THE SECONDARY BOUNDARY OPERATOR

BY J. H. C. WHITEHEAD

MAGDALEN COLLEGE, OXFORD

Communicated by S. Lefschetz, October 31, 1949

1. The Sequence $S(K)$. Let $H_n = H_n(K)$ be the nth homology group of a complex[1] K and let

$$\Pi_n = \Pi_n(K) = \pi_n(K) \qquad \Gamma_n = \Gamma_n(K) = i_n\pi_n(K^{n-1}),$$

where $i_n:\pi_n(K^{n-1}) \to \pi_n(K^n)$ is the injection ($n \geq 2$). Then a sequence of homomorphisms

$$S(K): \ldots \xrightarrow{\mathfrak{j}} H_{n+1} \xrightarrow{\mathfrak{b}} \Gamma_n \xrightarrow{\mathfrak{i}} \Pi_n \xrightarrow{\mathfrak{j}} H_n \xrightarrow{\mathfrak{b}} \Gamma_{n-1} \xrightarrow{\mathfrak{i}} \ldots,$$

terminating with $H_3 \to 0 \to \Pi_2 \to H_2 \to 0$, is defined as follows. $\mathfrak{j}_n$ is the natural homomorphism and $\mathfrak{i}_n = i_n' | \Gamma_n$, where $i_n':\pi_n(K^n) \to \Pi_n$ is the injection. We assume that H_n is defined as

$$H_n = Z_n - d_{n+1}C_{n+1} \qquad (n \geq 3),$$

where $C_n = \pi_n(K^n, K^{n-1})$, $Z_n = d_n^{-1}(0) \subset C_n$ and $d_{n+1}:C_{n+1} \to C_n$ is the resultant of the boundary homomorphism, $\beta_{n+1}:C_{n+1} \to \pi_n(K^n)$, followed by the injection $j_n:\pi_n(K^n) \to C_n$. Let $z \in Z_{n+1}$. Since $j_n\beta_{n+1}z = 0$ it follows from the exactness of the homotopy sequence of K^{n-1}, K^n that $\beta_{n+1}z \in \Gamma_n$. Also $\beta_{n+1}d_{n+2} = 0$, since $\beta_{n+1}j_{n+1} = 0$. Therefore $\beta_{n+1}|Z_{n+1}$ induces a homomorphism, $\mathfrak{b}_{n+1}:H_{n+1} \to \Gamma_n$, which is the one in $S(K)$.

THEOREM 1. *The sequence* $S(K)$ *is exact.*[2]

Let $m \geq 4$ and let

$$S_m(K):\ H_m \to \Gamma_{m-1} \to \ldots$$

be the part of $S(K)$ which begins with H_m. We write $S_\infty(K) = S(K)$, thus defining $S_m(K)$ for $m \leq \infty$.

By a *homomorphism* (*isomorphism*)[3]

$$F = (\mathfrak{h}, \mathfrak{g}, \mathfrak{f}):S_m(K) \to S_m(K'), \tag{1.1}$$

where K' is a given complex, we mean a family of homomorphisms

$$\mathfrak{h}_{n+1}:H_{n+1} \to H_{n+1}',\ \mathfrak{g}_n:\Gamma_n \to \Gamma_n',\ \mathfrak{f}_n:\Pi_n \to \Pi_n'$$

such that $\mathfrak{b}\mathfrak{h} = \mathfrak{g}\mathfrak{b}$, $\mathfrak{i}\mathfrak{g} = \mathfrak{f}\mathfrak{i}$, $\mathfrak{j}\mathfrak{f} = \mathfrak{h}\mathfrak{j}$, where $\mathfrak{b}:H_{n+1}' \to \Gamma_n'$, etc., are the homomorphisms in $S(K')$. Let $\mathfrak{K}$ be any category[4] of (simply connected) complexes and homotopy classes of maps $K \to K'$, for every pair of complexes $K, K' \in \mathfrak{K}$. Let $\mathfrak{S}_m$ be the category in which the objects and mappings are the sequences, $S_m(K)$, and all homomorphisms, $S_m(K) \to S_m(K')$, for every pair K, $K' \in \mathfrak{K}$. Then a homotopy class of maps, $K \rightarrow K'$, induces a unique homomorphism, $S_m(K) \rightarrow S_m(K')$, in such a way as to determine a functor $\mathfrak{K} \rightarrow \mathfrak{S}_m$. Thus $S_m(K)$ is a homotopy invariant and *a fortiori* a topological invariant of K. If a given homomorphism, $F: S_m(K) \rightarrow S_m(K')$, is the one induced by a map, $\phi: K \rightarrow K'$, we shall describe ϕ as a *geometrical realization* of F.

