Manifolds homotopy equivalent to $RP^4 \# RP^4$

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Abstract

Recent computations of $UNil$-groups by Connolly, Ranicki and Davis are used to study splittability of homotopy equivalences between 4-dimensional manifolds with infinite dihedral fundamental groups.

The problem of splitting a manifold into a connected sum is one of the most natural, yet one of the most difficult problems in manifold topology. It was extensively studied by S. Cappell [**4**–**9**], who also provided an elegant solution. To be more specific: Let $f : M^n \to X^n = X_1^n \# X_2^n$, $n \geq 5$, be a homotopy equivalence of closed topological manifolds, where X^n is a connected sum of manifolds X_i^n, $i = 1, 2$. The homotopy equivalence f is *splittable* if it is homotopic to a map (necessarily a homotopy equivalence which we continue to call f), transverse regular to the separating sphere $S^{n-1} \subset X^n$, such that the restriction of f to $Y = f^{-1}(S^{n-1})$, and the two components of $f^{-1}(X^n - S^{n-1})$ are homotopy equivalences.

Cappell's solution to the splitting problem for $f : M^n \to X^n$ is in terms of certain exotic $UNil$ groups introduced and studied by him. Despite the fact that these groups are very difficult to deal with, he was able to perform various computations which lead to many important topological results ([**7**, **8**]). The simplest example of a group G for which $UNil$ groups can be nontrivial is that of an infinite dihedral group $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$. In this case, Cappell was able to show that $UNil_0(\mathbb{Z}/2 * \mathbb{Z}/2) = 0$ and that $UNil_2(\mathbb{Z}/2 * \mathbb{Z}/2)$ contains an infinite dimensional vector space over $\mathbb{Z}/2$ (cf. [**11**]. We here use the notation $UNil_*(\mathbb{Z}/2 * \mathbb{Z}/2) = UNil_*(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$). The odd-dimensional $UNil$ groups $UNil_1(\mathbb{Z}/2 * \mathbb{Z}/2)$ and $UNil_3(\mathbb{Z}/2 * \mathbb{Z}/2)$ resisted calculations for quite some time. (It was known, however ([**13**]), that in this case the $UNil$-groups are either trivial or infinitely generated.) Recently F. Connolly and A. Ranicki, using rather complicated algebraic machinery, have shown that $UNil_1(\mathbb{Z}/2 * \mathbb{Z}/2) = 0$ and $UNil_3(\mathbb{Z}/2 * \mathbb{Z}/2)$ contains an infinite dimensional vector space over $\mathbb{Z}/2$ [**12**], and Connolly and Davis [**10**] have completed the calculation of $UNil$ in the case of $\mathbb{Z}/2 * \mathbb{Z}/2$ in all degrees and for all orientation characters.

The purpose of this paper is to discuss some geometric consequences of these results for

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the topology of 4-dimensional manifolds. The relevant group is then $UNil_5$, which is trivial in the orientable case and infinitely generated in the non-orientable cases ([**10**, Theorem 1.10 and semi-periodicity]). Our main results are contained in theorems 1 and 2 below. The first deals with various stabilizations of exotic homotopy $RP^4\#RP^4$-manifolds, but as will be clear from the proof, much of this will also be true for many other non-orientable manifolds with the same fundamental group.

THEOREM 1. *There exist infinitely many manifolds* M_i^4, $i = 0, 1, 2, 3, \dots$, *with* $M_0^4 := RP^4\#RP^4$, *such that*

(a) $M_i^4 \underset{top}{\not\approx} M_j^4$, *for* $i \neq j$, *and no* M_i^4 *splits topologically into a nontrivial connected sum for* $i \geq 1$.
(b) $M_i^4 \simeq M_0^4$ *for* $i = 1, 2, 3, \dots$
(c) $M_i^4 \times \mathbb{R}^k \underset{top}{\approx} M_0^4 \times \mathbb{R}^k$ *for* $i \geq 0$ *and* $k \geq 3$.
(d) $M_i^4 \times \mathbb{R}^k \underset{top}{\not\approx} M_j^4 \times \mathbb{R}^k$ *for* $i \neq j$ *and* $k \leq 2$
(e) $M_i^4 \times T^k \underset{top}{\not\approx} M_j^4 \times T^k$ *for* $i \neq j$, *where* $T^k = \underbrace{S^1 \times \ldots \times S^1}_{k-times}$ *and* $k \geq 1$
(f) *For every* $i = 1, 2, \dots$, *there is an integer* $k > 0$ *such that*

$$M_i^4\#k(S^2 \times S^2) \underset{top}{\approx} M_0^4\#k(S^2 \times S^2)\ .$$

In sharp contrast to Theorem 1, in the case of orientable 4-manifolds, we will show

