ANNALS OF MATHEMATICS
Vol. 41, No. 4, October, 1940

ON C^1-COMPLEXES

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(Received July 18, 1939)

1. This paper is supplementary to S. S. Cairns' work[1] on the triangulation of, and polyhedral approximations to manifolds of class C^1. Its aim is to provide a foundation for theorems[2] which involve both differential geometry and the theory of combinatorial equivalence.[3] Theorem 8, for example, states that two C^1-triangulations of any manifold of class[4] C^1 are combinatorially equivalent. Thus a manifold of class C^1 is like a recti-linear polyhedron in that it has a preferred class of combinatorially equivalent triangulations. This theorem depends on the definition of a C^1-complex given in §2 below, and does not apply, for example, to the algebraic complexes first considered by B. L. van der Waerden.[5] For though two "algebraic triangulations" of the same space have a common algebraic sub-division, it is not certain that an algebraic triangulation of a recti-linear n-simplex is combinatorially equivalent to an n-simplex. The C^1-complexes, though more closely allied to, also differ essentially from the complexes considered by Cairns. Therefore we do not use Cairns' results, though the main ideas in many of our theorems are due to him.

We shall use R^n to stand for n-dimensional Euclidean space, and it is to be understood that R^n is Euclidean not only in its topology, but also in the sense of metric geometry. By a complex K we shall always mean a recti-linear, simplicial complex in Euclidean space, and $\dot{K}$ will stand for the mod 2 boundary of K. We shall denote a recti-linear, simplicial subdivision of K, but not necessarily a normal sub-division, by K', and if K_0 is any sub-complex of K, then K_0' will be the sub-complex of K' covering K_0. By a simplex we shall always mean a closed simplex (i.e. a simplex with its boundary), and we shall use the letters A, B to stand for recti-linear simplexes. By an *isomorphism* $t(K_1) = K_2$ we shall mean a homeomorphism of K_1 on K_2 which maps each simplex of K_1 on the whole of one, and only one, simplex of K_2, and which is linear throughout each simplex in K_1. In §§2 and 3, theorem 5 excepted, K will always stand for a finite complex. We shall use the summation con-

[1] Annals of Math., 35(1934), 579-87 (triangulation); 37(1936), 409-15 (approximations).

[2] See, for example, J. H. C. Whitehead, Annals of Math. 41(1940), 825-832.

[3] Here we may take as a definition: two simplicial complexes K_1 and K_2, finite or infinite, are combinatorially equivalent if, and only if, recti-linear models of K_1 and K_2 have isomorphic recti-linear sub-divisions.

[4] See, for example, O. Veblen and J. H. C. Whitehead, *The Foundations of Differential Geometry*, Cambridge (1932), Chap VI. By an n-dimensional manifold we shall always mean one which is covered by a countable set of open n-cells.

[5] Math. Ann., 102(1929), 337-62.

vention in analytical formulae, with the additional convention that Roman indices take on the values $1, \cdots, n$.

2. A map $f(U) \subset R^n$, of a region $U \subset R^k$ (possibly $U \subset R^k \subset R^m$, where $m > k$), is said to be of class C^1, or a C^1-map, if and only if, it is given by equation of the form

$$y^i = f^i(x^1, \cdots, x^k) = f^i(x), \tag{2.1}$$

where $x^1, \cdots, x^k$ and $y^1, \cdots, y^n$ are rectangular Cartesian coordinates for R^k and for R^n, and the functions $f^i(x)$ have continuous derivatives at each point of U. The map f will be described as *non-degenerate* if, and only if, the Jacobian matrix of the transformation (2.1) is of rank k at each point of U. A map $f(A) \subset R^n$, of a k-simplex $A \subset R^k$ will be described as of class C^1 (non-degenerate) if, and only if, it can be extended throughout some open set in R^k, containing A, in which it is of class C^1 (non-degenerate).

Let $f_\alpha(A) \subset R^n$ ($\alpha = 1, 2$) be two C^1-maps of a simplex A, given by equations of the form (2.1). We shall describe f_2 as an (ϵ, ρ)-approximation to f_1 if, and only if,

$$\| f_2 - f_1 \| \leqq \epsilon \quad \text{and} \quad \| df_2 - df_1 \| \leqq \rho \| df_1 \|$$

for each $x \in A$ and every vector dx, where

$$df^i_\alpha = \frac{\partial f^i_\alpha}{\partial x^\lambda} dx^\lambda \qquad (\lambda = 1, \cdots, k)$$

and $\| y_2 - y_1 \| = \{(y^i_2 - y^i_1)(y^i_2 - y^i_1)\}^{\frac{1}{2}}$, $\| y \| = (y^i y^i)^{\frac{1}{2}}$. When we are only interested in $\| df_2 - df_1 \|$, or in $\| f_2 - f_1 \|$, we may refer to f_2 as an (∞, ρ)-, or as an (ϵ, ∞)-approximation to f_1. Let a be the origin of the coordinates y and let p_1 and p_2 be the extremities of the vectors df_1 and df_2, situated at a. If f_2 is an (∞, ρ)-approximation to f_1, it follows from the geometry of the triangle ap_1p_2 that $ap_2 \leqq ap_1 + p_1p_2 \leqq (1 + \rho)ap_1$, and $ap_1 \leqq ap_2 + p_2p_1 \leqq ap_2 + \rho ap_1$, whence $(1 - \rho)ap_1 \leqq ap_2$. Therefore, if $\rho \leqq \frac{1}{2}$, we have $ap_1 \leqq 2ap_2$ and f_1 is an $(\infty, 2\rho)$-approximation to f_2. If f_2 is an (∞, ρ_1)-approximation to f_1, f_3 an (∞, ρ_2)-approximation to f_2, and if ap_3 is the vector df_3, then,

$$p_1p_3 \leqq p_1p_2 + p_2p_3 \leqq \rho_1 ap_1 + \rho_2 ap_2 \leqq (\rho_1 + \rho_2 + \rho_1\rho_2)ap_1,$$

since $ap_2 \leqq (1 + \rho_1)ap_1$. Therefore f_3 is an (∞, ρ_3)-approximation to f_1, where $\rho_3 = \rho_1 + \rho_2 + \rho_1\rho_2$. Notice that $\rho_3 < 3\rho_1$ if $\rho_1 = \rho_2 < 1$. Combining these with the familiar relations for (ϵ, ∞)-approximations, we see that, if f_2 is an (ϵ_1, ρ_1)-approximation to f_1, where $\rho_1 \leqq \frac{1}{2}$, and f_3 is an (ϵ_2, ρ_2)-approximation to f_2, then f_1 is an $(\epsilon_1, 2\rho_1)$-approximation to f_2 and f_3 an $(\epsilon_1 + \epsilon_2, \rho_1 + \rho_2 + \rho_1\rho_2)$-approximation to f_1. Notice also that an (∞, ρ)-approximation, f_2, to a non-degenerate map, f_1, is itself non-degenerate if $\rho < 1$. For to say that f is non-degenerate is to say that $df \neq 0$ if $dx \neq 0$, and $\| df_2 \| \geqq (1 - \rho) \| df_1 \|$.

LEMMA 1. *If f_2 is a non-degenerate (∞, ρ)-approximation to a non-degenerate map f_1, then the angle*[6] *between the vectors df_1 and df_2 does not exceed $\pi\rho$.*

If $\rho \geqq 1$ or if $\rho = 0$ this is trivial. So we assume $0 < \rho < 1$. Then, with the same notation as before, if the lengths ap_1 and p_1p_2 are fixed, the angle θ = angle p_1ap_2 is greatest when ap_2 touches a circle of radius p_1p_2 and center p_1, in which case p_1p_2 is perpendicular to ap_2. Therefore $\sin\theta \leqq p_1p_2/ap_1 \leqq \rho$, whence $\theta \leqq \frac{1}{2}\pi \sin\theta < \pi\rho$ and the lemma is established.

By the *radius* $r(A)$, of a simplex $A \subset R^n$ we shall mean the distance from its centroid to its boundary. Let $f_\alpha(A) \subset R^n$ $(\alpha = 1, 2)$ be two linear, non-degenerate maps of a k-simplex $A \subset R^k$, let ϵ be the maximum of $\| f_2(x) - f_1(x) \|$ as x varies over A, and let $r = r\{f_1(A)\}$.

