1a), F_{k-1} is a σ-annihilator, and so are all the F_i which occur upon repeating the process. Applying ($*$) to $E_{k-1}\to P_{k-1}(\mathbb{C})$ instead of $E\to P_k(\mathbb{C})$, we get $\sigma(E_{k-1})=\sigma(E_{k-2})$, and by induction $\sigma(E_{k-1})=\sigma(E_0)=\sigma(F_0)=0$ and hence $\sigma(E)=\sigma(P_k(\mathbb{C})\times F)$, qed.

Proof of (2b). By the real analogue of what we have done with $P_k(\mathbb{C})$ in the proofs of (1b) and (2a), we get from (B):

$$\varepsilon(E)+\varepsilon(P_k(\mathbb{R})\times F)=\varepsilon(E_{k-1})+\varepsilon(S^k\times F), \tag{$**$}$$

where $E_{k-1} \to P_{k-1}(\mathbb{R})$ has fibre F_{k-1}, and F_{k-1} is fibred over S^1. Any fibration over S^1 is an ε-annihilator, and hence we obtain by induction: $\varepsilon(E_{k-1}) = \varepsilon(E_0) = \varepsilon(F_0) = 0$, thus (2b) follows from (**).

(3a) *σ is an invariant of the oriented bordism type and hence induces a homomorphism $\Omega_* \to G$.*

Proof of (3a). As proved by A. H. Wallace and also by J. Milnor, two closed oriented manifolds X and X' are cobordant if and only if there is a finite sequence of (oriented) spherical modifications (or surgeries) leading from X to X'. Hence we have to prove that $\sigma(X) = \sigma(X')$ if X' is obtained from X by a single surgery. To perform surgery on an n-dimensional manifold X means first to remove an imbedded $S^k \times D^{n-k}$ and then "replace" it by $D^{k+1} \times S^{n-k-1}$. Thus if we apply (A) to $M_0 = X - S^k \times \mathring{D}^{n-k}$, $M_1 = S^k \times D^{n-k}$ and $M_2 = D^{k+1} \times S^{n-k-1}$, we get

$$\sigma(X) - \sigma(S^n) - \sigma(X') = 0\,,$$

and hence $\sigma(X) = \sigma(X')$ by (1a), and thus (3a) is proved.

ε, of course, will in general not be a cobordism invariant, since the Euler characteristic e is not. Instead, we consider an invariant δ defined for n-dimensional manifolds X^n by

$$\delta(X^n) = \begin{cases} \varepsilon(X^n) & \text{if} \quad n \quad \text{is odd} \\ \varepsilon(X^{2k}) - \varepsilon(P_{2k}(\mathbb{R}))\, e(X^{2k}) & \text{if} \quad n = 2k\,. \end{cases}$$

Note that δ has property (B) in all dimensions, so that (1b) and (2b) hold for δ as well. $\delta = 0$ is exactly what we have to prove in part (b) of the theorem.

(3b) *δ is an invariant of the unoriented bordism type and hence induces a homomorphism $\mathfrak{N}_* \to G$.*

Proof of (3b). We apply the unoriented version of the Wallace-Milnor theorem. Let the n-dimensional manifold X' be obtained from X by a single surgery of type $(k+1, n-k)$. If we put $M_0 = (X - S^k \times \mathring{D}^{n-k}) + S^k \times D^{n-k}$ and $M_1 = S^k \times D^{n-k} + D^{k+1} \times S^{n-k-1}$, then by (B) we have

$$\delta(X) + \delta(S^n) = \delta(X') + \delta(S^k \times S^{n-k})\,.$$

If k is odd, $\delta(S^k \times S^{n-k})$ vanishes by (1b), and hence $\delta(X) + \delta(S^n) = \delta(X')$. If k is even, we choose M_1 as before but use $M_0' = (X - S^k \times \mathring{D}^{n-k}) + D^{k+1} \times S^{n-k-1}$ instead of M_0. Then (B) gives $\delta(X) + \delta(S^{k+1} \times S^{n-k-1}) = \delta(X') + \delta(S^n)$. Thus in either case, it suffices to prove that $\delta(S^n) = 0$. To do this, we apply (2b) to $E = S^n \to P_n(\mathbb{R})$, obtaining $\delta(S^n) + 2\delta(P_n(\mathbb{R})) = 2\delta(S^n)$, and since $\delta(P_n(\mathbb{R})) = 0$ by definition of δ for n even, this implies $\delta(S^n) = 0$ for n even. Since the odd case follows from (1b), this proves (3b).