THEOREM 2. *Let* $f : M^4 \to X^4 = X_1^4\#X_2^4$ *be a homotopy equivalence of orientable topological manifolds with* $\pi_1(X_i^4) \approx \mathbb{Z}/2$, $i = 1, 2$. *Then* f *is splittable.*

Remarks. (1) Lemma 4.2 in [**13**] implies that all the manifolds M_i, $i = 1, 2, \dots$ in Theorem 1 are finitely covered by the standard $M^4 = RP^4\#RP^4$.

(2) One consequence of Theorem 1 is a strong failure of the 4-dimensional version of the classical Kneser Theorem. Namely, there are 4-manifolds simple homotopy equivalent to a connected sum which do not admit (topologically) corresponding connected sum splitting. For other results on the 4-dimensional Kneser-conjecture, see [**16**] and [**17**].

(3) The proof of theorem 1 is purely topological, but (b) and (f) also mean that all the manifolds can be smoothed after product with $\mathbb{R}^3$ or connected sum with enough $S^2 \times S^2$'s. It would be very interesting to know if any of the exotic $RP^4\#RP^4$'s themselves can be given a smooth structure. At the end of the paper we indicate a possible geometric construction which may give smooth examples.

Proof of Theorem 1. Since $\mathbb{Z}/2 * \mathbb{Z}/2$ is "good" [**14**], there is a long, exact Sullivan-Wall surgery sequence for $M^4 = RP^4\#RP^4$.

$$\begin{aligned} \cdots \longrightarrow \left[\Sigma(M_+^4); G/\mathrm{Top}\right] &\xrightarrow{\theta} L_1^s(\mathbb{Z}/2^- * \mathbb{Z}/2^-) \xrightarrow{\gamma} S_{\mathrm{Top}}(M^4) \xrightarrow{\eta} \\ \longrightarrow \left[M^4; G/\mathrm{Top}\right] &\xrightarrow{\theta} L_0^s(\mathbb{Z}/2^- * \mathbb{Z}/2^-)\ . \end{aligned}$$

(The notation $\mathbb{Z}/2^-$ means the group $\mathbb{Z}/2$ with the nontrivial orientation character.) Now $L_1^s(\mathbb{Z}/2^- * \mathbb{Z}/2^-) \cong L_1^s(\mathbb{Z}/2^-) \oplus L_1^s(\mathbb{Z}/2^-) \oplus UNil_1(\mathbb{Z}/2^- * \mathbb{Z}/2^-)$ [**5**]. Since $L_1^s(\mathbb{Z}/2^-) \cong 0$ [**26**], it then follows from the semi–periodicity of $UNil$-groups [**5**] that $L_1^s(\mathbb{Z}/2^- * \mathbb{Z}/2^-) \cong UNil_3(\mathbb{Z}/2 * \mathbb{Z}/2)$. The action of $UNil_3(\mathbb{Z}/2 * \mathbb{Z}/2)$ on $S_{\mathrm{Top}}(M^4)$ is free, so we get infinitely many non-split homotopy equivalences $M_i \to M$ by acting on the identity map on M. (a) and (b) then follow if we prove that none of them can be realized by exotic self-equivalences of M:

LEMMA 1. *Every homotopy equivalence* $h : M \to M$ *is homotopic to a homeomorphism.*

