SIMPLICIAL SPACES, NUCLEI AND m-GROUPS

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1. *Introductory.*

This paper centres round a generalization of the notion of a group which may be briefly described as follows. We start with the definition of a symbolic complex K, as a set of sets, called simplexes, which contains each sub-set of any simplex in K†. Restricting ourselves to complexes in which the dimensionality of each simplex is finite, though the dimensionalities of the simplexes need not have an upper bound, we separate complexes into mutually exclusive equivalence classes by means of certain elementary transformations of "order m" and associate an abstract "m-group" with each class, where $m = -1, 0, 1, \ldots$. The interest begins with $m = 2$, and it appears that two connected complexes, finite or infinite, have the same 2-group if, and only if, they have the same fundamental group. Moreover, any group is isomorphic to the fundamental group of some complex. If an abstract group is taken to be an object associated with a class of mutually isomorphic groups‡, we may therefore identify 2-groups with abstract groups. That is to say, a geometric, or set-theoretic, representation of an abstract group by a complex may be regarded as equivalent to an algebraic representation by a set of elements with a multiplication table. From this point of view an m-group is seen to be an automatic generalization of an abstract group.

Two complexes have the same m-group for each value of m if they have the same homotopy type. In particular the m-group of a geometrical complex is a topological invariant. The converse applies to complexes of bounded dimensionality. In fact, two complexes of at most n dimensions are of the same homotopy type if they have the same $(n+1)$-group.

† *Cf.* P. Alexandroff und H. Hopf, *Topologie*, 1 (Berlin, 1935), p. 155.

‡ This is how one normally uses the word in referring, for example, to the 6-group.

The simplicial spaces and nuclei, referred to in the title, are also defined in terms of elementary transformations. There is nothing new about the former. Two symbolic complexes determine the same simplicial space if, and only if, they are combinatorially equivalent in the sense of J. W. Alexander† and M. H. A. Newman‡. Nuclei are defined in terms of what we call formal deformations. An elementary sub-division is a formal deformation, and the nucleus of a complex is therefore a combinatorial invariant, meaning that two complexes which are combinatorially equivalent have the same nucleus. Whether or no the nucleus is a topological invariant remains an open question, except in the case of a finite complex whose fundamental group satisfies a certain condition, described in § 11. Two finite complexes whose fundamental groups satisfy this condition have the same nucleus if they are of the same homotopy type.

In a section on manifolds it is proved that any complex K, imbedded in a manifold M, has a "regular neighbourhood" in M and that any two regular neighbourhoods of the same complex in the same manifold are combinatorially equivalent. Moreover, if p is large enough, regular neighbourhoods of two finite complexes in Euclidean p-space R^p are combinatorially equivalent if the complexes have the same nucleus. In particular, a regular neighbourhood of a finite n-dimensional complex K^n, in R^p ($p \geqslant 2n+5$), is a p-element if the (multiplicative) fundamental group of K^n is unity and all its (additive) homology groups are zero. If the fundamental group and homology groups of K^n are the same as those of an n-sphere its regular neighbourhood in R^p is the topological product of an n-sphere and a $(p-n)$-element.

The presentation may be summarized as follows. With one or two exceptions everything in §§ 3–9 is needed for the proof of Theorem 17, which states that two (finite) complexes are of the same homotopy type if, and only if, they have the same m-group for each value of m. The main exception is Theorem 12, stating that two connected complexes have the same fundamental group if, and only if, they have the same 2-group. Sections 10 and 11 lead up to Theorem 21, that, subject to the condition on the fundamental group stated in § 11, two finite complexes of the same homotopy type have the same nucleus. Section 12 is concerned with regular neighbourhoods of complexes in manifolds, and § 13 is an appendix to the sections on finite complexes. Section 14 is concerned with the combinatorial, and § 15 with the topological theory of infinite complexes.

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† *Annals of Math.*, 31 (1930), 292–320.

‡ *Akad. Wet. Amsterdam*, 29 (1926), 611–626; 627–641.

In the final § 16 it is shown how many of the earlier results may be extended from finite to infinite complexes.

Nuclei and m-groups are closely related to the homotopy groups discovered by W. Hurewicz†, and §§ 8 and 10 below may overlap with the more complete account which was announced in the first of his notes on homotopy groups. In particular, I learn from S. Eilenberg that Theorem 15, and from Shaun Wylie that the group of automorphisms $\psi_n(g)$ in § 11 are known to Hurewicz and others. But, since these are auxiliary to the main theorems, I have given full details without further reference except to what has already been published.

2. *Nuclei and m-groups.*

We start with an infinite aggregate of undefined vertices (*Eckpunktbereich*) $a, b, c, \ldots$. Any set of $n+1$ vertices ($n \geqslant -1$) will be called an *open* (*symbolic*) *n-simplex* ‡. A *closed* (*symbolic*) *n-simplex* will be the closure of an open n-simplex, consisting of a set of $n+1$ vertices together with all its subsets, including the empty set or (-1)-simplex. The closure $Cl(\Sigma)$, of a set of simplexes Σ, will consist of the closures of the simplexes in Σ, and Σ will be described as *closed* if $\Sigma = Cl(\Sigma)$. By a *symbolic complex* we shall mean any closed set of simplexes. As usual a complex will be described as finite or infinite according as it contains a finite or an infinite number of simplexes. Until the end of § 13 it is to be understood that, except where the contrary is stated, all the complexes referred to are finite. Our formalism is similar to that of J. W. Alexander except that, instead of his "mod 2" or nilpotent algebra, we use the idempotent algebra of logic. Thus $\Sigma_1+\Sigma_2$, $\Sigma_1-\Sigma_2$ and $\Sigma_1 . \Sigma_2$ will denote respectively the set of simplexes in either of two given sets Σ_1 and Σ_2, the set in Σ_1 but not in Σ_2, and the set common to both. We shall continue to use multiplication without the dot $\Sigma_1 \Sigma_2$ to stand for the join of the two sets; that is to say, for the totality of simplexes $(a_0, \ldots, a_n, b_0, \ldots, b_m)$, where§ $(a_0, \ldots, a_n) \epsilon \Sigma_1$ and $(b_0, \ldots, b_m) \epsilon \Sigma_2$. This operation is also to be idempotent, so that

$$\Sigma+\Sigma = \Sigma . \Sigma = \Sigma\Sigma = \Sigma.$$

† *Akad. Wet. Amsterdam*, 38 (1935), 112–119; 521–528; 39 (1936), 117–125; 215–223.

‡ The word "symbolic" will be omitted except when a contrast with geometrical simplexes and complexes is necessary. Also when it is obvious which is meant, or irrelevant, we shall refer to either an open or a closed simplex simply as a simplex, and shall use the same kind of letter, namely, A, B or C, to stand for both.

§ If A is an open simplex belonging to a set Σ we write $A \epsilon \Sigma$, but for a closed simplex we write $A \subset \Sigma$. We use $\bar{\epsilon}$ to stand for "not ϵ."

The empty, or (-1)-dimensional, simplex plays a part analogous to zero with respect to the multiplication $\Sigma_1 . \Sigma_2$ and to unity with respect to $\Sigma_1 \Sigma_2$. Following Alexander we shall denote it by 1. Thus†

$$\Sigma . 1 = 1, \quad \Sigma 1 = \Sigma.$$

Though the simplex 1 belongs to every complex, we shall say that two sets of simplexes meet each other if, and only if, they have a k-simplex in common, where $k \geqslant 0$.

The boundaries which appear in this paper are calculated with residue classes mod 2 as coefficients, and we follow Alexandroff and Hopf in using‡ $\dot{K}$ to stand for the boundary of a complex K. We adopt Alexander's convention that the boundary of a 0-simplex is 1 and the boundary of 1 is 0, the empty set of simplexes. The latter satisfies the conditions

$$\Sigma \pm 0 = \Sigma, \quad \Sigma . 0 = \Sigma 0 = 0,$$

where Σ is any set of simplexes, and is therefore analogous to zero in its relation to all four operations.

We now associate three kinds of abstract object, a *simplicial space*, a *nucleus*, and an *m-group* $(m = -1, 0, 1, \ldots)$ with every complex. By analogy with polyhedra we shall describe a complex as a *triangulation* of the corresponding simplicial space, but we shall refer to the nucleus and the m-group, like the fundamental group and other classical invariants, as properties of the complex. Thus, under appropriate conditions, we shall say that two complexes have the same nucleus or m-group.

1. *Simplicial spaces.* Two complexes will be described as triangulations of the same simplicial space if, and only if, they are combinatorially equivalent. Here we adopt Alexander's definition of combinatorial equivalence in terms of elementary sub-divisions. We recall that an elementary sub-division§ (A^k, a), of order k, is a transformation of the form

$$K = A^k P + Q \rightarrow a \dot{A}^k P + Q,$$

where A^k is a closed k-simplex in K, a is not in K, and $P = K_{A^k}$, the

† In general, $\Sigma + 1 \neq \Sigma$, but $K + 1 = K$ if K is a complex.

‡ We shall also use $\dot{A}$ to stand for the boundary of an open simplex A (*i.e.* for the boundary of its closure).

§ Superscripts will invariably denote dimensionality.

complement† of A^k in K. Two (finite) complexes are said to be combinatorially equivalent if, and only if, one is transformable into the other by a finite sequence of elementary sub-divisions and their inverses. A property which is unaltered by an elementary sub-division or its inverse is called a combinatorial invariant.

2. *Nuclei.* If

$$K_1 = K_0 + aA,$$

where A is a closed simplex such that $a\dot{A} \subset K_0$, $A \not\subset K_0$, the transformation $K_0 \to K_1$ will be called an *elementary expansion*, and $K_1 \to K_0$ will be called an *elementary contraction*‡. As a matter of convention we admit the identical transformation $K_0 \to K_0$ both as an elementary expansion and as an elementary contraction§. An elementary expansion or contraction will be called an *elementary deformation*, and the resultant of a finite sequence of elementary deformations a *formal deformation*. We shall denote a formal deformation by the letter D. Two (finite) complexes will be said to have the same *nucleus* if, and only if, one is transformable into the other by a formal deformation.

If the simplex aA is m-dimensional, we shall describe

$$K_0 \to K_1 = K_0 + aA,$$

or $K_1 \to K_0$, as an elementary expansion, or contraction, of *order m*.

3. *m-groups.* If

$$K_1 = K_0 + A^k \quad (k \geqslant 0),$$

where A^k is a k-simplex such that $\dot{A}^k \subset K_0$, $A^k \not\subset K_0$, we shall describe the transformation $K_0 \to K_1$ as a *filling of order k* and $K_1 \to K_0$ as a *perforation*

† *Cf.* Alexander, *loc. cit.* We shall always use K_A to stand for the complement of a (closed or open) simplex A in K, and, when we write K in the form $AP+Q$, it is to be understood that A is closed and $P = K_A$. As in Alexander's paper, vertices will always be denoted by small Roman letters.

‡ Transformations of this kind have been previously studied by I. Johansson, *Avhand. Norske Vidensk.-Akad.* (1932), No. 1.

§ The cases $K_0 = K_1 = 0$ or 1 are possible. The identity $K_0 \to K_0$ is the only elementary transformation of any kind which is applicable to the empty complex. The only elementary transformation which is applicable to 1 is a filling of order -1, defined below, which transforms 1 into a closed 0-simplex. It is to be assumed throughout that no " given complex " is the complex 0, which will appear only, in special cases, as a term in a calculation.

of order k. Two complexes will be said to have the same m-group if, and only if, one is transformable into the other by a finite sequence of elementary deformations and, possibly, fillings and perforations whose orders exceed m. Thus two complexes with the same nucleus have the same m-group for each value of m, and if $n > m$ two complexes with the same n-group have the same m-group.

3. *Formal deformations.*

If a complex L can be transformed into K by a sequence of elementary expansions we shall say that L *expands* into K and that K *contracts* into L. If L is a single vertex, we shall describe K as *collapsible*. The open simplexes aA and A can be removed by an elementary contraction of K if, and only if, $aA \epsilon K$ and aA is the only simplex in K having A on its boundary. If, in a contraction of K_0, an elementary contraction $K_i \rightarrow K_{i+1} = K_i - aA^p - A^p$ is immediately followed by $K_{i+1} \rightarrow K_{i+1} - bB^q - B^q$, where $q > p$, it follows that these two elementary contractions are interchangeable. For $bB^q \epsilon K_{i+1} \subset K_i$, and $B^q \bar{\epsilon} (aA^p)$ since $q > p$. Therefore bB^q is the only simplex in K_i with B^q on its boundary, and bB^q and B^q may be removed first and then A^p and aA^p. When we repeat this argument, it follows that the elementary contractions in a given contraction of K_0 may be so arranged that all those of order q precede those of order p if $p < q$. In particular, if K_0 is collapsible it follows that K_0 contracts first into a linear graph containing all the vertices† of K_0 and then into a vertex. The graph, being collapsible, is obviously a tree and therefore contracts into a given one of its vertices. Thus any collapsible complex contracts into a given one of its vertices.

LEMMA 1. *If $K . L_0 \subset L_q$ and if L_0 contracts into L_q, then $K + L_0$ contracts into $K + L_q$.*

Let the transformation $L_0 \rightarrow L_q$ be the resultant of elementary contractions $L_i \rightarrow L_{i+1} = L_i - a_i A_i - A_i$ $(i = 0, \ldots, q-1)$, where A_i and $a_i A_i$ are open simplexes. Since $K . L_0 \subset L_q \subset L_{i+1}$, the simplex A_i is not on the boundary of any simplex in K and it follows that the transformation $K + L_i \rightarrow K + L_{i+1}$ is an elementary contraction of $K + L_i$. Therefore the transformation $K + L_0 \rightarrow K + L_q$ is a contraction and the lemma is established.

† Strictly speaking, we should refer to the 0-simplexes, rather than to the vertices of a complex. But we shall usually refer to a 0-simplex as a vertex, and a vertex may mean either an open or a closed 0-simplex according to the context [if a is an open 0-simplex $Cl(a) = a + 1$]. The distinction is not a trivial one, since the join aK does or does not contain K according as the 0-simplex a is closed or open.

LEMMA 2. *If $L \subset K$ and A is any closed simplex which does not meet K ($A \neq 1$, possibly $L = 1$), the complex AK contracts into AL.*

First let A be a single vertex a. If $K = L$ there is nothing to prove. Otherwise let B be a principal† open simplex of $K-L$. Then the closure of aB meets $a(K-B)$ in $a\dot{B}$. Therefore aK contracts into $a(K-B)$ and the special case of the lemma follows from induction on the number of simplexes in $K-L$. In general, let $A = aA_1$. Then it follows from what we have already proved that $AK (= aA_1 K)$ contracts into $aA_1 L (= AL)$ and the lemma is established.

COROLLARY. *AK is collapsible ($A \neq 1$).*

For if $A = aA_1$ (possibly $A_1 = 1$) it follows from the lemma, with $L = 1$ and K replaced by $A_1 K$, that $aA_1 K$ contracts into a.

LEMMA 3. *If $K = AL_0 + Q$ ($L_0 \neq 1$) and L_0 contracts into L, then K contracts into $AL + \dot{A}L_0 + Q$.*

If $L_0 = L$, there is nothing to prove. Otherwise let

$$L_0 = L_1 + bB \quad (b\dot{B} \subset L_1),$$

where bB is a closed simplex and $L_0 \to L_1$ is the first step in some process of contracting L_0 into L. Then the simplex AbB meets $AL_1 + \dot{A}L_0 + Q$ in

$$Ab\dot{B} + \dot{A}bB = b(AB)^{\cdot},$$

and

$$\begin{aligned} AL_0 + Q &= AL_0 + \dot{A}L_0 + Q \\ &= A(L_1 + bB) + \dot{A}L_0 + Q \\ &= AL_1 + \dot{A}L_0 + Q + AbB. \end{aligned}$$

Therefore the transformation

$$AL_0 + Q \to AL_1 + \dot{A}L_0 + Q$$

is an elementary contraction and the lemma follows from induction on the number of simplexes in $L_0 - L$.

COROLLARY. *If L_0 is collapsible $AL_0 + Q$ contracts into $\dot{A}L_0 + Q$.*

For $AL_0 + Q$ contracts into $Ab + \dot{A}L_0 + Q$, where b is any vertex in L_0, and so into $\dot{A}L_0 + Q$ by Lemma 2 and Lemma 1.

† *I.e.* B is not on the boundary of any other simplex in K, though it need not be a simplex of maximum dimensionality.

Notice two special cases of the corollary: first, if L_0 is collapsible, aL_0+Q contracts into L_0+Q, where a is any vertex not in L_0+Q; secondly, if L_0 is a single closed simplex B, then $AB+Q$ contracts into $\dot{A}B+Q$.

THEOREM 1. *The nucleus of a complex is a combinatorial invariant.*

It is enough to show that K_0 and K_1 have the same nucleus, where K_1 is derived from K_0 by an elementary sub-division (A, a). Let

$$K_0 = AP+Q, \quad K_1 = a\dot{A}P+Q \quad (A \not\subset Q, a \not\subset K_0).$$

By Lemma 2 the star aAP, with a as centre, contracts into $a\dot{A}P$. Since $A \not\subset Q$ we have $AP \,.\, Q \subset \dot{A}P$, and it follows from Lemma 1 that the complex $aAP+Q$ contracts into K_1. But AP is collapsible, by the corollary to Lemma 2. Therefore $aAP+Q$ also contracts into K_0, by the corollary to Lemma 3, and the theorem follows.

By a *contractible neighbourhood* of a complex L we mean a complex N which contains L as a sub-complex and satisfies the conditions:

1. *N is a normal simplicial neighbourhood of L, meaning that every principal closed simplex in N meets L, but no open simplex in $N-L$ has all its vertices in L,*

2. *$L \,.\, N_A$ is collapsible, where A is any closed simplex in N which does not meet L.*

THEOREM 2. *Any contractible neighbourhood of L contracts into L.*

Let N be a contractible neighbourhood of L, and let R be the complex consisting of the closed simplexes in N which do not meet L. If $R = 1$ every vertex lies in L and, since N is a normal neighbourhood, it follows that $N = L$ and there is nothing to prove. Otherwise let A be a principal closed simplex of R and let $N = AP+Q$. If some simplex in P was not in L it would contain at least one vertex a, in R. We should then have $aA \subset R$, contrary to the fact that A is a principal simplex of R. Therefore $P \subset L$ and, since N is a contractible neighbourhood, P is collapsible. Therefore N contracts into $N^* = \dot{A}P+Q$, by the corollary to Lemma 3. Let $B \subset L \,.\, N_C$, where $C \subset R$ and $C \neq A$. Then $A \not\subset BC$, since $A \not\subset C$, and it follows that $BC \subset N^*$, whence $L \,.\, N_C^* = L \,.\, N_C$. Therefore the second condition for a contractible neighbourhood, and obviously the first, is satisfied by N^*, and the theorem follows from induction on the number of simplexes in R.

Let L be any sub-complex of a given complex K, let K_1 be the complex consisting of all the closed simplexes in K which do not meet L and let $R = K_1 . N(L, K)$, where $N(L, K)$ stands for the set of all closed simplexes in K which meet L. The transformation $K \to K^* = aR + K_1$, where a is any vertex not in K_1, will be described as the operation of *shrinking L into a point*, namely the vertex a. It is the result of formally identifying the vertices in L with the vertex a, which may or may not belong to L.

THEOREM 3. *If L is collapsible and $N(K, L)$ is a contractible neighbourhood of L the operation of shrinking L into a point is a formal deformation.*

Let $K^* = aR + K_1$, where R and K_1 mean the same as before and $a \not\subset K_1$. Since an elementary subdivision of order zero is a formal deformation we may assume $a \not\subset K$, and the proof is similar to the proof of Theorem 1. For the complex $N(L, K)$ is collapsible, since it contracts into the collapsible complex L, and the complex $aN(L, K) + K_1$ contracts both into K^* and into K. Therefore $K^* = D(K)$ and the theorem is established.

If K and L are any complexes we shall use $s_L K$ to stand for the sub-division of K which consists of starring every simplex in

$$K - L \quad (= K - K . L)$$

in order of decreasing dimensionality, and $s_L^2 K$ will mean $s_L(s_L K)$. We conclude this section with an existence lemma.

LEMMA 4. *If $N(L, K)$ is a normal neighbourhood of L $(L \subset K)$, then $N = N(L, s_L K)$ is a contractible neighbourhood of L. Moreover, $L . N_A$ is a single closed simplex, where A is any open simplex in $N - L$.*

Let $A_1, \ldots, A_p$ be the open simplexes in $K - L$, arranged so that† $\dim (A_\lambda) \leqslant \dim (A_\mu)$ if $\lambda < \mu$, and let a_λ be the internal vertex of $s_L A_\lambda$. By an easy extension of a known theorem‡, $s_L K$ contains the open simplex $A = Ba_{\lambda_0} \ldots a_{\lambda_r}$, where $B \in L$ and $\lambda_0 < \ldots < \lambda_r$, if, and only if,

$$B \in \dot{A}_{\lambda_0}, \quad A_{\lambda_i} \in \dot{A}_{\lambda_{i+1}} \quad (i = 0, \ldots, r-1),$$

and any simplex in $s_L K$ is of this form (possibly with $B = 1$ or $r = -1$, *i.e.* $a_{\lambda_0} \ldots a_{\lambda_r} = 1$). Therefore N is a normal neighbourhood of L. Any open simplex in N whose closure does not meet L is of the form

$$A^* = a_{\lambda_0} \ldots a_{\lambda_r} \quad (\lambda_0 < \ldots < \lambda_r)$$

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† $\dim (P)$ stands for the dimensionality of P.

‡ H. Seifert und W. Threlfall, *Lehrbuch der Topologie* (Berlin, 1934), 230, Theorem II.

and it follows that

$$L \, . \, N_{A^*} = L \, . \, \dot{A}_{\lambda_0}.$$

Since any simplex in K is contained in L if all its vertices are in L it follows that the closure, and hence the boundary, of any simplex in $K-L$ meets L, if at all, in a single closed simplex. Therefore, $L \, . \, N_{A^*}$ is a single closed simplex and N is a contractible neighbourhood of L.

Let $A = A^* B$ be any open simplex in $N-L$ whose closure meets L, where $A^* = a_{\lambda_0} \ldots a_{\lambda_r}$ and $B \in L$ $(B \neq 1)$. To say that $B_1 \in N_A$, where B_1 is any open simplex having no vertex in common with A, is to say that $AB_1 = A^* BB_1 \in N$, or that $BB_1 \in N_{A^*}$. If $B_1 \in L$ it follows that $BB_1 \in L$, since all its vertices are in L, and hence that $B_1 \in (L \, . \, N_{A^*})_B$. Conversely, if $B_1 \in (L \, . \, N_{A^*})_B$ it follows that $B_1 \in L \, . \, N_A$. Therefore $L \, . \, N_A = (L \, . \, N_{A^*})_B$, and since $L \, . \, N_{A^*}$ is a single closed simplex, so is $L \, . \, N_A$.

If L is any sub-complex of a given complex K, it follows from the first argument in the proof of Lemma 4 that $N(L, s_L K)$ is a normal neighbourhood of L, and we have the corollary:

COROLLARY. *$N = N(L, s_L^2 K)$ is a contractible neighbourhood of L. Moreover $L \, . \, N_A$ is a single closed simplex, where A is any open simplex in $N - L$.*

4. *Sub-division.*

We shall need some theorems concerning sub-division and its relation to formal deformation. By a stellar sub-division we shall mean the resultant of a sequence of elementary sub-divisions (A, a), and a stellar sub-division will always be represented by the letter σ. In dealing simultaneously with two or more complexes $K_1, K_2, \ldots$, it is always to be understood that no vertex introduced by a sub-division σK_1 belongs to any of the others. Thus σ may be regarded as operating simultaneously on all the complexes, with the convention that (A, a) leaves K unaltered if A does not belong to K. We admit the identical transformation as an elementary sub-division, which we denote by 1, and by $\sigma = 1$ in K we shall mean that σ is a product of elementary sub-divisions which leave K unaltered.

THEOREM 4. *If K contracts into L, then σK contracts into σL, where σ is any stellar sub-division of K.*

Using a double induction, first on the number of elementary sub-divisions in σ, and then on the number of elementary contractions in the transformation $K \to L$, we see that it is sufficient to consider the case where σ is a single elementary sub-division (A, a), and $K \to L$ a single elementary contraction. Let

$$K = L + bB,$$

where bB is a closed simplex meeting L in $b\dot{B}$. The theorem is then obvious unless $A \subset bB$ and, by Lemma 1, it is enough to prove that σbB contracts into $\sigma b\dot{B}$.

First let $A \subset B$. Then

$$\sigma bB = b(\sigma B), \quad \sigma b\dot{B} = b(\sigma \dot{B})$$

and σbB contracts into $\sigma b\dot{B}$, by Lemma 2. If $A \not\subset B$, let $A = bB_1$ and $B = B_1 B_2$. Then

$$\begin{aligned} \sigma bB &= a(b\dot{B}_1 + B_1) B_2 \\ &= ab\dot{B}_1 B_2 + aB, \\ \sigma b\dot{B} &= \sigma b(B_1 \dot{B}_2 + \dot{B}_1 B_2) \\ &= a(b\dot{B}_1 + B_1) \dot{B}_2 + b\dot{B}_1 B_2. \end{aligned}$$

Removing the open simplexes B and aB from σbB, we are left with

$$\begin{aligned} ab\dot{B}_1 B_2 + a\dot{B} &= ab\dot{B}_1 B_2 + a(B_1 \dot{B}_2 + \dot{B}_1 B_2) \\ &= ab\dot{B}_1 B_2 + aB_1 \dot{B}_2, \end{aligned}$$

since $a\dot{B}_1 B_2 \subset ab\dot{B}_1 B_2$. Clearly $ab\dot{B}_1 B_2$ and $aB_1 \dot{B}_2$ meet in $a\dot{B}_1 \dot{B}_2$, which is contained in $(a\dot{B}_1 B_2)^{\cdot}$. It follows from Lemmas 2 and 1, with b taking the place of the simplex A in Lemma 2, that $ab\dot{B}_1 B_2 + aB_1 \dot{B}_2$ contracts into

$$\begin{aligned} b(a\dot{B}_1 B_2)^{\cdot} + aB_1 \dot{B}_2 &= b\dot{B}_1(a\dot{B}_2 + B_2) + aB_1 \dot{B}_2 \\ &= a(b\dot{B}_1 + B_1) \dot{B}_2 + b\dot{B}_1 B_2 \\ &= \sigma b\dot{B}, \end{aligned}$$

and the theorem is established.

Let $P_1, \ldots, P_n$ be sub-complexes of a complex K and let Q_i be a sub-complex of P_i $(i = 1, \ldots, n)$ with the following property: If σ_1 is any stellar sub-division of P_i, some stellar sub-division $\sigma_2 \sigma_1 P_i$ contracts into $\sigma_2 \sigma_1 Q_i$. Then we have the corollary to Theorem 4:

COROLLARY. *There is a sub-division σK such that σP_i contracts into σQ_i for each value of i.*

The corollary is trivial if $n = 1$, and we shall prove it by induction on n. Assume that $\sigma_1 P_\lambda$ contracts into $\sigma_1 Q_\lambda$ $(\lambda = 1, \ldots, n-1)$, where σ_1 is some stellar sub-division of K. By hypothesis, there is a sub-division $\sigma_2 \sigma_1 P_n$ which contracts into $\sigma_2 \sigma_1 Q_n$. By Theorem 4, $\sigma_2 \sigma_1 P_\lambda$ contracts into $\sigma_2 \sigma_1 Q_\lambda$ ($\lambda = 1, \ldots, n-1$), whence σP_i contracts into σQ_i ($i = 1, \ldots, n$), where $\sigma = \sigma_2 \sigma_1$.

THEOREM 5. *If $K_q = D(K_0)$, there is a complex which contracts both into K_0 and into a stellar sub-division of K_q.*

Let $$K_\lambda = C_\lambda E_\lambda \ldots C_1 E_1(K_0) \quad (\lambda = 1, \ldots, q),$$

where $E_1, \ldots, E_q$ are expansions and $C_1, \ldots, C_q$ contractions†. If σ is any stellar sub-division it follows from Theorem 4 that σK_λ expands into $\sigma E_{\lambda+1}(K_\lambda)$ ($\lambda < q$) and that the latter contracts into $\sigma K_{\lambda+1}$, and from induction on $q - \lambda$ that

$$\sigma K_q = C_q' E_q' \ldots C_{\lambda+1}' E_{\lambda+1}'(\sigma K_\lambda),$$

where $E_{\lambda+1}', \ldots, E_q'$ are expansions and $C_{\lambda+1}', \ldots, C_q'$ contractions. Therefore the theorem will follow from induction on q if we can show that, when $q > 1$, K_0 expands into some complex which contracts into a stellar sub-division of K_2 [if $q = 1$ the complex $E_1(K_0)$ satisfies the required conditions].