Let us note that we now can improve (2b) to

(2b') *If the closed manifold E is fibred over $P_k(\mathbb{R})$ with fibre F, then $\delta(E) = \delta(P_k(\mathbb{R}) \times F)$.*

This is simply because by (3b) all elements $\delta(X)$ are at most of order two and $\delta(S^n \times F)$ vanishes. We also notice that (3b) finishes the proof of part (b) of the theorem for the case where G has no elements of order two.

The next two propositions, (4a) and (4b), are a preparation for the last stage of the proof, the evaluation of σ and δ on a set of generators of Ω_* and $\mathfrak{N}_*$ respectively.

(4a) $\sigma(X \times P_{n+2}(\mathbb{C})) = \sigma(X \times P_n(\mathbb{C}) \times P_2(\mathbb{C}))$ *for any closed oriented* X.

(4b) $\delta(X \times P_{n+2}(\mathbb{R})) = \delta(X \times P_n(\mathbb{R}) \times P_2(\mathbb{R}))$ *for any closed* X.

Proofs of (4a) *and* (4b). In the real as well as in the complex case, we denote by DN_k^m the normal disc bundle of $P_k \subset P_m$. As bundle over P_k, DN_k^m can be described by $S^{2k+1} \times_{S^1} D^{2m-2k}$ in the complex and $S^k \times_{Z_2} D^{m-k}$ in the real case. We will now prove (4) for the special case $X = \{pt\}$. The proof of the general case can then be obtained by simply writing "$X \times \cdots$" in front of everything.

By (2a) we have in the complex case: $\sigma(P_n \times P_2) = \sigma(S^{2n+1} \times_{S^1} P_2)$. Here the action of S^1 on P_2 or more generally on P_k can be described as follows: Consider S^1 as the centre of $U(k)$ and let $U(k)$ act on $P_k(\mathbb{C})$ by the usual imbedding $U(k) \subset U(k+1)$. The point $P_0(\mathbb{C}) = \{(0, \ldots, 0, 1)\}$ then becomes a fixed point of this action, and hence we can choose the dissection $P_k = DN_{k-1}^k \cup D^{2k}$ to be invariant under the action, and therefore we can dissect $S^{2n+1} \times_{S^1} P_2$ into $S^{2n+1} \times_{S^1} DN_1^2$ and $S^{2n+1} \times_{S^1} D^4 = DN_n^{n+2}$.

Now we put $M_0 = DN_1^{n+2}$, $-M_1 = DN_n^{n+2}$ and $M_2 = S^{2n+1} \times_{S^1} DN_1^2$. Then we have canonical φ_0 and φ_1 such that $M_0 \cup_{\varphi_0} -M_1 = P_{n+2}$ and $M_1 \cup_{\varphi_1} -M_2 = -S^{2n+1} \times_{S^1} P_2$. Therefore we can use (A) to determine the difference $\sigma(P_{n+2}) - \sigma(P_n \times P_2)$, provided we can evaluate σ on $M_2 \cup_{\varphi_2} -M_0$ for some φ_2.

It is $M_0 = S^3 \times_{S^1} D^{2n+2}$ and $M_2 = S^{2n+1} \times_{S^1} (S^3 \times_{S^1} D^2)$, and the action of S^1 on $S^3 \times_{S^1} D^2$ is simply induced by the usual action of S^1 on $S^3 \subset \mathbb{C}^2$. Therefore we get

$$M_2 = S^{2n+1} \times_{S^1} (S^3 \times_{S^1} D^2) = S^3 \times_{S^1} (S^{2n+1} \times_{S^1} D^2) = S^3 \times_{S^1} DN_1^{n+1},$$

and thus we can glue M_2 and $-M_0$ together to $S^3 \times_{S^1} P_{n+1}$, and by (A) we get

$$\sigma(P_{n+2}) - \sigma(P_n \times P_2) = -\sigma(S^3 \times_{S^1} P_{n+1}) = 0,$$

since $S^3 \times_{S^1} P_{n+1}$ is fibred over S^2 and hence has vanishing invariant σ by (1a). This completes the proof of (4a).