LEMMA 2. *Under these conditions $f_2(A)$ is an $(\epsilon, 2\epsilon/r)$-approximation to $f_1(A)$.*

By definition, $f_2(A)$ is an (ϵ, ∞)-approximation to $f_1(A)$, and we have only to prove that $\| df_2 - df_1 \| \leqq \rho \| df_1 \|$, where $\rho = 2\epsilon/r$. Since f_1 and f_2 are linear, $df_2 - df_1$ does not depend on the point $x \in A$, but only on the vector dx. Therefore we may take $x = \bar{x}$, the centroid of A, and we take $\bar{x} + dx \in \dot{A}$. Then $f_1(\bar{x})$ is the centroid of $f_1(A)$, since f_1 is linear, and $f_1 + df_1 = f_1(\bar{x} + dx) \in \{f_1(A)\}$; whence $\| df_1 \| \geqq r$. Therefore

$$\begin{aligned} \| df_2 - df_1 \| &= \| (f_2 + df_2) - (f_1 + df_1) - (f_2 - f_1) \| \\ &= \| \{f_2(\bar{x} + dx) - f_1(\bar{x} + dx)\} - \{f_2(\bar{x}) - f_1(\bar{x})\} \| \\ &\leqq \| f_2(\bar{x} + dx) - f_1(\bar{x} + dx) \| + \| f_2(\bar{x}) - f_1(\bar{x}) \| \leqq 2\epsilon \leqq \frac{2\epsilon}{r} \| df_1 \| \end{aligned}$$

and the lemma is established.

Let A be a k-simplex in R^k and let $f(A) \subset R^n$ be a non-degenerate C^1-map, given by equations of the form (2.1). If b is any point in A these equations may be rewritten as

$$y^i - c^i = a^i_\lambda(x^\lambda - b^\lambda) + \gamma^i(x), \tag{2.2}$$

where $c = f(b)$ and a^i_λ are the derivatives $\partial f^i/\partial x^\lambda$, calculated for $x = b$. The image of A in the linear transformation F_b, given by

$$y^i - c^i = a^i_\lambda(x^\lambda - b^\lambda), \tag{2.3}$$

will be called the *tangent simplex* to $f(A)$ at the point c. Since the derivatives $\partial f^i/\partial x^\lambda$ are continuous, and therefore uniformly continuous in the compact set A, it is an obvious consequence of lemma 2 that, given $\epsilon, \rho > 0$, there is a $\delta > 0$ such that F_b is an (ϵ, ρ)-approximation to $F_{b'}(A)$ provided $\| b' - b \| \leqq \delta$. Since, at the point $x = b'$, $df = dF_{b'}$, we have:

LEMMA 3. *Given $\epsilon, \rho > 0$, there is a positive δ such that F_b is an (ϵ, ρ)-approximation to f throughout the sub-set of A given by $\| x - b \| \leqq \delta$, for any $b \in A$.*

By the relative thickness,[7] $\tau(B)$, of a simplex $B \subset R^m$, we shall mean r/l,

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[6] By the angle between two vectors we mean the positive angle which does not exceed π.

[7] Cf. Cairns' definition of a θ-set (Triangulation, p. 583).

where $r = r(B)$ and l is the diameter,[8] $l(B)$, of B. Let $f(A) \subset R^n$ be a non-degenerate C^1-map of a k-simplex $A \subset R^k$, let $b_0, \cdots, b_p$ be any points in A and let B be the simplex $b_0 \cdots b_p$. Let $L(B) \subset R^n$ be the linear map of B such that $L(b_\alpha) = f(b_\alpha)$ $(\alpha = 0, \cdots, p)$.

LEMMA[9] 4. *Given $\epsilon, \rho, \sigma > 0$, there is a positive δ such that, if $\tau(B) \geqq \sigma$ and $l(B) \leqq \delta$, then $L(B)$ is an (ϵ, ρ)-approximation to $f(B)$.*

By lemma 3 there is a positive δ_1 such that $F_b(B)$ is an $(\epsilon/2, \rho/3)$-approximation to $f(B)$ for any $b \in B$, provided $l(B) \leqq \delta_1$. Therefore, assuming, as we obviously may, that $\rho < 1$, the lemma will follow, with $\delta = \min(\delta_1, \delta_2)$, if there is a positive δ_2 such that $L(B)$ is an $(\epsilon/2, \rho/3)$- approximation to $F_b(B)$ for any $b \in B$, provided $l(B) \leqq \delta_2$. The transformations $F_b(B)$ and $L(B)$ are given by

$$y^i - c^i = a^i_\lambda(b^\lambda_\alpha - b^\lambda)t^\alpha$$

$$y^i - c^i = a^i_\lambda(b^\lambda_\alpha - b^\lambda)t^\alpha + \gamma^i(b_\alpha)t^\alpha \qquad (\alpha = 0, \cdots, p),$$

where $0 \leqq t^\alpha \leqq 1$, $t^0 + \cdots + t^p = 1$, $b^1_\alpha, \cdots, b^k_\alpha$ are the coordinates of the vertex b_α, and a^i_λ and $\gamma^i(x)$ mean the same as in (2.2). Since the derivatives of $\gamma^i(x)$ vanish when $x = b$ there is a positive $\delta(\eta)$, such that $|\gamma^i(x)| \leqq \eta \| x - b \|$ provided $\| x - b \| \leqq \delta(\eta)$, for a given $\eta > 0$. If $l = l(B) = \delta(\eta)$ and if $b \in B$ it follows that $|\gamma^i(b_\alpha)| \leqq \eta l$. Since $t^\alpha \geqq 0$ and $t^0 + \cdots + t^p = 1$ we have $|\gamma^i(b_\alpha)t^\alpha| \leqq \eta l$, whence

$$\| L(x) - F_b(x) \| = \| \gamma(b_\alpha)t^\alpha \| \leqq \eta l n^{\frac{1}{2}}$$

for any point $x = b_\alpha t^\alpha$ in B. It follows from lemma 2 that $L(B)$ is an $(\eta l n^{\frac{1}{2}}, 2\eta l n^{\frac{1}{2}}/r_b)$-approximation to $F_b(B)$, where $r_b = r\{F_b(B)\}$.

Now let $x^1, \cdots, x^k$ in (2.3), be rectangular coordinates for $R^k \supset A$, and let $\zeta_1(b)$ be the smallest root of the equation

$$|a^i_\lambda a^i_\mu - \zeta\delta_{\lambda\mu}| = 0. \tag{2.4}$$

Then $\zeta_1(b)$ is a continuous function of b, and is positive since f is non-degenerate. Therefore $\zeta_1(b)$ has a positive lower bound, ω^2, as b varies in the compact set A, and

$$\| dF_b \| = (a^i_\lambda a^i_\mu \, dx^\lambda \, dx^\mu)^{\frac{1}{2}} \geqq \omega \| dx \|$$

for any $b \in A$. Therefore $r_b \geqq \omega r$, where $r = r(B)$, and $l/r_b \leqq l/\omega r \leqq 1/\omega\sigma$, since $r/l \geqq \sigma$, whence

$$2\eta l n^{\frac{1}{2}}/r_b \leqq 2\eta n^{\frac{1}{2}}/\omega\sigma.$$

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[8] $l(B)$ is the length of the longest side of B (P. Alexandroff and H. Hopf, *Topologie*, Berlin (1935), 607).

[9] Cf. Cairns (Approximations, §4).

Also $l \leqq l(A) = l_1$, say. Therefore $L(B)$ is an $(\epsilon/2, \rho/3)$-approximation to $F_b(B)$ provided $l \leqq \delta_2 = \delta(\eta)$, where

$$\eta = \min(\epsilon/2l_1n^{\frac{1}{2}}, \qquad \rho\omega\sigma/6n^{\frac{1}{2}}),$$

and the lemma is established.

By a (δ, σ)-*subdivision* of $K \subset R^m$ we shall mean a subdivision K', such that $l(B) \leqq \delta$, $\tau(B) \geqq \sigma$, where B is any simplex in K' and, as before, $l(B)$ and $\tau(B)$ stand for the diameter and the relative thickness of B.