(This will be proved after we finish the proof of Theorem 1.) Now let $f_i : M_i^4 \to M$ be one of these homotopy equivalences. Since f_i is equivalent to the identity on M by the action of $L_1^s(\mathbb{Z}/2^- * \mathbb{Z}/2^-)$, it maps trivially to $[M; G/\mathrm{Top}]$. This means that $g_i^* \nu_{M_i} \approx \nu_M$, where ν is the stable normal bundle, g_i is a homotopy inverse of f_i and $\approx$ means stable (linear) equivalence. Hence $f_i^* \nu_M \approx f_i^* g_i^* \nu_{M_i} \approx \nu_{M_i}$, and it follows that, stably, $f_i^* \tau(M) \approx (\tau(M_i) + f_i^* \nu_M) + f_i^* \tau(M) \approx \tau(M_i) + f_i^*(\nu_M + \tau(M)) \approx \tau(M_i)$, i. e. f_i is in fact a stable *tangential* homotopy equivalence. The classical result of B. Mazur ([**21**], cf. [**23**]) implies that

$$M_i^4 \times \mathbb{R}^k \underset{top}{\approx} M^4 \times \mathbb{R}^k \quad \text{for some } k > 0.$$

To improve this result to $k = 3$, we use the proper surgery theory of S. Maumary and L. Taylor ([**20**], [**25**], see also [**22**]). Namely, the long surgery exact sequence for $M^4 \times \mathbb{R}^3$:

$$\begin{aligned} \cdots \longrightarrow L_0^{s,open}(M^4 \times \mathbb{R}^3) \xrightarrow{\gamma} S_{\mathrm{Top}}^{s,open}(M^4 \times \mathbb{R}^3) \xrightarrow{\eta} \left[M^4 \times \mathbb{R}^3; G/\mathrm{Top}\right] \\ \xrightarrow{\theta} L_3^{s,open}(M^4 \times \mathbb{R}^3) \end{aligned}$$

has $L_*^{s,open}(M^4 \times \mathbb{R}^3) = 0$ ([**22**, p. 252]). Since by construction $\eta(f_i) = 0$, $\eta(f_i \times id_{\mathbb{R}^3}) = 0$ and hence $[f_i \times id_{\mathbb{R}^3}] = 0$ in $S_{\mathrm{Top}}^{open}(M^4 \times \mathbb{R}^3)$. This means that $f_i \times id_{\mathbb{R}^3}$ is (properly) homotopic to a homeomorphism — proving (c).

Let $T^k = \underbrace{S^1 \times \ldots \times S^1}_{k-times}$ be the k-dimensional torus. Assume that there is a homeomorphism

$$f : M_i^4 \times T^k \xrightarrow{\approx} M^4 \times T^k \qquad \text{for some } i, \text{ and } k > 0\ .$$

Consider the induced isomorphism

$$f_\# : \pi_1(M_i^4 \times T^k) \xrightarrow{\cong} \pi_1(M^4 \times T^k)$$

i.e.,

$$f_\# : (\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}^k \xrightarrow{\cong} (\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}^k\ .$$

Since the center of $(\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}^k$ is $\mathbb{Z}^k$, $f_\#$ induces an isomorphism of $\mathbb{Z}^k$ given by some matrix $A \in GL(\mathbb{Z}, k)$. Using a self-homeomorphism of T^k represented by A^{-1}, we can assume that the induced isomorphism $f_\#$ is of the form $h_\# \times id_{\pi_1(T^k)}$ for some $h_\# : \pi_1(M_i^4) \to \pi_1(M_i^4)$. As a consequence, the homeomorphism f lifts to a homeomorphism

$$\widetilde{f} : M_i^4 \times T^{k-1} \times \mathbb{R} \xrightarrow{\approx} M^4 \times T^{k-1} \times \mathbb{R}\ .$$

This leads to an h-cobordism between $M_i^4 \times T^{k-1}$ and $M^4 \times T^{k-1}$. We show that this h-cobordism is an s-cobordism (and hence a product). By iterating this procedure, we end up with $M_i^4 \underset{top}{\approx} M^4$, contradicting (a).