THEOREM 2. *Let dim K, dim $K' \leqq m$. Then $\phi: K \equiv K'$ if ϕ induces an isomorphism $F: S_m(K) \approx S_m(K')$.*

This follows at once from Theorem 3 in CH I.

Let $\lambda: S^3 \rightarrow S^2$ be a map which represents a fixed generator of $\pi_3(S^2)$ and let $\mu: S^2 \rightarrow K^2$ represent a given element $a \in \Pi_2$. Then $\mu\lambda: S^3 \rightarrow K^2$ represents an element $\lambda(a) \in \Gamma_3$. We shall describe equation (1.1) as a *proper* homomorphism (isomorphism) if, and only if,

$$\mathfrak{g}_3\lambda(a) = \lambda(\mathfrak{f}_2 a), \tag{1.2}$$

for every $a \in \Pi_2$. Let equation (1.1) have a geometrical realization $\phi: K \rightarrow K'$. Then equation (1.1) is a proper homomorphism because $\phi(\mu\lambda) = (\phi\mu)\lambda$.

THEOREM 3. *Let dim $K \leqq 4$. Then any proper homomorphism, $S_4(K) \rightarrow S_4(K')$, has a geometrical realization, $K \rightarrow K'$.*

We now anticipate the definition of $\Gamma(A)$ in §2 below and consider a purely algebraic (exact) sequence

$$S_4: \; H_4 \xrightarrow{\mathfrak{h}} \Gamma_3 \xrightarrow{\mathfrak{i}} \Pi_3 \xrightarrow{\mathfrak{j}} \ldots \xrightarrow{\mathfrak{j}} H_2 \xrightarrow{\mathfrak{h}} o,$$

in which the (Abelian) groups are arbitrary except that

$$\theta: \Gamma(\Pi_2) \approx \Gamma_3, \qquad \Gamma_2 = 0.$$

The isomorphism θ is to be included as a component part of S_4. Let S_4', with groups H_{n+1}', Γ_n', Π_n', be a similar sequence. A *proper homomorphism (isomorphism)*, $S_4 \rightarrow S_4'$, shall mean the same as before, with equation (1.2) replaced by the condition

$$\mathfrak{g}_3\theta = \theta\mathfrak{g}: \Gamma(\Pi_2) \rightarrow \Gamma_3',$$

where θ means the same in S_4' as in S_4 and

$$\mathfrak{g}: \Gamma(\Pi_2) \rightarrow \Gamma(\Pi_2')$$

is the homomorphism induced by $\mathfrak{f}_2: \Pi_2 \rightarrow \Pi_2'$. By a *geometrical realization* of S_4 we shall mean a complex, K, such that $S_4(K)$ is properly isomorphic to S_4.

THEOREM 4. *The sequence S_4 has a geometrical realization, which is*

(a) at most 4-dimensional if H_4 is free Abelian,

(b) a finite complex if each of H_2, H_3 and H_4 has a finite set of generators.

Theorems 2, 3 and 4 show that $S_4(K)$ can be used to replace the more complicated "extended" cohomology ring[5] of K. Moreover they apply to infinite complexes and hence to universal covering complexes. Therefore it seems reasonable to hope that these theorems, in conjunction with the cohomology theory of abstract groups, may lead to similar theorems in case $\pi_1(K) \neq 1$.

2. The Group $\Gamma(A)$. Let A be an additive Abelian group and let wA be any aggregate which is in a (1-1) correspondence, $w:A \to wA$, with A. We define an (additive) group, $\Gamma(A)$, by means of the symbolic generators $w(a) \in wA$ and the relations

$$w(a) \equiv w(-a) \tag{2.1a}$$

$$w(a + b + c) - w(b + c) - w(c + a) - w(a + b) + w(a) + w(b) + w(c) \equiv 0, \tag{2.1b}$$

together with the "trivial relations," $w(a) - w(a) \equiv 0$. On writing $a = b = c = 0$ in (2.1b) we have $w(0) \equiv 0$. Hence it follows from (2.1b), with $b = 0$, that $\Gamma(A)$ is Abelian. Let $b = (n - 1)a$, $c = -a$ ($n \geq 1$). Then it follows from equation (2.1) and induction on n that $w(na) \equiv n^2w(a)$.