LEMMA[10] 5. *There is a* (δ, σ)*-subdivision of* K *for an arbitrary* $\delta > 0$ *and some* $\sigma > 0$, *which does not depend on the choice of* δ.

Let the equations (2.3) now represent an arbitrary, non-singular, linear transformation $F(R^k) \subset R^n$. If ζ_1 and ζ_k are the smallest and greatest roots of the equation (2.4) we have

$$\zeta_1^{\frac{1}{2}} \| dx \| \leqq \| dy \| \leqq \zeta_k^{\frac{1}{2}} \| dx \|.$$

If B is any simplex in R^k and $C = F(B)$ it follows that $\tau(B) \geqq \kappa\tau(C)$, where $\kappa = (\zeta_1/\zeta_k)^{\frac{1}{2}}$, and $\kappa > 0$ since F is non-singular. Therefore, if $K = F(K_1)$, where $K_1 \subset R^N$ and F is an isomorphism, and if K_1' is any subdivision of K_1, there are constants ω_0, κ_0, such that $l(B) \leqq \omega_0 l(B_1)$, $\tau(B) \geqq \kappa_0\tau(B_1)$, where B_1 is any simplex in K_1' and $B = F(B_1)$. Therefore we may replace K by an isomorphic complex in R^N, and shall assume it to be a sub-complex of the simplex Δ_1, whose vertices have rectangular Cartesian coordinates $(0, \cdots, 0)$, $(1, 0, \cdots, 0)$, $\cdots$, $(0, \cdots, 0, 1)$, where $N + 1$ is the number of vertices in K. Let P be the polyhedral complex consisting of the convex cells into which R^N is divided by the hyperplanes

$$y^\beta = k \qquad\qquad (\beta = 0, \cdots, N; k = 0, \pm1, \pm2, \cdots),$$

where $y^1, \cdots, y^N$ are the coordinates for R^N and $y^0 = y^1 + \cdots + y^N$. Then P contains a sub-complex covering the simplex Δ_q, whose vertices are the points $(0, \cdots, 0)$, $(q, 0, \cdots, 0)$, $\cdots$, $(0, \cdots, 0, q)$, for any integral value of q. The complex P also contains a sub-complex Q, which covers the hyper-cube given by $0 \leqq y^\lambda \leqq 1$ $(\lambda = 1, \cdots, N)$, and each cell in P is congruent to a cell in Q under the group of translations. Let P' and Q' be the complexes obtained from P and Q by a normal subdivision, the new vertices being placed at the centroids of the corresponding cells. Then each simplex in P' is congruent to some simplex in Q' and its relative thickness is therefore at least σ, where σ is the minimum relative thickness of the simplexes in Q'. Let E_q be the sub-complex of P' covering Δ_q and let E_1 be the image of E_q in the transformation given by $\tilde{y}^\lambda = y^\lambda/q$. Then E_1 is a subdivision of Δ_1 and so contains a sub-complex K', which is a subdivision of K. The relative thickness of each simplex in K' is at least σ, since the relative thickness is an invariant of the

[10] Cf. Cairns (Triangulation, p. 585).

similarity group, and its diameter is less than $\frac{1}{q}N^{\frac{1}{2}}$. Taking $q \geqq \frac{1}{\delta}N^{\frac{1}{2}}$ the lemma is established.

By a *C^1-map, $f(K) \subset R^n$*, or a *map of class C^1*, we shall mean a map which is of class C^1 throughout each simplex in K. The map $f(K)$ will be described as non-degenerate if, and only if, it is non-degenerate throughout each simplex. We shall also describe $f(K)$ as a C^1-complex, or a complex of class C^1, and as a non-degenerate complex if, and only if, the map f is non-degenerate. By an *(ϵ, ρ)-approximation* to $f(K)$, we shall mean a C^1-map $f'(K') \subset R^n$, where K' is any sub-division of K, such that $f'(A)$ is an (ϵ, ρ)-approximation to $f(A)$ throughout each simplex $A \subset K'$. We shall use $L_f(K')$ to denote the map which is linear (possibly degenerate) throughout each simplex of K' and coincides with f at the vertices of K'. Notice that $L_f(K')$ is not, in general, the image of K' in $L_f(K)$.

THEOREM 1. *Given a non-degenerate C^1-complex $f(K) \subset R^n$, and $\epsilon, \rho, \sigma > 0$, there is a positive δ such that $L_f(K')$ is an (ϵ, ρ)-approximation to $f(K)$, where K' is any (δ, σ)-sub-division of K.*

By lemma 4 there is a $\delta(A) > 0$ such that, if K' is any $\{\delta(A), \sigma\}$-subdivision of K, then $L_f(A')$ is an (ϵ, ρ)-approximation to $f(A)$, where A' is the subcomplex of K' covering $A \subset K$. Taking $\delta = \min \delta(A)$, for any $A \subset K$, the theorem follows.

Let K_1 be a sub-complex of a given complex K and let K_1' be any subdivision of K_1. By an *extension* of K_1' throughout K we shall mean a subdivision K' of K, which coincides with K_1' in K_1. Let $f(K) \subset R^n$ be a non-degenerate C^1-map and let $f_1(K_1') \subset R^n$ be an (ϵ_1, ρ_1)-approximation to $f(K_1)$. By an *(ϵ, ρ)-extension* of $f_1(K_1')$ throughout K we shall mean an (ϵ, ρ)-approximation, $f'(K') \subset R^n$, to $f(K)$, which coincides with f_1 in K_1', where K' is an extension of the subdivision K_1'.

THEOREM 2. *Given a non-degenerate C^1-map, $f(K) \subset R^n$, a sub-complex $K_1 \subset K$ and $\epsilon, \rho > 0$, there are positive numbers ϵ_1, ρ_1, such that any (ϵ_1, ρ_1)-approximation to $f(K_1)$ has an (ϵ, ρ)-extension throughout K.*

This will follow from an obvious induction on the number of simplexes in $Cl(K - K_1)$, the closure of $K - K_1$, when we have proved it in case $K = A$, a single simplex, and $K_1 = \dot{A}$. Let a be the centroid of A, let x_0 be the mid point of the segment ax_1, for any $x_1 \epsilon \dot{A}$, and let A_0 be the simplex bounded by the locus of x_0 as x_1 describes $\dot{A}$. Let $\dot{A}'$ be any subdivision of $\dot{A}$ and let P be the polyhedral complex consisting of the convex cells $B \times x_0x_1$, swept out by the segment x_0x_1 as x_1 varies over the simplexes $B \subset \dot{A}'$. Let x_t be the point on x_0x_1 such that $x_0x_t:x_tx_1 = t:(1 - t)$ $(0 \leqq t \leqq 1)$ and let $x_1^1, \cdots, x_1^k, t$ be taken as coordinates for $B \times x_0x_1$, where $x_1^1, \cdots, x_1^k$ are Cartesian coordinates for any $B \subset \dot{A}'$. Let Δ be the maximum and δ the minimum attained by $|| df ||$ for any $x_t \epsilon P$ and any vector $(dx_1^1, \cdots, dx_1^k, dt)$ whose length is unity in terms of a Euclidean metric for A.

Now let $f_1(\dot{A}') \subset R^n$ be an (ϵ_1, ρ_1)-approximation to $f(\dot{A})$, and, treating $f(x_1)$

and $f_1(x_1)$ as vectors in R^n, let $\gamma(x_1) = f_1(x_1) - f(x_1)$. Then $\| \gamma \| \leqq \epsilon_1$ and, if dx_1 is a unit vector, $\| d\gamma \| \leqq \rho_1 \| df \| \leqq \rho_1 \Delta$, whence

$$\text{(2.5)} \qquad \begin{cases} | \gamma^i | \leqq \| \gamma \| \leqq \epsilon_1 \\ | d\gamma^i | \leqq \| d\gamma \| \leqq \rho_1 \Delta, \end{cases}$$

where the index i refers to some rectangular Cartesian coordinate system for R^n. Let $f'(P)$ be the C^1-map given, in vector notation, by

$$f'(x_t) = f(x_t) + t\gamma(x_1) \qquad (0 \leqq t \leqq 1;\, x_1 \in \dot{A}').$$

Then

$$\| f' - f \| = t \| \gamma \| \leqq \epsilon_1 \quad \text{and} \quad \| df' - df \| = \| t\, d\gamma + \gamma\, dt \|.$$

Taking (dx_1, dt) to be a unit vector, in which case $| dt |$ is bounded, it follows from (2.5) and the continuity of the function $\| y \|$ that there are positive numbers ϵ_1 and ρ_1 such that $\| df' - df \| \leqq \rho\delta \leqq \rho \| df \|$. If $\epsilon_1 \leqq \epsilon$ we have also $\| f' - f \| \leqq \epsilon$ and $f'(P)$ is an (ϵ, ρ)-approximation to $f(P)$. Finally we take $f' = f$ in A_0 and extend the sub-division A' by starring A_0 and each of the cells $B \times x_0x_1$, leaving $\dot{A}'$ untouched. The result is an (ϵ, ρ)-extension of f_1 throughout A, and the theorem is established.