Let $G \cong \mathbb{Z}/2 * \mathbb{Z}/2$. The Whitehead group $Wh(G \times \mathbb{Z}^k)$ can be computed via the iterated Bass-Heller-Swan decomposition (cf. [**1**])

$$Wh(G \times \mathbb{Z}^k) \xrightarrow{\cong} \sum_{j=0}^{k} \binom{k}{j} Wh_{1-j}(G) \oplus \widetilde{Nil}\text{-groups} .$$

Here the Whitehead groups are defined as

$$Wh_{1-j}(G) \cong \begin{cases} Wh(G) & , \text{ if } j = 0 \\ \widetilde{K}_0(G) & , \text{ if } j = 1 \\ K_{1-j}(G) & , \text{ if } j \geq 2 . \end{cases}$$

Now the decomposition results of Stallings and Gersten for the free products (cf. [**15**, **24**]) imply $Wh(G) \cong \widetilde{K}_0(G) \cong 0$. The corresponding free product decomposition for lower K-groups follows from lemmas in [**3**, pp. 19–20 (in particular, the Theorem on p. 20)]. Namely,

$$K_{1-j}(G) \cong K_{1-j}(\mathbb{Z}/2) \oplus K_{1-j}(\mathbb{Z}/2) \approxeq 0 \quad \text{for } j \geq 2 .$$

It turns out that a corresponding free product decomposition $\widetilde{Nil}$-terms also exists (cf. [**2**, Thm. 8.2, p. 677]). This and the fact that $\widetilde{Nil}(\mathbb{Z}/2) \cong 0$ (cf. [**18**]) give

$$Wh(G \times \mathbb{Z}^k) \cong 0 .$$

As a consequence, every h-cobordism is an s-cobordism and hence $M_i^4 \underset{top}{\approx} M^4$ which is a contradiction. This proves part (e) of Theorem 1.

Part (d) is proved by a method analogous to that in [**19**]. The case $k = 1$ has essentially been delt with in the proof of (e), so we let $k = 2$:

A homeomorphism $M_i^4 \times \mathbb{R}^2 \approx M^4 \times \mathbb{R}^2$ restricts to an embedding $M_i^4 \times D_1^2 \xrightarrow{\subset} M^4 \times \mathrm{int} D_2^2$ where D_1^2 and D_2^2 are two disks, and the difference is an h–cobordism $M_i^4 \times S^1$ and $M^4 \times S^1$. As in the proof of part (e) this is an s–cobordism, so we get $M_i^4 \times S^1 \underset{top}{\approx} M^4 \times S^1$, and hence, by (e), $M_i^4 \approx M^4$.

Finally, part (f) of Theorem 1 is true for any two 4–manifolds M and M' which are related by a homotopy equivalence $M' \to M$ obtained by the action of $L_5^s(\pi_1(M), w)$ (where π_1 is 'good') on the identity map of M, and follows by a standard argument from the usual description of this action (Cf. the proof of Theorem 11.3A in [**14**]). An element of $L_1^s(\pi_1(M), w)$ can be realized as the surgery obstruction of a normal map $W^5 \xrightarrow{F} M \times I$, where W^5 is obtained from $M^4 \times I$ by first attaching k trivial 2–handles and then k 3–handles, for some k. The resulting homotopy equivalence is the restriction of F to the 'other' boundary component M' of W. The middle stage is homeomorphic to $M \# k(S^2 \times S^2)$, and the reason $M' \to M$ is a homotopy equivalence is that the 3–cells are attached along spheres having transverse spheres. (The crucial tool here is Corollary 5.1B in [**14**].) But then we can, dually, think of $M \# k(S^2 \times S^2)$ as being obtained by surgery on *trivial* 1-spheres in M', hence it is also homeomorphic to $M' \# k(S^2 \times S^2)$. □

Proof of the Lemma. There are two useful ways of describing $M_0 = RP^4 \# RP^4$:

(1) Let $RP_0^4 = RP^4 - \mathrm{int}\, D^4$. Then RP_0^4 has double cover S^4 minus the interiors of *two* 4-disks. This is homeomorphic to $S^3 \times I$, hence M_0 has orientable double cover $S^3 \times S^1$, obtained by gluing two copies of $S^3 \times I$ along the boundaries. (In fact, M_0 can be obtained by identifying (x, y) and $(-x, \bar{y})$ in $S^3 \times S^1$.) Hence the first homotopy groups of M_0 are $\pi_1 \cong \mathbb{Z}/2 * \mathbb{Z}/2$, $\pi_2 = 0$, $\pi_3 \cong \mathbb{Z}$, generated by the inclusion of the connecting 3–sphere, and $\pi_4 \cong \mathbb{Z}/2$.