Let $$K_{01} = E_1(K_0), \quad K_{12} = E_2(K_1) = E_2 C_1 E_1(K_0),$$

let $K_{12}' = s_{K_1} K_{12}$, the new vertices introduced by s_{K_1} being, as usual, different from any of those in K_{01}, and let $e_1, \ldots, e_r$ be the elementary expansions of which E_2 is the resultant (e_λ applied before $e_{\lambda+1}$). If e_λ is the transformation $K^* \to K^* + aA$, let e_λ' stand for the transformation

$$s_{K_1} K^* \to s_{K_1}(K^* + aA),$$

which is an expansion by Theorem 4, and let $E_2' = e_r' \ldots e_1'$. If $C_2 = c_s \ldots c_1$, where $c_1, \ldots, c_s$ are elementary contractions, let c_λ' be similarly defined and let $C_2' = c_s' \ldots c_1'$. Then

$$s_{K_1} K_2 = C_2' E_2' C_1 E_1(K_0).$$

I say that the expansion E_2' is interchangeable with the contraction C_1. For, in general, let c be any elementary contraction $K \to L = K - aA - A$ and let e be an expansion $L \to L + bB + B$ ($b\dot{B} \subset L$, $B \bar{\in} L$). Then e and c are interchangeable if neither of the open simplexes aA and A coincides

† Either E_1 or C_q, or both, may be the identity.

with either† bB or B. For, if B is neither aA nor A, the transformation $K \to K+bB+B$ is an elementary expansion, since $b\dot{B} \subset L \subset K$ and $B \bar{\epsilon} K$. The only simplex in $(bB)^{\cdot}$ which is not in K is B and, if $B \neq A$, it follows that aA is the only simplex in $K+bB+B$ with A on its boundary. Therefore the transformation

$$\begin{aligned} K+bB+B \to (K+bB+B)-aA-A \\ = (K-aA-A)+bB+B \\ = L+bB+B \end{aligned}$$

is an elementary contraction. That is to say $ce = ec$. If C is any contraction of a given complex K, and E is an expansion of $C(K)$, it follows from an inductive argument that $EC = CE$ provided that none of the simplexes removed by C is restored by E. This condition is satisfied by C_1 and E_2'. For any open simplex A which is added by E_2' belongs to $K_{12}'-K_1$ and so contains at least one vertex introduced by s_{K_1}. According to our rule this vertex, and therefore A, does not belong to K_{01}. Therefore A is not one of the simplexes removed by C_1. Therefore $E_2' C_1 = C_1 E_2'$, and

$$\begin{aligned} s_{K_1} K_2 &= C_2' E_2' C_1 E_1(K_0) \\ &= C_2' C_1 E_2' E_1(K_0). \end{aligned}$$

Therefore the complex

$$K_{02} = E_2' E_1(K_0)$$

contracts both into K_0 and into $s_{K_1} K_2$, and the theorem is established.

We now give two definitions.

1. By the *order* of a deformation D will be meant the maximum order of the elementary deformations in D.

2. If $L \subset K_0 . K_q$ and if no simplex of L is removed by any of the elementary contractions in a deformation $K_q = D(K_0)$, we shall describe D as *relative* to L, and shall write

$$K_q = D(K_0) \quad \text{(rel. } L).$$

Two addenda follow from the proof of Theorem 5:

ADDENDUM 1. *If K_0 and K_q are at most n-dimensional and if the order of D does not exceed m, where $m \geqslant n$, there is a complex of at most m dimensions which contracts both into K_0 and into σK_q.*

† Actually $ec = ce$ if $A \neq B$.

ADDENDUM 2. *If* $K_q = D(K_0)$ (rel. L) *there is a complex which contracts both into* K_0 *and into* σK_q, *where* $\sigma = 1$ *in* L.

For each simplex added by E_λ' is contained in a sub-division of some simplex added by E_λ ($\lambda = 2, \ldots, q$) and its dimensionality does not exceed m. Therefore neither the order of E_λ' ($\lambda = 2, \ldots, q$) nor the dimensionality of K_{02} exceeds m and the first addendum follows from induction on q. The second addendum follows from induction on q and the fact that $L \subset K_1$, since no simplex of L is removed during the contraction C_1.

Notice that the complex which contracts both into K_0 and into σK_q is uniquely determined by the construction used in proving Theorem 5, except for the actual vertices introduced by the sub-division σ.

THEOREM 6. *If* K *is any complex, there is a sub-division* σK *such that* σE^m *contracts into* σE^{m-1} ($m > 0$), *where* E^m *is any* m*-element*† *in* K *and* E^{m-1} *any* $(m-1)$*-element in* $\dot{E}^m$, *the sub-division* σ *being independent of* m, *of* $E^m \subset K$ *and of* $E^{m-1} \subset \dot{E}^m$.

If K is 0-dimensional there is nothing to prove and, assuming the theorem for a complex of at most n dimensions ($n \geqslant 0$), we shall prove it by induction on n. First notice that each element σE^m is collapsible, where $E^m \subset K$ and σK is any sub-division which satisfies the conditions of the theorem. For, if $m > 0$, σE^m contracts into σA^{m-1}, where A^{m-1} is any $(m-1)$-simplex in $\dot{E}^m$. From the corollary to Lemma 2 and Theorem 4 it follows that σA^{m-1} and hence σE^m are collapsible.

There are in K, which we now take to be $(n+1)$-dimensional, only a finite number of elements. Therefore the theorem will follow from the corollary to Theorem 4 if we can prove that some sub-division σE^m contracts into σE^{m-1} ($0 < m \leqslant n+1$), where E^m is a given m-element and E^{m-1} a given $(m-1)$-element in $\dot{E}^m$. This follows from the hypothesis of the induction unless $m = n+1$, which we assume to be the case.

Let C^m ($m = n+1$) be a rectilinear, geometrical representation of the m-element $ab\dot{A}^n$, where A^n is an n-simplex. Some partition $\pi_0 \dot{C}^m$ is the image of a stellar sub-division $\sigma_0 \dot{E}^m$ in an isomorphic transformation f_0, such that‡ $f_0(\sigma_0 E^{m-1}) = \pi_0 a\dot{A}^n$. The sub-divisions $\pi_0 \dot{C}^m$ and $\sigma_0 \dot{E}^m$ and the transformation f_0 can be extended to sub-divisions $\pi_1 C^m$ and $\sigma_1 E^m$ and an isomorphism $f_1(\sigma_1 E^m) = \pi_1 C^m$, with $\sigma_1 = \sigma_0$ and $\pi_1 = \pi_0$ on $\dot{E}^m$ and

† Following Alexander and Newman, we use the terms m-element and $(m-1)$-sphere to mean complexes which are combinatorially equivalent to an m-simplex and to its boundary.

‡ J. H. C. Whitehead, *Proc. Cambridge Phil. Soc.*, 31 (1935), 69–75.

$\dot{C}^m$, and† $f_1 = f_0$ on $\sigma_0 \dot{E}^m$. Therefore we lose no generality in assuming that E^m was initially isomorphic to $\pi_1 C^m$ with E^{m-1} corresponding to $\pi_1 a\dot{A}^n$ and, assuming this to have been the case, we take E^m and E^{m-1} to be the geometrical complexes $\pi_1 C^m$ and $\pi_1 a\dot{A}^n$.

Let $p_0 = a, p_1, \ldots, p_r = b$ be the points on the rectilinear segment ab, arranged in this order, such that all the vertices of E^m lie in the (geometrical) n-elements $p_i \dot{A}^n$ $(i = 0, \ldots, r)$. By a construction used elsewhere‡, we can find a stellar sub-division σE^m which is a partition of the rectilinear m-element $(p_0p_1 + p_1p_2 + \ldots + p_{r-1}p_r)\dot{A}^n$ and is such that all its vertices are in the n-elements $p_i \dot{A}^n$ $(i = 0, \ldots, r)$. To economize our symbols, let us assume that E^m satisfied this condition initially, let E_i^n be the complex covering $p_i \dot{A}^n$, and let E_i^m be the complex covering the m-element

$$p_{i-1}p_i\dot{A}^n \quad (i = 1, \ldots, r).$$

If $r > 1$, assume that some stellar sub-division $\sigma_{r-1}(E_1^m + \ldots + E_{r-1}^m)$ contracts into $\sigma_{r-1} E^{m-1}$. If we can show that some stellar sub-division $\sigma^* \sigma_{r-1} E_r^m$ contracts into $\sigma^* \sigma_{r-1} E_{r-1}^n$, the theorem will be established directly, if $r = 1$ (taking $\sigma_{r-1} = 1$); and it will follow from Lemma 1, Theorem 4 and induction on r, if $r > 1$. The element $\sigma_{r-1} E_r^m$ has no internal vertices and, again simplifying our notation, it remains to prove the following: if E^m is a partition of $p_0 p_1 \dot{A}^n$ with no internal vertices, some stellar sub-division σE^m contracts into σE_0^n, where E_i^n is the sub-complex of E^m covering $p_i \dot{A}^n$ $(i = 0, 1)$.

Since E^m is a partition of $p_0 p_1 \dot{A}^n$ and since it has no internal vertices, the vertices of any simplex in E^m lie in $(p_0 + p_1) A^{n-1}$, where A^{n-1} is some closed simplex in $\dot{A}^n$. Since $p_0 A^{n-1}$ and $p_1 A^{n-1}$ are flat, every internal simplex in E^m is of the form $B_0 B_1$, where $B_i \subset E_i^n$ $(i = 0, 1$; possibly $B_0 . B_1 \neq 1$, being in $\dot{A}^n)$. Therefore E^m is a normal simplicial neighbourhood both of E_0^n and E_1^n.

Now let q be an inner point of the segment $p_0 p_1$ and apply the sub-division $s_L E^m$, where $L = \dot{E}^m = E_0^n + E_1^n$, placing each new vertex on the locus $q\dot{A}^n$. Since each simplex of E^m is contained in one of the closed simplexes $p_0 p_1 A^{n-1}$ $(A^{n-1} \subset \dot{A}^n)$, and since the locus qA^{n-1} is flat, it follows that $q\dot{A}^n$ is covered by a sub-complex F^n of $s_L E^m$. Let

$$F^m = N(E_0^n, s_L E^m)$$

be the sub-complex of $s_L E^m$ covering the closure of the region between

† Alexander, *loc. cit.*, Theorem 13.2, and Whitehead, *loc. cit.*, Theorem 2.

‡ Whitehead, *loc. cit.*, Theorem 1.

E_0^n and F^n. Clearly F^m is the same as it would be were $L = E_0^n$, instead of $E_0^n + E_1^n$, and, by Lemma 4 and Theorem 2, F^m contracts into E_0^n. It follows from an argument used by Newman† that $F^n . (s_L E^m)_B$ is an element, where B is any internal simplex of E_1^n. By the hypothesis of the original induction, and our preliminary observation, there is a stellar sub-division $\sigma_1 F^n$ such that each of the elements $\sigma_1\{F^n . (s_L E^m)_B\}$, with B inside E_1^n, is collapsible. Therefore σE^m, with $\sigma = \sigma_1 s_L$, is a contractible neighbourhood of $\sigma_1 F^m$ and contracts into $\sigma_1 F^m$, by Theorem 2. By what we have just proved and Theorem 4 it then contracts into $\sigma_1 E_0^n = \sigma E_0^n$ and the proof is complete.

5. *Geometrical deformation.*

If K_0 is any complex and

$$K_1 = K_0 + E^n, \tag{5.1}$$

where E^n is an n-element which meets K_0 in an $(n-1)$-element on $\dot{E}^n$, we shall describe the transformation $K_0 \to K_1$ as a *geometrical expansion* and $K_1 \to K_0$ as a *geometrical contraction*. We shall also say that a complex expands and contracts geometrically into any general sub-division‡ of itself. Finally, any sequence of geometrical expansions (contractions) will also be called a geometrical expansion (contraction). When a contrast is unnecessary, or when it is obvious from the context which kind is meant, we shall refer to either a formal or a geometrical expansion (contraction) simply as an expansion (contraction).

THEOREM 7. *If K contracts geometrically into L some stellar sub-division σK contracts formally into σL.*

As explained in § 14 below, a general sub-division of any sub-complex of K may be extended to the whole of K. It follows from a straightforward inductive argument that some general sub-division γK contracts geometrically into γL without further sub-division. That is to say $\gamma K \to \gamma L$ is the resultant of transformations of the form $K_1 \to K_0$, where K_0 and K_1 are related by (5.1). Clearly the same is true of $\gamma^* \gamma K$ and $\gamma^* \gamma L$, where γ^* is any general sub-division of γK, and there is a sub-

† *Journal London Math. Soc.*, 2 (1926), 56–64, Lemma 2. The double sub-division is not needed for Newman's argument provided that the $(n-1)$-element in question has a normal neighbourhood initially (*cf.* Lemma 10 below).

‡ See Newman, *Journal London Math. Soc.* (*loc. cit.*), also §14 below. We denote a general sub-division by the symbol γ.

division γ^* such that $\gamma^* \gamma K$ is a stellar sub-division† $\sigma_1 K$. It follows from Theorem 6 that there is a further stellar sub-division σ_2, such that σK contracts formally into σL, where $\sigma = \sigma_2 \sigma_1$, and the theorem is established.

COROLLARY. *If K contracts geometrically into L any sub-division γK contracts geometrically into γL.*

For some stellar sub-division σK contracts formally into σL. Clearly any sub-division $\gamma_1 \sigma K$ contracts geometrically into $\gamma_1 \sigma L$, and the corollary follows from the fact that γ and σ have a common sub-division $\gamma_2 \gamma = \gamma_1 \sigma$.

A complex which contracts geometrically into a single vertex may be described as *geometrically collapsible* and it follows from the corollary to Theorem 7 that the property of being geometrically collapsible is a combinatorial invariant. It also follows from Theorem 7 that geometrical expansions and contractions are formal deformations.

6. *Maps and homotopy.*

Let f be a simplicial map of a complex K in a complex L, where $K . L = 1$, meaning a transformation of vertices such that, if $A = a_0 \ldots a_n$ is any simplex in K, then $f(a_0), \ldots, f(a_n)$, which need not be distinct, are the vertices of a simplex $f(A)$ in L. As usual, we shall refer to f as a map of K on, as distinguished from in, L only if each simplex in L is the image of one or more simplexes in K. We shall describe f as (1-1) if no two simplexes in K have the same image, even if f is not a map on L, and a (1-1) simplicial map of K on L will be called an *isomorphism.*

We now define what we call the *mapping cylinder* $C_f(K)$, of a map f. For convenience we represent K and L as rectilinear, geometrical complexes, and we take f to be the semi-linear map determined by the given transformation of vertices. Let K_{01} be the simplicial complex derived from the topological product $K \times \langle 0, 1 \rangle$, by starring all the cells $A \times \langle 0, 1\rangle$ $(A \subset K)$, leaving $K \times 0$ and $K \times 1$ untouched. Then we define $C_f(K)$ as the simplicial complex obtained from K_{01} by identifying each simplex $A \times 0$, of $K \times 0$, with the corresponding simplex A, in K, and each simplex $A_1 = A \times 1$ in $K \times 1$ with the simplex $f(A)$ in L. If c is the centre of the star in K_{01} covering $A \times \langle 0, 1\rangle$ the simplex cA_1 is thus transformed into $cf(A)$, which may be of lower dimensionality than cA, and $C_f(K)$ is the image of K_{01} in a simplicial map ϕ, which is an isomorphism if, and only if, K is (1–1). If K^* is any sub-complex of K, and if K_{01}^* is the sub-complex of K_{01} covering $K^* \times \langle 0, 1\rangle$, then $\phi(K_{01}^*)$ is obviously the

† See the addendum to Lemma 16 in §14.

mapping cylinder $C_f(K^*)$ of the map $f(K^*) \subset L$. If A is any closed simplex in K and c the centre of the star covering $A \times \langle 0, 1 \rangle$ an easy inductive argument shows that†

$$C_f(A) = c[A + f(A) + C_f(\dot{A})].$$

We now prove two theorems which are analogous to Lemmas 2 and 3.

THEOREM 8. *If $f(K) \subset L$ is a simplicial map of K in L, and if K^* is any sub-complex of K (possibly $K^* = 1$), then $L + C_f(K)$ contracts formally into $L + C_f(K^*)$.*

If $K^* = K$ there is nothing to prove. Otherwise let A be the closure of any principal open simplex in $K - K^*$. Assuming that $C_f(A)$ contracts into $f(A) + C_f(\dot{A})$, we deduce the theorem from Lemma 1 and induction on the number of simplexes in $K - K^*$. If $\dim(A) = 0$ it is obvious that $C_f(A)$ contracts into $f(A) + C_f(\dot{A})$ $[= f(A)$ with the convention $C_f(1) = 1]$, and if $\dim(A) > 0$ we assume this to be true of any k-simplex for $k < \dim(A)$. Then, taking $K = \dot{A}$, $L = f(A)$, and $K^* = 1$, we deduce by our first argument that $f(A) + C_f(\dot{A})$ contracts into $f(A)$, and is therefore collapsible, $f(A)$ being a single closed simplex. If c is the centre of the star $C_f(A)$ we first remove the open simplexes A and cA from $C_f(A)$, leaving

$$c[f(A) + C_f(\dot{A})],$$

which contracts into $f(A) + C_f(\dot{A})$ by what we have already proved and the corollary to Lemma 3. Therefore, the fact that $C_f(A)$ contracts into $f(A) + C_f(\dot{A})$, and hence the theorem, follow from a second induction on $\dim(A)$.

Let E_0^n be an n-element which is completely inside an n-element E^n, meaning that $E_0^n \subset E^n - \dot{E}^n$. As a corollary to Theorem 8 we have:

COROLLARY. *$Cl(E^n - E_0^n)$ contracts geometrically both into $\dot{E}^n$ and into $\dot{E}_0^n$.*

For some sub-division σE^n may be represented as a partition of a geometrical simplex Δ^n in such a way that σE_0^n covers a "concentric" simplex Δ_0^n, contained in‡ Δ^n. The closure of the region between Δ^n and Δ_0^n is the geometrical mapping cylinder $C_t(\dot{\Delta}_0^n)$, where t is the projection of $\dot{\Delta}_0^n$ on $\dot{\Delta}^n$ from their common centre, and contracts geometrically into

† This property may be taken as the basis of an inductive definition of $C_f(K)$. As a matter of convention we take $f(1) = C_f(1) = 1$.

‡ Newman, *Journal London Math. Soc.*, 2 (1927), 64, Theorem 3.

$\dot{\Delta}^n$, by Theorem 8, and similarly into $\dot{\Delta}_0^n$. Therefore, $Cl(E^n - E_0^n)$ contracts geometrically both into $\dot{E}^n$ and into $\dot{E}_0^n$.

THEOREM 9. *If K contracts formally into K^* and if $f(K) \subset L$ is a simplicial map of K in L, then $L + C_f(K)$ contracts formally into*

$$K + L + C_f(K^*).$$

If $K^* = K$ there is nothing to prove. Otherwise let $K = K_1 + aA$ ($a\dot{A} \subset K_1$, $A \not\subset K_1$), where $K \to K_1$ is the first step in a contraction $K \to K^*$. The theorem will follow from Lemma 1 and induction on the number of simplexes in $K - K^*$ if we can show that $C_f(aA)$ contracts into

$$P = aA + f(aA) + C_f(a\dot{A}).$$

By Lemma 2, $C_f(aA)$ contracts into cP, where c is the centre of $C_f(aA)$. Clearly P contracts into $f(aA) + C_f(a\dot{A})$, which contracts into $f(aA)$, by Theorem 8, and is therefore collapsible. Therefore P is collapsible, cP contracts into P, by the corollary to Lemma 3, and the theorem is established.

The purpose of our next theorem is to establish a certain relation between homotopic maps $f_0(K_0) \subset L$ and $f_1(K_1) \subset L$, the complexes K_0 and K_1 being combinatorially equivalent. We shall say that two simplicial maps $f_0(K_0) \subset L$ and $f_1(K_1) \subset L$ are *equivalent* if, and only if, K_1 is the image of K_0 in an isomorphism t, such that $f_0 = f_1 t$ [*i.e.* $f_0(A_0) = f_1(A_1)$, where A_0 is any simplex in K_0 and $A_1 = t(A_0)$]. If we represent K_0, K_1 and L as geometrical complexes, the maps f_0 and f_1 will be described as homotopic† in L if, and only if, first, supposing that $K_0 . K_1 = 1$, there is an isomorphism $t(\pi_0 K_0) = \pi_1 K_1$, where π_i is a partition of K_i ($i = 0, 1$), and a map‡ of $C_t(\pi_0 K_0)$ in L which, regarded as a transformation of points, coincides with f_i in $\pi_i K_i$; secondly, if K_1 meets K_0 the maps f_0 and f_1 will be described as homotopic if $f_0(K_0)$ is homotopic to $f_2(K_2)$, where the latter is equivalent to $f_1(K_1)$ and $K_0 . K_2 = 1$. This definition obviously includes the ordinary definition of homotopy in case $K_0 = K_1$.

Let K, K_0 and K_1 be combinatorially equivalent complexes and let $K_0 . K_1 = 1$. By a *simple cylinder joining K_0 to K_1* we shall mean a (simplicial) complex containing the complexes K_0 and K_1 (*i.e.* K_0 and K_1 themselves, not merely sub-divisions of K_0 and K_1), some sub-division of which is isomorphic to a simplicial sub-division of $K \times \langle 0, 1 \rangle$ in a transformation which maps a sub-division of $K \times i$ ($i = 0, 1$) on a sub-division of K_i.

† *Cf.* Hurewicz, *loc. cit.* (2nd paper), 524.

‡ By a map of a geometrical complex we shall always mean a continuous transformation.

THEOREM 10. *If two simplicial maps $f_0(K_0) \subset L$ and $f_1(K_1) \subset L$ are homotopic in L there is a simple cylinder P, joining K_0 to K_1, which contracts formally both into K_0 and into K_1, and a simplicial map $f(P) \subset L$ which coincides with f_i in K_i ($i = 0, 1$), the map f being simplicial with respect to the complex P, not merely with respect to a sub-division of P.*

Let K_0, K_1 and L be represented as geometrical complexes. Since f_0 and f_1 are homotopic there is a map $\phi(C) \subset L$, which coincides, as a transformation of points, with f_i in $\pi_i K_i$ ($i = 0, 1$), where $C = C_t(\pi_0 K_0)$ and t is an isomorphism of a partition $\pi_0 K_0$ on a partition $\pi_1 K_1$. The map ϕ may be approximated, in the usual way, by a simplicial map $\phi'(\sigma C) \subset L$, where σ is a stellar sub-division of C. Since $\phi = f_i$ in $\pi_i K_i$ and f_i is simplicial we may assume that $\phi'(a)$ is a vertex of $f_i(A)$, where a is any vertex of $\pi_i A$ and $A \subset K_i$. By Theorem 6 there is a further sub-division $\sigma_1 \sigma C = \sigma' C$ such that each of the elements $\pi_i' A$ is collapsible, where $A \subset K_i$ and $\pi_i' = \sigma_1 \sigma \pi_i$. By Theorem 8 the complex C contracts formally into each of $\pi_0 K_0$ and $\pi_1 K_1$, and by Theorem 4 the complex $\sigma' C$ contracts formally into each of $\pi_0' K_0$ and $\pi_1' K_1$. The map ϕ' may be replaced by a simplicial map $\phi''(\sigma' C) \subset L$, which transforms each vertex in $\pi_i' A$ ($A \subset K_i$) into a vertex of $f_i(A)$.

We now form the topological product $K_i \times \langle 0, 1 \rangle$, taking $p \times 0 = p$, where p is any point in K_i. Let T_i be the polyhedral complex covering $K_i \times \langle 0, 1 \rangle$, which consists of the open simplexes $A \times 1$ ($A \epsilon K_i$), the open cells $A \times \langle 0, 1 \rangle$, and the simplexes in $\pi_i' K_i$. Let T_i^* be the simplicial complex derived from T_i by starring all the cells $A \times \langle 0, 1 \rangle$, leaving the complexes $K_i \times 1$ and $\pi_i' K_i$ untouched, and let

$$P_1 = T_0^* + \sigma' C + T_1^*.$$

Then P_1 is obviously a simple cylinder joining $K_0 \times 1$ to $K_1 \times 1$. It follows from Theorem 8, with trivial modifications, that T_i^* contracts formally into $K_i \times 1$, and also into $\pi_i' K_i$, since each of the elements $\pi_i' A$ is collapsible ($A_i \subset K$). Therefore P_1 contracts, first into $T_0^* + \sigma' C$, then into T_0^*, since $\sigma' C$ contracts formally into $\pi_0' K_0$, and finally into $K_0 \times 1$. Similarly it contracts formally into $K_1 \times 1$.

We now extend the map $\phi''(\sigma' C)$ to a map $\phi''(P_1) \subset L$ by taking $\phi''(A \times 1) = f_i(A)$ and $\phi''(c)$ to be any vertex of $f_i(A)$, where $A \subset K_i$ and c is the vertex of T_i^* which is inside $A \times \langle 0, 1 \rangle$. The map so defined is simplicial since each vertex in $\pi_i' A$ is transformed by ϕ'' into a vertex of $f_i(A)$. Finally we replace P_1 by an isomorphic complex $P = \psi(P_1)$, where $\psi(A \times 1) = A$ ($A \subset K_i$). Then P is a simple cylinder joining K_0 to K_1, it contracts formally both into K_0 and into K_1, and $\phi'' \psi^{-1}(P) \subset L$ is a simplicial map which coincides with f_i in K_i. Thus the theorem is established.

A map $f(S^n) \subset L$, where S^n is an n-sphere, will be called an n-spherical *map* in L, or simply a *spherical map* if the dimensionality is irrelevant or obvious from the context. If $f(\dot{E}^n)$ is a simplicial, spherical map in L, where E^n $(n > 0)$ is an n-element which does not meet $L + C_f(\dot{E}^n)$ except in $\dot{E}^n$, we shall describe

$$\mathfrak{E}^n = E^n + C_f(\dot{E}^n)$$

as a *simple membrane*† bounded by the spherical map $f(\dot{E}^n)$. Notice that $\mathfrak{E}^n$ is an n-sphere if $f(\dot{E}^n)$ is a single point, and that any n-sphere which meets L in a single point may be regarded as a simple membrane bounded by such a map. In dealing simultaneously with a set of simple membranes, of the same or different dimensionalities, and with any number of complexes, it is always to be understood that none of the membranes has an inner point in common with any of the others or with any of the complexes.

If t is an isomorphic map of $\dot{E}^n$ on an $(n-1)$-sphere which does not meet E^n, it can be proved without difficulty that $E^n + C_t(\dot{E}^n)$ is an n-element. From this and from the definition of a mapping cylinder it follows that $\mathfrak{E}^n = E^n + C_f(\dot{E}^n)$ is an n-cell‡ bounded by a map which is equivalent§ to f. For, if t is an isomorphism of $\dot{E}^n$ on an $(n-1)$-sphere S^{n-1}, we have $\mathfrak{E}^n = \phi(E_0^n)$, where $E_0^n = E^n + C_t(\dot{E}^n)$ and $\phi(A) = A$ if $A \subset E^n$, $\phi(A_1) = ft^{-1}(A_1)$ and $\phi(c_1) = c$, where $A_1 \subset S^{n-1}$ and c_1 and c are the vertices inside the cells $C_t(A)$ and $C_f(A)$ respectively. Since the vertices inside different cells $C_f(A)$ and $C_f(B)$ are distinct the map ϕ has no folds, which means that no two n-simplexes in E_0^n have the same image in $\mathfrak{E}^n$. We shall denote the boundary of a simple membrane, or of any cell Γ, by $F(\Gamma)$, remembering that $F(\Gamma)$ is a spherical map rather than a complex.

Let $\Gamma_i^n = f_i(E_i^n) \subset L$ $(i = 1, 2)$ be n-cells bounded by equivalent spherical maps $f_i(\dot{E}_i^n)$, and, replacing $f_2(E_2^n)$ by an equivalent map if necessary, let $\dot{E}_2^n = \dot{E}_1^n$ and $f_2 = f_1$ in $\dot{E}_1^n$. Let σE_2^n be an internal sub-division of E_2^n (*i.e.* $\sigma = 1$ in $\dot{E}_2^n$) which has no internal simplexes in

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† *Cf.* N. Aronszajn, *Akad. Wet. Amsterdam*, 40 (1937), 69–69.

‡ By a cell we shall always mean a map of an element, which (*i.e.* the map) may be singular or non-singular. It will be obvious from the context when a simple membrane is to be regarded as no more than a complex (which will usually be the case) and when it is to be regarded as a cell.

§ In speaking of the boundary of a cell we shall often refer to a map f when, strictly speaking, we mean a map which is equivalent to f.

common with E_1^n. Then $E_1^n+E_2^n=\phi(S^n)$, where $S^n=E_1^n+\sigma E_2^n$ and ϕ is a simplicial map such that $\phi(A)=A$ if $A\subset E_1^n$. We shall write

$$\Gamma_1^n+\Gamma_2^n=f(E_1^n+E_2^n)=f\phi(S^n),$$

where $f=f_i$ in E_i^n. Thus $\Gamma_1^n+\Gamma_2^n$ is a spherical map which is determined up to the choice of the sub-division σ, and what we shall later call a homotopic deformation rel. E_1^n (see §8).

THEOREM 11. *If $\mathcal{E}_0$ and $\mathcal{E}_1$ are simple membranes bounded by homotopic maps in a complex L, then*

$$L+\mathcal{E}_1=D(L+\mathcal{E}_0)\quad (\text{rel. } L).$$

Let
$$\mathcal{E}_i=E_i+C_{f_i}(\dot{E}_i)\quad (i=0,1),$$

where $F(\mathcal{E}_i)=f_i(\dot{E}_i)$, E_0 and E_1 being n-elements. We first dispose of two trivial cases. If $\dot{E}_0=\dot{E}_1$ and $f_0=f_1$, we may transform E_0 into E_1, and hence $L+\mathcal{E}_0$ into $L+\mathcal{E}_1$, by internal combinatorial transformations. The result is a formal deformation of $L+\mathcal{E}_0$, by Theorem 1, and it is obviously relative to L. Secondly, by elementary sub-divisions of order zero applied to the internal vertices of $\mathcal{E}_0$, which again are formal deformations relative to L, we may replace $\mathcal{E}_0$ by a simple membrane having no internal simplex in common with $\mathcal{E}_1$. Thus we may suppose, first that $E_i=a_i\dot{E}_i$ $(i=0,1)$, and secondly that our universal condition relative to $\mathcal{E}_0.\mathcal{E}_1$ is satisfied.