Let K_2 be the complex consisting of all the simplexes in K which do not meet K_1. As a corollary to theorem 2, replacing $f_1(K_1')$ by $f_2(K_1' + K_2)$ with $f_2 = f_1$ in K_1', $f_2 = f$ in K_2, we have the addendum:

ADDENDUM. *The extension $f'(K')$, referred to in theorem 2, may be chosen so that the subdivision K' leaves K_2 unaltered and $f' = f$ in K_2.*

If B is any simplex in K we shall use $N(B, K)$ to stand for the *stellar neighbourhood* of B in K, consisting of all the simplexes $AB \subset K$, where AB is the join[11] of A and B. If b is an internal point of B we shall also describe $N(B, K)$ as the *stellar neighbourhood*,[12] $N(b, K)$, of b. If $f(K) \subset R^n$ is any C^1-complex, the recti-linear complex in R^n which consists of the tangent simplexes at $f(b)$ to the simplexes in $f\{N(b, K)\}$, will be called the *tangent star at $f(b)$ to $f(K)$*. Thus the tangent star is the image of $N(b, K)$ in a simplicial transformation F_b, which coincides with the transformation $F_b(AB)$, defined by (2.3), throughout each of the simplexes AB, where b is internal to B. By a *non-singular C^1-complex, or map, $f(K)$* we shall mean a C^1-complex such that

1. *f is $(1 - 1)$ throughout K,*
2. *F_b is $(1 - 1)$ throughout $N(b, K)$ for each point $b \in K$.*

It follows from the second of these conditions that a non-singular map is non-degenerate.

THEOREM 3_ρ. *To any non-singular C^1-complex $f(K) \subset R^n$ correspond positive numbers ϵ, ρ, such that any (ϵ, ρ)-approximation to $f(K)$ is non-singular.*

According to a previous observation, any (ϵ, ρ)-approximation to f is non-

[11] Here we allow A to be 1, the empty simplex, in which case $AB = B$.

[12] In general $N(B, K) \subset$ but $\neq N(b, K)$ if $b \in \dot{B}$.

degenerate if $\rho < 1$. Thus, taking $\rho < 1$, we may confine ourselves to non-degenerate approximations. On this understanding we shall prove a similar theorem with less restrictive hypotheses. A non-degenerate map $f'(K') \subset R^n$ will be called an $| \epsilon, \alpha |$-approximation to $f(K)$ if, and only if, $|| f' - f || \leqq \epsilon$ and the angle between df and df' is at most α, for each $x \epsilon K'$ and non-zero vector dx, in any simplex of K'. Notice that this relation is symmetric between f and f_1, and if f_1 is an $| \epsilon_1, \alpha_1 |$-approximation to f and f_2 an $| \epsilon_2, \alpha_2 |$-approximation to f_1, then f_2 is an $| \epsilon_1 + \epsilon_2, \alpha_1 + \alpha_2 |$-approximation to f. By lemma 1 an (ϵ, ρ)-approximation is an $| \epsilon, \pi\rho |$-approximation, but an example of the form $y = x + \epsilon \sin \lambda x$ $(\epsilon, \lambda > 0; 0 \leqq x \leqq \pi/2\lambda)$ shows that an $| \epsilon, 0 |$-approximation need not be an (ϵ, ρ)-approximation for any given ρ. Our theorem is:

THEOREM 3_α. *To any non-singular C^1-complex $f(K) \subset R^n$ correspond positive numbers ϵ, α such that any $| \epsilon, \alpha |$-approximation to $f(K)$ is non-singular.*

First consider the special case in which $K = x_1x_0 + x_0x_2$, where x_0x_λ $(\lambda = 1, 2)$ are linear segments with no common point other than x_0, and $f(x_1x_0 + x_0x_2) = y_1y_0 + y_0y_2$ is linear throughout each of x_0x_λ. In this case any (non-degenerate) $| \infty, \theta |$-approximation $f'(K')$, to $f(K)$, is $(1 - 1)$ provided $2\theta <$ angle $y_1y_0y_2$. For let R^2 be the plane containing y_0, y_1 and y_2, or any plane through these points if they are collinear, and let $l \subset R^2$ be the external bisector of the angle $y_1y_0y_2$, or the line $y_1y_0y_2$ if these points are collinear. Then the inclination to y_0y_λ $(\lambda = 1$ or $2)$ of any direction in R^n perpendicular to l, is at least $\frac{1}{2}$ angle $y_1y_0y_2$. If $2\theta <$ angle $y_1y_0y_2$ it follows that the vector df' is never perpendicular to l, and its orthogonal projection on l points away from y_0. Therefore the orthogonal projection of $f'(K')$ on l is a non-singular image of K', whence f' is $(1 - 1)$.

Let A_1B and A_2B be simplexes in $R^m (A_\lambda \neq 1)$, let b_0 be the centroid of B, and let $\theta_0(A_1, A_2, B)$ be the minimum attained by the angle a_1b_0p for[13] $a_1 \epsilon A_1$, $p \epsilon A_2\dot{B}$, and let

$$\theta(A_1, A_2, B) = \min \{\theta_0(A_1, A_2, B), \pi/2\}.$$

It follows from a standard type of argument that $\theta_0(A_1, A_2, B)$, and hence $\theta(A_1, A_2, B)$ vary continuously with the vertices of A_1, A_2 and B, provided the simplexes A_1B and A_2B remain non-degenerate (and under less stringent conditions). Let $x_\lambda \epsilon A_\lambda B - B$ $(\lambda = 1, 2)$, let a_1 be the point in A_1 such that the simplex a_1B contains x_1, and let the line through x_1 parallel to a_1b_0 meet B in b. Then the line through b_0 parallel to bx_2 meets $A_2\dot{B}$ in p, say, and angle $x_1bx_2 =$ angle $a_1b_0p \geqq \theta(A_1, A_2, B)$. Notice that the construction for b, given x_1, is affine and so invariant under a linear transformation. We shall call b the A_1-projection of x_1 in B.

If A_1B and A_2B do not meet except in B we have $\theta = \theta(A_1, A_2, B) > 0$, and I say that $|| x_\lambda - b || \leqq || x_2 - x_1 || \operatorname{cosec} \theta$, where x_λ and b mean the same

[13] $A_2\dot{B} = A_2$ if B is a 0-simplex.

as before. For if angle $x_1bx_2 \geqq \pi/2$ we have $\| x_\lambda - b \| < \| x_2 - x_1 \| \leqq \| x_2 - x_1 \| \operatorname{cosec} \theta$, and if angle $x_1bx_2 < \pi/2$ we have

$$\| x_\lambda - b \| \leqq \| x_2 - x_1 \| \operatorname{cosec} (x_1bx_2) \leqq \| x_2 - x_1 \| \operatorname{cosec} \theta,$$

since $\| x_\lambda - b \|$ does not exceed the diameter of the circle through x_1, x_2 and b.