(2) We shall also need the following explicit cell structure of M_0:

Taking the connected sum of two copies of RP^3 can be thought of as first joining two copies of RP^3_0 in a point on the boundary, and then adding a 4-disk, attached by the map $S^3 \xrightarrow{\nabla} S^3 \vee S^3 \subset RP^4_0 \vee RP^4_0$. ($\nabla$ is the usual pinch map.) But $RP^4_0 \xrightarrow{p} RP^3$ by a map p restricting to the double cover on the boundary, so we get $M_0 \simeq (\widetilde{RP}^3 \vee RP^3) \cup D^4$, with attaching map $S^3 \xrightarrow{\nabla} S^3 \vee S^3 \xrightarrow{p \vee p} RP^3 \vee RP^3$.

We complete the cell structure using the standard cell filtration $* \subset RP^1 \subset RP^2 \subset RP^3$ on each RP^3.

Claim: *The automorphism group of* $\mathbb{Z}/2 * \mathbb{Z}/2$ *is generated by conjugations and the interchange* τ *of the two copies of* $\mathbb{Z}/2$.

To see this, we shall calculate the automorphism group of the *infinite dihedral group* $D_\infty = < s, u; s^2 = 1, sus = u^{-1} >$, which is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2 = < s, t; s^2 = t^2 = 1 >$ by $s \leftrightarrow s$ and $u \leftrightarrow st$. Then $\{u^i\}_{i \in Z} = [D_\infty, D_\infty] \approx \mathbb{Z}$, and the remaining elements in D_∞ are the elements $\{u^i s\}_{i \in Z}$, all of order two. It follows easily that an automorphism $\phi : D_\infty \xrightarrow{\approx} D_\infty$ must satisfy $\phi(u) = u^\varepsilon,\ \phi(s) = u^i s$, for some $\varepsilon \in \{\pm 1\},\ i \in \mathbb{Z}$, and there is a unique such $\phi = \phi_{(i,\varepsilon)}$ for every pair (i, ε).

Now it is atraightforward to check that $\phi_{(0,1)} = \mathrm{Id}$, $\phi_{(i,\varepsilon)}\phi_{(i',\varepsilon')} = \phi_{(i+\varepsilon i', \varepsilon\varepsilon')}$, and $u^i s^\eta \mapsto \phi_{(i,(-1)^\eta)}$ defines a *group isomorphism* $D_\infty \approx \mathrm{Aut}(D_\infty)$.

Furthermore, we can calculate the effect of conjugating by an element w, and we obtain:

- Conjugation by u^i is $\phi_{(2i,1)}$
- Conjugation by $u^i s$ is $\phi_{(2i,-1)}$

These automorphisms form (normal) subgroup (also isomorphic to D_∞) of $\mathrm{Aut}(D_\infty)$ with quotient $\mathbb{Z}/2$ generated by any of the remaining automorphisms — e. g. $\phi_{(-1,-1)}$. It now only remains to check that $\phi_{(-1,-1)}$ corresponds to the automorphism of $\mathbb{Z}/2 * \mathbb{Z}/2$ given by interchanging s and t.

But $t \in \mathbb{Z}/2 * \mathbb{Z}/2$ corresponds to $u^{-1}s \in D_\infty$, and we have $\phi_{(-1,-1)}(s) = u^{-1}s$ by definition and $\phi_{(-1,-1)}(u^{-1}s) = (\phi_{(-1,-1)}(u))^{-1}\phi_{(-1,-1)}(s) = uu^{-1}s = s$. This proves the claim.

Both the conjugations and τ can be realized by homeomorphisms of M, so it suffices to consider the case when h preserves that basepoint and induces the identity on π_1. To analyse h further, we use the cell filtration above, and since $\pi_2(M) = 0$, we may assume that h restricts to the inclusion on both RP^2's.