Let $\gamma(a)$ be the element of $\Gamma(A)$ which is represented by $w(a)$ and let

$$[a, b] = \gamma(a + b) - \gamma(a) - \gamma(b).$$

Then, given that addition is commutative, equation (2.1b) expresses the fact that $[a, b]$ is bilinear in a and b.

Let A be free Abelian and let $\{a_i\}$ be a set of free generators of A. Then $\Gamma(A)$ is freely generated by the elements $\gamma(a_i)$, $[a_j, a_k]$, for every a_i and every (unordered) pair of distinct elements $a_j, a_k \in \{a_i\}$.

Let A, generated by a_1, be a finite, cyclic group of order m. Then $\Gamma(A)$ is generated by $\gamma(a_1)$ and is of order m or $2m$, according as m is odd or even.

Let A be the weak direct sum of a set of groups $\{A_i\}$. Let Γ be the weak direct sum of the groups $\Gamma(A_i)$ and the tensor products $A_j o A_k$, for every A_1 and every (unordered) pair of distinct groups, A_j, A_k, in the set $\{A_i\}$ Then $\Gamma(A) \approx \Gamma$.

It follows that, if A is finitely generated, so is $\Gamma(A)$. Moreover the rank and invariant factors of $\Gamma(A)$ can be calculated from those of A and conversely. Also $a = 0$ if $[a, a'] = 0$ for every $a' \in A$. Therefore the pairing $(A, A) \to \Gamma(A)$, in which $(a, a') = [a, a']$, is orthogonal.

It follows from the form of the relations (2.1) that a homomorphism $f:A \to A'$, into an additive Abelian group A', *induces* a homomorphism, $g:\Gamma(A) \to \Gamma(A')$, which is given by $g\gamma(a) = \gamma(fa)$. If A admits a group, Π_1, as a group of operators, so does $\Gamma(A)$, according to the rule $x\gamma = \gamma x$ ($x \in \Pi_1$).

Let $A = \Pi_2(K)$ and let $\lambda(a)$ mean the same as in equation (1.2). Then $\lambda(a) = \lambda(-a)$ and[6]

$$\lambda(a + b) - \lambda(a) - \lambda(b) = [a, b],$$

where $[a, b] \in \Gamma_3(K)$ is the bilinear product, or commutator,[7] of a and b. Therefore the relations (2.1) are satisfied when w and $\equiv$ are replaced by λ and $=$. Therefore a homomorphism, $\theta:\Gamma(\Pi_2) \rightarrow \Gamma_3$, is defined by $\theta\gamma(a) = \lambda(a)$.

Let $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f}):S_m(K) \rightarrow S_m(K')$ be a proper homomorphism and let $\mathfrak{g}:\Gamma(\Pi_2) \rightarrow \Gamma(\Pi_2')$ be the homomorphism induced by $\mathfrak{f}_2$. Then it follows from equation (1.2) and the relations $\mathfrak{g}\gamma = \gamma\mathfrak{f}_2$, $\theta\gamma = \lambda$, that

$$\theta\mathfrak{g} = \mathfrak{g}_3\theta:\Gamma(\Pi_2) \rightarrow \Gamma_3', \tag{2.2}$$

where $\theta:\Gamma(\Pi_2') \rightarrow \Gamma_3'$ is also defined by $\theta\gamma = \lambda$. In particular let Π_1 be a group of homomorphisms of K onto itself (e.g., the covering group if K is a covering complex). Then each $x \in \Pi_1$ induces a proper automorphism $x:S_m(K) \approx S_m(K)$ and it follows from equation (2.2) that θ is an operator homomorphism.

THEOREM 5.[8] $\theta:\Gamma(\Pi_2) \approx \Gamma_3$.

3. $\Gamma(A)$ and Cohomology. Let X be any topological space and let $H_n(G)$ be the Cech cohomology group of X, which is defined in terms of the nerves of all finite open coverings, with G as the (discrete) group of coefficients (we could equally well take closed coverings). We define the cup-product, $\mathfrak{a} \cup \mathfrak{b} \in H^{2n}\{\Gamma(A)\}$, of elements $\mathfrak{a}, \mathfrak{b} \in H^n(A)$, by means of the pairing $(a, b) \rightarrow [a, b]$, where $a, b \in A$.

THEOREM 6. *Let n be even. Then there is a natural homomorphism,*

$$h:\Gamma\{H^n(A)\} \rightarrow H^{2n}\{\Gamma(A)\},$$

such that $h[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a} \cup \mathfrak{b}$ for any pair $\mathfrak{a}, \mathfrak{h} \in H^n(A)$.