The real case is proved analogously: We first use (2b′) to replace $P_n \times P_2$ by $S^n \times_{Z_2} P_2$, and then we dissect P_{n+2} into DN_1^{n+2} and DN_n^{n+2} and $S^n \times_{Z_2} P_2$ into $S^n \times_{Z_2} DN_1^2$ and $S^n \times_{Z_2} D^2 = DN_n^{n+2}$. Since $DN_1^{n+2} = S^1 \times_{Z_2} D^{n+1}$ and

$$S^n \times_{Z_2} DN_1^2 = S^n \times_{Z_2} (S^1 \times_{Z_2} D^1) = S^1 \times_{Z_2} (S^n \times_{Z_2} D^1) = S^1 \times_{Z_2} DN_n^{n+1},$$

we can apply (B) to get

$$\delta(P_{n+2}) + \delta(P_n \times P_2) = \delta(S^1 \times_{Z_2} P_{n+1}) + \delta(DN_n^{n+2} \cup DN_n^{n+2}).$$

The first summand of the right hand side vanishes because of (1 b), and the second vanishes since the double of a bounded manifold always bounds. Since all $\delta(X)$ are at most of order two, we get $\delta(P_{n+2})=\delta(P_n\times P_2)$, as desired. Thus (4 b) is proved.

Now we come to the final part of the proof, namely the evaluation of σ and δ on sets of generators of the cobordism groups. For convenience we will now introduce Δ analogously to δ as follows: If X^n is a closed oriented n-dimensional manifold, we put

$$\Delta(X^n)=\begin{cases}\sigma(X^n) & \text{if}\quad n\not\equiv 0 \bmod 4\\ \sigma(X^{4k})-\sigma(P_{2k}(\mathbb{C}))\tau(X^{4k}) & \text{if}\quad n=4k\,.\end{cases}$$

Then Δ still has the property (A), and it vanishes on even dimensional complex projective spaces. $\Delta=0$ is then exactly what we have to prove in part (a) of the theorem.

As usual, we represent the abelian group Ω as a direct sum $\tilde{\Omega}+T$ of a torsion free part $\tilde{\Omega}$ and the torsion subgroup T. We will divide our proof into the following three steps:

(5 a) $\Delta\,|\,\tilde{\Omega}=0$,

(5 b) $\quad\delta=0$,

(6 a) $\Delta\,|\,T=0$.

Proof of (5 a). $\tilde{\Omega}$ can be generated as an abelian group by cartesian products of (i) even dimensional complex projective spaces $P_{2k}(\mathbb{C})$ and (ii) Hypersurfaces $H_{(r,t)}$ of degree (1, 1) in $P_r(\mathbb{C})\times P_t(\mathbb{C})$ with $1\leqq r\leqq t$. These hypersurfaces are defined by

$$H_{(r,t)}=\{(x,y)\in P_r(\mathbb{C})\times P_t(\mathbb{C})\,|\,x_0y_0+\cdots+x_ry_r=0\}\,.$$

The canonical projection $H_{(r,t)}\to P_r(\mathbb{C})$ is in fact a fibering with fibre $P_{t-1}(\mathbb{C})$, and therefore by (2 a) and (4 a) we have

$$\Delta\big(\prod H_{(r_i,t_i)}\times\prod P_{2k_j}(\mathbb{C})\big)=\Delta\big(P_2(\mathbb{C})\times\cdots\times P_2(\mathbb{C})\times P_1(\mathbb{C})\times\cdots\times P_1(\mathbb{C})\big)\,.$$

If factors $P_1(\mathbb{C})$ do occur, then Δ vanishes by (1 a), otherwise we have $\Delta(P_2(\mathbb{C})\times\cdots\times P_2(\mathbb{C}))=\Delta(P_{2k}(\mathbb{C}))$ by (4 a), and this vanishes by definition of Δ. Thus (5 a) is proved.