By Theorem 10 there is a complex P, which contracts formally both into $\dot{E}_0$ and into $\dot{E}_1$, and a simplicial map $f(P)\subset L$, which coincides with f_i in $\dot{E}_i$. We may take $P.L=1$ and $C_{f_i}(\dot{E}_i)=C_f(\dot{E}_i)$, and it follows from Theorem 9 that $L+C_f(P)$ contracts into $L+P+C_f(\dot{E}_i)$, and hence into $L+C_f(\dot{E}_i)$. Let

$$K=L+C_f(P)+b(E_0+P+E_1),$$

where b is a new vertex. By Lemma 2 the complex $b(E_0+P+E_1)$ contracts into $b(E_0+P)$. Since P contracts into E_0 and $E=a_0\dot{E}_0$, it follows that E_0+P is collapsible, and by the corollary to Lemma 3 that $b(E_0+P)$ contracts into E_0+P. Therefore K contracts into

$$L+C_f(P)+E_0$$

and hence into $L+\mathcal{E}_0$, since $C_f(P)$ contracts into $C_f(\dot{E}_0)$. Similarly K contracts into $L+\mathcal{E}_1$, and it follows that

$$L+\mathcal{E}_1=D(L+\mathcal{E}_0),$$

where D is the resultant of the expansion $L+\mathcal{E}_0 \to K$, followed by the contraction $K \to L+\mathcal{E}_1$. Clearly D is relative to L, and the theorem is established.

The cylinder P is n-dimensional, $\dot{E}^n$ being $(n-1)$-dimensional. Therefore $K-L$ is $(n+1)$-dimensional and we have the addendum:

ADDENDUM. *The order of the deformation D in Theorem 10 need not exceed* $\dim(\mathcal{E}_i)+1$.

LEMMA 5. *If $\mathcal{E}^n$ is a simple membrane bounded by a spherical map in K, the complexes K and $K+\mathcal{E}^n$ have the same $(n-1)$-group. If $F(\mathcal{E}^n)$ is homotopic to a point in K they have the same n-group.*

Let $\mathcal{E}^n = E^n + C_f(\dot{E}^n)$, where $F(\mathcal{E}^n) = f(\dot{E}^n)$, and let A^n be an open n-simplex in E^n, none of whose vertices lie in $\dot{E}^n$. Then $K+\mathcal{E}^n$ has the same $(n-1)$-group as $K+\mathcal{E}^n - A^n$. By the corollary to Theorem 8, $K+\mathcal{E}^n - A^n$ contracts into $K+C_f(\dot{E}^n)$ and by Theorem 8 itself the latter contracts into K. Therefore $K+\mathcal{E}^n$ and K have the same $(n-1)$-group.

If $f(\dot{E}^n)$ is homotopic to a point in K it follows from Theorem 11 that

$$K+\mathcal{E}^n = D(K+S^n),$$

where S^n is a simple membrane bounded by a single point. That is to say, S^n is an n-sphere, which we may take to be $\dot{A}^{n+1}$, where A^{n+1} is a closed $(n+1)$-simplex meeting K in a single vertex. Then $K+\dot{A}^{n+1}$, and therefore $K+\mathcal{E}^n$, has the same n-group as $K+A^{n+1}$, which contracts into K. Therefore K and $K+\mathcal{E}^n$ have the same n-group, and the lemma is established.

Let $\mathcal{E}^n = E^n + C_f(\dot{E}^n)$ be a simple membrane bounded by a spherical map $f(\dot{E}^n)$ in a complex K, and let K contain a principal open $(n-1)$-simplex B^{n-1}, which is covered in the map f by one, and only one, open simplex A^{n-1}, in $\dot{E}^n$.

LEMMA 6. *$K+\mathcal{E}^n$ contracts into $K-B^{n-1}$.*

We first take away from $K+\mathcal{E}^n$ the open simplex B^{n-1} and the interior of the cell $C_f(A^{n-1})$. The resulting complex, namely

$$K-B^{n-1}+E^n+C_f(\dot{E}^n-A^{n-1}),$$

contracts into $$K-B^{n-1}+C_f(\dot{E}^n-A^{n-1}),$$

since E^n meets the latter in the $(n-1)$-element $\dot{E}^n-A^{n-1}$. The lemma now follows from Theorem 8.

7. m-*Groups.*

THEOREM 12. *Two connected complexes have the same 2-group if, and only if, they have the same fundamental group.*

Two connected complexes with the same 2-group obviously have the same fundamental group. For neither a formal deformation nor the addition or removal of an open m-simplex ($m > 2$) alters the fundamental group.

Conversely, let K and L be two connected complexes with the same fundamental group G. After removing the simplexes of higher dimensionality, if any, we may assume that K and L are at most 2-dimensional. After a familiar process of shrinking segments into a point we may further assume that K consists of oriented circuits $a_1, \ldots, a_p$, with a common point (no two meeting anywhere else), together with certain simple membranes $\mathcal{E}_1^2, \ldots, \mathcal{E}_k^2$, bounded by circuits which are represented in the usual way as products

$$R = a_{i_1}^{m_1} \ldots a_{i_s}^{m_s}. \tag{7.1}$$

Then K determines a system of generators and relations

$$a_1, \ldots, a_p; \quad R_1 = 1, \ldots, R_k = 1, \tag{7.2}$$

for the group G, where R_λ is the product of the form (7 . 1) corresponding to $F(\mathcal{E}_\lambda^2)$. The complex L may be treated similarly so as to determine a system of generators and relations

$$b_1, \ldots, b_q; \quad S_1 = 1, \ldots, S_l = 1. \tag{7.3}$$

If the two systems (7 . 2) and (7 . 3) are identical it is obvious from Theorem 11 that $L = D(K)$.

In general, the system (7 . 2) can be transformed into the system (7 . 3) by a finite sequence of transformations of the two following types†, and their inverses:

(1) adding a new generator a_0, together with a relation of the form

$$a_0^{-1} W(a) = 1,$$

where W is a product of the existing generators and their inverses;

(2) adding a new relation $R_0 = 1$, which is a consequence of the existing relations.

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† See K. Reidemeister, *Einführung in die kombinatorische Topologie* (Brunswick, 1932), 46–48.

Let $\mathcal{E}_0^2$ be a simple membrane corresponding to the relation

$$a_0^{-1} W(a) = 1$$

(a_0 being a new circuit) in the first case, and to the relation $R_0 = 1$ in the second case. In either case

$$K_1 = K + \mathcal{E}_0^2$$

is a complex which determines the new system of generators and relations. In the first case $\mathcal{E}_0^2$ may obviously be chosen so as to satisfy the conditions of Lemma 6, with $n = 2$ and B^1 in the circuit a_0. Clearly K expands into $K + a_0 - B^1$ and hence, by Lemma 6, into K_1. In the second case the boundary of $\mathcal{E}_0^2$ is homotopic to a point in K since the relation $R_0 = 1$ is redundant. Therefore K and K_1 have the same 2-group, by Lemma 5. Equally, if a generator a_0 and a relation $a_0^{-1} W = 1$, or a redundant relation $R_0 = 1$, are removed, K and K_1 have the same 2-group, where

$$K = K_1 + \mathcal{E}_0^2.$$

Therefore there is a sequence of complexes $K_0 = K$, K_1, ..., $K_r = L$, all of which have the same 2-group, and the theorem is established.

We now introduce a new kind of elementary transformation which we shall call a *special filling of order m*, and its inverse which we call a *special perforation of order m*. A special filling is a transformation of the form

$$K \to K + A^m,$$

where $\dot{A}^m$, but not A^m, belongs to K, and $\dot{A}^m$ is homotopic to a point in K. If L is derived from K by a special filling or perforation of order m it follows from Lemma 5 that K and L have the same m-group.

Let p^m be a perforation given by

$$K \to K_1 = (K - A^m) + \dot{A}^m,$$

where A^m is a principal closed simplex in K, and let σ be a stellar sub-division of K.

LEMMA 7. *The transformation $\sigma K \to \sigma K_1$ is the resultant of a perforation p_0^m followed by a formal contraction. If p^m is special so is p_0^m.*

As in Theorem 4 we may take σ to be a single elementary sub-division (A, a), where $A \subset A^m$. Let $A = bB_1$ and $A^m = bB$, where $B = B_1 B_2$. Then

$$\sigma A^m = a(b\dot{B}_1 + B_1) B_2$$
$$= ab\dot{B}_1 B_2 + aB,$$

and we remove the interior of the simplex aB by a perforation p_0^m. If $(bB)^{\cdot}$ is homotopic to a point in K_1 so obviously is $(aB)^{\cdot}$ in $(\sigma K - aB) + (aB)^{\cdot}$. Therefore p_0^m is special if p^m is special. It follows from the second part of the proof of Theorem 4 that $(\sigma K - aB) + (aB)^{\cdot}$ contracts into σK_1, and the lemma is established.

LEMMA 8. *If K and L are two complexes with the same n-group, each of which is at most n-dimensional, then K is transformable into L by special fillings and perforations of order n and elementary deformations whose orders do not exceed n.*

We first replace any elementary expansion

$$K \to K + bB^{m-1} \quad (m > n,\ b\dot{B}^{m-1} \subset K),$$

whose order exceeds n, by the filling

$$K \to K + B^{m-1}$$

followed by the filling

$$K + B^{m-1} \to K + bB^{m-1}.$$

Since $B^{m-1} = (bB^{m-1})^{\cdot}$ it is homotopic to a point in K. Therefore the first of these is a special filling and is permissible even if $m = n+1$. Similarly we replace any elementary contraction of order m $(m > n)$ by a perforation of order m followed by a special perforation of order $m-1$. Therefore we may suppose that K is transformed into L by elementary deformations whose orders do not exceed n and fillings and perforations whose orders exceed $n-1$, those of order n, if any, being special. Let k be the maximum order of the fillings. If $k = n$ there is nothing more to be said. For K is at most n-dimensional and if no simplexes of higher dimensionality are introduced none can be taken away. If $k > n$, let A^k be one of the open simplexes introduced by a filling f^k, of order k. Since $k > n$, A^k is subsequently removed. Let p^k be the first† perforation after f^k which removes A^k. Then p^k commutes with each of the elementary transformations between f^k and p^k. For none of the latter add or remove a simplex belonging to the closure of A^k, since A^k is present throughout, and, since k is the maximum order, no simplex is introduced having A^k on its boundary. Therefore p^k, which is the inverse of f^k, may be applied immediately after f^k and both may be omitted from the sequence. The lemma now follows from induction on the number of elementary transformations in the passage from K to L.

† It may happen that A^k is inserted and removed more than once.

By a *cluster* of simplexes (spheres) attached to a complex K will be meant a set of simplexes (spheres) with a vertex of K in common, which do not meet K or each other anywhere else.

THEOREM 13. *If K and L are connected complexes of at most n dimensions with the same n-group, then*

$$L+\sum_{\mu=1}^{s}\dot{B}_{\mu}^{n+1}=D\left(K+\sum_{\lambda=1}^{r}\dot{A}_{\lambda}^{n+1}\right),$$

where $A_1^{n+1}, \ldots, A_r^{n+1}$ and $B_1^{n+1}, \ldots, B_s^{n+1}$ are clusters of $(n+1)$-simplexes attached to K and L respectively†.

We may suppose that $K\to L$ by a transformation of the kind described in Lemma 8. It follows from Lemma 7 and an argument similar to the proof of Theorem 5 that, after a suitable sub-division, the elementary transformations in $K\to L$ may be arranged so that every filling and expansion precedes every perforation and contraction. Further, if e^k is an elementary expansion of order $k\leqslant n$ which follows immediately after a filling f^n, it is obvious that

$$e^k f^n = f^n e^k.$$

Similarly

$$p^n c^k = c^k p^n,$$

where c^k is an elementary contraction of order $k\leqslant n$ and p^n is a perforation of order n. Therefore we may exhibit the transformation $K\to L$ in the form

$$L = Cp_s^n \ldots p_1^n f_r^n \ldots f_1^n \, E(K),$$

where E is an expansion, f_λ^n and p_λ^n are special fillings and perforations of order n, and C is a contraction. That is to say

$$L_0 = p_s^n \ldots p_1^n f_r^n \ldots f_1^n \, (K_0),$$

where $K_0 = E(K)$, $L_0 = C^{-1}(L)$.

Let A_λ^n be the open simplex added by f_λ^n, let B_μ^n be the open simplex removed by p_μ^n, and let

$$K_\lambda = f_\lambda^n \ldots f_1^n (K_0).$$

Since $\dot{A}_{\lambda+1}^n$ is homotopic to a point in K_λ it is obvious that any spherical map in K_λ which is homotopic to a point in $K_\lambda + A_{\lambda+1}^n$ is homotopic to a point in K_λ. It follows from induction on λ that any spherical map in K_0 which

† If $R^n(P)$ stands for the n-th connectivity of P, it is obvious that

$$s-r = R^n(L)-R^n(K).$$

is homotopic to a point in K_r is homotopic to a point in K_0. Therefore each of $\dot{A}_1^n, \ldots, \dot{A}_r^n$ is homotopic to a point in K_0 and it follows from Theorem 11, as in the proof of Lemma 5, that

$$K_r = D_1\Big(K_0 + \sum_{\lambda=1}^{r} \dot{A}_\lambda^{n+1}\Big), \tag{7.5}$$

where $A_1^{n+1}, \ldots, A_r^{n+1}$ is a cluster of $(n+1)$-simplexes attached to K_0. Similarly

$$K_r = D_2\Big(L_0 + \sum_{\mu=1}^{s} \dot{B}_\mu^{n+1}\Big), \tag{7.6}$$

where $B_1^{n+1}, \ldots, B_s^{n+1}$ is a cluster of $(n+1)$-simplexes attached to L_0. Since K_0 and L_0 contract into K and L respectively we may assume that the vertices $A_\lambda^{n+1} . K_0$ and $B_\mu^{n+1} . L_0$ belong to K and L. Then

$$K_0 + \sum_{\lambda=1}^{r} \dot{A}_\lambda^{n+1} \quad \text{and} \quad L_0 + \sum_{\mu=1}^{s} \dot{B}_\mu^{n+1}$$

contract into

$$K + \sum_{\lambda=1}^{r} \dot{A}_\lambda^{n+1} \quad \text{and} \quad L + \sum_{\mu=1}^{s} \dot{B}_\mu^{n+1},$$

and the theorem follows from (7.5) and (7.6).

From the addendum to Theorem 11 we have the addendum:

ADDENDUM. *The order of the deformation D in Theorem* 13 *need not exceed* $n+1$.

From the first addendum to Theorem 5 we have the corollary:

COROLLARY. *If K and L are two complexes of at most n dimensions with the same n-group there is a complex of at most $n+1$ dimensions which contracts into*

$$K + \sum_{\lambda=1}^{r} \dot{A}_\lambda^{n+1}$$

and also into some sub-division of

$$L + \sum_{\mu=1}^{s} \dot{B}_\lambda^{n+1}.$$

8. *Retracts by deformation.*

Let $f_0(p)$ $(p \in P)$ be a map of a topological space P in a space Q and let P^* be any sub-space of P. We shall describe a deformation of f_0 into a map f_1, given by $f_t(p) = f(p, t)$ $(0 \leqslant t \leqslant 1)$, as *relative to* P^* if $f_t(p^*) = f_0(p^*)$ for every point p^* in P^* and every t in $\langle 0, 1 \rangle$.

We recall that a sub-space of a topological space, and in particular a sub-complex L of a geometrical complex K, is called a retract by deformation† of K if there is a deformation

$$p_t = \phi_t(p) \quad (p \in K,\ 0 \leqslant t \leqslant 1),$$

such that $p_0 = p$, $p_1 \in L$ and $p_1 = p_0$ if $p_0 \in L$. If the deformation ϕ_t is relative to L we shall describe L as a retract by deformation *relative to itself*. If K contracts into L it is obvious that L is a retract by deformation relative to itself. If

$$p = f(q) \quad (q \in Q)$$

is a map of a topological space Q, in K, the map

$$p_1 = \phi_1 f(q)$$

is uniquely determined up to homotopy in L. For if

$$p_t = \psi_t(p) \quad (0 \leqslant t \leqslant 1;\ \ p_0 = p)$$

is any deformation of K into L the map $\psi_1 f$ is homotopic to f and hence to $\phi_1 f$ in K. Therefore the two maps are homotopic in L since the latter is a retract of K. If K_1 and K_2 are complexes such that $K_2 = D(K_1)$, there is a complex K^* which contracts into K_1 and into a stellar sub-division σK_2, and, except for the choice of the new vertices, K^* is uniquely determined by the deformation D. If $f_1(Q)$ is a map of Q in K_1 it follows that D determines a unique homotopy class of maps $[f_2] = D[f_1]$ in K_2, given by

$$D[f_1] = [\sigma^{-1}(\psi f_1)],$$

where ψ is the final result of a deformation of K^* into σK_2, and $\sigma^{-1}(\psi f_1)$ is a map obtained from ψf_1 by a canonical displacement‡ of the vertices of σK_2 into the vertices of K_2. Let $\mathcal{E}_{i1}, \ldots, \mathcal{E}_{ip}$ be simple membranes bounded by (simplicial) spherical maps $f_{i1}(S_1), \ldots, f_{ip}(S_p)$ in K_i $(i = 1, 2)$, where $f_{2\lambda} \in D[f_{1\lambda}]$. Further let D be relative to L, where $L \subset K_1 . K_2$.

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† *Cf.* K. Borsuk, *Fundamenta Math.*, 21 (1933), 91–98.

‡ The purpose of the sub-division σ in Theorem 5 is to eliminate unwanted intersections. Therefore we cannot, in general, take K_2 and σK_2 to be complexes covering the same (polyhedral) point-set. However, each vertex of σK_2 is internal to just one element σA, where $A \subset K_2$, and a canonical displacement is a simplicial map in which every vertex inside σA corresponds to a vertex of A, for each $A \subset K_2$. If follows from a well-known argument that any two such maps of σK_2 on K_2 are homotopic in K_2.

THEOREM 14. *Under these conditions*

$$K_2+\sum_{\lambda=1}^{p}\mathcal{E}_{2\lambda}=D^*\left(K_1+\sum_{\lambda=1}^{p}\mathcal{E}_{1\lambda}\right) \quad (\text{rel. } L).$$

First assume that D is the resultant of an expansion $K_1\to K^*$, followed by a contraction $K^*\to K_2$. Then $F(\mathcal{E}_{1\lambda})$ is homotopic in K^* to $F(\mathcal{E}_{2\lambda})$ and it follows from Theorem 11 that

$$K^*+\sum_{\lambda=1}^{p}\mathcal{E}_{2\lambda}=D_1\left(K^*+\sum_{\lambda=1}^{p}\mathcal{E}_{1\lambda}\right),$$

where the deformation D_1 is relative to K^* and hence to L. Since $F(\mathcal{E}_{i\lambda})\subset K_i$, we have

$$K^*+\sum_{\lambda=1}^{p}\mathcal{E}_{i\lambda}=E_i\left(K_i+\sum_{\lambda=1}^{p}\mathcal{E}_{i\lambda}\right) \quad (i=1,\,2),$$

where E_i is an expansion, and the special case of the theorem follows.

In general the complex K^* contracts both into K_1 and into some stellar sub-division $K_1{}^*=\sigma K_2$, where $\sigma=1$ in L, by the second addendum to Theorem 5. So, if we begin again with $K_1{}^*$, the proof will be complete if we can prove the theorem in case $D=\sigma^{-1}$, where $\sigma=1$ in L. Using induction on the number of elementary sub-divisions in σ, we may take σ to be a single elementary sub-division $K_2=AP+Q\to a\dot{A}P+Q=K_1{}^*$. There is then a complex which contracts both into $K_1{}^*$ and into K_2, namely $aAP+Q$. Moreover, if a_0 is a vertex of A, the canonical displacement $a\to a_0$ may be realized in $aAP+Q$, by shrinking the edge aa_0 into the vertex a_0. Therefore a homotopic deformation of $aAP+Q$ into K_2 determines the given transformation of classes of maps in $K_1{}^*$ into classes of maps in K_2. Since $\sigma=1$ in L, it follows that $A\not\subset L$, whence $L\subset K_1{}^*.K_2$, and the theorem follows from what we have already proved.

Under the conditions of Theorem 14 we shall write

$$\mathcal{E}_{2\lambda}=D^*(\mathcal{E}_{1\lambda}).$$

From the addendum to Theorem 11 we have the addendum to Theorem 14:

ADDENDUM. *If D is of order m and if the membranes $\mathcal{E}_{1\lambda}$ are at most n-dimensional, the order of D^* need not exceed* max $(m,\,n+1)$.

We now quote for reference a lemma which is essentially a restatement of a familiar result. Let $p_t=f_t(p)$ $(0\leqslant t\leqslant 1;\ p_0=p)$ be a deformation of a connected complex K into itself.

LEMMA 9. *If p_0 is any point in K the circuits $f_1(C)$ constitute a geometrical basis for $\pi_1(K, p_1)$, the fundamental group of K with p_1 as a base point, where $\{C\}$ is the set of circuits beginning and ending at p_0.*

If s is the segment $p = p_t$ described by p_0 in the deformation, the singular circuits of the form $s+C+s$ (orientations ignored) constitute a geometrical basis for $\pi_1(K, p_1)$. Such a circuit is homotopic, rel. p_1, to $f_1(C)$.

Let L be a sub-complex of K. We specify four sets of conditions and shall show that each of them implies all the others. The first is

R. *L is a retract by deformation of K.*

R_L. *L is a retract by deformation of K relative to itself.*

A. *Any r-cell in K $(r = 0, 1, \ldots)$ with its boundary in L is homotopic, relative to its boundary, to an r-cell in L.*

Our final set, which we shall denote by B, contains three conditions B_1, B_2, and B_3, namely

B_1. *If p is any point in L the circuits in L beginning and ending at p constitute a geometrical basis for $\pi_1(K, p)$.*

B_2. *Any spherical map in K is homotopic to a spherical map in L.*

B_3. *Any spherical map in L which bounds a cell in K bounds a cell in L.*

THEOREM 15. *Each of the conditions R, R_L, A and B implies all the others.*

It is obvious that R_L implies each of the others. It follows from an argument used by Hurewicz†, in establishing this result when L is a single point, that A implies R_L, and therefore R and B. In the presence of Lemma 9 it is obvious that R implies B. Therefore the theorem will follow if we can show that B implies A.

Let pq be a segment in K whose end points, p and q, lie in L. According to B_3, there is a segment s joining p to q in L. According to B_1 the circuit $pq+s$ is homotopic, rel. p, to a circuit C, in L. Then the singular segment $s+C$ is homotopic to pq, rel. $(p+q)$, and the condition A is established in case $r = 1$.

Let Σ_0^n $(n > 0)$ be any n-spherical map in K containing a point p_0, in L. According to B_2, Σ_0^n is homotopic to a map Σ_1^n in L. Let $p_0 p_1$ be the

† Hurewicz, *Proc. Akad. Amsterdam* (*loc. cit.*), 2nd paper, § 6.

segment described by p_0 during a deformation of Σ_0^n into Σ_1^n. Then Σ_0^n is homotopic, rel. p_0, to a spherical map of the form† $p_0p_1+\Sigma_1^n$. By what we have just proved, p_0p_1 is homotopic, rel. (p_0+p_1), to a segment s in L. Therefore Σ_0^n is homotopic, rel. p_0, to a spherical map in L, namely $s+\Sigma_1^n$.

Finally, let Γ_1^n be any n-cell in K whose boundary lies in L. According to B_3, $F(\Gamma_1^n)$ bounds a cell Γ_2^n in L. By what we have just proved the spherical map $\Gamma_1^n+\Gamma_2^n$ is homotopic, rel. p_0, to a spherical map Σ^n in L, where p_0 is any point in Γ_2^n. As in the case $n=1$ it follows that Γ_1^n is homotopic, rel. $F(\Gamma_1^n)$, to the cell $\Gamma_1^n+\Sigma^n$, which lies in L, and the proof is complete.

THEOREM 16. *If L is a retract by deformation of K, then K and L have the same m-group for all values of m.*

Let K^p $(p>0)$ be the complex consisting of L together with all the simplexes in $K-L$ whose dimensionalities do not exceed p. Thus $K^p=K$ if K is n-dimensional and $p\geqslant n$. For a given value of p let us assume that

$$L_p = D_p(K^p) \quad (\text{rel. } L), \tag{8.1}$$

where

$$L_p = L+\sum_{i=1}^{r} \mathfrak{E}_i^{p-1}+\sum_{\lambda=1}^{s} \mathfrak{E}_\lambda^p, \tag{8.2}$$

$\mathfrak{E}_1^{p-1}, \ldots, \mathfrak{E}_r^{p-1}$ being simple membranes whose boundaries are in L, and $\mathfrak{E}_1^p, \ldots, \mathfrak{E}_s^p$ being simple membranes whose boundaries are in

$$L+\sum_{i=1}^{r} \mathfrak{E}_i^{p-1}. \tag{8.3}$$

By Theorem 14 there is a deformation of K^{p+1}, which we also denote by D_p, such that

$$D_p(K^{p+1}) = L_p+\sum_{\rho=1}^{t} \mathfrak{E}_\rho^{p+1} \quad (\text{rel. } L) \tag{8.4}$$

$$= L_p^*,$$

where‡ $\mathfrak{E}_\rho^{p+1}=D_p(A_\rho^{p+1})$ and $F(\mathfrak{E}_\rho^{p+1})\subset L_p$, $A_1^{p+1}, \ldots, A_t^{p+1}$ being the $(p+1)$-simplexes in K.

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† *Cf.* §10 below.

‡ If $p>n$ we have $t=0$, the corresponding sets of cells being empty, and $L_p^*=L_p$.

Since L is a retract by deformation of K, any simplicial $(p-1)$-cell in K^{p+1}, say $\tilde{\Gamma}^{p-1}$, whose boundary lies in L, is homotopic in K, rel. $F(\tilde{\Gamma}^{p-1})$, to a $(p-1)$-cell Γ^{p-1}, in L. The p-dimensional deformation cell may be deformed into K^{p+1}, holding Γ^{p-1} and $\tilde{\Gamma}^{p-1}$ fixed, and it follows that $\tilde{\Gamma}^{p-1}$ is homotopic to Γ^{p-1}, rel. $F(\tilde{\Gamma}^{p-1})$, in K^{p+1}. This property is obviously invariant under a formal deformation, rel. L, of K^{p+1}. Therefore $\mathscr{E}_i^{p-1}$ is homotopic, rel. $F(\mathscr{E}_i^{p-1})$, in $L_p{}^*$ to a $(p-1)$-cell Γ_i^{p-1}, in L. Let ${}_i\mathscr{E}^p$ be a simple membrane bounded by the spherical map $\Gamma_i^{p-1}+\mathscr{E}_i^{p-1}$. Then $F({}_i\mathscr{E}^p)$ is homotopic to a point in $L_p{}^*$ and it follows from Theorem 11 that

$$L_p{}^*+\sum_{i=1}^{r}{}_i\mathscr{E}^p=D^*\Big(L_p{}^*+\sum_{i=1}^{r}\dot{B}_i^{p+1}\Big)\quad(\text{rel. } L_p{}^*),$$

where $(B_1^{p+1}, \ldots, B_r^{p+1})$ is a cluster of $(p+1)$-simplexes attached to $L_p{}^*$. By Theorem 14 there is a deformation of

$$L_p{}^*+\sum_{i=1}^{r}B_i^{p+1}, \tag{8.5}$$

which we also denote by D^*, such that

$$D^*\Big(L_p{}^*+\sum_{i=1}^{r}B_i^{p+1}\Big)=L_p{}^*+\sum_{i=1}^{r}\Big(\mathscr{E}_{t+i}^{p+1}+{}_i\mathscr{E}^p\Big)\quad(\text{rel. } L_p{}^*), \tag{8.6}$$

where $\quad \mathscr{E}_{t+i}^{p+1}=D^*(B_i^{p+1})\quad$ and $\quad F(\mathscr{E}_{t+i}^{p+1})\subset L_p{}^*+\sum_{i=1}^{r}{}_i\mathscr{E}^p.$

Let $$L_p{}^*+\sum_{i=1}^{r}(\mathscr{E}_{t+i}^{p+1}+{}_i\mathscr{E}^p)=L_p^{**}.$$

Clearly $\mathscr{E}_i^{p-1}$ is a sub-complex of ${}_i\mathscr{E}^p$. Therefore $\mathscr{E}_i^{p-1}+{}_i\mathscr{E}^p={}_i\mathscr{E}^p$ and

$$L_p^{**}=L+\sum_{i=1}^{r}{}_i\mathscr{E}^p+\sum_{\lambda=1}^{s}\mathscr{E}_\lambda{}^p+\sum_{\rho=1}^{t+r}\mathscr{E}_\rho^{p+1}. \tag{8.7}$$

From (8.4), from the fact that $L_p{}^*$ expands into (8.5) and from (8.6) we have

$$L_p^{**}=D_0{}^*(K^{p+1})\quad(\text{rel. } L).$$

We now show that L_p^{**} is deformable, rel. L, into a complex L_{p+1}, given by an equation of the form (8.2) with p replaced by $p+1$. As explained in § 6, the cell $\mathscr{E}_i^{p-1}$ is defined by a map without folds. Therefore no two $(p-1)$-simplexes in the original $(p-1)$-sphere of the map

$$\Gamma_i^{p+1}+\mathscr{E}_i^{p-1}=F({}_i\mathscr{E}^p)$$

correspond to the same $(p-1)$-simplex in $\mathcal{E}_i^{p-1}$, and it follows from Lemma 6 that

$$L+\sum_{i=1}^{r}{}_i\mathcal{E}^p$$

contracts into

$$L+\sum_{i=1}^{r}(\mathcal{E}_i^{p-1}-B^{p-1}),$$

where B_i^{p-1} is an open $(p-1)$-simplex in $\mathcal{E}_i^{p-1}$, and so contracts into L†. It follows from Theorem 14 that there is a deformation of

$$L+\sum_{i=1}^{r}{}_i\mathcal{E}^p+\sum_{\lambda=1}^{s}\mathcal{E}_\lambda{}^p,$$

which we denote by $D_1{}^*$, such that

$$D_1{}^*\left(L+\sum_{i=1}^{r}{}_i\mathcal{E}^p+\sum_{\lambda=1}^{s}\mathcal{E}_\lambda{}^p\right)=L+\sum_{\lambda=1}^{s}\tilde{\mathcal{E}}_\lambda{}^p \quad (\text{rel. } L),$$

where

$$\tilde{\mathcal{E}}_\lambda{}^p=D_1{}^*(\mathcal{E}_\lambda{}^p) \quad \text{and} \quad F(\tilde{\mathcal{E}}_\lambda{}^p)\subset L.$$

Therefore it follows from (8.7) and yet another appeal to Theorem 14 that there is a deformation of L_p^{**}, which we also denote by $D_1{}^*$, such that

$$D_1{}^*(L_p^{**})=L+\sum_{\lambda=1}^{s}\tilde{\mathcal{E}}_\lambda{}^p+\sum_{\rho=1}^{t+r}\tilde{\mathcal{E}}_\rho^{p+1} \quad (\text{rel. } L),$$

where

$$\tilde{\mathcal{E}}_\rho^{p+1}=D_1{}^*(\mathcal{E}_\rho^{p+1}) \quad \text{and} \quad F(\tilde{\mathcal{E}}_\rho^{p+1})\subset L+\sum_{\lambda=1}^{s}\mathcal{E}_\lambda{}^p.$$

Therefore

(8.8) $$L_{p+1}=D_{p+1}(K^{p+1}) \quad (\text{rel. } L),$$

where $D_{p+1}=D_1{}^*D_0{}^*$ and

(8.9) $$L_{p+1}=L+\sum_{\lambda=1}^{s}\tilde{\mathcal{E}}_\lambda{}^p+\sum_{\rho=1}^{t+r}\tilde{\mathcal{E}}_\rho^{p+1}.$$

The equations (8.8) and (8.9) are similar to (8.1) and (8.2) with p replaced by $p+1$. Equations of the form (8.1) and (8.2) are obviously satisfied when $p=1$, taking the set $\mathcal{E}_1{}^0, \ldots, \mathcal{E}_r{}^0$ to be empty, and it follows by induction that they are satisfied for all values of p‡. By Lemma 5,

† *Cf.* the proof of Lemma 5.