Extensions to each RP^3 are parametrized by $\mathbb{Z} = \pi_3(M)$. Note that both inclusions $RP^3 \subset M$ induce isomorphisms $\pi_3(RP^3) \approx \pi_3(M)$, but the two isomorphisms differ by sign. (The gluing in the connected sum is by an orientation *reversing* homeomorphism.) If we act on one of the inclusions by $n \in \pi_3(M) \approx \pi_3(RP^3)$, we obtain the map $RP^3 \xrightarrow{\nabla} RP^3 \vee S^3 \xrightarrow{\mathrm{Id} \vee n} RP^3 \subset M$ which will take the generator to $(\pm 1 + 2n) \in \pi_3(M)$. It follows that to get a map $RP^3 \vee RP^3 \to M$ which extends to M, we must act by n on one and by $-n$ on the other copy of RP^3, for some n. Then the induced homomorphism on $\pi_3(M)$ is multiplication by $1 + 2n$, so to get an isomorphism we must have n equal to 0 or -1. Both may be realized by homeomorphisms of $RP^3 \vee RP^3$extending to homeomorphisms of M: $n = 0$ corresponds to the identity and $n = -1$ is realized by the homeomorphism induced by $A \times \mathrm{Id} : S^3 \times S^1 \to S^3 \times S^1$, where A is an orientation reversing isometry.

The possible extensions to all of M can now be constructed by action by $\pi_4(M) \approx \mathbb{Z}/2$, and the nontrivial element can be realized by a 1-parameter family of rotations of spheres in a collar neighborhood of the connecting 3-sphere. □

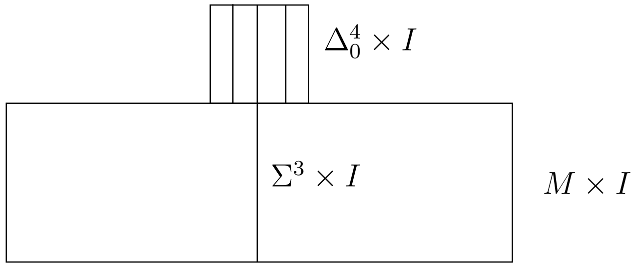

Fig. 1. N^5

Proof of Theorem 2. The simplest way to prove this is perhaps to compare the Sullivan–Wall sequence for X^4 with the direct sum of the sequences for X_1^4 and X_2^4 and use a version of the five–lemma. However, the geometric arguments in the following proof may be more illuminating.

Let $f : M^4 \to X^4 = X_1^4 \# X_2^4$ be a homotopy equivalence. Our first observation is that f splits *homologically* along the separating $S^3 \subset X^4$. This means that we may replace f by a homotopic map such that $f^{-1}(S^3) = \Sigma^3$ is a homology 3-sphere ($\mathbb{Z}$-coefficients). To see this, one can proceed directly, or one can apply the result of S. Weinberger (i.e., Theorem A in [**27**]). In either case one has to use the triviality of $UNil_1(\mathbb{Z}/2 * \mathbb{Z}/2)$. We now use a simple "neck exchange" trick to improve f to a splittable homotopy equivalence.

Let $J = [-1, 1]$ and identify a neighborhood of $\Sigma^3 \subset M^4$ with $\Sigma^3 \times J$. Similarly we can identify a neighborhood of the connecting S^3 in X^4 with $S^3 \times J$, and we may assume that via these identifications $f|(\Sigma^3 \times J) = (f|\Sigma^3) \times J$.

Let Δ^4 be a contractible manifold bounding Σ^3 and let Δ_0^4 be Δ^4 with the interior of a 4–disk removed. Then $\partial\Delta_0^4$ is a 4–manifold homotopy equivalent to S^3 with boundary components Σ^3 and S^3, and $\partial(\Delta_0^4 \times J) = (\Sigma \times J) \cup W^4$, where W^4 is a manifold which can be thought of as the (interior) connected sum of two copies of Δ^4, and with boundary two copies of Σ^3. Let N^5 be $M^4 \times I \cup \Delta_0^4 \times J$, where we glue $\Sigma^3 \times J \subset \Delta_0^4 \times J$ to $\Sigma^3 \times J \subset M^4 \times \{1\}$. Then N^5 has two boundary components $M = M^4 \times \{0\}$ and $\overline{M}$. $\overline{M}$ is clearly a connected sum, obtained by replacing the "neck" $\Sigma^3 \times J$ in M by W. (See fig. 1.)