Now let $A_\lambda B \subset K$, $A_\lambda \neq 1$ ($\lambda = 1, 2$; $A_1 \cdot A_2 = 0$) and let

$$\theta(p) = \theta\{F_p(A_1), F_p(A_2), F_p(B)\},$$

where $p \in B$. Since f is non-singular, $\theta(p)$ is a positive, continuous function of $p \in B$ and so attains a positive minimum. Let 7α be the least of these minima, calculated for every pair of simplexes in K which have a common point though neither is contained in the other, and let c be the greatest of the numbers cosec $\theta(A_1, A_2, B)$. Let $f'(K')$ be an $| \infty, \alpha |$-approximation to $f(K)$. We first show that the map F'_p is non-singular[14] for each $p \in K$. If F'_p were singular there would be two segments px_1 and px_2 in $N(p, K')$ with the same image under F'_p. Therefore it is enough to show that $F'_p(x_1) \neq F'_p(x_2)$ if $x_1 \neq x_2$, where x_1 and x_2 are arbitrarily near p. Let $A_\lambda B$ be the simplex of K (not K') containing x_λ as an inner point, where A_1B and A_2B do not meet except in B. If one of $A_\lambda B$ contains the other (i.e. if $A_1 = 1$ or $A_2 = 1$) let b be the mid-point of the segment x_1x_2, if not let b be the A_1-projection of x_1 in B. Since $\| x_\lambda - b \| \leqq c \| x_2 - x_1 \|$, and by lemmas 3 and 1, we may suppose x_1 and x_2 to be so near p that:

1. $x_\lambda b \subset N(p, K')$ ($\lambda = 1, 2$),
2. $F_p(x_1b + bx_2)$ *is an* $| \infty, \alpha |$*-approximation to* $f(x_1b + bx_2)$,
3. $F'_p(x_1b + bx_2)$ *is an* $| \infty, \alpha |$*-approximation to* $f'(x_1b + bx_2)$.

Since f' is an $| \infty, \alpha |$-approximation to f it follows that $F'_p(x_1b + bx_2)$ is an $| \infty, 3\alpha |$-approximation to $F_p(x_1b + bx_2)$. Since the angle between the segments $F_p(bx_1)$ and $F_p(bx_2)$ is at least 7α it follows from the special case of the theorem already proved that $F'_p(x_1) \neq F'_p(x_2)$. Therefore F'_p is $(1 - 1)$.

We now show that f' is locally $(1 - 1)$. By a familiar theorem and lemmas 3 and 1, there is a $\delta > 0$ such that:

1. $x_\lambda \in N(b, K)$ ($\lambda = 1, 2$), *for some* $b \in K$, if $\| x_2 - x_1 \| \leqq \delta$,
2. F_b *is an* $| \infty, \alpha |$*-approximation to* f *throughout the sub-set of* K *given by* $\| x - b \| \leqq c\delta$ *for any* $b \in K$.

This being so, I say that $f'(x_1) \neq f'(x_2)$ if $0 < \| x_2 - x_1 \| \leqq \delta$. For let $0 < \| x_2 - x_1 \| \leqq \delta$ and let A_1, A_2, B and b mean the same as before. Since $\| x_\lambda - b \| \leqq c \| x_2 - x_1 \| \leqq c\delta$, the map $f(x_1b + bx_2)$ is an $| \infty, \alpha |$-approximation to $F_b(x_1b + bx_2)$. Therefore $f'(x_1b + bx_2)$ is an $| \infty, 2\alpha |$-approximation to $F_b(x_1b + bx_2)$, and it follows from the special case of the theorem that $f'(x_1) \neq f'(x_2)$.

Finally we show that $f'(K')$ is $(1 - 1)$ throughout K' if it is an $| \epsilon, \alpha |$-approximation to $f(K)$, for a sufficiently small $\epsilon > 0$. The sub-set of the topological

[14] If f' is assumed to be recti-linear this step is unnecessary.

product $K^2 = K \times K$, for which $\| x_2 - x_1 \| \geqq \delta$ is compact, where δ means the same as in the preceding paragraph. Therefore the continuous function $\| f(x_2) - f(x_1) \|$ attains its minimum, say 3ϵ, on this sub-set, and $\epsilon > 0$ since f is $(1 - 1)$. Therefore, if $\| x_2 - x_1 \| \geqq \delta$ and if $f'(K')$ is an $| \epsilon, \alpha |$-approximation to $f(K)$, we have

$$\| f'(x_2) - f'(x_1) \| \geqq \| f(x_2) - f(x_1) \| - \| f(x_2) - f'(x_2) \| - \| f'(x_1) - f(x_1) \| \geqq \epsilon > 0,$$

whence f' is non-singular and the proof is complete.

As a corollary to lemma 5 and theorems 1 and 3 we have:

THEOREM 4. *Given a non-singular C^1-complex $f(K) \subset R^n$, and $\epsilon, \rho > 0$, there is a subdivision K', of K, such that $L_f(K')$ is a non-singular, (ϵ, ρ)-approximation to $f(K)$.*

3. Let M^n be an n-dimensional manifold of class C^1. Without loss of generality we assume M^n to be smoothly imbedded[15] in R^m, and (ϵ, ρ)-approximations to maps in M^n will be measured in terms of the Euclidean metric for R^m. Let $f(K) \subset U \subset M^n$ be a non-singular C^1-complex, where U is the domain of an allowable coordinate system for M^n. Then theorem 4 is valid if the term linear is interpreted in terms of the coordinates for U, provided the sub-division K' is so fine that $L_f(K') \subset U$. For $f(K)$ is compact, and the metric taken from R^n by the coordinates is continuous in terms of R^m, and the parallelism taken from R^n is a first approximation to the parallelism of R^m. If $f(K) \subset M^n$ is a C^1-complex such that $f(A) \subset U(A)$ for each $A \subset cl(K - K_1)$, where $U(A)$ is the domain of an allowable coordinate system for M^n, then the proof of theorem 2 applies to approximations in M^n, taking ϵ to be so small that $f'(A') \subset U(A)$.

By a *C^1-triangulation*[16] of M^n, we shall mean a non-singular, locally finite[17] C^1-complex $f(K) = M^n$, which covers M^n. By an (n-dimensional) *unbounded, formal manifold* we shall mean a simplicial complex K, such that the complement of each vertex is combinatorially equivalent to the boundary of an n-simplex.

THEOREM 5. *If $f(K)$ is a C^1-triangulation of M^n, then K is an unbounded formal manifold.*

Let $f(K) = M^n$ be a C^1-triangulation. Then K, being a homeomorph of M^n, is n-dimensional and is a pseudomanifold[18] (i.e. each $(n - 1)$-simplex is on the boundary of precisely two n-simplexes). Therefore the complement, K_b, of any vertex, b, is a pseudo-manifold and hence a finite $(n - 1)$-cycle (mod 2). Now $F_b\{N(b, K)\} = F_b(bK_b) = f(b)F_b(K_b) \subset R^n$, where R^n is the tangent flat n-space to M^n at $f(b)$. Also F_b is an isomorphic map of bK_b, since f is non-singular.

[15] Hassler Whitney, Annals of Math., 3(1936), 645–80.

[16] It follows very easily from theorems 4 and 2, by Cairns' piecemeal construction (cf. lemma 7, below) that M^n has a C^1-triangulation. As we shall see this also follows from our theorem 6.

[17] We recall this is the only passage in §§2 and 3 in which K may be infinite.

[18] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig (1934), 125.

Therefore the radial projection of $F_b(K_b)$ from $f(b)$ in the boundary of an n-simplex $A^n \subset R^n$, of which $f(b)$ is an inner point, is a semi-linear, topological transformation, $\pi F_b(K_b) \subset \dot{A}^n$. Since K_b is a finite $(n - 1)$-cycle it follows that $\pi F_b(K_b) = \dot{A}^n$, and the theorem is established.