‡ Notice that the step from $p=1$ to $p=2$ is achieved by (8.4), since $L_1{}^*$ is of the form L_2.

L_{m+2} and (8.3), with $p=m+2$, have the same $(m+1)$-group and (8.3) and L have the same m-group. Therefore K^{m+2} and L have the same m-group. But K and K^{m+2} obviously have the same m-group and the theorem is established.

From the addendum to Theorem 14 and induction on p we have the addendum to Theorem 16:

ADDENDUM. *The order of the deformation D_p in* (8.1) *need not exceed* $p+1$.

9. *Homotopy types.* Two topological spaces, P and Q, are said to belong to the same homotopy type if there is a map $f(P)$ in Q and a map $g(Q)$ in P such that the maps gf and fg, of P and Q into themselves, are each homotopic to the identity.

THEOREM 17. *Two complexes are of the same homotopy type if, and only if, they have the same m-group for each value of m.*

We first show that two given complexes K_1 and K_2, of at most n dimensions, belong to the same homotopy type if they have the same $(n+1)$-group. Let K_1 and K_2 have the same $(n+1)$-group. By Theorem 13

$$K_2+\Sigma_2 = D(K_1+\Sigma_1),$$

where Σ_i is a cluster of $(n+1)$-spheres attached to K_i at a vertex a_i $(i=1, 2)$. Two complexes with the same nucleus are obviously of the same homotopy type, and it follows that there are maps $f_i(K_i+\Sigma_i)$ in $K_j+\Sigma_j$ $(i=1, 2;$ $j=i+1 \bmod 2)$ such that each of f_2f_1 and f_1f_2 is homotopic to the identity. We may take f_i to be semi-linear, in which case, since K_i is at most n-dimensional, $f_i(K_i)$ does not cover the whole of Σ_j. If part of $f_i(K_i)$ lies in Σ_j, it may therefore be deformed into a_j, holding the rest of the map fixed. Therefore f_i and f_j may be deformed into maps f_i^* and f_j^* such that $f_i^*(K_i)\subset K_j$ and $f_j^*f_i^*(K_i)$ is homotopic to the identity in $K_i+\Sigma_i$. But K_i is obviously a retract (not by deformation) of $K_i+\Sigma_i$ and it follows that $f_j^*f_i^*(K_i)$ is homotopic to the identity in K_i. Therefore K_1 and K_2 are of the same homotopy type.

Conversely, let K and L be two complexes of the same homotopy type, which we may assume to be connected, and let $f(K)$ and $g(L)$ be maps of the kind described above. We may suppose that $K.L=1$ and, after a suitable sub-division of K, that the map f is simplicial. By Theorem 8 the mapping cylinder $C_f(K)$ contracts into L. Therefore the theorem will follow from Theorem 16 if we can show that K is a retract by deformation of $C_f(K)$, and hence from Theorem 15 if we can show that K satisfies the conditions B.

Let P be any complex and let $\phi(P) \subset K$ be a map of P in K. When we compare $C_f(K)$ with $K \times \langle 0, 1 \rangle$, it follows from the definition of $C_f(K)$, in § 6, that $\phi(P)$ is homotopic in $C_f(K)$ to the map $f\phi(P) \subset L$. Conversely, any map of the form $f\phi(P) \subset L$, where $\phi(P) \subset K$, is homotopic in $C_f(K)$ to $\phi(P)$. If P is a circuit and p is any point in P, the singular circuit $s+f\phi(P)+s$ is obviously homotopic, rel. p, to $\phi(P)$, where s is the segment $C_f\{\phi(p)\}$.

Let q_0 be any point in L, let $p_0 = g(q_0)$ and let $q_1 = f(p_0) = fg(q_0)$. By Lemma 9 the set of circuits $fg(\Sigma)$ constitutes a geometrical basis for $\pi_1(L, q_1)$, where Σ is any circuit in L beginning and ending at q_0. Since $C_f(K)$ contracts into L, the circuits $fg(\Sigma)$ also constitute a geometrical basis for $\pi_1\{C_f(K), q_1\}$. Therefore the circuits of the form $s+fg(\Sigma)+s$ constitute a geometrical basis for $\pi_1\{C_f(K), p_0\}$, where $s = C_f(p_0)$. But such a circuit is homotopic, rel. p_0, to the circuit $g(\Sigma)$. Therefore the circuits in K which contain p_0 constitute a geometrical basis for $\pi_1\{C_f(K), p_0\}$, and B_1 is satisfied.

Since L is a retract by deformation of $C_f(K)$, any spherical map in $C_f(K)$ is homotopic to a spherical map Σ, in L, and hence to $fg(\Sigma)$, since fg is homotopic to the identity. But $fg(\Sigma)$ is homotopic in $C_f(K)$ to $g(\Sigma) \subset K$ and it follows that B_2 is satisfied.

Finally, let Σ be any spherical map in K which bounds a cell Γ, in $C_f(K)$. Comparing $C_f(K)$ with $K \times \langle 0, 1 \rangle$, we see that L is a retract of $C_f(K)$ by a deformation f_t $(0 \leqslant t \leqslant 1)$ such that $f_1 = f$ in K. Therefore $f(\Sigma) \subset L$ bounds the cell $f_1(\Gamma) \subset L$ and $gf(\Sigma)$ bounds a cell in K, namely $gf_1(\Gamma)$. But gf is homotopic to the identity. Therefore Σ bounds a cell in K and B_3 is satisfied. Therefore K is a retract by deformation of $C_f(K)$ and the theorem is established.

10. *Certain questions: a ring.* In this section we ask certain questions and introduce a ring which is in many ways analogous to Reidemeister's homotopy ring†. The first question is:

Q. 1. *If two complexes of at most n dimensions have the same n-group and the same connectivities, have they the same nucleus?*

† K. Reidemeister, *Abhand. Math. Sem. Hamburg*, 10 (1934), 211–215; *Journal für Math.*, 173 (1935), 164–173, and other papers.

If K and L, being at most n dimensional, have the same n-group it follows from Theorem 13 that

$$L + \sum_{\mu=1}^{s} \dot{B}_{\mu}^{n+1} = D\Big(K + \sum_{\lambda=1}^{r} \dot{A}_{\lambda}^{n+1}\Big),$$

where $A_1^{n+1}, \ldots, A_r^{n+1}$ and $B_1^{n+1}, \ldots, B_s^{n+1}$ are clusters of $(n+1)$-simplexes attached to K and L. Then $r = s$ if $R^n(K) = R^n(L)$, where $R^n(P)$ stands for the n-th connectivity of P, and Q. 1 raises the question :

Q. 2. *If $K_2 = D(K_1)$, $K_2 + L_2 = D_0(K_1 + L_1)$ and if $K_i . L_i$ is geometrically collapsible ($i = 1, 2$), have L_1 and L_2 the same nucleus?*

An affirmative answer to Q. 2 carries with it an affirmative answer to Q. 1. We shall see, in § 12, that Q. 2 is equivalent to the apparently narrower question:

Q. 3. *If M^n, M_1^n, and M_2^n are bounded n-dimensional manifolds†, with connected boundaries, such that M^n meets M_i^n in an $(n-1)$-element on the boundary of both ($i = 1, 2$), and if the manifolds $M^n + M_1^n$ and $M^n + M_2^n$ are combinatorially equivalent, have M_1^n and M_2^n the same nucleus?*

We now leave these questions for the moment and turn to the ring. Let an n-spherical map in a connected‡ (geometrical) complex K be taken as a map $f(C_1^n)$ of a hyper-cube C_1^n, such that $f(\dot{C}_1^n)$ is a constant§ p_1, and let C_1^n be given by $-t_1 \leqslant x_i \leqslant t_1$ ($i = 1, \ldots, n$) in Cartesian space. Let p_0 be any point in K and let s be any oriented segment in K, beginning at p_0 and joining it to p_1, which is given by||

$$p_t = \phi(t) \quad (t_1 \leqslant t \leqslant t_0; \quad p_{t_0} = p_0, \ p_{t_1} = p_1).$$

With the map $f(C_1^n)$ we associate the map ${}_sf(C_0^n)$, such that ${}_sf = f$ in C_1^n and ${}_sf(\dot{C}_t^n) = p_t$ ($t_1 \leqslant t \leqslant t_0$), where C_t^n is given by

$$-t \leqslant x_i \leqslant t \quad (C_{t_0}^n = C_0^n, \quad C_{t_1}^n = C_1^n).$$

We shall denote the map ${}_sf(C_0^n)$ by

$$s + f(C_1^n) - s$$

if $n = 1$, and by

$$s + f(C_1^n)$$

† Here, as in § 12, a manifold is a complex M^n such that M_a^n is an $(n-1)$-sphere or $(n-1)$ element according as a is inside or on the boundary of M^n.

‡ In any discussion involving the homotopy groups of a complex it is always to be understood that the latter is connected.

§ *Cf.* Hurewicz, 2nd paper (*loc. cit.*), § 2.

|| Notice that the parameter t decreases as p_t describes s in the positive direction.

if $n > 1$. It is easy to verify that the transformation $f \to {}_s f$ determines an isomorphism of $\pi_n(K, p_1)$ on $\pi_n(K, p_0)$, where $\pi_n(P, p)$ $(p \in P)$ stands for the geometrical representation of the n-th homotopy group of a space P having p as its base point. If s_1 is a segment joining p_0 to p_1 which is homotopic, rel. (p_0+p_1), to s, it is obvious that ${}_{s_1}f(C_0^n)$ is homotopic, rel. $(\dot{C}_0^n+\dot{C}_1^n)$, to ${}_s f(C_0^n)$. Therefore the transformations $f \to {}_s f$ and $f \to {}_{s_1} f$ determine the same isomorphism $\pi_n(K, p_1) \to \pi_n(K, p_0)$.

If $p_0 = p_1$, the isomorphisms just described are automorphisms of the group $\pi_n(K, p_1)$. To each element g of the fundamental group $\pi_1(K, p_1)$ corresponds an automorphism $\psi_n(g)$, given by

$$f(C_1^n) \to c + f(C_1^n) - c$$

if $n = 1$, and by

$$f(C_1^n) \to c + f(C_1^n)$$

if $n > 1$, where c is an oriented circuit representing the element g. The transformation $g \to \psi_n(g)$ is obviously a homomorphism of $\pi_1(K, p_1)$ in the group of automorphisms of $\pi_n(K, p_1)$, which, in case $n = 1$, is the familiar homomorphism of $\pi_1(K, p_1)$ on its group of inner automorphisms. This homomorphism is invariant, relative to a change of base point, in the same sense that a tensor (not to be confused with its components in any one coordinate system) is invariant under a transformation of coordinates. For if $g' \to \psi'_n(g')$ is the homomorphism defined as above with p_0 as a base point, and if T_k is the isomorphism of $\pi_k(K, p_1)$ on $\pi_k(K, p_0)$ $(k = 1, 2, \ldots)$ determined by a segment joining p_0 to p_1, it may be verified either formally or geometrically that the transformation law of $\psi_n(g)$ is

$$\psi_n'(g') = T_n \psi_n(g) T_n^{-1},$$

where $g' = T_1(g)$. In particular, if $p_0 = p_1$ and $T_k = \psi_k(g_0)$, we have

$$\begin{aligned} \psi_n'(g') &= \psi_n(g_0)\psi_n(g)\psi_n^{-1}(g_0) \\ &= \psi_n(g_0 g g_0^{-1}) \\ &= \psi_n(g'). \end{aligned}$$

With the group of automorphisms of the form $\psi_n(g)$ is associated a ring† $\mathfrak{R}_n = \mathfrak{R}_n(K, p)$, consisting of homomorphisms of $\pi_n = \pi_n(K, p)$ into itself $(n > 1)$. If we rewrite $\psi_n(g)\,a$ as ga, the elements of the ring are

† *Cf.* B. L. van der Waerden, *Moderne Algebra*, 1 (Berlin, 1930), 133.

transformations of the form†

$$a \to ra = \Sigma_i a_i g_i a,$$

where $a \in \pi_n$, $g_i \in \pi_1$, and $a_1, a_2, \ldots$ are rational integers, all but a finite number of which are zero. The addition and multiplication $(r+r')a$ and $rr'a$ are defined in the usual way. If $a_1, \ldots, a_k$ are fixed elements in π_n, the set of all elements of the form

$$r_1 a_1 + \ldots + r_k a_k \quad (r_i \in \mathfrak{R}_n),$$

is a sub-group of π_n, which we shall denote by $r(a_1, \ldots, a_k)$.

Let $K^* = K + \mathfrak{E}_1^{n+1} + \ldots + \mathfrak{E}_k^{n+1}$ $(n > 1)$, where $\mathfrak{E}_i^{n+1}$ is an oriented simple membrane bounded by a (simplicial) map representing the element a_i in π_n, and let $\pi_n^* = \pi_n(K^*, p)$. If a is any element in π_n, represented by an oriented map $f(S^n)$ in K, and if a^* is the element in π_n^* which is represented by the same map $f(S^n)$, the transformation $a \to a^* = \phi(a)$ is obviously a homomorphism of π_n in π^*. It is also obvious that any map $f^*(S^n)$ in K^* is homotopic, rel. q, to a map in K, where $q \in S^n$ and $f^*(q) = p$. Therefore ϕ is a homomorphism of π_n on π_n^*.

THEOREM 18. *The kernel of the homomorphism ϕ is*‡ $r(a_1, \ldots, a_k)$.

If $a \in r(a_1, \ldots, a_k)$, it is obvious that $\phi(a) = 0$. Conversely, let $\phi(a) = 0$, where a is a given element in π_n, let $f(\dot{E}^{n+1})$ be an oriented map in K representing a and $\phi(a)$, and let $f(E^{n+1})$ be an oriented $(n+1)$-cell in K^* bounded by $f(\dot{E}^{n+1})$.

Let $\mathfrak{E}_i^{n+1} = E_i^{n+1} + C_{f_i}(\dot{E}_i^{n+1})$ and let A_i^n be any closed n-simplex in $\dot{E}_i^{n+1}$ whose image $f_i(A_i^n)$ contains p, which we take to be a vertex of K. Let B_i^{n+1} be any principal open simplex in $C_i = C_{f_i}(A_i^n)$ of which p is a vertex. I say that $C_i - B_i^{n+1}$ contracts into $\dot{C}_i = A_i^n + f_i(A_i^n) + C_{f_i}(\dot{A}_i^n)$. For, if $f_i(A_i^n)$ is n-dimensional and $B_i^{n+1} = c_i f_i(A_i^n)$, where c_i is the centre of C_i, this follows from an argument used in proving Theorem 8. Otherwise $B_i^{n+1} = c_i B_i^n$, where B_i^n is a principal open simplex of $C_{f_i}(A_i^{n-1})$ $(A_i^{n-1} \subset \dot{A}_i^n)$. In this case the assertion follows from induction on n, Lemma 3, and an argument used in proving Theorem 9. Therefore $C_i - B_i^{n+1}$ contracts into $\dot{C}_i$ and, by an argument used in proving Lemma 6, $\mathfrak{E}_i^{n+1} - B_i^{n+1}$ contracts

† The (commutative) homotopy groups π_n $(n > 1)$ will always be written additively and the fundamental group with multiplication.

‡ That is to say, $\phi^{-1}(0) = r(a_1, \ldots, a_k)$.

into $F(\mathcal{E}_i^{n+1})$. Let the map $f(E^{n+1})$ be simplicial, $f(\dot{E}^{n+1})$ being the spherical map defining α, and let $A_{i1}^{n+1}, \ldots, A_{ir_i}^{n+1}$ be the open simplexes in E^{n+1} which fall on B_i^{n+1}. Let B_i^{n+1} be oriented so that $\dot{B}_i^{n+1}$ is homotopic in $\mathcal{E}_i^{n+1}-B_i^{n+1}$, with regard to sense, to the oriented map $F(\mathcal{E}_i^{n+1})$, representing α_i, and let $A_{i\lambda}^{n+1}$ be oriented so as to cover B_i^{n+1} positively.

If the set of simplexes $A_{i\lambda}^{n+1}$ is empty, $f(E^{n+1})$ is deformable into a cell in K bounded by $f(\dot{E}^{n+1})$, and it follows that $\alpha = 0$. We suppose, therefore, that the set of simplexes $A_{i\lambda}^{n+1}$ is not empty. After a suitable sub-division† of E^{n+1} we may suppose that no two of the simplexes $A_{i\lambda}^{n+1}$ meet $\dot{E}^{n+1}$ or have a vertex in common. This being so, let q and $q_{i\lambda}$ be points in $\dot{E}^{n+1}$ and $\dot{A}_{i\lambda}^{n+1}$ such that $f(q)=f(q_{i\lambda})=p$. Let $t_{i\lambda}$ be an oriented segment in E^{n+1}, beginning with q and joining it to $q_{i\lambda}$, such that $t_{i\lambda}$ does not meet $Cl(A_{i\lambda}^{n+1})$ except in $q_{i\lambda}$ and does not meet $t_{j\mu}+Cl(A_{j\mu}^{n+1})$ except in q $(t_{j\mu}\neq t_{i\lambda})$. Then

$$E^{n+1}-\sum_{i,\lambda}(t_{i\lambda}+\epsilon_{i\lambda}A_{i\lambda}^{n+1}) \quad (\epsilon_{i\lambda}=\pm 1)$$

may be regarded as a singular cell bounded by the oriented singular n-sphere

$$\dot{E}^{n+1}-\sum_{i,\lambda}(t_{i\lambda}+\epsilon_{i\lambda}\dot{A}_{i\lambda}^{n+1}).$$

Therefore $f(\dot{E}^{n+1})$ is homotopic, rel. q, in

$$K_0{}^{*}=K^{*}-\sum_{i=1}^{k}B_i^{n+1},$$

to

$$\sum_{i,\lambda}(s_{i\lambda}+\epsilon_{i\lambda}\dot{B}_i^{n+1}),$$

where $s_{i\lambda}=f(t_{i\lambda})$, and hence to

$$\sum_{i,\lambda}\{s_{i\lambda}+\epsilon_{i\lambda}F(\mathcal{E}_i^{n+1})\},$$

since $\mathcal{E}_i^{n+1}-B_i^{n+1}$ contracts into $F(\mathcal{E}_i^{n+1})$. Since $\mathcal{E}_i^{n+1}-B_i^{n+1}$ contracts into $F(\mathcal{E}_i^{n+1})$, the complex $K_0{}^{*}$ contracts into K. Therefore each of the circuits $s_{i\lambda}$ is homotopic in $K_0{}^{*}$, rel. q, to a circuit $c_{i\lambda}$ in K, and $f(\dot{E}^{n+1})$ is homotopic in K, rel. q, to

$$\sum_{i,\lambda}\{c_{i\lambda}+\epsilon_{i\lambda}F(\mathcal{E}_{i\lambda}^{n+1})\}.$$

† We can modify E^{n+1} and the map f by applying a sub-division (A, a) to E^{n+1}, making the new vertex a correspond to $f(b)$, where b is a vertex of A. Repeating this process we can isolate the simplexes $A_{i\lambda}^{n+1}$ from $\dot{E}^{n+1}$ and from each other.

Therefore
$$a = \sum_{i,\lambda} \epsilon_{i\lambda} g_{i\lambda} a_i,$$

where $g_{i\lambda}$ is the element in $\pi_1(K, p)$ corresponding to the circuit $c_{i\lambda}$, whence $a \epsilon r(a_1, \ldots, a_k)$ and the theorem is established.

COROLLARY. *π_n^* is isomorphic to the residue group $\pi_n - r(a_1, \ldots, a_k)$.*

Let $\mathfrak{E}_1^{n+1}, \ldots, \mathfrak{E}_k^{n+1}$ be simple membranes bounded by oriented spherical maps in K, which do not necessarily contain p, and let

$$K^* = K + \sum_{i=1}^{k} \mathfrak{E}_i^{n+1}.$$

Let s_i be an oriented segment beginning with p and joining it to some point in $F(\mathfrak{E}_i^{n+1})$, and let a_i be the element in $\pi_n(K, p)$ corresponding to the map $s_i + F(\mathfrak{E}_i^{n+1})$. Then the group $r(a_1, \ldots, a_k)$ is independent of the particular segments s_i. For, if s_i is replaced by $\bar{s}_i$, the map $\bar{s}_i + F(\mathfrak{E}_i^{n+1})$ is homotopic to

$$(\bar{s}_i - s_i + s_i) + F(\mathfrak{E}_i^{n+1}) = (\bar{s}_i - s_i) + \{s_i + F(\mathfrak{E}_i^{n+1})\},$$

and the corresponding element $\bar{a}_i$ is given by $\bar{a}_i = g_i a_i$, where g_i is the element in $\pi_1(K, p)$ corresponding to the circuit $\bar{s}_i - s_i$. It follows that

$$r(\bar{a}_1, \ldots, \bar{a}_k) = r(a_1, \ldots, a_k) = r(f_1, \ldots, f_k), \quad \text{say}.$$

We can deform $\mathfrak{E}_i^{n+1}$ into a simple membrane bounded by the map $s_i + f_i$, without altering any of the groups concerned, and Theorem 18, with its corollary, may be restated in terms of the group $r(f_1, \ldots, f_k)$.

Now let $f_i = f_i(S^n)$ and $\bar{f}_i = \bar{f}_i(S^n)$ $(i = 1, \ldots, k)$ be oriented maps in K and let a_i and $\bar{a}_i$ be the elements in π_n corresponding to the maps $s_i + f_i$ and $\bar{s}_i + \bar{f}_i$, where s_i and $\bar{s}_i$ are segments joining p to points of f_i and $\bar{f}_i$. Also let $\mathfrak{E}_i^{n+1}$ and $\bar{\mathfrak{E}}_i^{n+1}$ be simple membranes bounded by f_i and $\bar{f}_i$, and let

$$(10.1) \qquad K^* = K + \sum_{i=1}^{k} \mathfrak{E}_i^{n+1}, \quad \bar{K}^* = K + \sum_{i=1}^{k} \bar{\mathfrak{E}}_i^{n+1}.$$

Then it follows from Theorem 18 that each map $\bar{f}_i$ is homotopic to a point in K^*, and f_i is homotopic to a point in $\bar{K}^*$, if, and only if, $\bar{a}_i \epsilon r(a_1, \ldots, a_k)$ and $a_i \epsilon r(\bar{a}_1, \ldots, \bar{a}_k)$. That is to say, if, and only if, there are elements r_{ij} and $\bar{r}_{ij}$ in $\mathfrak{R}_n$ such that

$$(10.2) \qquad \begin{cases} \bar{a}_i = \sum_{j=1}^{k} r_{ij} a_j, \\ a_i = \sum_{j=1}^{k} \bar{r}_{ij} \bar{a}_j. \end{cases}$$

If these conditions are satisfied, K^* and $\bar{K}^*$ have the same $(n+1)$-group. For, by Lemma 5, they both have the same $(n+1)$-group as

$$K^*+\bar{K}^* = K+\sum_{i=1}^{k}(\mathfrak{E}_i^{n+1}+\bar{\mathfrak{E}}_i^{n+1}).$$

On the other hand, if two complexes of at most $(n+1)$-dimensions have the same $(n+1)$-group and if their $(n+1)$-st connectivities are the same, it follows from arguments used in proving Theorem 13 that one is transformable into the other by formal deformations and a transformation of the form† $K^* \to \bar{K}^*$, where K^* and $\bar{K}^*$ are given by (10.1), subject to the conditions (10.2). Therefore, if $n > 1$, the question Q. 1 is equivalent to the following:

Q. 4. *Have K^* and $\bar{K}^*$ the same nucleus if the conditions* (10.2) *are satisfied, K being at most $(n+1)$-dimensional?*

We conclude this section with an example of complexes K^* and $\bar{K}^*$ satisfying the conditions (10.2), though there seems to be no easy method for finding out whether or no they have the same nucleus. Let P be a 2-dimensional complex such that $\pi_1(P)$ is a cyclic group of order five. Let A^{n+1} $(n > 1)$ be an oriented $(n+1)$-simplex meeting P in the single point p, and let $K = P+\dot{A}^{n+1}$. Let α be the element in $\pi_n(K, p)$ which is represented by $\dot{A}^{n+1}$ and let g be a generator of $\pi_1(K, p)$. Let $\bar{\mathfrak{E}}^{n+1}$ be a simple membrane bounded by a representative of the element

$$\bar{\alpha} = (1-g-g^4)\,\alpha.$$

It may be verified that $(1-g^2-g^3)(1-g-g^4) = 1$ as a consequence‡ of $g^5 = 1$, whence

$$\alpha = (1-g^2-g^3)\,\bar{\alpha}.$$

If $$K^* = K+A^{n+1} \quad \text{and} \quad \bar{K}^* = K+\bar{\mathfrak{E}}^{n+1},$$

it follows that K^* and $\bar{K}^*$ satisfy the conditions of Q. 4. Since

$$K^* \quad (= K+A^{n+1} = P+A^{n+1})$$

contracts into P, the question is: "Have $\bar{K}^*$ and P the same nucleus?"

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† Replace n by $n+1$ in Theorem 13, take the complex $K_0 - \sum_{\lambda=1}^{r} B_\lambda^{n+1}$ of Theorem 13 for K in (10.1), the simplexes B_λ^{n+1} and A_λ^{n+1} for the membranes $\mathfrak{E}_i^{n+1}$ and $\bar{\mathfrak{E}}_i^{n+1}$ and the complexes K_0 and L_0 of Theorem 13 for K^* and $\bar{K}^*$ in (10 . 1).

‡ I am indebted to Prof. L. J. Mordell for this example.

11. *A special class of groups.* In this section we show that two complexes have the same nucleus if they are of the same homotopy type, provided that their fundamental groups satisfy a certain condition. This condition will be stated in terms of the integral ring $\mathfrak{R}(G)$ of a group G, whose elements are linear forms

$$r = \Sigma_i a_i g_i,$$

where $g_i \in G$ and $a_1, a_2, \ldots$ are rational integers, almost all of which are zero. If $G = \pi_1(K, p)$ and if $\psi_n(r)$ stands for the transformation $\alpha \rightarrow r\alpha$ in $\mathfrak{R}_n(K, p)$ $(n > 1)$, the transformation $r \rightarrow \psi_n(r)$ is obviously a homomorphism of $\mathfrak{R}(\pi_1)$ on $\mathfrak{R}_n = \mathfrak{R}_n(K, p)$. If $\alpha_i \in \pi_n(K, p)$, an element

$$r_1 \alpha_1 + \ldots + r_k \alpha_k$$

in the group $r(\alpha_1, \ldots, \alpha_k)$ may be regarded as a linear form in† $\alpha_1, \ldots, \alpha_k$ with coefficients in $\mathfrak{R}(\pi_1)$, and the elements $\alpha_1, \ldots, \alpha_k$ will be described as *linearly independent* if

$$r_1 \alpha_1 + \ldots + r_k \alpha_k = 0$$

implies $r_1 = \ldots = r_k = 0$. Notice that $\alpha_1, \ldots, \alpha_k$ are linearly independent with coefficients in $\mathfrak{R}(\pi_1)$ if, and only if, the elements $g\alpha_i$ $(g \in \pi_1)$ are linearly independent with integral coefficients; that is to say if $r(\alpha_1, \ldots, \alpha_k)$ is freely generated, with commutative addition, by the elements $g\alpha_i$. If there is a linearly independent set of elements in π_n the homomorphism $r \rightarrow \psi_n(r)$ is an isomorphism. For, if $\alpha_1, \ldots, \alpha_k$ is a linearly independent set, $r_1 \alpha_1 \neq 0$ unless $r_1 = 0$.