To finish the proof of the theorem, we need the following two assertions:

(1) f extends to a map $F : N^5 \to X^4$ such that $F^{-1}(S^3) = \Sigma^3 \times I \cup_{\Sigma \times \{1\}} \Delta_0^4$. Hence the map $\overline{f} = F|\overline{M} : \overline{M} \to X$ splits.

(2) N^5 is an s-cobordism. Then, since the fundamental group is "good", N^5 is a product by Freedman's 5-dimensional h-cobordism theorem [**14**]. Hence $\overline{M} \approx M$ and F becomes a homotopy from f to a split map.

Assertion (1) is the simplest: Using that $\Delta_0^4 \simeq S^3$ it is easy to extend $f|\Sigma^3 : \Sigma^3 \to S^3$ to, say, $f' : \Delta_0^4 \to S^3$. Then define F to be $f \circ \mathrm{proj}_M$ on $M \times I$ and $f' \times J$ on $\Delta_0^4 \times J$.

(2) needs more work. First observe that since $\pi_1(\Sigma^3)$ is perfect and the commutator subgroup of $\mathbb{Z}/2 * \mathbb{Z}/2$ is $\mathbb{Z}$, the map induced on π_1 by the inclusion $\Sigma^3 \subset M^4$ must be trivial. By the van Kampen theorem it then follows that $\pi_1(N^5) \cong \pi_1(M^4)$. A Mayer–Vietoris argument (arbitrary local coefficients) then shows that $M^4 \subset N^5$ also induces an isomorphism in homology with all coefficients — hence is a homotopy equivalence. Lefschetz duality then shows that $\overline{M} \subset N^5$ also induces an isomorphism on all homology,

so it only remains to prove that it induces an isomorphism on π_1. This is the same as showing that $\overline{f}$ induces an isomorphism on π_1, Since it has degree one, we know that it induces an *epimorphism*, hence it suffices to prove that $\pi_1(\overline{M}) \cong \mathbb{Z}/2 * \mathbb{Z}/2$. (Any epimorphism of this group on itself is clearly an isomorphism.)

Write $\overline{M} = \overline{M}_1 \cup_{S^3} \overline{M}_2$ such that $\overline{f}$ maps $\overline{M}_i$ to X_i, and similarly $M = M_1 \cup_{\Sigma^3} M_2$. By construction, $\overline{M}_i = M_i \cup_{\Sigma^3} \Delta^4_0$. If $H \to G$ is a homomorphism of groups, we denote by $G//H$ the quotient of G by the normal closure of the image of H. Then, by the van Kampen theorem, $\pi_1(\overline{M}_i) \cong \pi_1(M_i)//\pi_1(\Sigma^3)$.

We now use the following algebraic observation:

Let $H \to G_i$, $i = 1, 2$ *be two group homomorphisms and form the pushout* $G = G_1 *_H G_2$. *If the composed map* $H \to G$ *is trivial, then*

$$G \cong G_1//H * \; G_2//H\,.$$

(In fact, by universal properties there is always a canonical homomorphism $G \to G_1//H * G_2//H$. The assumption implies that there is also one going the other way, and they are easily seen to be inverses of each other.) We apply this to the two homomorphisms $\pi_1(\Sigma^3) \to \pi_1(M_i)$: by van Kampen the pushout is $\pi_1(M)$, and by the observation above $\pi_1(\Sigma^3) \to \pi_1(M^4)$ is trivial. $\square$

Remarks. It would be interesting to have a more explicit construction of the exotic homotopy $RP^4 \# RP^4$–manifolds in Theorem 1. One possible attempt could be based on the construction of exotic 4-dimensional s-cobordisms of Cappell and Shaneson [**9**]. $RP^4 \# RP^4$ can be represented as a union of two D^2-bundles over the Klein bottle K. Each such bundle can also be represented as an $(S^1 \times D^2)$-bundle over S^1. Now, in the spirit of [**9**], one might try to replace the fiber (in one or both bundles) by a complement of an appropriately selected knot in S^3 and try to recover the right homotopy type of the constructed manifold by a 4-dimensional topological surgery. Of course the main problem left to deal with would be to show the exoticity of these constructed manifolds via some invariants from classical knot theory. This problem is of considerable interest in its own right, since the geometric aspects of the computation of $UNil_3(\mathbb{Z}/2 * \mathbb{Z}/2)$ are rather mysterious as of now.

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