Proof of (5 b). Let $P(m,n)$ be the quotient manifold of the free involution $(x,z)\to(-x,\bar{z})$ on $S^m\times P_n(\mathbb{C})$. In [4] Dold proves that $\mathfrak{N}$ is additively generated by products of manifolds $P(m,n)$. The canonical projection $P(m,n)\to P_m(\mathbb{R})$ is a fibre bundle with fibre $P_n(\mathbb{C})$, so by (2 b′) we may restrict our attention to products of real and complex projective spaces. It is well known that $P_n(\mathbb{C})$ is mod 2 cobordant to $P_n(\mathbb{R})\times P_n(\mathbb{R})$, and hence it remains to show that $\delta(P_2(\mathbb{R})\times\cdots\times P_2(\mathbb{R})\times P_1(\mathbb{R})\times\cdots\times P_1(\mathbb{R}))$ vanishes – which it does: By (1 b) if a factor $P_1(\mathbb{R})$ does occur, and by $\delta(P_2\times\cdots\times P_2)=\delta(P_{2k})=0$ otherwise. Thus (5 b) is proved.

Proof of (6a). A little more care is required to prove $\Delta | T = 0$. In [9], Wall gives the following description of a set of generators for T:

Let $P(m, n)$ denote again the Dold manifolds, and let $\alpha: P(m, n) \to P(m, n)$ be the diffeomorphism induced by the map $(x, y) \to (x', y)$ on $S^m \times P_n(\mathbb{C})$, where $x \to x'$ is the reflection at $x_{m+1} = 0$. Identifying the two boundary components of $P(m, n) \times [0, 1]$ by α defines a bundle $Q(m, n)$ over S^1.

Any positive integer which is not a power of 2 can be uniquely written as $a = 2^{r-1}(2s+1)$, and Wall defines X_{2a} to be the element of $\mathfrak{N}$ represented by $Q(2^r - 1, 2^r s)$. Then T can be generated additively by products of elements of the form $\partial_3(X_{2a_1} \cdot \cdots \cdot X_{2a_n})$, in Wall's terminology.

Recall the definition of ∂_3: If M is a closed manifold and $\tilde{M}$ the orientation covering, then one can always find a closed 1-codimensional submanifold V of M such that $\tilde{M}$ is trivial over $M - V$. Let us call V an "orientation submanifold", because it represents the orientation line bundle in $H^1(M, Z_2)$. Wall proves that if V can be chosen orientable with trivial normal bundle, then V represents an element $[V] = -[V]$ in Ω, which only depends on the class $[M] \in \mathfrak{N}$. Given these circumstances, he defines $\partial_3[M] = [V] \in \Omega$. ∂_3 is then defined on a subgroup $\mathfrak{W}$ of $\mathfrak{N}$.

In our case, M is a product of manifolds $Q(m, n)$. First let this product consist of a single factor $Q(m, n)$. Since $m = 2^r - 1$ is odd $(r - 1 \geqq 0)$, the fibre $P(m, n)$ of the bundle $Q(m, n) \to S^1$ is orientable, but $Q(m, n)$ itself is not, because the defining diffeomorphism α is orientation reversing. Thus the orientation covering of Q, which of course is also fibred over S^1, has fibre $P + (-P)$:

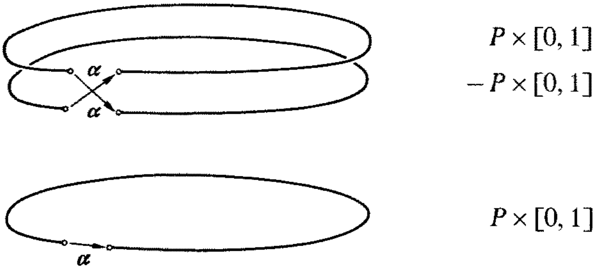

Fig.1

and is defined as bundle over S^1 by $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$, one could say. Hence $\partial_3[Q(m, n)]$ is represented by $P(m, n)$. Therefore, since $P(m, n)$ is fibred over $P_m(\mathbb{R})$, we have: If one of the factors of a generator of T is of the form $\partial_3[Q(m, n)]$, then this generator can be represented by a bundle over an odd dimensional real projective space $P_{2k+1}(\mathbb{R})$. But any such bundle is of the form $M \cup_\varphi -M$, and hence has the same invariant Δ as the double of M, namely zero. To see this, dissect $P_{2k+1}(\mathbb{R})$ into the normal disc bundles DN_k^{2k+1} and $DN_k'^{2k+1}$ of two complementary projective subspaces P_k and P'_k. If $E \to P_{2k+1}$ is the bundle in question, $M = E|DN_k^{2k+1}$ and $M' = E|DN_k'^{2k+1}$ are (at least unorientedly) diffeomorphic, and hence E is of the form $M \cup_\varphi M$ or $M \cup_\varphi -M$. There is no need to decide which, because M admits an orientation reversing diffeomorphism as follows: Choose a non-vanishing section of the normal bundle of $P_k \subset P_{2k+1}$. Then the reflection in each fibre at $f(x)^\perp$ defines an orientation reversing diffeomorphism of DN_k^{2k+1} which leaves the zero section pointwise fixed, and since $E|DN_k^{2k+1}$ is induced from $E|P_k$, this map can be lifted to an orientation reversing diffeomorphism of $M = E|DN_k^{2k+1}$.

Now consider a factor $\partial_3(X_{2a_1} \cdot \dots \cdot X_{2a_n})$ in which $n \geqq 2$. How can one construct an orientation submanifold of a product $M_1 \times M_2$ if one has orientation submanifolds V_1 and V_2 with trivial normal bundles for the factors M_1 and M_2? Start with the non-manifold $M_1 \times V_2 \cup V_1 \times M_2 \subset M_1 \times M_2$ and proceed as indicated in the following picture.

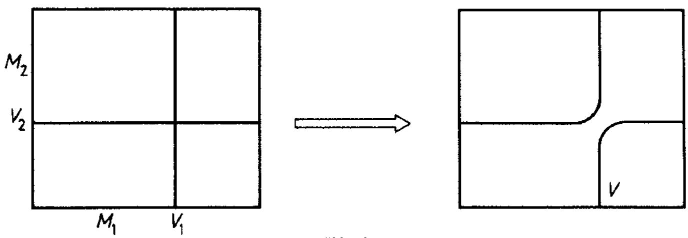

Fig.2

Then V is in fact an orientation submanifold of $M_1 \times M_2$, because first Stiefel-Whitney classes behave additively under direct sum of bundles. In our application, M_1 is $Q(m_1, n_1)$ and $M_2 = \prod_{i=2}^{n} Q(m_i, n_i)$. Since each $Q(m, n)$ is fibred over S^1 with orientable fibre and orientation reversing defining diffeomorphism, it is not difficult to show by induction:

Proposition. *If* $M = \prod_{i=1}^{r} Q(m_i, n_i)$ *and all* m_i odd, *then one can find an orientable orientation submanifold V with trivial normal bundle which is fibred over the* $(r-1)$*-dimensional torus.*

In particular, for $r \geqq 2$, V is fibred over S^1 and therefore is a Δ-annihilator, which finishes the proof of the theorem.

A few remarks shall be made about "additive" invariants for manifolds with boundaries. Let σ now denote an invariant of the oriented diffeomorphism type of oriented compact bounded manifolds with values in G, and let ε denote an invariant of the diffeomorphism type of compact bounded manifolds. Then we say that σ has property (A′), if σ is additive with respect to disjoint union and $\sigma(M \cup_\varphi -M') = \sigma(M) - \sigma(M')$ for any orientation preserving diffeomorphism $\varphi : \partial M \to \partial M'$, and we say that σ has property (A″), if $\sigma(M \cup_\varphi -M')$

$=\sigma(M)-\sigma(M')$ for any orientation preserving diffeomorphism φ of a union of boundary components of M onto a union of boundary components of M'. Similarly we define (B′) and (B″) for ε by $\varepsilon(M\cup_{\varphi}M')=\varepsilon(M)+\varepsilon(M')$. Clearly (A″)⇒(A′)⇒(A) and (B″)⇒(B′)⇒(B).