Two maps, $f(K)$ and $f^*(K^*)$, will be described as *equivalent* if, and only if, K^* is the image of K in an isomorphism t, such that $f = f^*t$. This is obviously an equivalence relation in the technical sense (i.e. it is symmetric and transitive) and we shall now identify any two C^1-complexes which are given by equivalent C^1-maps. Thus $f_1(K_1) = f_2(K_2)$ will mean that the maps $f_1(K_1)$ and $f_2(K_2)$ are equivalent and the complexes $f_1(K_1)$ and $f_2(K_2)$ identical. If[19] $K = K_1 + \cdots + K_q$ we shall describe a non-singular C^1-complex $f(K)$ as the *non-singular union*, $f(K_1) + \cdots + f(K_q)$, of its sub-complexes $f(K_1), \cdots, f(K_q)$. Conversely, a set of non-singular C^1-complexes $f_1(K_1), \cdots, f_q(K_q)$ in M^n, will be said to have a non-singular union, $f_1(K_1) + \cdots + f_q(K_q)$, if, and only if, there is a non-singular C^1-complex $f^*(K^*) = f_1(K_1) + \cdots + f_q(K_q)$, such that $K^* = K_1^* + \cdots + K_q^*$ and each map $f^*(K_\lambda^*)$ is equivalent to $f_\lambda(K_\lambda)$ $(\lambda = 1, \cdots, q)$. Notice that, if $K = K_1 + \cdots + K_q$ and if a given map $f(K)$ is non-singular throughout each of $K_1, \cdots, K_q$, then $f(K_1), \cdots, f(K_q)$ may have a non-singular union even if $f(K)$ is singular. But in this case $f(K_1) + \cdots + f(K_q) \neq f(K)$. The following lemma is an obvious consequence of these definitions.

Lemma 6. *If $f_1(K_1)$ and $f(K)$ are non-singular C^1-complexes with a non-singular union, and if $K = K_2 + K_3$, then $f_1(K_1)$ and $f(K_2)$ have a non-singular union and*

$$f_1(K_1) + f(K) = \{f_1(K_1) + f(K_2)\} + f(K_3).$$

Two complexes $f_1(K_1)$ and $f_2(K_2)$ in M^n will be said to *intersect in a common sub-complex* if, and only if, their intersection, as point sets, coincides with $f_1(K_{10}) = f_2(K_{20})$, where $K_{\alpha 0}$ is a sub-complex of K_α $(\alpha = 1, 2)$ and the map $f_1(K_{10})$ is equivalent to the map $f_2(K_{20})$. Let this be the case and, without altering the notation, let us star each simplex A_1, if there are any, belonging to $Cl(K_1 - K_{10})$ but not to K_{10} and such that $f_1(A_1)$ has the same vertices as a simplex $f_2(A_2) \subset f_2(K_2)$. Then, replacing K_α by an isomorphic complex,[20] if necessary, we may first separate K_1 from K_2 and then identify each simplex $A \subset K_{10}$ with $tA \subset K_{20}$, where $t(K_{10}) = K_{20}$ is an isomorphism such that $f_1 = f_2 t$ in K_{10}. The result is a complex $K^* = K_1^* + K_2^*$, where K_α^* is the image of K_α in an isomorphism t_α such that $t_1 = t_2 t$ in K_{10}. Therefore $f_1(K_1)$ and $f_2(K_2)$ are sub-complexes of $f^*(K^*)$, where $f^* = f_\alpha t_\alpha^{-1}$ in K_α^*. If each of the maps $f_\alpha(K_\alpha)$ is $(1 - 1)$ so is $f^*(K^*)$, since $f_1(K_1)$ and $f_2(K_2)$ do not meet except in $f_1(K_{10})$. Let $p_1 \in K_{10}$, $p_2 = tp_1$, $p^* = t_\alpha p_\alpha$ and let $N_\alpha = N(p_\alpha, K_\alpha)$, $N^* =$

[19] Here addition is used as in the theory of sets. K_λ and K_μ may have simplexes in common and may even coincide.

[20] e.g. a sub-complex of a k-simplex for an arbitrarily large k.

$N(p^*, K^*)$. Then, subject to the above conditions, it is clear that $f_1(K_1)$ and $f_2(K_2)$ have a non-singular union, namely $f^*(K^*)$, if, and only if,

1. *$f_1(K_1)$ and $f_2(K_2)$ are non-singular,*

2. *$f_1(N_1)$ and $f_2(N_2)$ have a non-singular union, namely $f^*(N^*)$, for each $p_1 \in K_{10}$.*

In general the tangent star at $f^*(p^*)$ to $f^*(K^*)$ may be singular for some point $p^* \in t_1K_{10}$ even if $f_1(K_1)$ and $f_2(K_2)$ are both non-singular.

Let $K = K_0 + E$ and let $f(K) \subset M^n$ be a C^1-complex such that $f(K_0)$ and $f(E)$ are non-singular. Also let $f(E) \subset U$, where U is the domain of an allowable coordinate system, which we regard as map, $x(D) = U$, of a region $D \subset R^n$ on U.

LEMMA 7. *Under these conditions, given $\epsilon, \rho > 0$ there is an (ϵ, ρ)-approximation, $f'(K'') \subset M^n$, to $f(K)$, such that $f'(K_0'')$ and $f'(E'')$ are non-singular and have a non-singular union.*

Let $H \subset K$ be the sub-complex consisting of all the simplexes in K whose images in f meet $f(E)$, let $K_1 = H \cdot K_0$ and let $K_2 = Cl(K - H)$. Then $K = H + K_2$, $H = E + K_1$, and $K_0 = K_1 + K_2$. Without altering our notation we assume, after a suitable sub-division, that $f(H) \subset U$ and also $f(A) \subset U$, where $A \subset K$ is any simplex which meets H. By theorem 4, given $\epsilon_1, \rho_1 > 0$, there is an (ϵ_1, ρ_1)-approximation, $f'(H') \subset U$, to $f(H)$, such that f' is "x-linear" throughout each simplex in H' (i.e. $f'(A) = x(B)$ for each $A \subset H'$, where $B \subset D$ is recti-linear). We partially extend f' by writing $f' = f$ throughout each simplex which does not meet H. Then, given $\epsilon_2, \rho_2 > 0$ and assuming ϵ_1 and ρ_1 to be sufficiently small, it follows from theorem 2 that the approximation $f'(H')$ has an (ϵ_2, ρ_2)-extension, $f'(K')$, throughout K. Since $f(E)$ and $f(K_0)$ are non-singular, it follows from theorem 3 that $f'(E')$ and $f'(K_0')$ are non-singular if ϵ_2 and ρ_2 are sufficiently small, which we assume to be the case. We also take ϵ_2 to be so small that $f'(K_2')$ does not meet $f'(E')$. Finally we take $\epsilon_2 \leqq \epsilon$, $\rho_2 \leqq \rho$, in which case $f'(K'')$ is an (ϵ, ρ-approximation to $f(K)$, where K'' is any sub-division of K'.

The sub-set $f'(H') \subset M^n$, besides being the image of H' in f' is the homeomorph, $x(P)$, of a polyhedron $P = F + P_1 \subset D$, where

$$F = x^{-1}f'(E'), \qquad P_1 = x^{-1}f'(K_1'),$$

$x^{-1}f'$ being an isomorphism throughout each of E' and K_1' since $K_1' \subset K_0'$ and $f'(E')$ and $f'(K_0')$ are non-singular. Let $P' = F' + P_1'$ be a triangulation of P, F' and P_1' being recti-linear sub-divisions of F and P_1 which intersect in a common sub-complex. Since the map $x^{-1}f'\ (A) \subset P$ is non-degenerate for each simplex $A \subset H'$, the triangulation P' determines a sub-division $H'' = E'' + K_1''$, of H', such that F' and P_1' are isomorphic in $x^{-1}f'$ to E'' and K_1''. Therefore $x(F') = f'(E'')$ and $x(P_1') = f'(K_1'')$. Let $K'' = H'' + K_2'' = K_0'' + E''$ be an extension of the sub-division H'' throughout K'.

Since $x(P') = x(F') + x(P_1') = f'(E'') + f'(K_1'')$, and $f'(E'') \cdot f'(K_2'') = 0$, the (non-singular) C^1-complexes $x(P')$ and $f'(K)_2''$ intersect in a common sub-complex, namely $f'(K_1'' \cdot K_2'')$. Also any simplex $A \subset P'$, such that $x(A)$ meets

$f'(K_2'')$, belongs to P_1' and not to F'. Therefore, if $p \in K_1''\cdot K_2''$ and $q = x^{-1}f'(p)$, then $N(q, P') = N(q, P_1')$. Since $x(P_1') = f'(K_1'')$ and $f'(K_0'') = f'(K_1'') + f'(K_2'')$ it follows that $x\{N(q, P')\}$ and $f'\{N(p, K_2'')\}$ have a non-singular union, namely $f'\{N(p, K_0'')\}$. Therefore $x(P')$ and $f'(K_2'')$ have a non-singular union. But

$$x(P') = f'(E'') + f'(K_1'')$$

and it follows from lemma 6 that

$$x(P') + f'(K_2'') = f'(E'') + \{f'(K_1'') + f'(K_2'')\} = f'(E'') + f'(K_0''),$$

and the lemma is established.