Let
$$K^* = K + \sum_{i=1}^{k} \mathcal{E}_i^n \quad (n > 1),$$

where $\mathcal{E}_1^n, \ldots, \mathcal{E}_k^n$ are simple membranes such that $F(\mathcal{E}_i^n) = F(\Gamma_i^n)$, where Γ_i^n is an n-cell in K. Join $\Gamma_1^n, \ldots, \Gamma_k^n$ to a base point p in K and let α_i be the element in $\pi_n^* = \pi_n(K^*, p)$, corresponding to the spherical map $\mathcal{E}_i^n + \Gamma_i^n$, oriented either way. As a complement to Theorem 18 we have

THEOREM‡ 19. $$\pi_n^* = \pi_n + r(\alpha_1, \ldots, \alpha_k),$$

and the elements $\alpha_1, \ldots, \alpha_k$ *are linearly independent.*

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† *Cf.* Reidemeister, *Abhand. Math. Sem. Hamburg* (*loc. cit.*).

‡ If P is a retract of P^* (not necessarily by deformation) $\pi_n = \pi_n(P, p)$ is isomorphic to a sub-group of $\pi_n^* = \pi_n(P^*, p)$, corresponding elements being represented by the same map in P. For simplicity of statement we shall identify each element in π_n with the corresponding element in π_n^*. Here K is a retract of K^*. For, since $F(\mathcal{E}_i^n) = F(\Gamma_i^n)$, the polyhedron $\mathcal{E}_i^n$ may be mapped on Γ_i^n so that each point in $F(\Gamma_i^n)$ corresponds to itself. Actually, as will appear in the proof, no generality is lost in taking Γ_i^n to be a single point.

We have to show, first that any element α^* in π_n^* can be expressed as a sum of the form $r_1\alpha_1+\ldots+r_k\alpha_k+\alpha$, where $\alpha\in\pi_n$, secondly that

$$r_1\alpha_1+\ldots+r_k\alpha_k+\alpha=0$$

implies $r_1\alpha_1+\ldots+r_k\alpha_k=\alpha=0$, and thirdly that $r_1\alpha_1+\ldots+r_k\alpha_k=0$ implies $r_1=\ldots=r_k=0$.

A formal deformation of K^*, rel. K, obviously does not alter

$$\pi_n,\ \pi_n^*,\ r(\alpha_1,\ \ldots,\ \alpha_k)$$

or the relations between them. Therefore we may suppose $\mathcal{E}_i^n$ to be an n-sphere which meets K in the single vertex p. Then K is obviously a retract of K^*, and the projection $\psi(K^*)=K$, such that $\psi(p^*)=p^*$ if $p^*\in K$ and $\psi(p^*)=p$ if $p^*\in\mathcal{E}_i^n$, determines a homomorphism ϕ, of π_n^* on π_n, such that $\phi(\alpha^*)=\alpha^*$ if $\alpha^*\in\pi_n$ and $\phi(\alpha^*)=0$ if $\alpha^*\in r(\alpha_1,\ \ldots,\ \alpha_k)$. Therefore $\phi(r_1\alpha_1+\ldots+r_k\alpha_k+\alpha)=\alpha$, and $r_1\alpha_1+\ldots+r_k\alpha_k+\alpha=0$ implies $\alpha=0$, and hence $r_1\alpha_1+\ldots+r_k\alpha_k=0$.

To prove that any element α^* in π_n^* is a sum of the form

$$r_1\alpha_1+\ldots+r_k\alpha_k+\alpha,$$

let $\mathcal{E}_i^n=\dot{E}_i^{n+1}$, where E_i^{n+1} does not meet K or E_j^{n+1} except in p $(j\neq i)$, and let

$$K^{**}=K^*+\sum_{i=1}^{k}E_i^{n+1}.$$

Then the projection $\psi(K^*)=K$ may be realized in K^{**} by shrinking each of $E_1^{n+1},\ \ldots,\ E_k^{n+1}$ into the point p. It follows first that $\pi_n^{**}=\pi_n$, where $\pi_n^{**}=\pi_n(K^{**},\ p)$, and secondly that the homomorphism $\phi(\pi_n^*)=\pi_n^{**}$ of Theorem 18, with the complexes K and K^* of Theorem 18 replaced by K^* and K^{**}, is the same as the homomorphism $\phi(\pi_n^*)=\pi_n$ of the last paragraph. Since $\phi^2=\phi$ we have

$$\phi\{\alpha^*-\phi(\alpha^*)\}=\phi(\alpha^*)-\phi(\alpha^*)=0,$$

whence $\alpha^*-\phi(\alpha^*)\in r(\alpha_1,\ \ldots,\ \alpha_k)$, by Theorem 18. Since $\phi(\alpha^*)\in\pi_n$, it follows that $\alpha^*=r_1\alpha_1+\ldots+r_k\alpha_k+\alpha$, where α^* is any element in π_n^* and $\alpha\in\pi_n$. Therefore $\pi_n^*=\pi_n+r(\alpha_1,\ \ldots,\ \alpha_k)$.

If $\pi_1(K)=1$, the independence of $\alpha_1,\ \ldots,\ \alpha_k$ follows from one of Hurewicz's theorems†, or directly from the corresponding theorem con-

† *Loc. cit.* (3rd paper), p. 120, Theorem 1.

cerning homology groups. For a spherical map representing the element $m_1 a_1 + \ldots + m_k a_k$, where $m_1, \ldots, m_k$ are integers, is homologous to

$$m_1 \mathfrak{E}_1^n + \ldots + m_k \mathfrak{E}_k^n;$$

and, if $m_1 a_1 + \ldots + m_k a_k = 0$, it follows that $m_1 = \ldots = m_k = 0$. For this argument it is unnecessary that K and k should be finite, or that $\mathfrak{E}_1^n$, $\mathfrak{E}_2^n$, ..., should all meet K in the same point p. Therefore, when $\pi_1(K)$ is arbitrary, the linear independence of $a_1, \ldots, a_k$ with coefficients in $\mathfrak{R}(\pi_1)$ follows by a standard type of argument from the isomorphism between π_n^* and $\pi_n(\tilde{K}^*)$, where $\tilde{K}^*$ is the general covering complex of K^*. Thus the proof is complete.

We now state a certain condition on a group G. A square matrix $\|r_{ij}\|$ $(i, j = 1, \ldots, k)$, whose elements belong to $\mathfrak{R}(G)$, will be said to have a *left inverse*, namely $\|r_{ij}^*\|$, if

$$\sum_{t=1}^{k} r_{it}^* r_{tj} = \delta_{ij}.$$

Our condition on G is that any square matrix with a left inverse can be transformed into the empty matrix, having no rows and columns, by a finite sequence of operations which consist either of:

(1) multiplying each element in a row or column by $\pm g$, where $g \in G$;

(2) interchanging two rows or columns;

(3) adding a "left multiple" of one row to another, the multiplier being any element in $\mathfrak{R}(G)$ (*i.e.* $\rho_i \to \rho_i + \lambda \rho_j$, where $j \neq i$, $\lambda \in \mathfrak{R}(G)$ and ρ stands for the t-th row);

(4) adding a row $(r_{00}, \ldots, r_{0k})$ and a column $(r_{00}, \ldots, r_{k0})$ such that $r_{00} = 1$, $r_{10} = \ldots = r_{k0} = 0$;

or of

(5) removing such a row and column†.

The group consisting of the unit element alone satisfies this condition. For then $\mathfrak{R}(G)$ is isomorphic to the ring of rational integers and a matrix

† Notice that these operations allow us to add a right multiple of one column to another. For we can add an extra row and column such that

$$r_{00} = 1, \quad r_{0j} = -1, \quad r_{0i} = \lambda \quad \text{and} \quad r_{0t} = 0 \quad [t \neq 0,\ i,\ \text{or}\ j;\ \lambda \in \mathfrak{R}(G)],$$

and then add $r_{sj} \rho_0$ to ρ_s for each $s = 1, \ldots, k$. We can then remove the row ρ_0 and the column containing r_{0j} and move the column $r'_{s0} = r_{sj}$ back to the j-th place. The final result is to replace r_{si} by $r_{si} + r_{sj} \lambda$ $(s = 1, \ldots, k)$, leaving the other columns unaltered.

with an inverse is unimodular. It is proved elsewhere† that the condition is satisfied by at least two other groups, namely the cyclic groups of order two and three.

THEOREM 20. *If $\pi_1(K)$ satisfies the above condition and if L is a retract by deformation of K, then*

$$L = D(K) \quad (\text{rel. } L).$$

Let K be n-dimensional and, after an expansion if necessary, let $n > 2$. With the notation used in proving Theorem 16,

$$L_n = L + \sum_{i=1}^{k} \mathcal{E}_i^{n-1} + \sum_{\lambda=1}^{l} \mathcal{E}_\lambda{}^n = D_n(K) \quad (\text{rel. } L),$$

and L is a retract by deformation of L_n. Since L is a retract by deformation of L_n, the connectivities of L_n, calculated mod L in the sense of Lefschetz‡, are all zero. If $l \neq k$, some cycle (mod L), composed either of the cycles (mod L) $\mathcal{E}_i^{n-1}$ or of the chains $\mathcal{E}_\lambda{}^n$, would fail to bound mod L. Therefore $l = k$. Since L is a retract of L_n, the map $F(\mathcal{E}_i^{n-1})$ bounds a cell Γ_i^{n-1} in L. Let $f_i(S^{n-1})$ be the spherical map $\mathcal{E}_i^{n-1} + \Gamma_i^{n-1}$, and let s_i be a segment in L joining some point p to a point $f_i(q)$ in $f_i(S^{n-1})$, where $q \in S^{n-1}$. Let

$$L^* = L + \sum_{i=1}^{k} \mathcal{E}_i^{n-1},$$

and let a_i be the element in $\pi_{n-1}^* = \pi_{n-1}(L^*, p)$ corresponding to the map $s_i + f_i(S^{n-1})$, oriented either way. Join p to a point in $F(\mathcal{E}_i^n)$ by a segment t_i in L and let a_i^* be the element in π_{n-1}^* corresponding to the map $t_i + F(\mathcal{E}_i^n)$. By Theorem 19, $\pi_{n-1}^* = \pi_{n-1} + r(a_1, \ldots, a_k)$, whence

$$a_i^* = \sum_{j=1}^{k} r_{ij} a_j + \beta_i^* \quad [\beta_i^* \in \pi_{n-1} = \pi_{n-1}(L, p)],$$

which we write as

$$(11.1) \qquad a_i^* = \sum_{j=1}^{k} r_{ij} a_j \quad (\text{mod } \pi_{n-1}).$$

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† See a forthcoming paper by G. Higman.

If G is Abelian, the determinant $|r_{ij}|$ can be calculated in the ordinary way, and the elementary transformations can only alter $|r_{ij}|$ by a factor $\pm g$ $(g \in G)$. Therefore the above condition is not satisfied if $\mathfrak{R}(G)$ has a unit ϵ, other than $\pm g$, as one sees by taking $k = 1$ and $r_{11} = \epsilon$ (*e.g.* $r_{11} = 1 - g - g^4$, where $g^5 = 1$).

‡ *Topology* (New York, 1930), p. 17.

Since L is a retract by deformation of L_n each map $f_i(S^{n-1})$ is homotopic in L_n, rel. q, to a map $\bar{f}_i(S^{n-1})$ in L. If β_i is the element in π_{n-1} corresponding to the map $s_i+\bar{f}_i(S^{n-1})$, it follows from Theorem 18 that

$$\alpha_i-\beta_i=\sum_{=1}^{k} r_{ij}^{*}\alpha_j^{*},$$

i.e. that

$$(11.2) \qquad \alpha_i=\sum_{j=1}^{k} r_{ij}^{*}\alpha_j^{*} \pmod{\pi_{n-1}}.$$

From (11.1) and (11.2) we have

$$(11.3) \qquad \Sigma_j(\Sigma_t r_{it}^{*} r_{tj}-\delta_{ij})\alpha_j=0 \pmod{\pi_{n-1}}.$$

Since $\pi_{n-1}^{*}=\pi_{n-1}+r(\alpha_1, \ldots, \alpha_k)$, the left-hand side of (11.3) is zero absolutely, and since $\alpha_1, \ldots, \alpha_k$ are linearly independent, we have

$$\sum_{t=1}^{k} r_{it}^{*} r_{tj}=\delta_{ij}.$$

Therefore the matrix $\|r_{ij}\|$ has a left inverse and by our condition on π_1 it can be reduced to nothing by the operations described above.

If the matrix $\|r_{ij}\|$ has no rows or columns, the sets of membranes $\mathcal{E}_1^{n-1}, \ldots, \mathcal{E}_k^{n-1}$ and $\mathcal{E}_1^{n}, \ldots, \mathcal{E}_k^{n}$ being empty, we have $L_n=L$. Therefore it is enough to show that each of the five kinds of elementary transformation $\|r_{ij}\|\to\|r'_{ij}\|$ can be copied by a formal deformation of L_n, rel. L, which transforms it into a complex L_n' with the matrix $\|r'_{ij}\|$. We take them in order.

(1) The i-th row (column) is multiplied by -1 if we change the orientation of $\mathcal{E}_i^n$ ($\mathcal{E}_i^{n-1}$) and by g if we replace t_i (s_i) by $c+t_i$ ($c+s_i$), where c is a circuit corresponding to g.

(2) To interchange two rows (columns) we merely re-order $\mathcal{E}_i^n(\mathcal{E}_i^{n-1})$.

(3) For the transformation $\rho_i\to\rho_i+\lambda\rho_j$ ($j\neq i$), let $K_i=L_n-I(\mathcal{E}_i^n)$, where $I(\mathcal{E}_i^n)$ is the interior of $\mathcal{E}_i^n$, and let $\bar{\mathcal{E}}_i^n$ be a simple membrane whose boundary is a map representing the element $\alpha_i^{*}+\lambda\alpha_j^{*}$. Then $F(\mathcal{E}_i^n)$ is homotopic to $F(\bar{\mathcal{E}}_i^n)$ in K_i, since $\mathcal{E}_j^n\subset K_i$, and it follows from Theorem 11 that

$$K_i+\bar{\mathcal{E}}_i^n=D(L_n) \quad (\text{rel. } L).$$

Then $L_n'=K_i+\bar{\mathcal{E}}_i^n$ is a complex with the required matrix $\|r'_{ij}\|$.

(4, 5) Since (4) is the inverse of (5), it is sufficient to consider a transformation of type 5. If $r_{i1} = \delta_{i1}$ $(i = 1, \ldots, k)$, it follows from Theorem 19 that the map $F(\mathcal{E}_i^n)$ is homotopic to one which covers an open simplex B^{n-1}, in $\mathcal{E}_1^{n-1}$, just once if $i = 1$ and not at all if $i = 2, \ldots, k$. Therefore we may assume that the maps $F(\mathcal{E}_i^n)$ have this property, in which case L_n contracts into $L_n - I(\mathcal{E}_1^n) - B^{n-1}$, by Lemma 6, and then into

$$L_n' = L_n - I(\mathcal{E}_1^n) - I(\mathcal{E}_1^{n-1}),$$

as in the proof of Lemma 5. Then L_n' has the required matrix

$$\|r_{\lambda\mu}\| \quad (\lambda, \mu = 2, \ldots, k).$$

Thus each type of algebraic transformation can be copied geometrically and the theorem is established.

From the addenda to Theorems 16 and 11 we have the addendum:

ADDENDUM. *The order of the deformation D in Theorem* 20 *need not exceed* $n+1$ $(n > 2)$.

From Theorem 20, with its addendum, and the proof of Theorem 17, we have

THEOREM 21. *If two complexes K and K^* are of the same homotopy type and if $\pi_1(K)$ satisfies the above condition, then*

$$K^* = D(K).$$

If K and K^* are at most n-dimensional†, *the order of the deformation D need not exceed* $n+2$.

12. *Regular neighbourhoods in manifolds.* Let M be an n-dimensional manifold in the sense of Alexander and Newman. That is to say, the complement of any vertex in M is an $(n-1)$-sphere or an $(n-1)$-element according as the vertex in question is inside M or in $\dot{M}$. We recall that this implies the more general condition: the complement of any k-simplex in M is an $(n-k-1)$-sphere or an $(n-k-1)$-element according as the simplex in question is inside M or in $\dot{M}$. We also recall the relation‡ $(M_A)^{\cdot} = \dot{M}_A$, both sides being zero unless $A \in \dot{M}$.

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† Here we need not require $n > 2$. For the relevant dimensionality in Theorem 17 is $\dim \{C_f(K)\} = \dim(K)+1$, and, if $n = 0$ or 1, Theorem 21 may be verified directly.

‡ For let $B \in M_A$. Then $(M_A)_B = M_{AB}$ and $B \in (M_A)^{\cdot}$ if, and only if, $(M_A)_B$ is an element, and $B \in \dot{M}_A$ if, and only if, $AB \in \dot{M}$, i.e. if M_{AB} is an element.

If M is bounded, a transformation of the form

$$M \to M + E^n,$$

where E^n is an n-element which meets M in an $(n-1)$-element on the boundary of both, or the resultant of a finite sequence of such transformations, will be called a *regular expansion* of M. The inverse of a regular expansion will be called a *regular contraction*. As a matter of convention it is to be understood that M expands and contracts regularly into itself. If $M \to M^*$ is a regular expansion or contraction, M^* is combinatorially equivalent† to M.

Let Σ be any set of simplexes, not necessarily closed, in a complex K. By $O(\Sigma, K)$ and $N(\Sigma, K)$ [$= Cl\, O(\Sigma, K)$] we shall mean respectively the set of all open simplexes in K whose closures contain one or more simplexes in Σ, other than 1 in case $1 \in \Sigma$, and the set of all closed simplexes which contain one or more simplexes in Σ, other than 1. Clearly $O(\Sigma, K)$ is open in the sense that $K - O(\Sigma, K)$ is a complex. Notice also that $A \in K_B$ if $O(A, K) . K_B \neq 0$, A being any open simplex in K. For K_B is a complex, and, if A is not in K_B, it is not on the boundary of any simplex in K_B. The distinction between closed and open simplexes is important when the symbols $O(A, K)$ and $N(A, K)$ are used. For, if A is an open simplex, $O(A, K)$ is its open star, while, if A is closed, $O(A, K)$ is the sum of the open stars of all the vertices of A.

Let M be a bounded n-dimensional manifold and let Σ be a set of open simplexes in $\dot{M}$ such that:

(1) if $A \in \Sigma$, all the internal simplexes of M_A are inside M (expressed formally: if $A \in \Sigma$ and $B \in \dot{M} . M_A$, then $AB \in \dot{M}$),

(2) if B is inside M and $A_1 + A_2 \subset \Sigma . M_B$, then $A_1 A_2 \in M_B$.

LEMMA 10. *Under these conditions, the transformation*

$$M \to M - O(\Sigma, M)$$

is a regular contraction.

If $\Sigma = 0$ or 1, there is nothing to prove. Otherwise let $A \neq 1$ be any open simplex in Σ. Then it follows easily enough from the first condition that the boundary of $N(A, M)$ does not meet $\dot{M}$ except in the $(n-1)$-element $N(A, \dot{M})$, and hence that the transformation

$$M \to M^* = M - O(A, M)$$

† Newman, *Proc. Akad. Amsterdam*, 29 (1926), 635, Theorem 8a, or Alexander (*loc. cit.*), 317, Theorem 14.3.

is a regular contraction. Clearly $M - O(\Sigma, M) = M^* - O(\Sigma^*, M^*)$, where $\Sigma^* = \Sigma - A$, and, if we can show that the above conditions are also satisfied by M^* and Σ^*, the lemma will follow by induction on the number of simplexes in Σ.

Consider the second condition first. If B is inside M^*, the complement $M_B{}^*$ is a sphere and so cannot be a proper sub-set of M_B. Therefore

$$M_B = M_B{}^*,$$

and, if $A_1 + A_2 \subset \Sigma^* . M_B{}^* \subset \Sigma . M_B$, it follows from the second condition on M and Σ that $A_1 A_2 \in M_B$. Since $M_B = M_B{}^*$, the second condition is satisfied by M^* and Σ^*. To verify the first condition let $B \in \dot{M}^* . M^*_{A^*}$, where $A^* \in \Sigma^*$, and first let $B \in \dot{M}$. Then $B \in \dot{M} . M_{A^*}$, since $M^* \subset M$, and it follows from the first condition on M and Σ that $A^* B \in \dot{M}$, whence $A^* B \in \dot{M}^*$. If $B \in \dot{M}^* - \dot{M}$, then M_B is a sphere and $M_B{}^*$ is an element. Therefore $M_B{}^*$ is a proper sub-set of M_B, and it follows that

$$O(A, M) . M_B \neq 0,$$

i.e. that $A \in M_B$. But $A^* \in M_B$, since $B \in M^*_{A^*} \subset M_{A^*}$, and it follows that $A + A^* \subset \Sigma . M_B$. Since $B \in M - \dot{M}$, it follows from the second condition on M and Σ that $AA^* \in M_B$, or that $AA^* B \in M$. Therefore $A \in M_{A^* B}$ and $M^*_{A^* B}$ is a proper sub-set of $M_{A^* B}$. It is therefore an element, rather than a sphere, and $A^* B \in \dot{M}^*$. Therefore both conditions are invariant under the transformation $M \to M^*$ and the lemma is established.

Let M_1 be an n-dimensional manifold such that $M . M_1$ consists of one or more unbounded $(n-1)$-dimensional manifolds in $\dot{M} + \dot{M}_1$ (in particular $M . M_1 = \dot{M} = \dot{M}_1$ if $\dot{M}$ and $\dot{M}_1$ are connected). Then the following corollary follows immediately from the proof of the lemma:

COROLLARY 1. *If $\Sigma \in \dot{M} . \dot{M}_1$ and M and Σ satisfy the above conditions, the transformation $M_1 \to M_1 + N(\Sigma, M)$ is a regular expansion.*

It follows from Lemma 4 that $s_{\dot{M}} M$ and $\dot{M}$ satisfy the conditions of Lemma 10 provided that no simplex inside M has all its vertices in $\dot{M}$. Therefore† $s_{\dot{M}'} M'$ and $\dot{M}'$ satisfy them. If K is any complex and $L \subset K$, the sub-division $K \to K'$ is the resultant of $s_L K$ followed by a regular sub-division of L. Also it is obvious that if $M_1 \to M_2$ is a regular contraction (expansion), so is $\sigma M_1 \to \sigma M_2$, where σ is any sub-division of M_1 (of M_2).

† Throughout this section K' and K'' will stand for the first and second derived complexes of a given complex K. In this section it is to be understood that the vertices of K are unaltered by a sub-division of the form $s_L K$.

Finally, if M and Σ satisfy the conditions of Lemma 10, so do M and any sub-set of Σ. Therefore we have a second corollary to Lemma 10:

COROLLARY 2. *The transformation* $M'' \to M'' - O(\Sigma'', M'')$ *is a regular contraction, where* Σ *is any set of simplexes in* $\dot{M}$. *If* M *and* M_1 *satisfy the conditions of Corollary* 1 *and* $\Sigma \in \dot{M} . \dot{M}_1$, *the transformation*

$$M_1'' \to M_1'' + N(\Sigma'', M'')$$

is a regular expansion.

We now come to the main purpose of this section. Let K be a sub-complex of an n-dimensional manifold M. By a *regular neighbourhood* of K in M we shall mean a sub-complex $U(K, M)$, of M, such that:

(1) $U(K, M)$ *is an n-dimensional manifold,*

(2) $U(K, M)$ *contracts geometrically into* K.

It follows from a well-known theorem on sub-division that the first of these conditions, and from the corollary to Theorem 7 that the second, are invariant under a general sub-division of M. The two main theorems are:

THEOREM 22_n. *If* $K \subset M$, *where* M *is an n-dimensional manifold,* $N(K, s_K^2 M)$ *is a regular neighbourhood of* K $(K \neq 1)$.

THEOREM 23_n. *Any two regular neighbourhoods of* K *in* M *are combinatorially equivalent,* M *being any n-dimensional manifold.*

These theorems are trivial if $n = 0$ and we prove them together by induction on n, assuming Theorems 22_{n-1} and 23_{n-1}. First notice the corollary to Theorem 23_n.

COROLLARY 1_n. *If* K *is geometrically collapsible* $U(K, M)$ *is an n-element.*

For if K contracts geometrically into L then $U(K, M)$ is also a regular neighbourhood of L. Taking L to be a single vertex, $N(L, M)$ is an n-element and, being a star, contracts into L. It is, therefore, a regular neighbourhood of L. By Theorem 23_n, $U(K, M)$ is combinatorially equivalent to $N(L, M)$ and is therefore an n-element.

Let $K \subset M$ be any complex other than 1 such that:

(12.1) (*a*) *no simplex in* $M - K$ *has all its vertices in* K,

(*b*) *if* $A \in M - K$, *then* $K . M_A$ *is a single closed simplex (possibly* 1).

If B is any open simplex in M, the complexes $K . M_B$ and M_B also satisfy these conditions. For (12.1*a*) is obviously satisfied, and if $A \in M_B - K$, then $AB \epsilon M-K$ and $K . M_{AB}$ is a single closed simplex. Therefore (12.1b) is satisfied, since $M_{AB}=(M_B)_A$ and $(K . M_B).(M_B)_A = K.(M_B)_A$. I say that, under the conditions (12.1), $N=N(K, M)$ is a regular neighbourhood of K. This will be proved by induction on n, being trivial if $n=0$. By Theorem 2, N contracts into K, and it remains to prove that N_b is an $(n-1)$-sphere or an $(n-1)$-element, where b is any vertex in N. This is certainly the case if $b \subset K$. For then $N_b = M_b$. If $b \not\subset K$ a closed simplex Ab meets K if, and only if, A meets K. Therefore $N_b = N(B, M_b)$, where B is the closed simplex $K . M_b$. By the hypothesis of the induction, $N(B, M_b)$ is a regular neighbourhood of B, and is therefore an $(n-1)$-element by Theorem 23_{n-1}, Corollary 1_{n-1}. Therefore N is a manifold, and hence a regular neighbourhood of K.

By the corollary to Lemma 4 the conditions (12.1) are satisfied by K and $s_K{}^2 M$, where K is any sub-complex of M, and Theorem 22_n follows from Theorem 23_{n-1}.

We shall need two observations for the proof of Theorem 23_n. First, if M is an unbounded n-dimensional manifold and K is any sub-complex of M, then $N(K', M')$ is the aggregate of closed cells in the dual cell-structure (*Zellteilung*) which are dual to the simplexes in K. Therefore $N(K', M')$ consists of the closed n-cells which are dual to the vertices $a_1, \ldots, a_m$ of K. Therefore $N(K', M') = N(a_1+\ldots+a_m, M')$. If we regard $N(K', M')$ as the cell-complex consisting of these dual cells, its $(n-1)$-cells are the duals of the edges in M which have an extremity in K. Such an $(n-1)$-cell is inside $N(K', M')$ if both extremities of the dual edge are in K and on the boundary (mod 2) if only one extremity is in K. If L is any sub-complex of M which does not meet K, it follows that $N(K', M')$ meets $N(L', M')$, if at all, in the aggregate of closed $(n-1)$-cells which are dual to the edges of M with one end in K and the other in L.

Secondly, no simplex in $M'-K'$ has all its vertices in K'. Therefore K' and $s_{K'} M'$ satisfy the conditions (12.1), by Lemma 4, and $N(K', s_{K'} M')$ is a regular neighbourhood of K'. As we have already remarked, the sub-division $M \rightarrow M''$ is the resultant of $s_{K'} M'$, followed by a regular sub-division of K'. Therefore $N(K'', M'')$ is a regular neighbourhood of K'' in M''.

Theorem 23_n will now follow without difficulty from the following lemma. Let M be an n-dimensional, unbounded manifold and K_p a sub-complex of M which contracts formally into K_0.

LEMMA 11. *The manifold $N(K_0'', M'')$ expands regularly into*

$$N(K_p'', M'').$$

Let $K_0, \ldots, K_p$ be a sequence of complexes such that the transformation $K_i \rightarrow K_{i+1}$ is an elementary expansion ($i = 0, \ldots, p-1$). Let

$$K_1 = K_0 + A,$$

where $A = aB$, $a\dot{B} \subset K$, $B \not\subset K$, and let x and y be the vertices of M' which are internal to A' and B' respectively. Then the vertices of K_1' are the vertices of K_0' together with x and y. Therefore, according to the first observation,

$$N(K_1'', M'') = N(K_0''+x+y, M'')$$
$$= N(K_0'', M'')+N(x, M'')+N(y, M'').$$

We shall prove that $N(K_0'', M'')$ expands regularly into

$$N(K_0'', M'')+N(x, M'')$$

and that the latter expands regularly into $N(K_1'', M'')$. The lemma will then follow by induction on p.