We write $h\gamma = \mathfrak{p}:H^n(A) \rightarrow H^{2n}\{\Gamma(A)\}$ and call $\mathfrak{p}\mathfrak{a}$ the *Pontrjagin square* of $\mathfrak{a} \in H^n(A)$ (n is even). We have

$$\mathfrak{p}(\mathfrak{a} + \mathfrak{b}) = \mathfrak{p}\mathfrak{a} + \mathfrak{p}\mathfrak{b} + \mathfrak{a} \cup \mathfrak{b}, \qquad 2\,\mathfrak{p}\mathfrak{a} = \mathfrak{a} \cup \mathfrak{a}. \tag{3.1}$$

Thus $-\mathfrak{a} \cup \mathfrak{b}$ is a factor set, which measures the error made in supposing $\mathfrak{p}$ to be a homomorphism. Let $g:\Gamma(A) \rightarrow \Gamma(A')$ be the homomorphism induced by a homomorphism, $f:A \rightarrow A'$, into an additive Abelian group A'. Then f, g induce homomorphisms

$$f^n:H^n(A) \rightarrow H^n(A'), \qquad g^{2n}:H^{2n}\{\Gamma(A)\} \rightarrow H^{2n}\{\Gamma(A')\} \tag{3.2}$$

such that $\mathfrak{p}\mathfrak{f}^n = g^{2n}\mathfrak{p}$. If X is a finite polyhedron and if A is cyclic of even order, then $\mathfrak{p}$ is the same as in "SCP."

Let $X = K$ and let I_m be the group of integers, reduced mod. m. Let

$$H_n(m) = H_n(K, I_m), \qquad A_m = A - mA \qquad (m \geq 0).$$

Then the pairing $(A, I_m) \to A_m$, in which $(a, 1) \in A_m$ is the residue class containing $a \in A$, determines a homomorphism

$$u^n(m): H^n(A) \to \text{Hom}\,\{H^n(m), A_m\}. \tag{3.3}$$

If K has no $(n - 1)$ dimensional torsion $u^n(0)$ is an isomorphism (onto).

Now take $A = H_n$ and let K be without $(n - 1)$-dimensional torsion. Then

$$u^n(0): H^n(H_n) \approx \text{Hom}\,(H_n, H_n)$$

and we identify each element $\mathfrak{a} \in H^n(H_n)$ with $u^n(0)\mathfrak{a}$. Thus $H^n(H_n)$ becomes the additive group of the ring, E_n, of endomorphisms of H_n. Let $f^n(e), g^{2n}(e)$ denote f^n, g^{2n} in equation (3.2) when $A = A' = H_n$ and $f = e \in E_n$. Then it follows from the way in which e induces $f^n(e)$ that $f^n(e)e' = ee'$ $(e' \in E_n)$. Since $\mathfrak{p}f^n = g^{2n}\mathfrak{p}$ we have

$$\mathfrak{p}(ee') = g^{2n}(e)\mathfrak{p}e', \qquad \mathfrak{p}e = g^{2n}(e)\mathfrak{p}(1), \tag{3.4}$$

where $1 \in E_n$ is the identity. Thus $\mathfrak{p}$ is determined by the map $e \to g^{2n}(e)$ and by $\mathfrak{p}(1)$.

Now let K be a finite (simply connected) complex of arbitrary dimensionality. We make the natural identification $\Pi_2 = H_2$ and, using Theorem 5, we identify each $\gamma \in \Gamma(H_2)$ with $\theta\gamma \in \Gamma_3$. Also K has no 1-dimensional torsion and we identify each $\mathfrak{a} \in H^2(H_2)$ with $u^2(\text{O})\mathfrak{a} \in E_2$. Then equation (3.3), with $n = 4$, $A = \Gamma_3$, becomes

$$u(m) = u^4(m): H^4(\Gamma_3) \to \text{Hom}\,\{H_4(m), \Gamma_{3,\,m}\}.$$

The homomorphism $\gamma: C_4(K) \to \Gamma_3$, which is defined on page 85 of "SCP," induces what we call the secondary *modular* boundary homomorphism,[9]

$$\mathfrak{b}(m) \in \text{Hom}\{H_4(m), \Gamma_{3,\,m}\},$$

and $\mathfrak{b}(0)$ is the same as $\mathfrak{b}_4$ in $S_4(K)$.