We now come to the main theorem.

THEOREM 6. *Given ϵ, $\rho > 0$, and non-singular C^1-complexes $f_\lambda(K_\lambda) \subset M^n$ $(\lambda = 1, \cdots, q)$, there are (ϵ, ρ)-approximations in M^n to $f_\lambda(K_\lambda)$, which have a non-singular union.*

If $q = 1$ the theorem is trivial and we shall prove it by induction on the total number of simplexes in $K_2, \cdots, K_q$, after an initial sub-division such that $f_\lambda(A)$ is in the domain of an allowable coordinate system, for each simplex $A \subset K_\lambda$ and each $\lambda = 2, \cdots, q$. Let $K_q = K_{q0} + A$, where A is a principal simplex in K_q and[21] $K_{q0} = Cl(K_q - A)$, and let U be the domain of an allowable coordinate system, which contains $f_q(A)$. By the hypothesis of the induction, given $\epsilon_1, \rho_1 > 0$ there are non-singular (ϵ_1, ρ_1)-approximations $f_\alpha'(K_\alpha'), f_q'(K_{q0}') \subset M^n$ to $f_\alpha(K_\alpha)$ and $f_q(K_{q0})$ $(\alpha = 1, \cdots, q - 1)$, such that $f_1'(K_1'), \cdots, f_q'(K_{q0}')$ have a non-singular union. By theorem 2, given $\epsilon_2, \rho_2 > 0$ and provided ϵ_1 and ρ_1 are sufficiently small, there is an (ϵ_2, ρ_2)-extension, $f_q'(K_q')$, of $f_q'(K_{q0}')$, where $K_q' = K_{q0}' + A'$ and $f_q(A') \subset U$. We take ϵ_2 and ρ_2 to be so small that $f_q'(K_q')$ is non-singular, according to theorem 3. We also take $\epsilon_1 \leqq \epsilon_2$, $\rho_1 \leqq \rho_2$, so that $f_\lambda'(K_\lambda')$ is an (ϵ_2, ρ_2)-approximation to $f_\lambda(K_\lambda)$ for each value of $\lambda = 1, \cdots, q$. Replacing $f'(K_\lambda')$ $(\lambda = 1, \cdots, q)$ by equivalent maps, if necessary, and taking care that no internal simplex of A' coincides with a simplex of[22] K_α' $(\alpha = 1, \cdots, q - 1)$ we may, without altering our notation, represent $f_q'(K_q')$ and the union $f_1'(K_1') + \cdots + f_q'(K_{q0}')$ as non-singular sub-complexes of a C^1-complex $g(K)$, where $K = K_1' + \cdots + K_q'$ and $g = f_\lambda'$ in K_λ'. Then $K = K_0 + A'$, where $K_0 = K_1' + \cdots + K_{q0}'$, and $g(K_0) = f_1'(K_1') + \cdots + f_q'(K_{q0}')$. Since $g(A') \subset U$ it follows from lemma 7 that, given $\epsilon_3, \rho_3 > 0$, there is an (ϵ_3, ρ_3)-approximation, $g'(K')$, to $g(K)$, such that $g'(K_0')$ and $g'(A'')$ have a non-singular union, where $K' = K_0' + A'' = K_1'' + \cdots + K_q''$. Then $g'(K_0')$, and hence $g'(K_\alpha'')$ $(\alpha = 1, \cdots, q - 1)$, are non-singular, and since $g'(K_q'')$ is an (ϵ_3, ρ_3)-approximation to the non-singular complex $g(K_q')$ $(= f_q'(K_q'))$ we may

[21] K_{q0} and $f_q(K_{q0})$ are empty if $K_q = A$.

[22] This may require an internal sub-division of A' if there are internal simplexes with all their vertices in the boundary. However, if the sub-division A' is given by the construction in the proof of theorem 2 there are no such simplexes.

take ϵ_3 and ρ_3 to be so small that $g'(K_q'')$ is also non-singular. This being so $g'(K_q'') = g'(K_{q0}'') + g'(A'')$, and by lemma 6 we have

$$g'(K_0') + g'(A'') = \{g'(K_1'') + \cdots + g'(K_{q0}'')\} + g'(A'')$$
$$= g'(K_1'') + \cdots + g'(K_q'').$$

Finally $g(K_\lambda')$ $(= f_\lambda'(K_\lambda'))$ is an (ϵ_2, ρ_2)-approximation to $f_\lambda(K_\lambda)$ and $g'(K_\lambda'')$ is an (ϵ_3, ρ_3)-approximation to $g'(K_\lambda')$. Therefore, taking $\epsilon_2 + \epsilon_3 \leqq \epsilon$, $\rho_2 + \rho_3 + \rho_2\rho_3 \leqq \rho$, the theorem is established.

Let V be any open sub-set of M^n and let $f_\lambda(K_{\lambda 0})$ be the sub-complex consisting of all the simplexes in $f_\lambda(K_\lambda)$ which meet $Cl(V)$. From the proof of lemma 7, and by adding to the hypotheses of the induction in theorem 6, we have the addendum:

ADDENDUM: *If $f_1(K_{10}), \cdots, f_q(K_{q0})$ have a non-singular union the approximations in theorem* 6 *may be chosen so as not to disturb the part of this union which lies in V.*

For in the proof of lemma 7 it is only necessary to sub-divide K, or to alter the map f, in those simplexes which meet H. If $f(E) \subset M^n - V$ no simplex $f(A) \subset M^n$ which meets $f(E)$ is contained in V. If $f(E) \subset M^n - Cl(V)$ we may therefore assume, after an initial sub-division which leaves A unaltered if $f(A) \subset V$, that $f(H) \subset M^n - V$. Then $A \cdot H = 0$ if $f(A) \subset V$.

We now require M^n to be closed, a restriction which we remove later.

THEOREM 7. *There is a C^1-triangulation of M^n.*

Since M^n is closed it can be covered by the interiors of a finite set of non-singular, n-dimensional C^1-simplexes $f_1(A_1), \cdots, f_q(A_q)$. By theorem 6, given $\epsilon > 0$, there are (ϵ, ∞)-approximations, $f_\lambda'(A_\lambda')$ $(\lambda = 1, \cdots, q)$, to $f_\lambda(A_\lambda)$, which have a non-singular union $f(K) = f_1'(A_1') + \cdots + f_q'(A_q')$. It follows from well known theorems[23] that, provided ϵ is sufficiently small, each point of M^n is internal to at least one of the cells $f_\lambda'(A_\lambda')$. Therefore $f(K)$ covers M^n and is a C^1-triangulation.

THEOREM 8. *If $f_1(K_1)$ and $f_2(K_2)$ are two C^1-triangulations of M^n, then K_1 and K_2 are combinatorially equivalent.*

For, by theorem 6, there are non-singular approximations $f_\lambda'(K_\lambda') \subset M^n$ to $f_\lambda(K_\lambda)$ $(\lambda = 1, 2)$, which have a non-singular union. Since K_λ is a pseudo-manifold it is a cycle (mod 2). Therefore K_λ' is a cycle (mod 2), and since the map f_λ' is topological it follows that M^n is completely covered by $f_\lambda'(K_\lambda')$. Therefore $f_1'(K_1') = f_2'(K_2')$, since $f_1'(K_1')$ and $f_2'(K_2')$ intersect in a common sub-complex, and K_1' is isomorphic to K_2'.

