V, and suppose that the 4-cycle corresponding to any algebraic surface on V is dependent on these cycles. We may suppose the cycles arranged so that $\Gamma_4^{(i)}$ $(i = 1, \ldots, \rho_1)$ is homologous to a cycle in the Riemannian submanifold corresponding to a prime section of V_1 and that no combination of the cycles $\Gamma_4^{(i)}$ $(i > \rho_1)$ is homologous to such a cycle. Finally we may arrange that

$$(\Gamma_4^{(i)} \Gamma_4^{(j)}) = 0$$

$(i \leqslant \rho_1;\ j > \rho_1)$. Then, if the orientation is such that the effective intersection of two surfaces corresponds to an intersection of the 4-cycles to which a positive sign is attached, *there is one positive term in the signature of the intersection matrix of the cycles* $\Gamma_4^{(i)}$ $(i = 1, \ldots, \rho_1)$ *and* ρ_2 *positive terms in the signature of the intersection matrix of the cycles*

$$\Gamma_4^{(i)} \quad (i = \rho_1 + 1, \ldots, \rho_1 + \rho_2).$$

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ON DOUBLED KNOTS

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1. In this section we say what we mean by a doubled knot and give certain linkages $c + f_\rho(t)$ $(\rho = 0, \pm 1, \pm 2, \ldots)$, in Euclidean space, the residual spaces of which are topologically equivalent, though c and $f_0(t)$ are unknotted circuits while $f_\rho(t)$ is knotted if $\rho \neq 0$. In § 2 we prove a general theorem on the group of a knot which we use to show that, with the exception of a "simply doubled" unknotted circuit, and with certain hypothetical exceptions‡, every doubled circuit is knotted. It appears that every simply doubled knot belongs to the class of knots described by H. Seifert§, for which Alexander's polynomial|| $\Delta(x)$ is 1. Thus every one of a certain (infinite) sub-class of Seifert's knots is shown to be knotted¶.

† Received 29 September, 1936; read 12 November, 1936.

‡ These exceptions only appear if Dehn's lemma is false for circuits in Euclidean space [M. Dehn, *Math. Annalen*, 69 (1910), 137–68. See also I. Johansson, *Math. Annalen*, 110 (1934), 312–20].

§ *Math. Annalen*, 110 (1934), 571–92.

|| J. W. Alexander, *Trans. American Math. Soc.*, 30 (1928), 275–306.

¶ Seifert gives a process for constructing all knots with a given $\Delta(x)$ and, using a special method, proves that a particular knot for which $\Delta(x) = 1$ is actually knotted.

We describe the process of doubling in terms of a recent paper† by M. H. A. Newman and the present author. Let T_σ, l_σ, and m_σ ($\sigma = 0, 1$) mean the same as in N.W., except that T_0 and T_1, regarded as sets of points, shall be open regions. We recall that $\bar{T}_0$ is a solid tube and T_1 is a tubular neighbourhood (here an open neighbourhood) of a self-linking‡ circuit t, in T_0. Let k be any simple circuit in a Euclidean 3-space E, let T be an open tubular neighbourhood of k, and let l and m be circuits on βT which cut in a single point, m being a meridian and l a longitudinal circuit which bounds§ in $E-T$. Let f_ρ ($\rho = 0, \pm 1, \pm 2, \ldots$) be a semi-linear‖ topological transformation of $\bar{T}_0$ into $\bar{T}$ such that

$$m = f_\rho(m_0), \quad f_\rho(l_0) \sim l + \rho m$$

on βT. Then $f_\rho(t)$ will be described as a circuit obtained by *doubling* k (see Fig. 1, where k is unknotted). In particular, we say that k is *simply doubled* to obtain $f_0(t)$.

If k is unknotted, it follows from an argument given in N.W. that $f_0(t)$ is unknotted¶. If k is unknotted and $\rho \neq 0$ the circuit $f_\rho(t)$, with a suitable convention as to sense, is represented by one or other of the diagrams in Fig. 1, according as ρ is positive or negative. In each case there are ρ complete twists, the total number of crossings being $2\rho+2$.