If we wish to determine all invariants with properties (A″) and (B″), then by our theorem it suffices to determine those which have these properties and vanish on closed manifolds. And just as in [6] (Bemerkung 1), these invariants can all be obtained as follows: (i) Oriented case: For each oriented diffeomorphism class X^{n-1} of closed connected manifolds choose an element $\alpha(X^{n-1})\in G$ with the only requirement that always $\alpha(X)=-\alpha(-X)$. Then define $\sigma(M^n)$ to be the sum of the α's of the boundary components. (ii): Unoriented case: For each diffeomorphism class X^{n-1} of closed connected manifolds choose an element $\beta(X)\in G$ with $2\beta(X)=0$. Then define $\varepsilon(M^n)$ to be the sum of the β's of the boundary components.

(A′) and (B′) do not imply (A″) and (B″) in general, as the following example shows. Let M^n be an oriented connected bounded manifold and $X_1,\ldots,X_r$ its boundary components. We choose a subset $X_{i_1},\ldots,X_{i_k}$ of the set of boundary components in which any (unoriented) diffeomorphism type of $X_1,\ldots,X_r$ occurs exactly once. Then we define $\sigma(M^n)=[X_{i_1}]+\cdots+[X_{i_k}]\in\mathfrak{N}_{n-1}$. For nonconnected manifolds, we define σ to be the sum of the σ's of the components. Then σ satisfies (A′) but not (A″), and in particular is not of the form described above, although it vanishes on closed manifolds. The same example works in the unoriented case.

If however G has no elements of order two, the (A′)⇒(A″) and (B′)⇒(B″), which can be proved as for τ in [6], p. 35.

3. "Additive" Invariants for Involutions

In this section we consider compact oriented differentiable manifolds with orientation preserving involutions. Let me introduce the following notations: By $\mathrm{conj}\colon P_{2k}(\mathbb{C})\to P_{2k}(\mathbb{C})$ we denote the involution on the $2k$-dimensional complex projective space which is induced by complex conjugation. If X is an oriented manifold, then $\mathrm{triv}\colon 2X\to 2X$ denotes the trivial interchanging of the two copies in the disjoint union $2X=X+X$. If X is an oriented manifold and $T\colon X\to X$ an orientation preserving involution, then the components of the fixed point set must have even codimension, and we denote by F_k the union of the $2k$-codimensional components of Fix T.

Definition. A real valued invariant ϱ of the oriented equivariant diffeomorphism type of orientation preserving involutions on closed oriented manifolds has the property $(\mathrm{A}Z_2)$, if for any three orientation preserving involutions $T_i\colon M_i\to M_i$ on bounded manifolds and equivariant orientation preserving diffeomorphisms $\varphi_i\colon\partial M_i\to\partial M_{i+1}$ we have

$$\varrho(M_0\cup_{\varphi_0}-M_1,\,T_0\cup T_1)+\varrho(M_1\cup_{\varphi_1}-M_2,\,T_1\cup T_2)+\varrho(M_2\cup_{\varphi_2}-M_0,\,T_2\cup T_0)=0\,.$$

In particular, if $\tau(M, T)$ denotes the signature of an orientation preserving involution $T: M \to M$, (i.e. $\tau(M, T) = 0$ if $\dim M \not\equiv 0$ (4) and $\tau(M^{4k}, T)$ = signature of the symmetric bilinear form on $H_{2k}(M^{4k}, R)$ given by $(x, y) \to x \circ T_* y$), then τ has this property (AZ_2). (See [1], Prop. 7.1.)

Theorem 2. *If ϱ has property* (AZ_2), *then*

$$\varrho(X, T) = \begin{cases} 0 & \text{if} \quad \dim X \not\equiv 0\,(4) \\ a_n \tau(X) + \sum\limits_k a_n^k \tau(F_k \circ F_k) & \text{if} \quad \dim X = 4n\,, \end{cases}$$

where a_n, a_n^k only depend on ϱ, n and k, namely

$$a_n = \tfrac{1}{2}\, \varrho(2P_{2n}(\mathbb{C}), \mathrm{triv})\,,$$
$$a_n^k = (-1)^k\, \varrho(P_{2(n-k)}(\mathbb{C}) \times P_{2k}(\mathbb{C}), \mathrm{Id} \times \mathrm{conj}) - \tfrac{1}{2}\, \varrho(2P_{2n}(\mathbb{C}), \mathrm{triv})\,.$$

Proof. Define

$$\lambda(X, T) = \begin{cases} \varrho(X, T) \\ \varrho(X, T) - a_n \tau(X)\,. \end{cases}$$