Since M' and K_0' satisfy (12.1a), any edge xb, in M', lies in K_1', if $b \in K_1'$. Since A is a principal simplex of K_1 it follows, if $b \in K_1'$, that $b \in \dot{A}'$ and $xb \in A'$. If $b \in K_0'$, it also follows that $b \in (a\dot{B})'$. Therefore $N(K_0'', M'') . N(x, M'') = E^{n-1}$, say, is the aggregate of closed cells in the dual of M' which are dual to $xb_1, \ldots, xb_k$, where $b_1, \ldots, b_k$ are the vertices in $(a\dot{B})'$. By a familiar property of regular sub-division, E^{n-1} is isomorphic to

$$(12.2) \qquad N[\{(a\dot{B})'\}', (M_x')'].$$

The complexes M' and $(a\dot{B})'$, and *a fortiori* M_x' and $(a\dot{B})'$, satisfy (12.1a). It follows from Lemma 4 and the proof of Theorem 22_{n-1} that

$$N\{(a\dot{B})', s_{(a\dot{B})'} M_x'\}$$

is a regular neighbourhood of $(a\dot{B})'$. Therefore (12.2) is a regular neighbourhood of $(a\dot{B})''$ and, by the corollary to Theorem 23_{n-1}, (12.2) and E^{n-1} are $(n-1)$-elements. By our first observation E^{n-1} is in the boundary of $N(K_0'', M'')$ and of $N(x, M'')$, and, since the latter is an n-element, $N(K_0'', M'')$ expands regularly into

$$N(K_0'', M'')+N(x, M'') = N(K_0''+x, M'').$$

By our first observation

$$N(K_0''+x, M'') = N[\{K_0'+x(a\dot{B})'\}', M''],$$

and it follows from the same argument as before, with a replaced by x and x by y, that $N(K_0''+x, M'')$ expands regularly into

$$N(K_0''+x, M'')+N(y, M'') = N(K_1'', M''),$$

and the lemma is established.

Now let $U_1 = U_1(K, M)$ and $U_2 = U_2(K, M)$ be two regular neighbourhoods of K in M. Since the conditions for a regular neighbourhood are intrinsic to $U(K, M)$ we may replace M, in case it is bounded, by an unbounded manifold $M+M^*$, where $M^*.M = \dot{M} = \dot{M}^*$. So, without loss of generality, we assume that $\dot{M} = 0$. Since U_1 contracts geometrically into K there is, by Theorem 7, a stellar sub-division $\sigma_1 M$ such that $\sigma_1 U_1$ contracts formally into $\sigma_1 K$. By the corollary to Theorem 7, $\sigma_1 U_2$ contracts geometrically into $\sigma_1 K$, and by Theorem 7 itself there is a further stellar sub-division $\sigma_2\sigma_1 M$ such that $\sigma_2\sigma_1 U_2$ contracts formally into $\sigma_2\sigma_1 K$. By Theorem 4, $\sigma_2\sigma_1 U_1$ contracts formally into $\sigma_2\sigma_1 K$ and it follows that σU_i $(i = 1, 2)$ contracts formally into σK, where $\sigma = \sigma_2\sigma_1$. So we may assume, after an initial sub-division, that each of the neighbourhoods U_1 and U_2 contracts formally into K, the manifold M being unbounded.

By the second corollary to Lemma 10, U_i'' expands regularly into $N(U_i'', M'')$. Since K expands formally into U_i, it follows from Lemma 11 that $N(K'', M'')$ expands regularly into $N(U_i'', M'')$. Therefore, with the sign of congruence denoting combinatorial equivalence,

$$U_i = U_i'' \equiv N(U_i'', M'') \equiv N(K'', M'').$$

Therefore $U_1 \equiv U_2$ and the theorem is established.

Let K be a given sub-complex of a manifold M, let

$$U_i = U(\gamma_i K, \gamma_i M) \quad (i = 1, 2)$$

be a regular neighbourhood of $\gamma_i K$ in a general sub-division $\gamma_i M$, and let $\gamma_1^*\gamma_1 = \gamma_2^*\gamma_2 = \gamma$ be a common sub-division of γ_1 and γ_2. Since the property of being a regular neighbourhood is invariant under sub-division, $\gamma_i^* U_i$ $(i = 1, 2)$ is a regular neighbourhood of γK in γM. Therefore, $\gamma_1^* U_1 \equiv \gamma_2^* U_2$ and hence $U_1 \equiv U_2$. Therefore the simplicial space associated with a regular neighbourhood of $\gamma_0 K$ in $\gamma_0 M$, where γ_0 is a suitable sub-division of M (*e.g.* $\gamma_0 M = M''$), is uniquely determined by the given complexes M and K. We shall denote it by $\Sigma(K, M)$.

Let P be a (finite) polyhedron imbedded in an n-dimensional, polyhedral manifold† M. A regular neighbourhood of P in M may be defined

† *I.e.* any rectilinear triangulation of M satisfies the combinatorial condition for a manifold.

geometrically as an n-dimensional, polyhedral manifold $U(P, M)$, contained in M and containing P, which contracts geometrically into the latter. This is obviously equivalent to the combinatorial definition stated in terms of a rectilinear triangulation of M with sub-complexes covering P and $U(P, M)$. Therefore Theorems 22 and 23 may be restated geometrically in the form of the corollary:

COROLLARY 3. *Any polyhedron P, in a polyhedral manifold M, has a regular polyhedral neighbourhood in M, and any two such neighbourhoods are semi-linearly homeomorphic.*

Let K be a symbolic complex, let Σ be the simplicial space determined by K and let P be a polyhedron covered by a rectilinear, simplicial complex which is isomorphic to K. In addition to the symbolic complexes which are combinatorially equivalent to K, we shall admit as a representative of Σ any polyhedron which is semi-linearly homeomorphic to P. If P is imbedded in a polyhedral manifold M we shall use $\Sigma(P, M)$ to stand for the simplicial space determined by a regular neighbourhood of P in M. So long as P and the regular neighbourhoods are finite, M may obviously be infinite. In particular M may be Euclidean space.

THEOREM 24. *If P^n and Q^n are semi-linearly homeomorphic, n-dimensional polyhedra in Euclidean p-space R^p, then $\Sigma(P^n, R^p) = \Sigma(Q^n, R^p)$ provided $p \geqslant 2n+3$.*

Let $f(P^n) = Q^n$ be a semi-linear, topological map of P^n on Q^n and let Γ^{n+1} be the locus swept out by the linear segment $pf(p)$ as p describes P^n. Then Γ^{n+1} is an image, which may be singular, of the mapping cylinder $C_f(P^n)$, where we first assume that P^n does not meet Q^n. Then, since $p \geqslant 2n+3$, the singularities in Γ^{n+1} may be removed by simplicial subdivision and slight displacement of the vertices which do not lie on P^n or Q^n. The resulting complex C^{n+1} is a semi-linear topological image of $C_f(P^n)$ and, since f is (1-1), C^{n+1} contracts both into P^n and into Q^n. Therefore a regular neighbourhood of C^{n+1} in R^p is a regular neighbourhood both of P^n and of Q^n. Therefore $\Sigma(P^n, R^p) = \Sigma(Q^n, R^p)$ in case P^n and Q^n do not meet each other. If they do meet, we have

$$\Sigma(P^n, R^p) = \Sigma(P_1{}^n, R^p) = \Sigma(Q^n, R^p),$$

where $P_1{}^n$ is any semi-linear topological image of P^n in R^p which does not meet P^n or Q^n, and the theorem is established.

Any n-dimensional (simplicial) complex K^n may be imbedded in R^p if $p \geqslant 2n+1$, and if $p \geqslant 2n+3$ it follows from Theorem 24 that $\Sigma(K^n, R^p)$ is independent of the way in which K^n is imbedded in R^p. It is also the same for combinatorially equivalent complexes. Therefore we have the corollary:

COROLLARY. *If* $p \geqslant 2n+3$, *the simplicial space* $\Sigma(K^n, R^p)$ *is a combinatorial invariant of* K^n.

Notice that $\Sigma(S^n, R^p)$ is the topological product $S^n \times E^{p-n}$, where S^n is an n-sphere, E^{p-n} a $(p-n)$-element and $p \geqslant 2n+3$. For it is obvious that $\Sigma(S^n, R^p) = \Sigma(S^n, S^p)$, where S^p is a p-sphere, and we can take

$$S^p = (E^{n+1} \times E^{p-n})^{\cdot}$$

$$= \dot{E}^{n+1} \times E^{p-n} + E^{n+1} \times \dot{E}^{p-n},$$

where $\dot{E}^{n+1} = S^n$, and $\dot{E}^{n+1} \times E^{p-n}$ is clearly a regular neighbourhood of $\dot{E}^{n+1}$.

THEOREM 25. *If* $L^m = D(K^n)$, *where* K^n *and* L^m *are complexes of* n *and* m *dimensions and* D *is a formal deformation of order* k, *then*

$$\Sigma(K^n, R^p) = \Sigma(L^m, R^p)$$

provided that $p \geqslant 2l+1$, *where* $l = \max(m+1, n+1, k)$.

By Theorem 5, Addendum 1, there is an l-dimensional complex K^*, which contracts formally into both K^n and a sub-division of L^m. Imbedding K^* in R^p, where $p \geqslant 2l+1$, it follows, as in Theorem 24, that

$$\Sigma(K^n, R^p) = \Sigma(K^*, R^p) = \Sigma(L^m, R^p),$$

and the theorem is established.

From Theorem 21 we have the corollary to Theorem 25:

COROLLARY 1. *If* $\pi_1(K^n)$ *satisfies the condition imposed in* § 11 *and if* K^n *and* L^m *are of the same homotopy type, then* $\Sigma(K^n, R^p) = \Sigma(L^m, R^p)$ *provided that* $p \geqslant 2\max(m, n)+5$.

As a special case of Corollary 1 we have:

COROLLARY 2. *If* $\pi_1(K^n)$ *satisfies the condition imposed in* § 11, *the simplicial space* $\Sigma(K^n, R^p)$ *is a topological invariant of* K^n *provided that*

$$p \geqslant 2n+5.$$

Combining Corollary 1 with Hurewicz's results we have:

COROLLARY 3. *If* $\pi_1(K^n) = 1$ *and all the homology groups* $\beta_i(K^n)$ *vanish* $(i = 2, \ldots, n)$, *then* $U(K^n, R^p)$ *is a* p-*element* $(p \geqslant 2n+5)$.

From a previous remark we have also:

COROLLARY 4. *If* $\pi_1(K^n)=1$, $\beta_i(K^n)=\beta_i(S^n)$ $(i=2, \ldots, n)$, S^n *being an n-sphere, then* $U(K^n, R^p) = S^n \times E^{p-n}$ $(p \geqslant 2n+5)$.

We now return to the questions in § 10. If K and L are sub-complexes of a manifold H^n, which is either an n-sphere or an n-element, we shall say that K can be *freed isotopically* from L if there is an n-element in some sub-division σH^n, which contains σK and does not meet σL.

LEMMA 12. *If* $K, L \subset H^n$, $K . L = 1$ *and* $n > r+s+1$, *where* $r = \dim(K)$ *and* $s = \dim(L)$, *then K can be freed isotopically from L.*

If H^n is an n-sphere, it contains an n-element containing K and L. Therefore we may suppose that $H^n = E^n$ and, after a suitable sub-division, E^n may be represented as a simplicial covering of a rectilinear simplex Δ^n. Let p_0 be a point in Δ^n whose position is general with respect to K and L. Since K does not meet L and $r+s+1<n$, the cone C, swept out by the segment $p_0 p$ as p varies over K, does not meet L. After a further sub-division we may suppose that C is covered by a sub-complex of E^n, which we also denote by C. Then $N = N(C'', E''^n)$ does not meet L''. But N is a regular neighbourhood of C'', and C'', being a sub-division of the star $p_0 K$, contracts into p_0. Therefore N is an n-element, by Theorem 23, Corollary 1, and the lemma is established.

If two bounded, connected, n-dimensional manifolds M_1 and M_2 meet in an $(n-1)$-element on the boundary of both, we shall describe the simplicial space associated with M_1+M_2 as a *topological sum* of the simplicial spaces associated with M_1 and M_2. Let K_1 and K_2 be two sub-complexes of an n-dimensional manifold M, which meet in a single vertex b.

THEOREM 26. *If* K_{1b}, *the complement of b in* K_1, *can be freed isotopically from* K_{2b} *in* M_b, *the simplicial space* $\Sigma(K_1+K_2, M)$ *is a topological sum of the simplicial spaces* $\Sigma(K_1, M)$ *and* $\Sigma(K_2, M)$.

Adding an n-element of the form bE^{n-1} to M, if necessary, we may suppose that b is inside M. After a suitable sub-division we also assume that M_b does not meet $\dot{M}$, also that $K_{1b} \subset E_1^{n-1} \subset M_b$, where E_1^{n-1} does not meet K_{2b}. Let $E_2^{n-1} = Cl(M_b - E_1^{n-1})$ and let

$$M_i = Cl(M - bE_j^{n-1}) \quad (i=1, 2;\ j=i+1 \bmod 2).$$

Then $K_i \subset M_i$ and M_i is a manifold since M_b does not meet $\dot{M}$. Let

$$N_i = N(K_i'', M_i'').$$

Then N_i is a regular neighbourhood of K_i'' in M_i'' and therefore in M'', since the conditions for a regular neighbourhood are intrinsic. Therefore N_i is a triangulation of the simplicial space $\Sigma(K_i, M)$. Moreover, N_1+N_2 consists of all the closed simplexes in $M_1''+M_2''=M''$ which meet $K_1''+K_2''$. That is to say $N_1+N_2=N(K_1''+K_2'', M'')$, and N_1+N_2 is a triangulation of $\Sigma(K_1+K_2, M)$. The intersection $N_1 . N_2$ consists of the closed simplexes in $M_1''. M_2''$ which meet $K_1''. K_2''=b$. But

$$M_1 . M_2 = E+\{M-O(b, M)\},$$

where $E=b\dot{E}_1^{n-1}=b\dot{E}_2^{n-1}$. Therefore $N_1 . N_2$ is an $(n-1)$-element, namely $N(b, E'')$. Finally $E \subset \dot{M}_1 . \dot{M}_2$, whence $N_1 . N_2 \subset \dot{N}_1 . \dot{N}_2$, and the theorem is established.

In Theorem 26 the manifold M may be infinite, K_1 and K_2 being finite. In particular M may be a triangulation of R^n.

THEOREM 27. *The questions* Q. 2 *and* Q. 3 *of* § 10 *are equivalent.*

It is clear that an affirmative answer to Q. 2 carries with it an affirmative to Q. 3. It remains to show that an affirmative to Q. 3 implies an affirmative to Q. 2. Let $K_2=D(K_1)$, $K_2+L_2=D_0(K_1+L_1)$ and let $P_i=K_i . L_i$ be geometrically collapsible. After a suitable sub-division we may assume that P_i is formally collapsible and that $N(P_i, K_i)$, $N(P_i, L_i)$ and therefore $N(P_i, K_i+L_i)$ are contractible neighbourhoods of P_i. By Theorem 3 the operation of shrinking P_i into a point is a formal deformation of K_i, L_i and of K_i+L_i. Therefore we may take $K_i . L_i$ to be a single vertex b_i. This being so, let l be the maximum of m_i+1, n_i+1 and the orders of the deformations D and D_0, where $n_i=\dim(K_i)$ and $m_i=\dim(L_i)$. After a suitable sub-division, K_i+L_i may be imbedded in R^p, as a sub-complex of some triangulation M^p, of R^p, where $p \geqslant 2l+1$. Since $p-1>(m_i-1)+(n_i-1)+1$ it follows from Lemma 12 that K_{ib_i} may be separated isotopically from L_{ib_i} in $M_{b_i}^p$, and from Theorem 26 that $\Sigma(K_i+L_i, R^p)$ is a topological sum of $\Sigma(K_i, R^p)$ and $\Sigma(L_i, R^p)$, Since $p \geqslant 2l+1$, it follows from Theorem 25 that $\Sigma(K_1, R^p)=\Sigma(K_2, R^p)$ and that $\Sigma(K_1+L_1, R^p)=\Sigma(K_2+L_2, R^p)$. Moreover the boundaries of regular neighbourhoods $U(K_i, R^p)$ and $U(L_i, R^p)$ are connected. For these neighbourhoods contain no non-bounding $(p-1)$-cycles since $m_i, n_i<p-1$. If the answer to Q. 3 is "yes" it follows that any two regular neighbourhoods $U(L_1, R^p)$ and $U(L_1, R^p)$ have the same nucleus and, since $U(L_i, R^p)$ contracts into L_i, that L_1 and L_2 have the same nucleus. Therefore an affirmative answer to Q. 3 implies an affirmative answer to Q. 2, and the theorem is established.

13. *Newman's moves.* The object of this section is to prove two theorems which are similar to Theorems 4 and 6 with elementary expansions and reductions replaced by Newman's moves of Type 1 and 2. We recall that a move of type 1, applied to an n-dimensional manifold M_0, is a regular expansion of the form

$$M_0 \rightarrow M_1 = AB + M_0,$$

where AB is n-dimensional, $\dot{A}B \subset \dot{M}_0$, $A \not\subset M_0$. A move of Type 2 is the inverse of a move of type 1. For $M \rightarrow Cl(M-C)$ to be a move of type 2 it is necessary and sufficient that $C = AB \subset M$, where

1. $B = M_A$,

2. B is internal to M.

An n-element E^n will be described as *regularly collapsible* if $E^n \underset{2}{\rightarrow} A^n$, where A^n is a closed n-simplex and the symbol $M \underset{\alpha}{\rightarrow} M$ ($\alpha = 1$ or 2) means that the transformation indicated is a product of moves of type α. As a matter of convention $M \underset{\alpha}{\rightarrow} M$, so that a closed simplex is regularly collapsible.

LEMMA 13. *If $M = AE + M^*$ and if $E \underset{2}{\rightarrow} E_q$, then $M \underset{2}{\rightarrow} AE_q + M^*$, provided that every internal simplex in E is inside M.*

If $E_q = E$ there is nothing to prove. Otherwise let $E = CB + E_1$, where $E \rightarrow E_1$ is the first move in the transformation $E \underset{2}{\rightarrow} E_q$ and $B = E_C$, B being inside E. Then $B = M_{AC}$ and B is inside M. Therefore

$$M \rightarrow Cl(M - ABC) = AE_1 + M^*$$

is a move of type 2, and the lemma follows by induction on the number of moves in $E \underset{2}{\rightarrow} E_q$.

COROLLARY. *If E is regularly collapsible, $M \underset{2}{\rightarrow} M^*$.*

For if $E = B$, a single closed simplex which is inside M, the transformation $AB + M \rightarrow M^*$ is a move of type 2.

LEMMA 14. *If $E \underset{2}{\rightarrow} E_q$, where E is an element, then $AE \underset{2}{\rightarrow} AE_q$, where A is a closed simplex which does not meet E.*

For, with the notation used in proving Lemma 13, the simplex AB is inside AE and is the complement of C. Therefore $AE \rightarrow AE_1$ is a move of type 2 and the lemma follows from induction on the number of moves in $E \underset{2}{\rightarrow} E_q$.

COROLLARY. *If E is regularly collapsible so is AE.*

LEMMA† 15. *If A and B are (closed) simplexes which do not meet each other ($A \neq 1$, $B \neq 1$), $A\dot{B}$ is regularly collapsible.*

If B is 0-dimensional, $A\dot{B} = A$. Otherwise let $B = bB_1$ ($B_1 \neq 1$). Then

$$A\dot{B} = A(b\dot{B}_1 + B_1).$$

The simplex A is inside $A\dot{B}$ and is the complement of B_1 in $A\dot{B}$. Therefore the transformation $A\dot{B} \to Ab\dot{B}_1$ is a move of type 2. Writing $Ab = A_1$, and assuming that $A_1 \dot{B}_1$ is regularly collapsible, we deduce the lemma by induction on dim (B).

THEOREM 28. *If $M \underset{2}{\to} M_1$, then*

$$\sigma M \underset{2}{\to} \sigma M_1,$$

where σ is any stellar sub-division of M.

This will follow by an inductive argument similar to the one used in proving Theorem 4 if we can prove it in case $M \underset{2}{\to} M_1$ is a single move and σ is an elementary sub-division (A, a).

Let $$M = CB + M_1,$$

where $B = M_C$ and B is inside M. As in Theorem 4 the result is obvious unless $A \subset BC$. So let $A = B_1 C_1$, where $B = B_1 B_2$ and $C = C_1 C_2$ (possibly $B_1 = 1$ or $C_1 = 1$). Then

$$\begin{aligned} \sigma M &= a(C_1 \dot{B}_1 + \dot{C}_1 B_1) B_2 C_2 + \sigma M_1 \\ &= Ca\dot{B}_1 B_2 + aC_2 B\dot{C}_1 + \sigma M_1, \end{aligned} \tag{13.1}$$

and $a\dot{B}_1 B_2 = (\sigma M)_C$, since $C \not\subset M_1$ and hence $C \not\subset \sigma M_1$. The simplex BC_1 is inside M, being incident with B, and

$$\begin{aligned} \sigma(BC_1) &= \sigma(B_1 B_2 C_1) \\ &= \sigma(AB_2) \\ &= a\dot{A}B_2. \end{aligned}$$

Therefore aB_2 is inside $\sigma(BC_1)$ and hence inside σM. The internal simplexes of $a\dot{B}_1 B_2$ are those incident with aB_2 and are therefore inside σM. By

† *Cf.* Newman, *Proc. Akad. Amsterdam* (second paper, *loc. cit.*), 619, Theorem 20.

Lemma 15, $a\dot{B}_1 B_2$ is regularly collapsible and, by the corollary to Lemma† 13,

$$\sigma M \underset{2}{\rightarrow} aC_2 B\dot{C}_1 + \sigma M_1 = M^*, \tag{13.2}$$

say.

If $C_1 = 1$, we have $M^* = \sigma M_1$ and the proof is complete. So we assume that $C_1 \neq 1$. Then $B \subset \sigma M$ and B is inside σM, since it is inside M. Also $M_B^* = (\sigma M)_B$, since $B \not\subset Ca\dot{B}_1 B_2$. It follows that B, and therefore the internal simplexes of $B\dot{C}_1$, are inside M^*. Also $B_1 C \not\subset M_1$, since $C \not\subset M_1$, and

$$\begin{aligned} \sigma(B_1 C) &= \sigma(B_1 C_1 C_2) \\ &= \sigma(AC_2) \\ &= a\dot{A} C_2. \end{aligned}$$

Therefore aC_2 is inside $\sigma(B_1 C)$ and it follows that $aC_2 \not\subset \sigma M_1$. Therefore $M^*_{aC_2} = B\dot{C}_1$ and is regularly collapsible by Lemma 15. Since the internal simplexes of $B\dot{C}_1$ are inside M^* it follows from the corollary to Lemma 13 that $M^* \underset{2}{\rightarrow} \sigma M_1$. Therefore $\sigma M \underset{2}{\rightarrow} \sigma M_1$ and the theorem is established.

In order to state the next theorem we shall extend our notation by writing

$$E^n \underset{2}{\rightarrow} E^{n-1} \quad (n > 0)$$

if $E^{n-1} \subset \dot{E}^n$ and

$$M + E^n \underset{2}{\rightarrow} M,$$

where M is any n-dimensional manifold such that $M \,.\, E^n = \dot{M} \,.\, \dot{E}^n = E^{n-1}$.

THEOREM 29. *If K is any complex, there is a stellar sub-division σK such that $\sigma E^m \underset{2}{\rightarrow} \sigma E^{m-1}$ $(m > 0)$, where E^m is an arbitrary m-element in K, and E^{m-1} is an arbitrary $(m-1)$-element in $\dot{E}^m$, the sub-division σ being independent of any particular choice of m or of E^m and E^{m-1}.*

There are in K only a finite number of elements and the theorem will follow from Theorem‡ 28 if we can show that there is a sub-division σE^m, such that $\sigma E^m \underset{2}{\rightarrow} \sigma E^{m-1}$, where E^m is a given m-element and E^{m-1} a given $(m-1)$-element in $\dot{E}^m$. Assuming that this is true when $0 < m \leqslant n$ we shall prove the following corollary for $0 \leqslant m \leqslant n$.

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† Notice that the first term on the right-hand side of (13 . 1) is absent if $B_1 = 1$. In this case $M = M^*$.

‡ *Cf.* the corollary to Theorem 4.

Corollary. *Some stellar sub-division of any m-element is regularly collapsable.*

The corollary is trivial if $m = 0$ and we therefore take $m > 0$. After a preliminary sub-division, if necessary, we may assume that the internal simplexes of E_b^m are inside E^m, where E^m is a given m-element and b is a vertex in $\dot{E}^m$. Then

$$E_0^m = E^m - O(b, E^m)$$

is an m-element and it follows from the above hypothesis that there is a stellar sub-division σ_1 such that $\sigma_1 E_0^m \underset{2}{\rightarrow} \sigma_1 E_b^m$. Let us assume that some sub-division $\sigma_2 \sigma_1 E_b^m = \sigma E_b^m$ is regularly collapsible. Then

$$b(\sigma E_b^m) = \sigma(b E_b^m)$$

is regularly collapsible, by the corollary to Lemma 14, and by Theorem 28 $\sigma E^m \underset{2}{\rightarrow} \sigma(b E_b^m)$, where $\sigma = \sigma_2 \sigma_1$. Therefore σE^m is regularly collapsible and the corollary follows by induction on m for $m \leqslant n$.

Now let $m = n+1$ and let E^m be a partition of the geometrical simplex $p_0 p_1 A^n$ $(0 \leqslant n = m-1)$ having no internal vertices, and let E_i^n be the sub-complex of E^m covering $p_i A^n$ $(i = 0, 1)$. I say that $\sigma E^m \underset{2}{\rightarrow} \sigma E_0^n$, where σ is some stellar sub-division of E^m. For let $L = \dot{E}^m = E_0^n + E_1^n$ and let F^n and F^m mean the same as in the proof of Theorem 6. If we assume Theorem 29 for n-dimensional complexes, it follows from the corollary that there is a stellar sub-division $\sigma_1 F^n$ such that $\sigma_1 E$ is regularly collapsible, where E is any element in F^n. As in the proof of Theorem 6, $F^n . (s_L E^m)_B$ is an element, where B is any internal open simplex of E_1^n. Therefore $\sigma\{F^n . (s_L E^m)_B\} = \sigma_1 F^n . (\sigma E^m)_B$ is regularly collapsible, where $\sigma = \sigma_1 s_L$. If we take the open simplexes inside $\sigma_1 E_1^n (= \sigma E_1^n)$ in order of decreasing dimensionality, it follows from the corollary to Lemma 13 that the open stars $O(B, \sigma E^m)$ $(B \epsilon \sigma E_1^n - \sigma \dot{E}_1^n)$ may be removed successively by moves of type 2. Therefore $\sigma E^m \underset{2}{\rightarrow} \sigma_1 F^m$. Let M and M_1 be any m-dimensional manifolds such that $M . M_1 = \dot{M} . \dot{M}_1$ and let $E^n (n = m-1)$ be an $(m-1)$-element in $\dot{M} . \dot{M}_1$. If $M \rightarrow Cl(M - AB)$ is a move of type 2, where A is inside E^n and B is inside M, the transformation $M_1 \rightarrow M_1 + AB$ is a move of type 1. If we take $M = \sigma E^m$ and $E^n = \sigma E_1^n$, it follows that $M_1 \underset{1}{\rightarrow} M_1 + \sigma_1 F_1^m$, where $F_1^m = Cl(s_L E^m - F^m)$. Therefore

$$M_1 + \sigma_1 F_1^m \underset{2}{\rightarrow} M_1 \quad \text{i.e.} \quad \sigma_1 F_1^m \underset{2}{\rightarrow} \sigma E_1^n.$$

But the construction of F^n and σ_1 is symmetrical between E_0^n and E_1^n.

Therefore $\sigma_1 F^m \underset{2}{\rightarrow} \sigma E_0{}^n$, and since $\sigma E^m \underset{2}{\rightarrow} \sigma_1 F^m$ we have $\sigma E^m \underset{2}{\rightarrow} \sigma E_0{}^n$, as stated. The rest of the proof is the same as the proof of Theorem 6.

COROLLARY. *If M contracts regularly into M^* there is a stellar sub-division σ such that $\sigma M \underset{2}{\rightarrow} \sigma M^*$.*

14. *Infinite complexes.* Starting with a given aggregate of vertices, we now define a complex as in § 2, with the single condition that the dimensionality of each simplex is to be finite. For example, if we take the vertices to be real numbers, the totality of finite sets of real numbers is a permissible complex. In order that sub-divisions and expansions shall be applicable to such a complex, we allow ourselves to create new vertices if and when they are needed. More precisely, we assume that a given aggregate of vertices can be duplicated at any stage in an argument, and the duplicate set combined with the original into a single set. We rely on the axiom of choice but, except at one stage in the proof of Theorem 37, only so far as sub-sets of the given vertices are concerned. For, if the original aggregate of vertices is well-ordered, choice can be eliminated from the combinatorial constructions by lexicographical and other standard devices.

We proceed to extend the definitions of equivalence to infinite complexes†. For combinatorial equivalence we shall use the idea of general sub-division rather than elementary transformations. We recall that a complex γK is a *general sub-division* of K if the simplexes of γK are grouped into k-elements γA^k ($k = -1, 0, 1, \ldots$; $\gamma 1 = 1$), which are in a (1-1), incidence-preserving correspondence $\gamma A^k \rightarrow A^k$ with the closed simplexes in K. We shall write $\gamma = 1$ if γ is the identical sub-division, given by $\gamma A = A$ for each closed simplex in K. It is always to be understood that no vertex in γK belongs to K, or to any complex which is being considered simultaneously with K, unless it is a 0-simplex A^0, in K, and $\gamma A^0 = A^0$. If L is any sub-complex of K, the cells in γK which correspond to the simplexes in L constitute a sub-division of L, which we denote by γL. Thus γ may be regarded as a transformation which operates on each sub-complex of K. Conversely, let γL be given, where L is a sub-complex of K. We define γK as follows. Let K^n be the complex consisting of

† A little care is needed here, as the following example shows. Let K_i be a triangulation of the topological product $A^n \times \langle i, \infty \rangle$ ($i = 0, 1, \ldots$), together with the rest of the cylinder $C = \dot{A}^n \times \langle 0, \infty \rangle$. Let c_i be the geometrical contraction $K_i \rightarrow K_{i+1}$. Then the infinite sequence of contractions $c_0, c_1, \ldots$ transforms K_0 into C, and therefore alters the homotopy group $\pi_{n-1}(K_0)$.