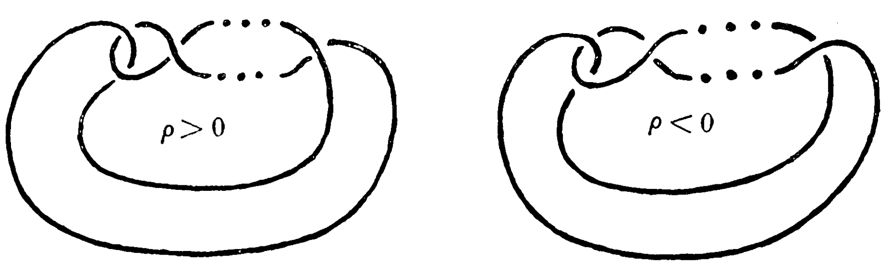

Fig. 1.

The absolute value of the determinant†† of $f_\rho(t)$ is seen to be $|4\rho+1|$ and it follows that $f_\rho(t)$ is knotted if $\rho \neq 0$.

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† *Quart. J. of Math.* (Oxford), in the press. This paper will be referred to as N.W. As in N.W. a meridian circuit on the boundary of a tube means one which bounds a 2-element inside the tube, but does not bound on the boundary, and βK stands for the boundary of a complex K. We use $\bar{R}$ to stand for the closure of a region R.

‡ *Cf.* J. H. C. Whitehead, *Quart. J. of Math.* (Oxford), 6 (1935), 268–79; *Proc. Nat. Academy of Sciences*, 21 (1935), 364–6.

§ *Cf.* Dehn, *loc. cit.*, 154.

‖ It is to be understood that everything in this note refers to semi-linear analysis situs.

¶ See the second foot-note in § 3 of N.W.

†† See K. Reidemeister, *Knotentheorie* (Berlin, 1932), Chap. II, § 4. Alternatively, it is easy to see that the group of $f_\rho(t)$ is not a free cyclic group if $\rho \neq 0$. (*Cf.* § 3, below.)

Let Q be a point in T and let f_ρ be such that $Q = f_\rho(Q_0)$ for every value of ρ, where Q_0 is a fixed point in T_0, not on t. Let E be converted into a 3-sphere S by the addition of an ideal point at infinity. If k is unknotted the region $R = S - \bar{T}$ may be regarded as an open tubular neighbourhood of an unknotted circuit c, and we further require f_ρ to be such that all the circuits $f_\rho(t)$ lie outside some open tubular neighbourhood of $\bar{R}$, say R^*, which, we may assume, does not contain Q. Then there is a semi-linear topological transformation ϕ, of $T = S - \bar{R}$ into $S - c$, which leaves fixed each point of $S - R^*$, and hence Q and every circuit $f_\rho(t)$. Let g_ρ be the resultant of f_ρ, operating on T_0, followed by ϕ, and let S be again represented as a Euclidean space E^0, by taking Q as the point at infinity. Then

$$E^0 - c - f_\rho(t) = g_\rho(T_0 - t - Q_0)$$

$$= g_\rho g_0^{-1}\{E^0 - c - f_0(t)\}.$$

Collecting these results, we have

THEOREM 1. *If k is unknotted the residual space of the linkage $c + f_0(t)$ is equivalent, in the sense of semi-linear analysis situs, to the residual space of $c + f_\rho(t)$ $(\rho = \pm 1, \pm 2, \ldots)$, though the former consists of two unknotted circuits while $f_\rho(t)$ is knotted if $\rho \neq 0$.*

2. Using the same notation as in § 1, let the point in which l cuts m be taken as the base point of the fundamental group of $E - k$ and let A be the element corresponding to the circuit l, arbitrarily oriented.

LEMMA†. *If $A^r = 1$ $(r \neq 0)$, then $A = 1$.*

If $A^r = 1$ we have $A^{-r} = 1$, and we may therefore suppose that $r > 0$. If $r = 1$ there is nothing to prove; we prove the lemma by induction on r.

Let βT be cut along l and m to form a rectangle R, of which μ and μ' are the sides corresponding to m. Let $p_1, \ldots, p_r$ $(r > 1)$ be a series of points on m, let π_i and π_i' be the images of p_i on μ and μ' respectively, and let $\pi_1, \ldots, \pi_r$ lie in this order on μ. If $A^r = 1$, the circuit on βT corresponding to the rectilinear segments $\pi_i \pi_{i+1}'$ $(i = 1, \ldots, r-1)$, $\pi_r \pi_1'$ bounds a 2-cell e_2, in $E - T$. Let the singularities on e_2 be normalized in the way described by

† *Cf.* H. Kneser, *Jahresb. d. Deut. Math.-Verein.*, 38 (1929), 248–60, §§ 1 and 2. We give a separate proof, since the first paragraph on p. 251 of Kneser's paper does not seem to be conclusive.