Then λ still has property (AZ_2), and $\lambda(2P_{2n}(\mathbb{C}), \mathrm{triv}) = 0$. We have then to show:

$$\lambda(X, T) = \begin{cases} 0 \\ \sum\limits_k a_n^k \tau(F_k \circ F_k)\,. \end{cases}$$

We first note that for any X we have

$$\lambda(2X, \mathrm{triv}) = 0\,. \tag{1}$$

This follows from Theorem 1, because $\lambda(2X, \mathrm{triv}) = \sigma(X)$ depends only on X and has property (A), hence

$$\sigma(X) = \begin{cases} 0 \\ \sigma(P_{2n}(\mathbb{C}))\tau(X) \end{cases} = 0\,,$$

since $\lambda(2P_{2n}(\mathbb{C}), \mathrm{triv}) = \sigma(P_{2n}(\mathbb{C})) = 0$.

Proceeding as in [7], the next step would be to show that λ vanishes for all fixed point free involutions. Now, an orientation preserving free involution determines a real line bundle over an oriented manifold and vice versa. Hence instead of considering λ for free involutions, we may as well consider an invariant for real line bundles over oriented closed manifolds, satisfying a corresponding additivity condition. For later use in the proof of Theorem 2, we shall consider a more general situation:

Definition. An invariant μ for real m-dimensional vector bundles over oriented closed manifolds has property $(\mathrm{A}^{\sim})$, if for any three m-dimensional vector bundles E_i over oriented bounded manifolds Y_i and any vector bundle isomorphisms $\varphi_i: \partial E_i \to \partial E_{i+1}$, inducing orientation preserving diffeomorphisms $\varphi_i: \partial Y_i \to \partial Y_{i+1}$, we have

$$\mu(E_0 \cup_{\varphi_0} -E_1) + \mu(E_1 \cup_{\varphi_1} -E_2) + \mu(E_2 \cup_{\varphi_2} -E_0) = 0.$$

In particular, if $m=1$, then μ defined by $\mu(E) = \lambda(SE, -1)$ has property (A').

Proposition 1. *If μ has property* (A'), *then μ is a bordism invariant.*

Proof of Proposition 1. Let $E \to W$ be an m-dimensional vector bundle over an oriented bounded manifold W with $\partial W = Y_1 + (-Y_2)$. To show $\mu(E|Y_1) = \mu(E|Y_2)$, we make use of the fact (see [8], p. 43) that any oriented bordism is the "trace" of a finite sequence of (oriented) surgeries. So we may as well assume that W is the trace of a single surgery leading from Y_1 to Y_2. But then $E|Y_1$ and $E|Y_2$ "differ by a bundle over a sphere" in the sense that there are bounded submanifolds $A_i \subset Y_i$ such that $E|(Y_1 - \mathring{A}_1)$ and $E|A_2$ can be glued together to give $E|Y_2$, and $E|A_1$ and $E|A_2$ can be glued together to give a bundle over a sphere. But it is an immediate consequence of (A'), that μ vanishes for any bundle over a sphere. Therefore, again by (A'), we get $\mu(E|Y_1) = \mu(E|Y_2)$, which proves Proposition 1.

Proposition 2. *Let $m=2k$, let μ have property* (A') *and E be a $2k$-dimensional real vector bundle over the closed oriented manifold Y. Then if $Y \times \mathbb{R}^{2k}$ denotes the trivial bundle over Y, we have*

$$\mu(E) = \mu(Y \times \mathbb{R}^{2k}).$$

Proof of Proposition 2. By the same argument as in the proof of (2a) in Section 2 we obtain:

If Y is fibred over $P_r(\mathbb{C})$ and F_0 is one of the fibres, then

$$\mu(E) = \mu(P_r(\mathbb{C}) \times (E|F_0)). \tag{$*$}$$

Now, by Lemma (3.3) on p. 96 in [3], $\Omega_*(BO(2k))/Torsion$ can be generated (additively) by bundles of the form

$$X \times \eta_{2i_1} \times \cdots \times \eta_{2i_k},$$

where X is a closed manifold and $\eta_{2i} \to P_{2i}(\mathbb{C})$ is the canonical complex line bundle over $P_{2i}(\mathbb{C})$. Clearly, by $(*)$ we get

$$\mu(X \times \eta_{2i_1} \times \cdots \times \eta_{2i_k}) = \mu(X \times P_{2i_1}(\mathbb{C}) \times \cdots \times P_{2i_k}(\mathbb{C}) \times \mathbb{R}^{2k}),$$

thus Proposition 2 follows from Proposition 1.