With suitable restrictions, similar theorems to theorems 7 and 8 may be proved for a bounded manifold $M_0^n \subset M^n$. For example, let M^n and also the frontier, M^{n-1}, of M_0^n be manifolds of class C^3, and let M^n be given a Riemannian metric $ds^2 = g_{ij}\, dx^i\, dx^j$, where the functions g_{ij} are of class C^2 in allowable coordinate systems for M^n. Then, for some $\delta > 0$, no two of the geodesic seg-

[23] See, for example, Alexandroff and Hopf (loc. cit.), pp. 100 (theorem IV) and 459 (Rouché's theorem).

ments pq, of length δ, will meet each other, where $pq \subset M_0^n$ is normal at $p \in M^{n-1}$ to M^{n-1}. If $f(K) \subset M^{n-1}$ is a C^1-triangulation of M^{n-1} the sub-set of M^n covered by the segments pq is a non-singular C^1-image of the polyhedral complex $K \times \langle 0, 1 \rangle$, which may be triangulated by a normal sub-division. It is now easy to show that, without disturbing $f(K)$, some approximation to this triangulation may be extended throughout M_0^n.

Assuming only that M^n and M^{n-1} are of class C^1, let $f_1(K_1)$ and $f_2(K_2)$ be two C^1-triangulations of M_0^n. By theorem 6, applied to the sub-complexes $f_1(K_{10})$, $f_2(K_{20})$ covering M^{n-1}, by theorem 2, and since $f_\lambda(K_{\lambda 0})$ is the point-set frontier of $f_\lambda(K_\lambda)$, we may assume that $f_1(K_{10}) = f_2(K_{20})$. By adding to the hypotheses of the induction in theorem 6 we see that, if the maps $f_\lambda(K_\lambda)$ ($\lambda = 1, \cdots, q$) are equivalent to each other throughout mutually isomorphic sub-complexes $K_{\lambda 0} \subset K_\lambda$, then the approximations $f'_\lambda(K'_\lambda) \subset M^n$, which have a non-singular union, may be chosen so that $f'_1(K'_{10}) = \cdots = f'_q(K'_{q0})$. In the case of the triangulations $f_\lambda(K_\lambda) = M_0^n$ ($\lambda = 1, 2$), with $K_{\lambda 0} = \dot{K}_\lambda$ (mod 2), it follows that $f'_1(K'_1) = f'_2(K'_2)$, whence K'_1 is isomorphic to K'_2.

4. We conclude by showing how many of these results can be extended to infinite complexes and open manifolds. An infinite complex $f(K) \subset M^n$ is to be such that only a finite number of simplexes $f(A)$ ($A \subset K$) meet any compact sub-set of M^n. A manifold $M^n \subset R^m$ is to be a closed, but not necessarily compact, sub-set of R^m. An (ϵ, ρ)-approximation to $f(K)$ shall mean the same as before, except that ϵ and ρ may now be any non-negative functions, $\epsilon(p)$ and $\rho(p)$, which are defined for each $p \in K$, and $\epsilon, \rho > 0$ is to mean that $\epsilon(p)$ and $\rho(p)$ have positive lower limits in each compact sub-set of K. It is often convenient to define such a function in terms of a particular covering of K by compact sub-sets $[F]$ (e.g. the simplexes or stellar neighbourhoods), only a finite number of which meet any one compact sub-set, and a positive function of sets, $\bar{\eta}(F)$, defined for each set in the covering. Then $\eta(p)$ may be defined as the minimum of $\bar{\eta}(F)$ for F containing p. Conversely, given $\eta(p)$, the function $\bar{\eta}(F)$ may be defined as the lower limit of $\eta(p)$ for $p \in F$. For example, in the proof of theorem 2, with K finite or infinite, we may take $\epsilon(p)$, $\rho(p)$ to be defined in terms of given functions $\bar{\epsilon}(A)$, $\bar{\rho}(A)$, where A is any simplex in K, and $\epsilon_1(p)$, $\rho_1(p)$ to be not greater than suitably chosen functions $\bar{\epsilon}_1(\dot{A})$, $\bar{\rho}_1(\dot{A})$ if $p \in \dot{A}$. The theorem then follows by induction on the dimensionality of $K - K_1$ and the same construction as in the finite case. In proving theorem 3 we may define α in terms of a function $\bar{\alpha}(N) > 0$, where N is the stellar neighbourhood of any vertex in K. If $\bar{\alpha}(N)$ is suitably chosen, any $| \infty, \alpha |$-approximation to a given non-singular C^1-map is locally non-singular and it is easily shown, as in the finite case, that an $| \epsilon, \alpha |$-approximation is non-singular for a suitable $\epsilon(p)$.

In the absence of lemma 5, which seems to be comparatively difficult if K is infinite, we replace theorem 4 by the less explicit theorem:

THEOREM 9. *Given $\epsilon, \rho > 0$, there is a non-singular, recti-linear (ϵ, ρ)-approximation to any non-singular C^1-complex $f(K) \subset R^n$.*

This may be proved in the same way as the extension of theorem 6 to infinite sets of (possibly infinite) complexes $f_\lambda(K_\lambda) \subset M^n$ ($\lambda = 1, 2, \cdots$), only a finite number of which meet any one compact sub-set of M^n. To prove this let M^n be a closed set in R^m, referred to Cartesian coordinates $y^1, \cdots, y^m$. Let V_r be the sub-set of M^n for which $\| y \| < r$ and, after a suitable sub-division, let each simplex of $f_\lambda(K_\lambda)$ which meets $Cl(V_\mu)$ lie in $V_{\mu+1}$, for each $\lambda, \mu = 1, 2, \cdots$. By theorems 6 and 2 we may assume, after a suitable (ϵ_r, ρ_r)-approximation to $f_\lambda(K_\lambda)$ ($\lambda = 1, 2, \cdots$; $\epsilon_r(p) < \epsilon(p)$, $\rho_r(p) < \rho(p)$), that the maximal sub-complexes of $f_1(K_1)$, $f_2(K_2)$, $\cdots$ whose simplexes all meet $Cl(V_r)$ have a non-singular union, for some[24] $r = 1, 2, \cdots$. If follows from theorem 6 and its addendum that, by a suitable (ϵ_r', ρ_r')-approximation to the first (ϵ_r, ρ_r)-approximation, this condition can be maintained with r replaced by $r + 1$, without disturbing the part of the union, say $g(P_r)$, which lies in V_r. The result will be an $(\epsilon_{r+1}, \rho_{r+1})$-approximation to $f_\lambda(K_\lambda)$, for each $\lambda = 1, 2, \cdots$, where $\epsilon_{r+1} = \epsilon_r + \epsilon_r'$, $\rho_{r+1} = \rho_r + \rho_r' + \rho_r\rho_r'$. Since $\epsilon_r < \epsilon$, $\rho_r < \rho$, we may choose ϵ_r', ρ_r' so that $\epsilon_{r+1} < \epsilon$, $\rho_{r+1} < \rho$ and the induction is complete. In the succeeding stages of the construction we may take $P_r \subset P_{r+1} \subset \cdots$ and the required union in $g(P)$, where $P = P_1 + P_2 \cdots$ This theorem carries with it theorems 7 and 8 for open manifolds.[25]

Finally, if M^n is a manifold of class C^k ($k = 2, \cdots, \infty$ or ω), a C^k-complex $f(K) \subset M^n$ may be defined in the same way as a C^1-complex, and we have:

THEOREM 10. *Given $\epsilon, \rho > 0$ and a non-degenerate C^1-complex $f(K) \subset M^n$, there is an (ϵ, ρ)-approximation to $f(K)$ which is of class C^k.*

If M^n be imbedded as a class C^k manifold in R^m, the flat $(m - n)$-spaces normal to M^n form a system of class C^{k-1}. It is, however, possible[26] to define a class C^k system of flat $(m - n)$-spaces approximately normal to M^n. By means of such spaces, we can project back into M^n a recti-linear (ϵ', ρ')-approximation to $f(K)$, thus obtaining a proof of Theorem 10, provided (ϵ', ρ') are chosen sufficiently small.

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[24] Notice that we only need theorem 6 for finite complexes and theorem 2 for finite $K - K_1$. We may take $\epsilon_r(p)$, $\rho_r(p)$ to be any constants less than the lower limits of $\epsilon(p)$, $\rho(p)$ for $p \epsilon K_\lambda \cdot Cl(V_{r+1})$, and $\epsilon_r(p) = \rho_r(p) = 0$ for $p \epsilon K_\lambda - K_\lambda \cdot Cl(V_{r+1})$.

[25] Here again Cairns' method leads to a more direct proof of the triangulation theorem.

[26] Hassler Whitney, loc. cit., §25.