Dehn†, and first suppose that there are no branch points. Let x_i be the double point on βe_2 corresponding to the point ξ_i, in which $\pi_i \pi'_{i+1}$ cuts $\pi_r \pi_1{}'$, and let $x_1 y$ be the double edge of e_2 which is incident with x_1. After a subdivision, if necessary, we suppose that y does not lie on βe_2, and that $x_1 y$ does not contain a triple point. We transform e_2 into a singular surface c_2, by cutting‡ along the edge $x_1 y$ in such a way as to disconnect βe_2. The segment $\pi_1 \xi_1$ in R will then be joined to $\xi_1 \pi'_1$, the segment $\pi_r \xi_1$ to $\xi_1 \pi_2{}'$, and one of the two circuits in βc_2 will be represented by the segment $\pi_1 \xi_1 \pi_1{}'$ and the other by the segments $\pi_\lambda \pi'_{\lambda+1}$ ($\lambda = 2, \ldots, r-1$, if $r > 2$), $\pi_r \xi_1 \pi_2{}'$. Let E_2 be the non-singular image, or *pattern* of e_2, and let $X^\alpha Y^\alpha$ ($\alpha = 1, 2$) be the two images of the edge $x_1 y$. To construct a pattern for c_2 we cut $X^\alpha Y^\alpha$ into $X^{\alpha 1} Y^\alpha$ and $X^{\alpha 2} Y^\alpha$ and identify the segment $X^{11} Y^1 X^{12}$ with $X^{21} Y^2 X^{22}$ in such a way as to form two circuits out of βE_2. Thus C_2, the pattern of c_2, is a cylinder, rather than a Möbius band, and c_2 is a singular cylinder. If $r > 2$ notice that both the images of x_σ ($\sigma = 2, \ldots, r-1$) lie on the same circuit in βC_2.

The vertex y is a branch on c_2, and is the only one, since there was no branch point on e_2. Therefore the double segment d, beginning at y, terminates at one of the points $x_2, \ldots, x_{r-1}$ and $r > 2$. If d cuts itself we prolong the cut until we are left with a double segment which does not cut itself and joins the branch point to x_σ, say§. We now complete the cut in such a way as to disconnect the circuit in βc_2 which contains x_σ, transforming it into s and s'. Since the two images of x_σ lie on the same circuit of βC_2 it follows that the final cut severs C_2 into another cylinder and a 2-element. Therefore c_2 is severed into another cylinder and a 2-cell $e_2{}^*$.

The segment $\pi_\sigma \xi_\sigma$ in R will finally be joined to the segment $\xi_\sigma \xi_1 \pi_2{}'$, the segment $\pi_r \xi_\sigma$ to $\xi_\sigma \pi'_{\sigma+1}$, and one of the circuits s and s' will be represented in R by the segments $\pi_\lambda \pi'_{\lambda+1}$ ($\lambda = 2, \ldots, \sigma-1$, if $\sigma > 2$) and $\pi_\sigma \xi_\sigma \xi_1 \pi_2{}'$, and the other by $\pi_j \pi'_{j+1}$ ($j = \sigma+1, \ldots, r-1$, if $\sigma < r-1$) and $\pi_r \xi_\sigma \pi'_{\sigma+1}$. Therefore one of s and s' is homotopic on βT to $(\sigma-1)\, l$ and the other to $(r-\sigma)\, l$. But $\beta e_2{}^* = s$ or s'. Therefore $A^m = 1$, where $m = \sigma - 1$ or $r - \sigma$, and, since $1 < \sigma < r$, it follows from induction on r that $A = 1$.

If e_2 contains n branch points we cut along a segment beginning at any

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† *Loc. cit.*, 147–8. After this normalization, brought about by slight deformation, the singularities consist of double lines, along which two sheets cross, triple points, at which three sheets cut, and, in general, branch points. A point y is called a branch point if e_2 cuts a small sphere with y as its centre in a single singular circuit.

‡ *Cf.* M. Dehn, *loc. cit.*, 149 (our "cut" is the same as Dehn's "Umschaltung"). See also E. Pannwitz, *Math. Annalen*, 108 (1933), 629–72, § 3.

§ See E. Pannwitz, *loc. cit.* The connectivity of C_2 is obviously irrelevant to this argument.

branch point and complete the cut in such a way as to disconnect e_2. Let the final cut be along a double segment d. If d joins two branch points the lemma follows from a second induction on n. If d joins a branch point to one of the points $x_1, \ldots, x_{r-1}$ the lemma follows from induction on r and an argument similar to one used above. Thus the proof is complete.