L together with all the simplexes of K whose dimensionalities do not exceed n. Assuming that $\gamma_n K^n$ has been defined and that $\gamma_n = \gamma$ in L (*i.e.* $\gamma_n A = \gamma A$ if $A \subset L$) and $\gamma_n A = A$ if $\gamma = 1$ in $\dot{A}.L$, we define $\gamma_{n+1} K^{n+1}$ by the conditions $\gamma_{n+1} = \gamma_n$ in K^n, and

$$\gamma_{n+1} A^{n+1} = A^{n+1} \quad \text{if} \quad \gamma_n = 1 \quad \text{in} \quad \dot{A}^{n+1}$$
$$= a\gamma_n \dot{A}^{n+1} \quad \text{otherwise,}$$

where A^{n+1} is the closure of any open simplex in $K^{n+1} - L$ and a is a new vertex. If we begin with $\gamma_{-1} = \gamma$ in $K^{-1} = L$, the sub-division $\gamma_n K^n$ is thus defined inductively for all values of n. Clearly $\gamma_m = \gamma_n$ in K^n if $m > n$, and γK is defined by the condition

$$\gamma A = \gamma_n A \quad \text{if} \quad A \subset K^n,$$

this definition being unique except for the choice of the new vertices. A given set of complexes L, L^*, ..., may be combined to form a single complex K. If γL is given, it follows from what we have said that γ may be treated as an operator which is applicable to K, and therefore to any complex in the set. If γ is initially defined as a sub-division of L and if $\gamma = 1$ in LL^*, notice that $\gamma = 1$ in L^*.

A sub-division γK will be described as a partition† πK, or a stellar sub-division, if it is a partition, or a stellar sub-division, of each finite sub-complex of K. With the notation explained in the last paragraph, if a given sub-division γK (or γL) is a partition, or a stellar sub-division, it is obvious that γL (or γK) is also a partition, or a stellar sub-division, where L is any sub-complex of‡ K. If K is infinite, the sub-division $s_L K$ cannot be defined as a sequence of elementary sub-divisions since there are, in general, no simplexes of highest dimensionality with which to start. We define $s_L K$ inductively by the construction used in extending γL to γK, starting with K^0 and a sub-division $\gamma K^0 = s_L K^0$ ($s_L = 1$ in L). If $A^0 \subset K^0 - L$, it is to be a matter of choice whether or no $s_L A^0 = A^0$.

LEMMA 16. *Two sub-divisions $\gamma_1 K$ and $\gamma_2 K$ have a common sub-division $\gamma_1^* \gamma_1 = \gamma_2^* \gamma_2$.*

Let K^n be the *n-dimensional skeleton* (*Gerüst*) of K, that is the set of all simplexes whose dimensionalities do not exceed n. Assume that there

† *Cf.* Whitehead, *loc. cit.*

‡ If σL was given as a sequence of elementary sub-divisions this would differ from the natural definition of σK. But we shall adhere to the single definition of γK, whether γ is a stellar sub-division or not.

are sub-divisions γ_1^n and γ_2^n such that

$$(14.1) \qquad \begin{cases} \gamma_1^n \gamma_1 = \gamma_2^n \gamma_2 & \text{in} \quad K^n, \\ \gamma_i^n = \gamma_i^{n-p} & \text{in} \quad \gamma_i K^{n-p} \quad (i = 1, 2;\ p \geqslant 0). \end{cases}$$

Since $$\gamma_1^n \gamma_1 \dot{A}^{n+1} = \gamma_2^n \gamma_2 \dot{A}^{n+1},$$

where A^{n+1} is any $(n+1)$-simplex in K, there are sub-divisions γ_1^{n+1} and γ_2^{n+1} such that†

$$\gamma_1^{n+1} \gamma_1 A^{n+1} = \gamma_2^{n+1} \gamma_2 A^{n+1},$$

where $\gamma_i^{n+1} = \gamma_i^n$ in $\gamma_i \dot{A}^{n+1}$ and therefore in $\gamma_i K^n$. Extending γ_i^n in this way throughout all the $(n+1)$-simplexes in K we arrive at sub-divisions which satisfy (14.1) with n replaced by $n+1$. If we begin with $\gamma_i^{-1} = \gamma_i = 1$ in K^{-1}, it follows by induction on n that there are sub-divisions satisfying (14.1) for all values of n. The required sub-divisions γ_1^* and γ_2^* are given by

$$\gamma_i^* \gamma_i A = \gamma_i^n \gamma_i A \quad \text{if} \quad A \subset K^n,$$

and the lemma is established.

From the sharper results proved in my paper on sub-divisions we have the addendum:

ADDENDUM. *The sub-divisions γ_1^* and γ_2^* of Lemma* 16 *may be chosen so that a given one of them is a stellar sub-division and the other is a partition*‡. *If $\gamma_1 = \gamma_2 = \gamma$ in some sub-complex L, they may be chosen so that also $\gamma_1^* = \gamma_2^* = \gamma$ in L.*

We now define two complexes K_1 and K_2 as *combinatorially equivalent* if, and only if, they have a common sub-division $\gamma_1 K_1 = \gamma_2 K_2$. It follows from Lemma 16 that two complexes are combinatorially equivalent to each other if each is combinatorially equivalent to a third. Therefore the equivalence classes are mutually exclusive.

We now come to formal deformations and fillings and perforations. Our method, which we have already used in proving Lemma 16, is to replace "long" sequences of individual elementary transformations (*i.e.* sequences with high ordinal numbers) by countable sequences of

† Alexander, *loc. cit.*, Theorem 13.2, and Whitehead, *loc. cit.*, Theorem 2.

‡ Taking $\gamma_2 = 1$ and $\gamma_1^* = \sigma$, $\gamma_2^* = \pi$, we have $\sigma\gamma_1 = \pi$, and taking $\gamma_1^* = \pi$, $\gamma_2^* = \sigma$, we have $\pi\gamma_1 = \sigma$.

"composite" transformations. Though each composite set may have any cardinal number, the individual transformations contained in it can all be applied at once, and there is no need to consider questions of order within the set.

We shall denote the elementary expansion $K \to K+aA+A$, where A and aA are open simplexes, by aA and it is to be understood that the transformation aA is defined in the abstract though it is not applicable to every complex. It is applicable to a complex K if, and only if, K contains $a\dot{A}$ but not A. When aA denotes an elementary expansion we shall allow A to be 0, the empty set of simplexes, in which case aA will be the identical transformation, operating on every complex and transforming it into itself. Two elementary expansions aA and bB will be described as *independent* if, and only if†,

(14.2) $$A \bar{\in} Cl(bB), \quad B \bar{\in} Cl(aA).$$

In particular the identity is independent of every elementary expansion. The conditions (14.2) are obviously equivalent to the conditions

(1) *aA and bB are both applicable to some one complex K* [*e.g.* $Cl(a\dot{A}+b\dot{B})$],

(2) *bB is applicable to $K+aA+A$*;

and, if these are satisfied, the transformation

$$K \to K+aA+bB+A+B$$

will be described as the composite expansion due to the simultaneous application of aA and bB. More generally, let $\{aA\}$ be any set of elementary expansions, finite or infinite, each of which is applicable to a given complex K and any two of which are independent. The transformation

$$K \to K+\Sigma,$$

where Σ is the totality of open simplexes aA and A, will be called the *composite expansion* due to the *simultaneous application* of the elementary expansions aA. A composite expansion $\{aA\}$ will be described as applicable to K if, and only if, each of the elementary expansions aA is applicable to K, and $\{aA\}$ and $\{bB\}$ will be described as independent if, and

† Notice that, in dealing with elementary expansions, the question is not so much "are two given transformations interchangeable?" as "can one be applied both before and after the other?"

only if, each elementary expansion aA is independent of each bB. From this definition it follows that a necessary and sufficient condition for the independence of two composite expansions is that one of them shall be applicable to some complex both before and after the other. Any set of mutually independent composite expansions can obviously be combined into a single composite expansion, and any sub-set of the elementary expansions in a composite expansion may be applied simultaneously to form a composite expansion. Though we shall be working entirely with expansions, notice that composite contractions may be similarly defined; also composite deformations, consisting of expansions and contractions.

Now let $\bar{e}_1$ be a composite expansion† of a given complex K_0 and let $K_{n+1} = \bar{e}_{n+1} K_n$, where $\bar{e}_1, \bar{e}_2, \ldots$ is an enumerable sequence of composite expansions. Let Σ_{ω_0} be the aggregate of simplexes added by all the elementary expansions in $\bar{e}_1, \bar{e}_2, \ldots$, and let

$$K_{\omega_0} = K_0 + \Sigma_{\omega_0}.$$

Proceeding by transfinite induction, let $K_p = K_0 + \Sigma_p$, where p is any ordinal number and Σ_p is the aggregate of simplexes added by a given transfinite sequence of composite expansions $\{\bar{e}_j\}$ $(j \prec p+1)$, such that $K_{i+1} = \bar{e}_{i+1} K_i = \bar{e}_{i+1}(K_0 + \Sigma_i)$ $(i \prec p)$.

THEOREM 30. *The transformation $K_0 \to K_p$ is the resultant of a countable sequence of composite expansions.*

If an elementary expansion in $\bar{e}_j$ is applicable to K_i, where $i+1 \prec j$, it may be transferred from $\bar{e}_j$ to $\bar{e}_{i+1}$. We do this whenever possible, so that each elementary transformation is applied at the first opportunity. Then the elementary expansions in $K_0 \to K_p$ form a countable set of composite expansions. For if not, there is an elementary expansion aA in the (ω_0+1)-th set. But the complex $Cl(a\dot{A})$, being finite, is in K_n for some finite value of n, and $A \bar{\epsilon} K_n$ since $A \bar{\epsilon} K_{\omega_0}$ and $K_n \subset K_{\omega_0}$. Therefore aA is applicable to K_n, contrary to the fact that it is applied as soon as possible.

It follows from this theorem that we need only consider countable sequences of composite expansions. The resultant of such a sequence will be called an *expansion*.

Geometrical expansions, defined as in § 5, may be treated in the same way as elementary expansions. A geometrical expansion Σ is the addition of a set of open simplexes $\Sigma = E^n - E^{n-1}$, where E_n is an n-element and

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† When we write $\bar{e}K$, it is to be understood that $\bar{e}$ is applicable to K.

E^{n-1} is an $(n-1)$-element in $\dot{E}^n$. We shall say that Σ is *applicable* to a complex K if, and only if,

$$(14.3) \qquad Cl(\Sigma)-\Sigma \subset K, \quad \Sigma . K = 0,$$

and two geometrical expansions Σ_1 and Σ_2 will be described as *independent* if, and only if,

$$(14.4) \qquad \Sigma_1 . Cl(\Sigma_2) = \Sigma_2 . Cl(\Sigma_1) = 0.$$

All that we have said about elementary expansions, including Theorem 30, obviously applies with minor alterations to geometrical expansions. A product of composite geometrical expansions will also be called a geometrical expansion and, when a contrast is necessary, a product of composite formal expansions will be called a formal expansion.

THEOREM† 31. *If K_0 expands geometrically into K, it expands formally into some stellar sub-division σK, where $\sigma = 1$ in K_0.*

Our proof depends on the lemma:

LEMMA 17. *There is an internal stellar sub-division σE^n $(n > 0)$ which contracts formally into E^{n-1}, where E^n is any n-element and E^{n-1} is an $(n-1)$-element in $\dot{E}^n$.*

The lemma is trivial if $n = 1$ and will be proved by induction on n. First notice that, when we assume the lemma for k-elements if $k < n$, some internal stellar sub-division of a k-element is collapsible. For some internal stellar sub-division contracts into a closed $(k-1)$-simplex on the boundary. Now let $\Sigma = \dot{E}^n - E^{n-1}$, let $\sigma_0 = s^2_{\dot{E}^n}$ and let

$$E_1{}^n = \sigma_0 E^n - O(\Sigma, \sigma_0 E^n).$$

As proved by Newman‡, $\sigma_0 E^n$ is transformed into $E_1{}^n$ by a sequence of regular contractions of the form

$$A^{n-k-1} E^k + \tilde{E}^n \to \tilde{E}^n \quad (E^k \subset \dot{\tilde{E}}^n),$$

where E^k is a k-element $(k < n)$ whose internal simplexes are inside $A^{n-k-1} E^k + \tilde{E}^n$. By the corollary to the inductive hypothesis and Theorem 4, there is an internal stellar sub-division $\sigma_1 \sigma_0 E^n$ such that each of the elements $\sigma_1 E^k$ is collapsible. Then $\sigma_1 \sigma_0 = 1$ in $\dot{E}^n$, and $\sigma_1 \sigma_0 E^n$

† Notice that this theorem is sharper than the analogous Theorem 7. Because of Lemma 17, proved below, it is unnecessary to sub-divide K_0.

‡ *Journal London Math. Soc.* (*loc. cit.*), 509, Lemma 2.

contracts into $\sigma_1 E_1^n$ by Lemma 3. Let $L = E^{n-1}$ and let

$$E_2^n = s_L^2 \sigma_1 E_1^n - O(E^{n-1}, s_L^2 \sigma_1 E_1^n)$$

$$= \sigma_2 E_1^n - O(E^{n-1}, \sigma_2 E_1^n),$$

where $\sigma_2 = s_L^2 \sigma_1$. Then $\sigma_2 = 1$ in $\dot{E}^n$ and E_2^n does not meet $\dot{E}^n$. Since each internal simplex of $s_L \sigma_1 E_1^n$ is starred by s_L, it follows that E^{n-1} and $\sigma_2 E_1^n$ satisfy the first condition imposed on Σ and M in Lemma 10. By the corollary to Lemma 4 they also satisfy the second condition, and it follows from Lemma 10 that E_2^n is an n-element. Let us assume for the moment that

$$N = N(E^{n-1}, \sigma_2 E_1^n)$$

meets E_2^n in an $(n-1)$-element on $\dot{E}_2^n$. Then it follows from Theorem 6 that some stellar sub-division $\sigma_3 E_2^n$ contracts into $\sigma_3(N \,.\, E_2^n)$, and $\sigma_3 = 1$ in $\dot{E}^n$ since E_2^n does not meet $\dot{E}^n$. By Theorem 2, N contracts into E^{n-1}, and, by Theorem 4, $\sigma_3 N$ also contracts into E^{n-1}. Therefore $\sigma_3 \sigma_2 E_1^n$ contracts first into $\sigma_3 N$ and then into E^{n-1}, and it follows that σE^n contracts into E^{n-1}, where $\sigma = \sigma_3 \sigma_2 \sigma_0$ and $\sigma = 1$ in $\dot{E}^n$.

It remains to prove that $N \,.\, E_2^n \subset \dot{E}_2^n$ and is an $(n-1)$-element. An open simplex A $(A \neq 1)$ in $\sigma_2 E_1^n$ belongs to $N \,.\, E_2^n$ if, and only if, $(\sigma_2 E_1^n)_A$ meets E^{n-1}, since $A \in N$, but $Cl(A)$ does not, since $A \in E_2^n$. If $A \in N \,.\, E_2^n$, it follows that some simplex in $(\sigma_2 E_1^n)_A$ is absent from E_{2A}^n. Therefore E_{2A}^n is a proper sub-set of $(\sigma_2 E_1^n)_A$ and so cannot be a sphere. Therefore $A \in \dot{E}_2^n$, whence $N \,.\, E_2^n \subset \dot{E}_2^n$. Moreover, N is an n-element, by Theorem 22 and Theorem 23, Corollary 1. If $A \in N \,.\, E_2^n$, then $Cl(A)$ does not meet E^{n-1} and it follows from an argument in the proof of Theorem 22 that N_A is an element. Therefore $N \; E_2^n \subset \dot{N} - O(E^{n-1}, \dot{N})$. But $Cl(A) \,.\, E^{n-1} = 1$ if $A \in \dot{N} - O(E^{n-1}, N)$, whence $\dot{N} - O(E^{n-1}, \dot{N}) \subset E_2^n$. Therefore $N \,.\, E_2^n = \dot{N} - O(E^{n-1}, \dot{N})$. If $Cl(A)$ meets E^{n-1} $(A \in \sigma_2 E_1^n)$, we have $N_A = (\sigma_2 E_1^n)_A$, whence A is in both $\dot{N}$ and $\sigma_2 \dot{E}_1^n$ if it is in either. Therefore $O(E^{n-1}, \dot{N}) = O(E^{n-1}, \sigma_2 \dot{E}_1^n)$ and is the interior of $N(E^{n-1}, \sigma_2 \dot{E}_1^n)$, which is an $(n-1)$-element by Lemma 10, Corollary 2. Therefore $N \,.\, E_2^n$ is an $(n-1)$-element and the lemma is established.

The theorem now follows by an inductive argument similar to the one used in proving Lemma 16. Let $\bar{e}_1, \bar{e}_2, \ldots$ be a countable sequence of geometrical expansions which transform K_0 into K, and let $K_r = \bar{e}_r K_{r-1}$. Assume that K_0 expands formally into some stellar sub-division

$$\sigma_i K \quad (i = 0, \ldots, m),$$

where $\sigma_j = \sigma_i$ in K_i if $i < j$. If $K_m \to K_m + E^n$ is any one of the individual expansions in $\bar{e}_{m+1}$, where $E_n . K_m = \dot{E}^n . K_m = E^{n-1}$, it follows from Lemma 17 that $\sigma_m E^{n-1}$ expands formally into $\sigma^* \sigma_m E^n$, where σ^* is some internal stellar sub-division of $\sigma_m E^n$. Therefore $\sigma_m K_m$ expands formally into $\sigma_{m+1} K_{m+1}$, where σ_{m+1} is the resultant of σ_m followed by all the sub-divisions σ^* corresponding to the various elements added by $\bar{e}_{m+1}$. Since σ^* is internal to $\sigma_m E^n$ it leaves $\sigma_m E^{n-1}$, and hence $\sigma_m K_m$, unaltered. Therefore $\sigma_{m+1} = \sigma_m$ in K_m and it follows that $\sigma_{m+1} = \sigma_i$ in K_i if $i < m$. If we begin with $\sigma_0 = 1$, it follows inductively that there is a sub-division σ_m satisfying the above conditions for each value of m. Moreover, $\sigma_m = \sigma_0 = 1$ in K_0. The required sub-division σK is defined by the condition

$$\sigma A = \sigma_m A \quad \text{if} \quad A \subset K_m,$$

and the theorem is established.

If K_0 expands formally into K, it is obvious that γK_0 expands geometrically into γK, where γ is any sub-division. Therefore we have the corollary:

COROLLARY. *If K_0 expands formally into K and γ is any sub-division, then γK_0 expands formally into some stellar sub-division $\sigma\gamma K$, where $\sigma = 1$ in γK_0.*

We now define a *formal deformation* of an infinite complex as a transformation of the form

$$D = \gamma_2^{-1} E_2^{-1} E_1 \gamma_1,$$

where γ_1 and γ_2 are general sub-divisions and E_1 and E_2 are formal expansions. That is to say

$$K_1 = D(K_0)$$

if, and only if, there are sub-divisions γ_1 and γ_2 and formal expansions E_1 and E_2, such that

$$E_1(\gamma_1 K_0) = E_2(\gamma_2 K_1).$$

If $\gamma_1 = \gamma_2 = 1$ in L, where $L \subset K_0 . K_1$, we shall write

$$K_1 = D(K_0) \quad (\text{rel. } L).$$

THEOREM 32. *If $K_1 = D_1(K_0)$ (rel. L) and $K_2 = D_2(K_1)$ (rel. L), then $K_2 = D(K_0)$ (rel. L), where $L \subset K_0 . K_1 . K_2$.*

Let $$E_1(\gamma_1 K_0) = E_2^*(\gamma_2 K_1) = K_{01} \quad (\gamma_1 = \gamma_2 = 1 \text{ in } L),$$

and $$E_2(\gamma_2^* K_1) = E_3(\gamma_3 K_2) = K_{12} \quad (\gamma_2^* = \gamma_3 = 1 \text{ in } L).$$

By Lemma 16 and its addendum, we have $\sigma\gamma_2 K_1 = \gamma\gamma_2^* K_1 = K_1^*$, say, where σ is some stellar sub-division and γ a general sub-division and $\sigma = \gamma = 1$ in L. By Theorem 4, which is obviously true of infinite complexes, $\sigma\gamma_1 K_0$ and K_1^* expand formally into σK_{01}. By the corollary to Theorem 31, K_1^* expands formally into some stellar sub-division $\sigma_1\gamma K_{12}$, where $\sigma_1 = 1$ in K_1^*, and hence in L. By Theorem 4, $\sigma_1\sigma\gamma_1 K_0$ and K_1^* expand formally into $\sigma_1\sigma K_{01}$. Finally, $\sigma_1\gamma\gamma_3 K_2$ expands formally into some stellar sub-division $\sigma_2\sigma_1\gamma K_{12}$ ($\sigma_2 = 1$ in $\sigma_1\gamma\gamma_3 K_2$ and hence in L), $\sigma_2\sigma_1\sigma\gamma_1 K_0$ expands formally into $\sigma_2\sigma_1\sigma K_{01}$ and $\sigma_2 K_1^*$ expands formally both into $\sigma_2\sigma_1\sigma K_{01}$ and into $\sigma_2\sigma_1\gamma K_{12}$. Therefore we may assume that $\gamma_2^* = \gamma_2$, after preliminary sub-divisions if necessary.

Assuming that $\gamma_2^* = \gamma_2$, let $K'_{12} = s_P K_{12}$, where $P = \gamma_2 K_1$, the new vertices being, as usual, different from any of those in K_{01}. Then $(K_{01} - \gamma_2 K_1) . (K'_{12} - \gamma_2 K_1) = 0$, whence $A \bar{\epsilon} K_{01}$ if $A \epsilon K'_{12} - \gamma_2 K_1$ and $A \bar{\epsilon} K'_{12}$ if $A \epsilon K_{01} - \gamma_2 K_1$. Let $\gamma_3' = s_P \gamma_3$ and let E_2' and E_3' be the expansions $\gamma_2 K_1 \to K'_{12}$ and $\gamma_3' K_2 \to K'_{12}$, defined as in the proof of Theorem 5. Since $A \bar{\epsilon} K_{01}$ if $A \epsilon K'_{12} - \gamma_2 K_1$ and $A \bar{\epsilon} K'_{12}$ if $A \epsilon K_{01} - \gamma_2 K_1$, it follows from (14.2) that each elementary expansion in E_2' is independent of every elementary expansion in E_2^*. Therefore E_2' is applicable to $E_2^*(\gamma_2 K_1) = K_{01}$ and E_2^* is applicable to $E_2'(\gamma_2 K_1) = K'_{12}$. But

$$E_2'(K_{01}) = E_2^*(K'_{12}) = K_{01} + K'_{12}.$$

Therefore

$$E_2' E_1(\gamma_1 K_0) = E_2^* E_3'(\gamma_3' K_2) \quad (\gamma_1 = \gamma_3' = 1 \text{ in } L),$$

and the theorem follows from Theorem 30.

As a special case of Theorem 32 we have the corollary:

COROLLARY. *If two sub-divisions of K_1 expand into K_{01} and into K_{12} respectively, then*

$$K_{12} = D(K_{01}).$$

The relation of equivalence under formal deformations is reflexive and symmetric by definition, and it follows from Theorem 32 that it is also transitive. Therefore the equivalence classes are mutually exclusive and with each of these classes we associate an abstract *infinite nucleus*, or simply a *nucleus*. It is an immediate consequence of the definition that the nucleus of a complex is a combinatorial invariant.

Fillings, perforations, and m-groups may be treated in the same way as expansions, contractions, and nuclei. Let us describe either an elementary expansion or a filling whose order exceeds some given m as an *elementary addition*. Then an addition Σ consists of adding a set of open simplexes Σ to a complex, the set containing one simplex in the case of a filling and two in the case of an expansion. The conditions (14.3) are necessary and sufficient for an elementary addition Σ to be applicable to a complex K, and two elementary additions Σ_1 and Σ_2 will be described as *independent* if, and only if, the conditions (14.4) are satisfied. As in the case of expansions, any set of mutually independent elementary additions may be combined into a *composite addition*. All that was said about expansions, up to and including Theorem 30, obviously applies, with minor alterations, to additions. As the analogue of the corollary to Theorem 31 we have:

THEOREM 33. *If K_0 is transformed into K by a countable sequence of composite additions, and if γ is a given sub-division, then γK_0 is transformed into some stellar sub-division $\sigma\gamma K$ by a countable sequence of composite additions ($\sigma = 1$ in γK_0).*

The proof is similar to the proof of Theorem 31, with Lemma 17 supported by the auxiliary lemma:

If E^n is a given n-element, there is an internal sub-division σE^n, such that the transformation $\dot{E}^n \to \sigma E^n$ is the resultant of a formal expansion followed by a filling of order n.

Let A^{n-1} be an open $(n-1)$-simplex in $\dot{E}^n$, let aA^{n-1} be the open simplex in E^n with A^{n-1} on its boundary and, after a suitable internal sub-division, let a be inside E^n. Let $E^{n-1} = \dot{E}^n - A^{n-1}$ and $E_1{}^n = E^n - aA^{n-1}$. By Lemma 17, E^{n-1} expands formally into some stellar sub-division $\sigma E_1{}^n$, where $\sigma = 1$ in $\dot{E}_1{}^n$ and therefore in $\dot{E}^n$. Since $\sigma = 1$ in $a\dot{A}^{n-1}$, the required transformation is the resultant of the expansion

$$\dot{E}^n = A^{n-1} + E^{n-1} \to A^{n-1} + \sigma E_1{}^n,$$

followed by the filling $A^{n-1} + \sigma E_1{}^n \to A^{n-1} + \sigma E_1{}^n + aA^{n-1} = \sigma E^n$, and the lemma is established.

The theorem now follows from the proof of Theorem 31 with trivial modifications.

Two complexes K_0 and K_1 will be said to have the same m-group if, and only if, there are sub-divisions γ_1 and γ_2 and additions T_1 and T_2 (containing no fillings whose orders do not exceed m) such that

$$T_1(\gamma_1 K_0) = T_2(\gamma_2 K_1).$$

THEOREM 34. *If K_0 has the same m-group as K_1, and K_1 the same m-group as K_2, then K_0 has the same m-group as K_2.*

This follows from the proof of Theorem 32, with Theorem 31 replaced by Theorem 33, Theorem 4 supported by Lemma 7, and the appropriate changes in wording.

As with nuclei, it follows that complexes fall into mutually exclusive classes, two complexes having the same m-group if, and only if, they belong to the same class.

In the next section we shall discuss the relation between the formal and the topological theory of infinite complexes. It will appear that any two complexes have the same m-group, for each value of m, if they have the same homotopy type, in a sense to be defined. The converse holds for complexes of finite dimensionality. In particular, two finite complexes K_0 and K_1 have the same homotopy type if they have the same m-group for $m \geqslant \dim (K_i)+1$, with infinite additions allowed. It follows from Theorem 17 that they have the same m-group in the strictly finite sense of §§ 2–13. The analogous question for nuclei remains open, namely:

If two finite complexes have the same infinite nucleus, can they be interchanged by finite sequences of elementary deformations?

15. *The topology of infinite polyhedra.* By a *closed convex n-cell*† $C^n (n \geqslant -1)$ we shall mean a set of undefined points in a (1-1) correspondence f with the interior and boundary of a convex, polyhedral n-cell $f(C^n)$, in Cartesian n-space X^n, the set C^n being empty if $n = -1$. Two transformations, $f(C^n)$ and $f^*(C^n)$, will be said to determine the same convex n-cell if, and only if, $f^* = Tf$, where T is an affine transformation of X^n into itself. Thus C^n has the affine structure of its image $f(C^n)$ and the usual terms (interior, boundary, simplex, etc.) will mean the same when applied to C^n as to $f(C^n)$. In particular the boundary $\dot{C}^n$ of C^n consists of certain closed convex $(n-1)$-cells. The "cells on" C^n (*Seiten*), defined inductively, consist of C^n by itself if $n = -1$, and of C^n together with the cells on the $(n-1)$-cells in $\dot{C}^n$ if $n \geqslant 0$. Thus C^{-1} is on every cell. By a *polyhedral complex* K we shall mean a set of closed convex cells satisfying

† *Cf.* O. Veblen, *Analysis Situs* (New York, 1931), 76.

the first two conditions given by Alexandroff and Hopf for a cell-complex†, namely:

(1) *each cell on any cell in K is also in K,*

(2) *the intersection of two closed cells in K is a closed cell on both of them.*

We do not require the dimensionality of the cells in K to have a finite upper bound, and the set of cells which are incident with a given one may have any cardinal number. A partition of K (*i.e.* a rectilinear sub-division), and in particular a simplicial sub-division, is defined as when K is finite.

A simplicial polyhedral complex is obviously isomorphic to some symbolic complex and conversely. Moreover, it follows from Lemma 16 and its addendum that two geometrical complexes which are isomorphic to equivalent symbolic complexes are geometrically equivalent, meaning that they have isomorphic partitions. Also it follows from an inductive argument similar to the proof of Lemma 16, and Theorem 1 in my paper on sub-divisions, that two partitions of the same geometrical complex have a common partition. Thus the elementary theorems on which combinatorial analysis situs is based apply equally well to polyhedral complexes in general as to finite, or locally finite, complexes. Except when a contrast with symbolic complexes is necessary we shall refer to K simply as a complex and, unless the contrary is implied, it is to be understood that any complex to which we refer is simplicial.