Let us now continue with the proof of Theorem 2. Let $T: X \to X$ be fixed point free. Then by a result of Burdick [2] (see also [5], Section 2), the disjoint union of two copies of (X, T) is cobordant, as an orientation preserving free involution, to a trivial involution. Hence by Proposition 1 and by (1) we have $2\lambda(X, T) = 0$, and thus

(2) *If T is fixed point free, then $\lambda(X, T) = 0$.*

This can now be generalized to

(3) *If the normal bundle* N *of* Fix T *admits a non-zero section* f, *then* $\lambda(X,T)=0$.

Proof of (3). As in the proof of Lemma 1 in [7], we dissect X into DN and Y, where the disc bundle DN is regarded as an invariant tubular neighborhood of Fix T. Let $\Psi: SN \to SN$ be defined by reflection at $f(x)^{\perp}$ in each fibre. Then $2\lambda(X,T)=\lambda(DN \cup_{\Psi} DN, T)+\lambda(Y \cup_{\Psi} Y, T)$. But reflection at $f(x)^{\perp}$ also induces an orientation reversing equivariant diffeomorphism of (DN, T) onto itself, and therefore $\lambda(DN \cup_{\Psi} DN, T)=\lambda(DN \cup_{\Psi} -DN, T)=0$ by (AZ_2), and $\lambda(Y \cup_{\Psi} Y, T)=0$ by (2).

Let (3′) be those technical generalization of (3) which is formulated in [7] as Lemma 2 for τ. (3′) can be reduced to (3) in the same manner in which we just reduced (3) to (2).

Now let $T: X \to X$ be any orientation preserving involution on a closed manifold and F_k the $2k$-codimensional part of Fix T. Choose sections f_k for each $N \mid F_k$ which are transverse regular at the zero section. Then $B_k = \{x \mid f_k(x)=0\}$ represents $F_k \circ F_k$. We denote by E_k the restriction of the normal bundle of F_k in X to B_k and by $\tilde{E}_k$ the associated $O(2k)$-principal bundle (with respect to some bundle metric).

Then one shows exactly as in [7]:

$$\lambda(X,T)=\sum_k (-1)^k \lambda(\tilde{E}_k \times_{O(2k)} (P_{2k}(\mathbb{C}), \mathrm{conj})), \tag{4}$$

where $O(2k)$ acts on $P_{2k}(\mathbb{C})$ via $O(2k) \subset U(2k) \subset U(2k+1)$. (The $(-1)^k$ does not appear in [7] because there we used a non-canonical orientation of $P_{2k}(\mathbb{C})$.)

Now define for $2k$-dimensional real vector bundles E over oriented closed manifolds Y:

$$\mu(E)=\lambda(\tilde{E} \times_{O(2k)} (P_{2k}(\mathbb{C}), \mathrm{conj})).$$

Then (AZ_2) implies that μ has property (A˜), and therefore, by Proposition 2, does the value of $\mu(E)$ only depend on Y; and μ, now considered as an invariant for Y, has property (A). Hence by Theorem 1 we get

$$\mu(E)=\begin{cases} 0 & \text{if} \quad \dim Y \not\equiv 0(4) \\ \mu(P_{2(n-k)}(\mathbb{C}) \times \mathbb{R}^{2k})\,\tau(Y) & \text{if} \quad \dim Y = 4n-4k, \end{cases}$$

or

$$\lambda(\tilde{E} \times_{O(2k)} (P_{2k}(\mathbb{C}), \mathrm{conj}))=\begin{cases} 0 \\ (-1)^k a_n^k \tau(Y) \end{cases}$$

and thus by (4):

$$\lambda(X,T)=\begin{cases} 0 \\ \sum_k a_n^k \tau(F_k \circ F_k), \end{cases}$$

what is what we wanted to prove.

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