By an *ordinary circuit* we shall mean a simple circuit which is either unknotted or is such that† $A \neq 1$, where A means the same as in the lemma. If A means the same as in the lemma and if b is the element in the group corresponding to the circuit m, arbitrarily oriented, we have

THEOREM 2. *If k is an ordinary knotted circuit, the sub-group of its group generated by A and b is a free Abelian group, freely generated by A and b.*

The sub-group generated by A and b is Abelian, since l and m lie on a torus in $E-k$, and we have to show that

$$A^p b^q = 1 \tag{2.1}$$

implies $p = q = 0$. The Abelian group associated with (Abelsch gemachte Gruppe) the group of k is the free cyclic group generated by b. Also $A = 1$ if multiplication is commutative, since $l \sim 0$ in $E-k$. Therefore (2.1) implies $q = 0$ at least. Therefore (2.1) implies $A^p = 1$, and if $p \neq 0$ it follows from the lemma that $A = 1$, contrary to hypothesis. Therefore $p = 0$ and the theorem is established.

THEOREM 3. *Except in the case of an unknotted circuit, simply doubled, every circuit $f_\rho(t)$, obtained by doubling an ordinary circuit k, is knotted. Moreover $f_\rho(t)$ is an ordinary knot and its group contains a sub-group which is (simply) isomorphic to the group of k.*

Using the same notation as in § 1, let G_1 be the fundamental group of the region $\overline{T}_0 - T_1$, and let the intersection of l_0 and m_0 be taken as the base point for G_1. According to N.W., § 2, G_1 is generated by a_0 and b_1, subject to the relation

$$[a_0, b_0] = 1, \tag{2.2}$$

where $[x, y] = xyx^{-1}y^{-1}$ and $b_0 = [a_0^{-1}, b_1]\,[a_0^{-1}, b_1^{-1}]$, the elements a_0 and b_0 correspond to the circuits l_0 and m_0 respectively, and b_1 corresponds to a certain circuit which links t.

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† The Dehn lemma, for Euclidean space, can be stated in the form "every simple circuit is an ordinary circuit".

First let k be unknotted. In this case we have seen that $f_\rho(t)$ is knotted unless $\rho = 0$. Also the group of k is a free cyclic group and the group of any knot contains a free cyclic group as a sub-group. Therefore the group of $f_\rho(t)$ contains a sub-group which is isomorphic to the group of k.

It remains to show that $f_\rho(t)$ is an ordinary knot when $\rho \neq 0$. Since the circuit l bounds a 2-cell in $E-k$ and the group of k is the free cyclic group generated by b, the group Γ, of $f_\rho(t)$, is generated by a_0 and b_1, subject to (2.2) and the additional relation†

$$(2.3) \qquad a_0 b_0^{-\rho} = 1.$$

This relation implies (2.2). Therefore we have to show that $a_1 \neq 1$ in consequence of (2.3), where‡

$$a_1 = b_1^{-1} a_0 b_1 a_0^{-1} b_1^{-1} a_0^{-1} b_1 a_0$$

$$= [b_1^{-1} a_0 b_1, \ a_0^{-1}].$$

The element a_1 belongs to the self-conjugate sub-group of Γ which is generated by $b_1{}^n a_0 b_1^{-n}$ $(n = 0, \pm 1, \pm 2, \ldots)$. The latter is isomorphic to the group Γ^*, generated by $a_{0/n}$, subject to the relations§

$$(2.4) \qquad a_{0/n} b_{0/n}^{-\rho} = 1,$$

where $b_{0/n} = a_{0/n}^{-1} a_{0/n+1} a_{0/n}^{-1} a_{0/n-1}$, and $a_{0/n}$ and $b_{0/n}$ are the elements in Γ^* corresponding to the elements $b_1{}^n a_0 b_1^{-n}$ and $b_1{}^n b_0 b_1^{-n}$ in Γ. Thus we have to show that $a_{1/n} \neq 1$ in Γ^*, where $a_{1/n}$, given by

$$a_{1/n} = [a_{0/n-1}, \ a_{0/n}^{-1}],$$

is the element corresponding to $b_1{}^n a_1 b_1^{-n}$.