By a *topological polyhedron* $P(K)$ we shall mean the set of points in a polyhedral complex K, with the topology defined by the conditions:

(1) *each closed cell in K has the topology natural to its affine geometry,*

(2) *a set of points in K is closed if, and only if, its intersection with each closed cell is closed.*

We enumerate some of the more obvious consequences of this definition. A set of points in K is closed (open) if, and only if, its intersection with each sub-complex $L \subset K$ is closed (open) relative to the topology of $P(L)$. The sum of a finite number, and the intersection of any number, of closed sets is closed. Any sub-complex of K is closed. If $O(X, K)$ is the set of open cells in K (*i.e.* the interiors of closed cells) whose closures meet a given set of points X, then $K-O(X, K)$ is a complex and $O(X, K)$, regarded as a set of points, is open. The definition of closed sets is invariant under partition. The following is not so obvious, namely a topological polyhedron

† *Topologie*, 126.

is a metric space if, and only if, it is locally finite. For let K be a given complex, which we may suppose to be connected and simplicial. Since it is connected, any two vertices in K are joined by a finite polygonal segment, consisting of edges of K. If K is locally finite, it follows without difficulty that it contains at most an enumerable infinity of vertices, and hence that $P(K)$ is a metric space†. Conversely, let some vertex a be incident with infinitely many simplexes in K, and let $A_1, A_2, \ldots$ be an enumerable sequence of simplexes with a on their boundaries. Assuming that $P(K)$ is metricized, let p_r be a point inside A_r, whose distance from a is less than $1/r$. According to the metric the sequence of points $p_1, p_2, \ldots$ converges to a. But the dimensionality of each simplex in K is finite and no open simplex contains more than one of the points p_r. Therefore the number of these points in any closed simplex is finite and $\{p_r\}$ is a closed set according to the topology of $P(K)$. Therefore $P(K)$ is not a metric space. A similar argument shows that any compact set of points in $P(K)$ is contained in a finite sub-complex of K. Therefore a map in $P(K)$ of any compact space is contained in a finite sub-complex of K, as when K is locally finite.

It is now to be understood that all our polyhedral complexes have this topological structure, in addition to the rectilinear geometry of each cell and the combinatorial structure of the incidence relations.

LEMMA 18. *A transformation $f(K)$, of a complex K into any topological space P, is continuous if, and only if, it is continuous throughout each closed cell in K.*

This follows at once from the definition of closed sets and a standard definition of continuous transformations, namely: $f(K) \subset P$ is continuous if, and only if, $f^{-1}(X)$ is a closed set in K, where X is any closed set in P.

Certain fundamental theorems may now be extended from finite and locally finite complexes to infinite complexes in general.

THEOREM 35. *If K is covered by a given set of open sets, there is a stellar sub-division σK such that $N(a, \sigma K) \subset U(a)$, where a is any vertex in σK and $U(p)$ denotes any one of the given open sets containing a point p.*

We first prove a sharper theorem for finite complexes. Let K be any finite complex which is covered by a given set of open sets and let

$$N(b_\lambda, L) \subset U_\lambda \quad (\lambda = 1, \ldots, m),$$

† See, for example, Lefschetz, *Topology* (*loc. cit.*), 292.

where L is a sub-complex of K ($L \neq 1$), $b_1, \ldots, b_m$ are the vertices of L, and U_λ is a particular one of the sets $U(b_\lambda)$. The auxiliary theorem is:

Under these conditions there is a sub-division σ, leaving L unaltered, such that $N(a, \sigma K) \subset U(a)$, where a is any vextex in σK, and

$$N(b_\lambda, \sigma K) \subset U_\lambda \quad (\lambda = 1, \ldots, m).$$

First assume that $N(b_\lambda, K) \subset U_\lambda$ for each value of $\lambda = 1, \ldots, m$. Since every open simplex A in $O(L, K)$ is incident with at least one of the vertices $b_1, \ldots, b_m$, it follows that $N(A, K)$ is contained in one or more of the neighbourhoods $U_1, \ldots, U_m$. Therefore, if a is any vertex of a sub-divison σK, which lies in the open region $O(L)$, covered by $O(L, K)$, the closed star $N(a, \sigma K)$ is contained in one or more of the sets $U_1, \ldots, U_m$. In particular $N(b_\lambda, \sigma K) \subset U_\lambda$. Let $N = N(L, s_L K)$ and let $K_0 = s_L K - O(L, s_L K)$. Then $N \subset O(L)$ and K_0 does not meet L. If $p \in K_0$, the set $U(p) . K_0$ is an open neighbourhood of p in K_0 and, by a standard theorem, there is a stellar sub-division $\sigma_1 K_0$ such that $N(a, \sigma_1 K_0) \subset U(a)$, where a is any vertex in $\sigma_1 K_0$. Also $\sigma_1 = 1$ in L, since L does not meet K_0. Then $N(a, \sigma K)$, with $\sigma = \sigma_1 s_L$, is contained in some U_λ if $a \in \sigma_1 N$, and $N(b_\lambda, \sigma K) \subset U_\lambda$. On the other hand, if $a \in \sigma K - \sigma_1 N$ we have $N(a, \sigma K) = N(a, \sigma_1 K_0) \subset U(a)$. Therefore $N(a, \sigma K) \subset U(a)$ if a is any vertex in σK, $N(b_\lambda, \sigma K) \subset U_\lambda$ and $\sigma = 1$ in L.

It remains to show that there is a sub-division σ_0, leaving L unaltered, such that $N(b_\lambda, \sigma_0 K) \subset U_\lambda$ ($\lambda = 1, \ldots, m$). We shall assume, for convenience, that none of the sets U_λ coincides with K. This involves no loss of generality. For, if $K = L$ the theorem is trivial, and, if $K \neq L$, $U_\lambda = K$, we remove a point of $K - L$ from U_λ. After a preliminary sub-division, if necessary, let $N(L, K)$ be a normal simplicial neighbourhood of L. Then each closed simplex in K is of the form AB, where $A . L = 1$ and $B \subset L$ (possibly $A = 1$ or $B = 1$). This being so, let K be imbedded in Euclidean metric space R^n. If $B = b_\lambda B_0$ is any closed simplex in L (other than 1), the Euclidean distance $\delta(B, K - U_\lambda)$, from B to the closed set $K - U_\lambda$, is positive. Let ρ be the least of these distances, calculated for all the vertices of B and all the simplexes in L. Then

$$0 < \rho \leqslant \delta(B, K - U_\lambda)$$

if B contains b_λ. Now apply the sub-division s_L to K, placing the centre of each star $s_L(AB)$, in which $B \neq 1$, at a distance less than ρ from B. Let $A' B = A' b_\lambda B_0$ be any principal closed simplex in $N(b_\lambda, s_L K)$, where $A' . L = 1$ and $B_0 \subset L$. If $A' = 1$ we have $A' B = B \subset U_\lambda$. Otherwise

$A' = a_0 \dots a_k$, where a_i is the centre of a star $s_L(A_i B)$. Since $\delta(a_i, B) < \rho$, all the vertices of $A'B$ are contained in the sub-set of R^n given by $\delta(p, B) < \rho$. But the latter is a convex region† and therefore contains $A'B$. That is to say, $\delta(p, B) < \rho$ if $p \in A'B$, and since $\rho \leqslant \delta(B, K - U_\lambda)$ we have $A'B \subset U_\lambda$. Therefore $N(b_\lambda, s_L K) \subset U_\lambda$ and the auxiliary theorem is established.

In proving the main theorem we assume that we can select a particular one from those of the given neighbourhoods containing each closed set which is in at least one of them‡. This being so, let K^n be the n-dimensional skeleton of K and assume that, if $n \leqslant m$ $(m \geqslant -1)$, there is a stellar sub-division $\sigma_n K^n$ such that:

(1) $\sigma_p = \sigma_n$ in K^p if $p < n$,

(2) $N(a, \sigma_m K^m) \subset U_a$, where a is any vertex in $\sigma_m K^m$ and U_a is a particular one of the given sets $U(a)$.

If A^{m+1} is any closed $(m+1)$-simplex in K^{m+1}, the sets $U(p) \,.\, A^{m+1}$ are open relative to A^{m+1}. It follows from the auxiliary theorem, with $\sigma_m A^{m+1}$ and $\sigma_m \dot{A}^{m+1}$ taking the place of K and L, that there is a sub-division $\sigma^* \sigma_m A^{m+1}$, leaving $\sigma_m \dot{A}^{m+1}$, and hence $\sigma_m K^m$, unaltered, such that

$$N(a, \sigma^* \sigma_m A^{m+1}) \subset U(a)$$

if $a \in \sigma^* \sigma_m (A^{m+1} - \dot{A}^{m+1})$, and $N(b, \sigma^* \sigma_m A^{m+1}) \subset U_b$ if $b \in \sigma_m \dot{A}^{m+1}$. Let σ_{m+1} be the resultant of σ_m, followed by all the sub-divisions σ^* corresponding to the various $(m+1)$-simplexes in K^{m+1}. Then $\sigma_{m+1} = \sigma_m$ in K^m, whence $\sigma_{m+1} = \sigma_n$ in K^n if $n < m$. If $b \in \sigma_{m+1} K^m$, any closed simplex bA, in $N(b, \sigma_{m+1} K^{m+1})$, is contained in $\sigma_{m+1}(K^m + A^{m+1})$, where A^{m+1} is some closed simplex in K^{m+1} (possibly $bA \subset \sigma_{m+1} K^m$, in which case A^{m+1} may be arbitrary). Therefore $bA \subset U_b$, and it follows that $N(b, \sigma_{m+1} K^{m+1}) \subset U_b$ if $b \in \sigma_{m+1} K^m$. Finally, with each vertex a, in $\sigma_{m+1} K^{m+1} - \sigma_{m+1} K^m$, we associate a particular neighbourhood U_a, selected from the neighbourhoods $U(a)$ which contain $N(a, \sigma_{m+1} K^{m+1})$.

The conditions are now as before, with m replaced by $m+1$. If we begin with $\sigma_{-1} = 1$, it follows inductively that they are satisfied by some sub-

† Alexandroff and Hopf, *Topologie*, 598.

‡ This assumption is justified, without an appeal to the axiom of choice, if no point is in an infinite number of the given sets. For in this case the intersection of each sub-set of the given sets. s open. Let all these intersections be included among the given sets. From those containing any closed set which is in at least one of them we then select that one of the given sets which is contained in all the others. This condition is satisfied in the corollary to Theorem 35 and in Theorem 36 below.

division σ_m for all values of m. Taking σ to be the sub-division given by $\sigma A = \sigma_n A$ if $A \subset K^n$, let a be any vertex in σK. Any closed simplex aA, in $N(a, \sigma K)$, is contained in $\sigma_n K^n$ for $n = \dim(aA)$. Therefore $aA \subset U_a$. It follows that $N(a, \sigma K) \subset U_a$, which is one of the given open sets, and the theorem is established.

Let X_1 and X_2 be closed sets in a complex K and let $U_i = K - X_i$. If X_1 and X_2 have no common point, each point in K is either in U_1 or U_2 or both. By Theorem 35 there is a sub-division σK such that $N(a, \sigma K) \subset U_i$ ($i = 1$ or 2) and hence does not meet both X_1 and X_2, where a is any vertex in σK. That is to say $N(X_1, \sigma K)$ does not meet $N(X_2, \sigma K)$, where $N(X_i, \sigma K)$ stands for the totality of closed simplexes in σK which meet X_i. Clearly $N(X_i, \sigma K)$ is the closure of the open neighbourhood $O(X_i, \sigma K)$, and we have the corollary to Theorem 35:

COROLLARY. *A topological polyhedron is a normal topological space*†.

THEOREM 36. *Any map of a complex K in a simplicial complex L is homotopic to a simplicial map of σK in L, where σ is a suitable sub-division of K.*

If $f(K) \subset L$ is a given map, $f^{-1}\{O(B, L)\}$ is an open set in K, where B is any open simplex in L. Each point in K is contained in at least one of these sets and, by Theorem 35, there is a sub-division σK such that $N(a, \sigma K) \subset U(a)$, where a is any vertex in σK and $U(a)$ is one of the sets $f^{-1}\{O(B, L)\}$. It follows from the argument used in the finite case that f is deformable into a simplicial map of σK in L, and from Lemma 18 that the deformation is continuous.

As in the finite case, the deformation f_t $(0 \leqslant t \leqslant 1)$ of a given map $f_0(K) \subset L$ into a simplicial map $f_1(\sigma K)$ may be chosen so that the "trajectory" $p_t = f_t(p_0)$, of any point p_0, is a rectilinear segment in the closure of the open simplex which contains $f_0(p_0)$.

THEOREM 37. *Let $f_0(K)$ be a map of a complex K in any topological space P, let L be a sub-complex of K and let g_t be a deformation of the map $f_0(L)$ into $g_1(L)$ $(0 \leqslant t \leqslant 1;\ g_0 = f_0$ in $L)$. Then there is a deformation of the complete map $f_0(K)$ which coincides with g_t in L.*

Let K^n be the complex consisting of L together with all the simplexes in $K - L$ whose dimensionalities do not exceed n. It follows from the same

† Alexandroff and Hopf, *Topologie*, 68.

argument as when K is finite† that $g_t(L)$ $[=f_t^{-1}(K^{-1})]$ can be extended to a deformation $f_t^n(K^n)$ $(n=-1, 0, 1, \ldots)$, such that $f_t^m = f_t^n$ in K^n if $m > n$. The function $f(p, t)$, defined by

$$f(p, t) = f_t^n(p) \text{ if } p \in K^n,$$

may be regarded as a transformation of the product complex

$$K_{01} = K \times \langle 0, 1 \rangle$$

into P. This transformation is continuous in $K^n \times \langle 0, 1 \rangle$ for each value of n, and is therefore continuous throughout each closed cell in K_{01}. Therefore it is continuous throughout K_{01}, by Lemma 18, and $f_t(K)$, given by $f_t(p) = f(p, t)$ $(p \in K)$, may be taken as the required deformation of $f_0(K)$.

As a corollary to Theorem 37 we have:

COROLLARY. *Theorem* 15 (§ 8 *above*) *applies equally well to infinite as to finite complexes.*

For, in the presence of Theorem 37, Hurewicz's argument, referred to in the proof of Theorem 15, shows that the condition A implies R_L. That is to say, if every cell in K whose boundary lies in L is homotopic, with its boundary fixed, to a cell in L (condition A), then L is a retract by deformation relative to itself (condition R_L). The remaining implications: R_L implies all the other conditions, R implies B and B implies A, are valid if K and $L \subset K$ are any topological spaces, the first two obviously and the last one by the argument given in § 8, which does not depend on the special nature of K and L.

LEMMA 19. *If a complex K_0 expands into K it is a retract by deformation of K.*

Let the transformation $K_0 \to K$ be the resultant of a sequence of composite expansions $\bar{e}_1, \bar{e}_2, \ldots$, and let $K_n = \bar{e}_n K_{n-1}$ $(n = 1, 2, \ldots)$. It is obvious that K_0 is a retract by deformation of K_1 and hence, by induction on n, of K_n. Therefore any cell in K_n whose boundary lies in K_0 is homotopic, with its boundary fixed, to a cell in K_0. Any cell in K, being a map of a compact space, is contained in a finite sub-complex of K, and hence in K_n for some value of n. Therefore any cell in K whose boundary lies in K_0 is homotopic, with its boundary fixed, to a cell in K_0, and the lemma follows from the corollary to Theorem 37.

† Alexandroff and Hopf, *Topologie*, 501.

16. *Extension of previous results.* It will now be clear, with a few indications, that many of the main results in §§3–12 can be extended to infinite complexes. Everything in §3 can be taken over provided that the theorems and proofs are restated in terms of expansion rather than contraction. In particular, a collapsible complex is a complex of the form $E(a)$, where a is a single vertex and E is an expansion. Theorem 4 is true of infinite complexes, as we have already observed. We drop Theorem 6, and the rest of §§4 and 5 is replaced by §14. Everything in §6 can be extended with trivial modifications, and we come to Theorem 12 in §7. First notice that, unless $K^{m+1} = K^m$, the transformation $K^m \rightarrow K^{m+1}$ is a composite filling of order $m+1$, where K^n is the n-dimensional skeleton of a given complex K. Therefore K^m and K^{m+1} have the same m-group, and in extending Theorem 12 we may assume that our complexes are at most 2-dimensional. Secondly, in any connected graph g there is a tree which contains all the vertices of g. For, since g is connected, any two vertices in g are contained in at least one finite, connected sub-graph. Let us define the distance between them as the minimum number of edges in such a graph, this minimum being attained by a simple segment. Let T_n be a tree in g containing those, and only those, vertices of g whose distances from a given vertex do not exceed n. Let each vertex whose distance from some vertex in T_n is unity be joined to T_n by a single edge, and let T_{n+1} be the graph consisting of T_n together with these edges. Then T_{n+1} is obviously a tree satisfying the same conditions as T_n, with n replaced by $n+1$, and a tree $T = T_1 + T_2 + \ldots$ containing all the vertices in g is defined by induction on n. Moreover, this argument shows that any tree is collapsible.

The first part of the extended Theorem 12, namely that all complexes in the same 2-group have the same fundamental group, follows from an argument similar to the proof of Lemma 19. To prove the converse, let T be a tree containing all the vertices in K ($T \subset K$). By the corollary to Lemma 4 and Theorem 2 we may shrink T into a point in $s_T^2 K$, obtaining a system of generators and relations for $\pi_1(K)$. Tietze's method of transforming two systems of generators and relations of the same abstract group into the same system can obviously be extended from finite to infinite systems, and in all remaining details the proof is the same as in the finite case.

As a complementary theorem to Theorem 12 we have

THEOREM 38. *Any group is isomorphic to the fundamental group of some complex.*

Any group G can be represented by a system of generators and relations. For we can take every element in G as a generator and as a set of relations we can take the "multiplication table". Then a complex having G as its fundamental group can be constructed by the method used when the set of relations is finite†.

We proceed to Lemma 8 and Theorem 13. Let K_0 and K_1 be two complexes of at most n dimensions which have the same n-group. Then

$$T_0(\gamma_0 K_0) = T_1(\gamma_1 K_1) = K_{01}, \text{ say},$$

where T_0 and T_1 consist of expansions and of fillings whose orders exceed n. As in Lemma 8, we replace each elementary expansion of the form

$$K \to K + aA^{m-1},$$

where $a\dot{A}^{m-1} \subset K$, $A^{m-1} \not\subset K$ and $m > n$, by the special filling $K \to K + A^{m-1}$ of order $m-1$, followed by the filling $K + A^{m-1} \to K + aA^{m-1}$. If $m = n+1$, the n-element $a\dot{A}^{m-1}$ is contained in K^n, the n-dimensional skeleton of K. Therefore the transformation $K^n \to K^n + A^{m-1}$ is a special filling of K^n. Similarly $K^n \to K^n + aA^{k-1}$ $(k \leqslant n)$ is an expansion of K^n if $K \to K + aA^{k-1}$ is an expansion of K. After modifying the transformations T_0 and T_1 in the way just described, we find that, omitting all the fillings whose orders exceed n,

$$(16.1) \qquad T_0{}^n(\gamma_0 K_0) = T_1{}^n(\gamma_1 K_1) = K_{01}^n,$$

where K_{01}^n is the n-dimensional skeleton of K_{01} and $T_i{}^n$ consists of expansions whose orders do not exceed n and of special fillings of order n. This is the generalization of Lemma 8.

If an elementary expansion of order $k \leqslant n$ is applicable after a filling f^n, of order n, it is applicable before f^n. Therefore (16.1) may be exhibited in the form

$$F_0{}^n\, E_0(\gamma_0 K_0) = F_1{}^n\, E_1(\gamma_1 K_1) = K_{01}^n,$$

where E_i is an expansion whose order does not exceed n and $F_i{}^n$ consists of special fillings of order n. Replacing each filling $K \to K + A^n$ in F_i by the expansion $K \to K + aA^n$, where $a\dot{A}^n \subset K$, we have an expansion of $E_i(\gamma_i K_i) = K_i{}^*$, say, into a complex K_i^{n+1}. By Lemma 19, $K_i{}^*$ is a retract of K_i^{n+1} and, since $K_i{}^* \subset K_{01}^n \subset K_i^{n+1}$, the complex $K_i{}^*$ is a retract of K_{01}^n, in general not by deformation. If $K \to K + A^n$ is any special filling in $F_i{}^n$ it follows that $\dot{A}^n$, being $(n-1)$-dimensional and hence in $K_i{}^*$, bounds a cell in $K_i{}^*$, since it bounds a cell in K_{01}^n. Therefore

$$K_{01}^n = D_0{}^*(K_0{}^* + \Sigma_0{}^*) = D_1{}^*(K_1{}^* + \Sigma_1{}^*),$$

† O. Veblen, *Analysis Situs* (*loc. cit.*), chap. V. § 24.

as in Theorem 13, where Σ_i^* is a cluster of n-spheres, in general infinite, attached to K_i^*. Clearly

$$K_i^* + \Sigma_i^* = D_i(K_i + \Sigma_i),$$

where Σ_i is a cluster of n-spheres attached to K_i, and we have

$$K_1 + \Sigma_1 = D(K_0 + \Sigma_0),$$

as in Theorem 13. Moreover, the order of the deformation D need not exceed $n+1$.

Everything in § 8 applies, with minor alterations, to infinite complexes. In particular, if L is a retract by deformation of K it has the same m-group as K for every value of m. Half of Theorem 17, in § 9, applies to infinite complexes in general, namely the theorem that two complexes have the same m-group if they are of the same homotopy type. The converse applies to infinite complexes of finite dimensionality.

In the definition of the ring $\mathfrak{K}_n(K, p)$ and in Theorems 18 and 19 (§§ 10 and 11) K may be not only an infinite complex, but any connected, locally 0-connected† topological space. On the other hand, the proof of Theorem 20 seems to break down completely if K is infinite, even if it is locally finite. For, if the matrix $\|r_{ij}\|$ has infinitely many rows and columns, the process of reduction involves an infinite sequence of formal deformations, which, so far as I can see, might require an infinite sequence of contractions, each of which could be applied only after its predecessors.

Theorem 21 is a consequence of Theorem 20 and conversely. Therefore the two further questions left open are:

(1) *Are two complexes of infinite dimensionality of the same homotopy type if their m-groups are the same for all values of m?*

(2) *Have two infinite complexes the same nucleus if they are of the same homotopy type and if their fundamental group satisfies the condition imposed in* § 11?

Theorems 22 and 23 in § 12 can be extended, in a modified form, to locally finite sub-complexes of infinite manifolds. An n-dimensional manifold, finite or infinite, is defined as before and may be bounded or unbounded. An infinite unbounded manifold will be described as *open* (by contrast with a closed, or finite unbounded manifold). If K is a sub-complex of a manifold M (K may be infinite if M is infinite), a sub-complex

† Lefschetz (*loc. cit.*), 91.

$U(K, M)$ of M will be called a regular neighbourhood of K if it is a manifold and if γK expands into $\gamma U(K, M)$, where γ is a suitable sub-division of $U(K, M)$. In general $U(K, M)$ is a bounded manifold and Theorem 23, as it stands, is false if the complexes are infinite. For example, if K is a point in Euclidean space R^n, a regular neighbourhood $U(K, R^n)$ might be an n-element, or some infinite bounded manifold, or the whole of R^n. However, if we take M to be a geometrical complex of the kind described in § 15, the following modification of Theorem 23 is true:

THEOREM 39. *The interiors*† *of two regular neighbourhoods of the same sub-complex of the same manifold are in a* (1-1) *semi-linear correspondence.*

In the first corollary to Theorem 23, and elsewhere, n-elements must be replaced by open (semi-linear) n-cells. Then Theorem 39 and the generalization of Theorem 22 follow from the arguments in § 12, supported by the following lemma:

LEMMA 20. *Let $M_0, M_1, \ldots$ be an infinite sequence of n-dimensional polyhedral manifolds such that $M_i \to M_{i+1} = M_i + E_i^n$ is a regular expansion. Then $I(M_0)$ is in a* (1-1) *semi-linear correspondence with*

$$M = \sum_{i=0}^{\infty} I(M_i),$$

where $I(M_i)$ stands for the interior of M_i.

It will be sufficient to outline a proof, the details of which do not involve anything new. Let f_i be a (1-1) semi-linear map of M_i on M_0. Then M_{i+1} can be mapped on M_0 in a (1-1) semi-linear transformation f_{i+1}, which coincides with f_i except in $\dot{E}_i^n + U_i$, where U_i is an arbitrarily chosen regular neighbourhood in M_i of $E_i^{n-1} = M_i . E_i^n = \dot{M}_i . \dot{E}_i^n$. Therefore we may assume that $f_{i+1} = f_i$ in L_i^n, where L_i^n is an open manifold in M_i, such that, treating M and $I(M_0)$ as polyhedra, we have

(1) $L_i^n \subset L_{i+1}^n$,

(2) $M = \sum_{i=0}^{\infty} L_i^n$ and $I(M_0) = \sum_{i=0}^{\infty} f_i(L_i^n)$.

If we begin with f_0 as the identical map of M_0 on itself and

$$L_0^n = M_0'' - O(\dot{M}_0'', M_0''),$$

† The interior $I(U)$ of a bounded polyhedral manifold U is obviously an open manifold in the sense that any rectilinear, simplical covering of $I(U)$ satisfies the combinatorial condition for an open manifold.

the map f_i, subject to the condition $f_i = f_j$ in L_i^n if $i < j$, is defined inductively for each i, and the required map, $f(M) = I(M_0)$, is given by $f(p) = f_i(p)$ if $p \in L_i^n$.

If $\Sigma(K, M)$ is taken to mean the simplicial space determined by some complex covering the interior of $U(K, M)$, the generalization of Theorem 24 presents no difficulty. Theorem 25 is true of Euclidean complexes† K and L, provided that they have the same "Euclidean nucleus", meaning that sub-divisions of K and L expand into the same Euclidean complex (it follows from the proof of Theorem 32 that this relation between K and L is transitive). The corollaries to Theorem 25 depend on Theorem 21, concerning which we are ignorant. Lemma 12 and Theorem 26 generalize automatically. If the questions Q. 2 and Q. 3 of § 10 are restated in terms of Euclidean complexes and nuclei, and of the interiors of bounded manifolds M, M_i and $M + M_i$, such that $M . M_i = \dot{M} . \dot{M}_i = E_i^{n-1}$, Theorem 27 follows from the arguments in § 12 and from

THEOREM 40. *A bounded manifold and its interior have the same Euclidean nucleus.*

If M is any bounded n-dimensional manifold the theorem will follow if we can show that some Euclidean complex expands both into a sub-division of M and into a rectilinear complex covering $I(M)$. After an initial subdivision, if necessary (*e.g.* $s_{\dot{M}}$), let $N(\dot{M}, M)$ be a normal simplicial neighbourhood of $\dot{M}$. Then, if we write M' for the first derived complex of M, $N(\dot{M}', M')$ is geometrically equivalent to the topological product $\dot{M}'_{01} = \dot{M}' \times \langle 0, 1 \rangle$. For let A_1^k, A_2^k, ... be the k-simplexes in $\dot{M}$ $(k = 0, \ldots, n-1)$ and let E_i^{n-k} and F_i^{n-k-1} be the closed cells dual to A_i^k in the complexes M^* and $\dot{M}^*$, dual to M and $\dot{M}$ (as usual the cells of M^* are composed of simplexes in M'). Let f_0 be the map of $\dot{M}'$ on $\dot{M}' \times 1$ given by $f_0(p) = p \times 1$ $(p \in \dot{M}')$, and let f_{r+1} $(0 \leqslant r \leqslant n-1)$ be a (1-1) semi-linear map of

$$N_{r+1} = \dot{M}' + E_1^{r+1} + E_2^{r+1} + \ldots \quad \text{on} \quad (\dot{M}' \times 1) + (\dot{M}'^r \times \langle 0, 1 \rangle),$$

where $\dot{M}'^r = F_1^r + F_2^r + \ldots$, which coincides with f_0 in $\dot{M}'$ and is such that $f_{r+1}(E_j^s) = F_j^{s-1} \times \langle 0, 1 \rangle$ $(1 \leqslant s \leqslant r+1;\ j = 1, 2, \ldots)$. If $r < n-1$, it follows from theorems on sub-division to which we have previously referred‡ that f_{r+1} can be extended throughout each cell E_i^{r+2} to give a map f_{r+2},

† *I.e.* locally finite complexes whose dimensionalities are finite.

‡ *Cf.* the proof of Theorem 6, § 4.

satisfying the same conditions as before with r replaced by $r+1$. It follows from induction on r that such a map exists for $r=n-1$, and f_n is the required map of $N(\dot{M}', M)$ on $\dot{M}'_{01}$.

Let $M_0 = M' - O(\dot{M}', M')$. Then it follows from the last paragraph that we may replace M' by $M_0 + \dot{M}'_{01}$, where each point p in $\dot{M}_0$ is identified with $p \times 0$ in $\dot{M}'_{01}$. Then M_0 expands geometrically into $M_0 + \dot{M}'_{01}$, by Theorem 8. Let

$$M_i = M_0 + \{\dot{M}_0' \times \langle 0, i/(i+1) \rangle\} \quad (i = 1, 2, \ldots).$$

By Theorem 8 each of the transformations $M_i \rightarrow M_{i+1}$ is a geometrical expansion. Their resultant is an expansion of M_0 into a polyhedral complex covering the interior of $M_0 + \dot{M}'_{01}$, and the theorem is established.

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