THEOREM 7. $\mathfrak{b}(m) = u(m)\mathfrak{p}(1) \qquad (m \geq 0).$

4. *The Calculation of $S_4(K)$.* The group Π_3, in $S_4(K)$, is an extension of H_3 by $G = \Gamma_3 - \mathfrak{b}H_4$ and $S_4(K)$ is determined, up to a proper isomorphism, by H_2, H_3, H_4, the homomorphism $\mathfrak{b}_4$ and the element of $H^2(H_3, G)$ which determines the equivalence class of the extension Π_3. Let K be a finite, simplicial complex. Then it will be shown how these items may be calculated (constructively) with the help of Theorems 5 and 7. This construction does not provide a finite algorithm for deciding whether or not $S_4(K)$ is properly isomorphic to $S_4(K')$. Some of the difficulties inherent in this question are indicated on page 88 of "SCP."

5. *A_n^2-polyhedra.* Let $\pi_r(K) = 0$ for $r = 1, \ldots, n - 1$, where $n > 2$. In this case we may identify Γ_{n+1} with[10] $H_n(2)$ and $\mathfrak{h}_{n+2}$ determines a homomorphism, $\mathfrak{h}(2):H_{n+2}(2) \rightarrow H_n(2)$, which is the dual of[11] $Sq_{n-2}:H^n(2) \rightarrow H^{n+2}(2)$. The structure of Π_{n+2}, as an extension of H_{n+1} by $H_n(2)$, is determined by $\mathfrak{h}(2)$. Thus $S_{n+2}(K)$ is determined, up to a proper isomorphism, by the co-homology system $H(K)$, or by the analogous system of homology groups,[12] in which $\mathfrak{h}(2)$ plays the part of Sq_{n-2}.

[1] All our complexes will be simply connected CW-complexes, as defined in §5 of J. H. C. Whitehead, "Combinatorial Homotopy I," *Bull. Am. Math. Soc.*, **55**, 213–245 (1949). This paper will be referred to as CH I.

[2] In the light of this theorem a $J_m$complex, K, as defined in CH I, is seen to be one such that $\mathfrak{j}_n:\pi_n(\tilde{K}) \approx H_n(\tilde{K})$, if $n \leq m$, and $\mathfrak{j}_{m+1}$ is onto, where $\tilde{K}$ is the universal covering complex of K. This, and the other theorems stated here, will be proved in a paper which is to appear in the *Annals of Mathematics*.

[3] An isomorphism will always mean an isomorphism onto.

[4] Cf. Eilenberg, S., and MacLane, Saunders, "General Theory of Natural Equivalences," *Trans. Am. Math. Soc.*, **58**, 231–294 (1945).

[5] See Whitehead, J. H. C., "On Simply Connected, 4-Dimensional Polyhedra," *Comm. Math. Helvetici*, **22**, 48–92 (1949). This paper will be referred to as "SCP."

[6] See equation (7.3) in Whitney, Hassler, "Relations Between the Second and Third Homotopy Groups of a Simply Connected Space," *Ann. Math.*, **50**, 180–202 (1949).

[7] Cf. Fox, R. H., "Homotopy Groups and Torus Homotopy Groups," *Ibid.*, **49**, 471–510 (1948).

[8] Cf. Hirsch, G., "Sur le troisieme groupe d'homotopie des polydères simplement connexe," *C. R. Acad. Sci. Paris*, **228**, 1920–1922 (1949), in case K is finite and without 2-dimensional torsion. Hirsch's representation of $\Gamma_3 - \mathfrak{h}H_4$ can be obtained from Theorem 5 and Theorem 7 below.

[9] In the forthcoming paper $\mathfrak{h}(m)$ is defined more generally and is shown to be natural.

[10] See Whitehead, J. H. C., "The Homotopy Type of a Special Kind of Polyhedron," *Ann. Soc. Polonaise Math.*, **21**, 176–186 (1949); also Whitehead, G. W., "On spaces with vanishing low-dimensional homotopy groups", *Proc. Nat. Acad. Sci.*, **34**, 207–211 (1948).

[11] Steenrod, N. E., "Products of Cocycles and Extensions of Mappings," *Ann. Math.*, **48**, 290–320 (1947).

[12] See a forthcoming paper by P. J. Hilton.

THEOREMS ON QUADRATIC PARTITIONS

BY ALBERT LEON WHITEMAN

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA

Communicated by H. S. Vandiver, November 23, 1949

If p is a prime of the form $3f + 1$, the diophantine equation $4p = a^2 + 3b^2$ has a unique solution in a and b with $a \equiv 1 (\text{mod } 3)$ and $b \equiv 0 (\text{mod } 3)$. About forty years ago von Schrutka[1] derived the formula $a = 1 + \phi_3(4)$, where $\phi_k(n)$ is the Jacobstahl sum defined by