Let Γ_n be the group generated by $a_{0/n-1}$, $a_{0/n}$, and $a_{0/n+1}$ subject to the relation (2.4), with a given value of n. Then it follows from the Dehn-Magnus‖ "Freiheitsatz" that $a_{0/n-1}$ and $a_{0/n}$ are free generators of a free sub-group of Γ_{n-1} and also of Γ_n. Therefore Γ^* is the infinite free product¶

$$\ldots \circ \Gamma_{-1} \circ \Gamma_0 \circ \Gamma_1 \circ \Gamma_2 \circ \ldots,$$

† H. Siefert and W. Threlfall, *Lehrbuch der Topologie* (Leipzig, 1934), § 52.

‡ N.W., § 2.

§ W. Magnus, *Math. Annalen*, 105 (1931), 52–74, Theorem 4, and 106 (1932), 295–307.

‖ W. Magnus, *Journal für Math.*, 163 (1930), 141–65.

¶ Here we use the term "free product" in the sense explained in a foot-note at the end of § 3 in N.W.

and $a_{0/n-1}$ and $a_{0/n}$ are free generators of a free sub-group of Γ^*. Therefore $a_{1/n} \neq 1$ in Γ^*, and $f_\rho(t)$ is an ordinary knot.

In general, k being knotted, if G is the group of k it follows from Theorem 2 above, and from N.W., Theorem 2, that the group of $f_\rho(t)$ is the free product $G \odot G_1$, with the identification of sub-groups determined by

$$a_0 = Ab^\rho, \quad b_0 = b,$$

the sub-group (A, b) of G coinciding with the sub-group (a_0, b_0) of G_1. Therefore $b_0 \neq 1$ in the group of $f_\rho(t)$, since $b_0 \neq 1$ in G_1. Therefore $f_\rho(t)$ is knotted, since its group would otherwise be Abelian and b_0 would reduce to 1. Also $a_1 \neq 1$ in G_1, by N.W., Theorem 2. Therefore $a_1 \neq 1$ in the group of $f_\rho(t)$ and $f_\rho(t)$ is an ordinary knot. Finally, the group of $f_\rho(t)$ contains a sub-group which is isomorphic to G, since it is the free product $G \odot G_1$ with identification of sub-groups, and the theorem is established.

3. It is easy to verify that a doubled knot $f_\rho(t)$ may be represented by a diagram† of the form indicated in Fig. 2, where $a_2 = k$ is the original

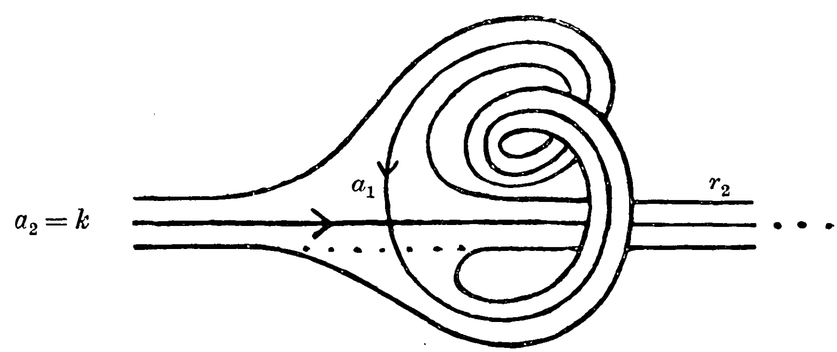

Fig. 2.

knot and the ribbon r_2, bordering a_2, is twisted‡ through a suitable multiple of 2π. The knot $f_\rho(t)$ is the boundary of the "punctured torus" bordering the circuits a_1 and a_2, and the lower edge of the ribbon r_2, together with the dotted segment in the neighbourhood of the intersection $a_1 . a_2$, is isotopic to $f_\rho(l_0)$ in $E - k$. Since $f_\rho(l_0) \sim l + \rho m$, and since l does not link k, we have

$$L\{f_\rho(l_0), k\} = \rho,$$

† *Cf.* H. Seifert, *Math. Annalen*, 110 (1934), 571–92. We refer to Seifert's paper as H.S.

‡ The twists may be replaced by loops, as indicated in H.S., fig. 7.

where $L(x, y)$ stands for the looping coefficient of circuits x and y and the orientations are such that $L(m, k) = 1$. By Seifert's method of calculation, it follows that $v_{22} = \rho$, where v_{ij} $(i, j = 1, 2)$ mean the same as in H.S. (p. 585). Also $v_{11} = -1$, $v_{12} = v_{21}+1 = 0$, and from H.S., pp. 286–7, we have

THEOREM 4. *The polynomial of a doubled knot $f_\rho(t)$ does not depend on the original knot k, but only on ρ. It is given by*

$$\left\{\begin{array}{lll} \pm\Delta(x) = \rho-(2\rho+1)x+\rho x^2 & \text{if} & \rho \neq 0, \\ \Delta(x) = 1 & \text{if} & \rho = 0. \end{array}\right. \tag{3.1}$$

We conclude with an alternative method of calculating $\Delta(x)$. Let Γ be the group of any knot, Γ' and Γ'' its first and second commutator groups†, and let Γ_c' be the commutative factor group Γ'/Γ''. We shall calculate Γ_c' for a doubled knot $f_\rho(t)$.

Let the group G, of a given knot k, be generated by $b, u, \ldots, v$, subject to relations

$$R_\lambda(b, u, \ldots, v) = 1, \tag{3.2}$$

where b is the same element as in § 2 and $u, \ldots, v$ belong to the commutator group‡ of G. Let

$$A = \psi(b, u, \ldots, v),$$

where A means the same as in § 2. If we write s instead of b_1, the group Γ, of $f_\rho(t)$, is generated by $b, u, \ldots, v, s$, and a_0, subject to the relations (3.2), (2.2), with b_1 replaced by s, and

$$a_0 = Ab^\rho, \quad b_0 = b.$$

If we eliminate the generator a_0 by means of the relation $a_0 = Ab^\rho$ and replace b_0 by b, the relation (2.2) is replaced by $[A\, b^\rho, b] = 1$, which is a consequence of (3.2) since $[A, b] = 1$ in G. Therefore Γ is generated by

† Γ' consists of the elements whose representative circuits bound in the residual space, *i.e.* do not link the knot. If Γ is indexed as in § 10 of Alexander's paper (*loc. cit.*), it follows that $\Gamma^* = \Gamma'$, where the asterisk means the same as in Alexander, § 9.

‡ We may suppose $u, \ldots, v$ to be of the form cb^{-1}, where c is the element determined by a circuit which links k once in the same sense as m.

$b, u, \ldots, v$, and s, subject to (3.2) and the relation

$$(3.3) \qquad \begin{aligned} b &= b_0 \\ &= [a_0^{-1}, s][a_0^{-1}, s^{-1}] \\ &= b^{-\rho} A^{-1} s A b^{\rho} s^{-1} b^{-\rho} A^{-1} s^{-1} A b^{\rho} s. \end{aligned}$$

If we make multiplication commutative (3.3) gives $b = 1$ and from (3.2) we have $u = \ldots = v = 1$, since $u, \ldots, v$ belong to the commutator group of G. Therefore Γ' is generated by $b, u, \ldots, v$ and their transforms by s. Writing

$$b_n = s^n b s^{-n}, \quad u_n = s^n u s^{-n}, \ \ldots, \ v_n = s^n v s^{-n},$$

$$\begin{aligned} A_n &= s^n A s^{-n} \\ &= \psi(b_n, u_n, \ldots, v_n), \end{aligned}$$

we see that Γ' is generated by $b_n, u_n, \ldots, v_n$ $(n = 0, \pm 1, \pm 2, \ldots)$, subject to the relations

$$(3.4) \qquad \begin{cases} (a) & R_\lambda(b_n, u_n, \ldots, v_n) = 1, \\ (b) & b_n = b_n^{-\rho} A_n^{-1} A_{n+1} b_{n+1}^{\rho} b_n^{-\rho} A_n^{-1} A_{n-1} b_{n-1}^{\rho}. \end{cases}$$

If multiplication is made commutative in G, we have $u = \ldots = v = A = 1$ in consequence of (3.2). If multiplication is made commutative in Γ', it follows that $u_n = \ldots = v_n = A_n = 1$ in consequence of (3.4a), and that (3.4b) is equivalent to

$$(3.5) \qquad b_n^{2\rho+1} = b_{n+1}^{\rho} b_{n-1}^{\rho}.$$

Thus Γ_c' is generated by b_n $(n = 0, \pm 1, \pm 2, \ldots)$ with commutative multiplication, subject to the relations (3.5).

If we replace commutative multiplication by commutative addition, and write $x^n b$ for b_n, (3.5) becomes

$$x^{n-1}\{\rho - (2\rho+1)x + \rho x^2\} b = 0,$$

in agreement with (3.1).

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