subbundles such that $\tilde{E}^s_x \subset C^s_\epsilon(x)$, $\tilde{E}^u_x \subset C^u_\epsilon(x)$ for all $x \in T^3$. Observing that $(Tf)E_2 = E_2$ and defining $\tilde{E}^c = E_2$ we have a continuous Tf-invariant splitting $T(T^3) = \tilde{E}^s \oplus \tilde{E}^c \oplus \tilde{E}^u$ and a foliation $\mathcal{F}^c$ tangent to $\tilde{E}^c$ with all leaves dense.

Since $\mathcal{F}^c$ is a normally hyperbolic foliation for f it follows[4] that f is plaque expansive. Finally since f_1 satisfies (a)–(e) with respect to the constants $\lambda, \mu, \delta_1, \ldots, \delta_5$ and the sets U_0, U_1 it follows from (1) that taking ϵ small enough f satisfy again conditions (a)–(e).

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I.M.P.A. Rio de Janeiro

Topology, Vol. 17, pp. 397–405
© Pergamon Press Ltd., 1978. Printed in Great Britain
0040–9383/78/1201–0397/$02.00/0

QUADRATIC FORMS AND STEENROD SQUARES

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(Received 20 June 1978)

INTRODUCTION

LET M and N be $2n$-dimensional compact smooth manifolds. Let νM be the stable normal bundle of M and let $f: (M, \partial M) \to (N, \partial N)$ be a degree one map covered by a bundle map $b: \nu M \to \xi$ where ξ is some bundle over N. In the terminology of Browder[3, p. 31] $(f, b): (M, \partial M) \to (N, \partial N)$ is a degree one normal map. Because f has degree one there exists a homomorphism $\alpha: H^*(M, \partial M) \to H^*(N, \partial N)$ such that $\alpha f^* = 1$. (All homology and cohomology groups will have $Z/2$ coefficients.) Using α there is an isomorphism

$$H^*(M, \partial M) \cong K^*(f, b) \oplus f^*H^*(N, \partial N)$$

where $K^*(f, b) = \ker(\alpha: H^*(M, \partial M) \to H^*(N, \partial N))$. Furthermore the action of the Steenrod squares on $H^*(M, \partial M)$ preserves this splitting.

Using the methods of Browder[2, 3], and Brown[4], it is possible to define a quadratic form $q: K^n(f, b) \to Z/2$, that is a function q satisfying the following formula:

$$q(x + y) = q(x) + q(y) + x \,.\, y$$

where $x \,.\, y$ is the mod 2 intersection number of x and y. This quadratic form is discussed in more detail in §2. The main result of this paper is a formula for $q(Sq^k y)$ valid when the integers n and k satisfy certain conditions and M and N satisfy a condition concerning their characteristic classes.

This paper is set out as follows: §1 contains the statement of the main theorem and the deduction of some corollaries one of which is required in [8], another concerns immersions; §2 contains a discussion of the quadratic form; and §3 an explanation of how to reduce the proof of the main theorem to two technical results. These results are proved in §4. Finally §5 contains two examples, motivated by [9], of the use of the main theorem.

It is a great pleasure to thank Elmer Rees for many helpful observations and much encouragement. He also pointed out that Corollary 1.4 in §1 was an immediate consequence of the main theorem and that this corollary should have an application to immersions.

§1. THE MAIN THEOREM AND SOME COROLLARIES

To state the main theorem requires some notation. Let W be a d-dimensional compact manifold. From the Poincaré duality theorem the homomorphism $Sq^i: H^{d-i}(W, \partial W)$ is given by $Sq^i x = x \,.\, v_i(W)$ for a unique class $v_i(W) \in H^i(W)$. By definition $v_i(W)$ is the i-th Wu-class of W.

Let (n, k) be a pair of integers, and suppose that 2^i divides $n + k + 1$, but 2^{i+1} does not divide $n + k + 1$:

(1.1) (n, k) is an *exceptional pair* if $k < 2^i$.

It is now possible to state the main theorem.

THEOREM 1.2. *Let* $q: K^n(f, b) \to Z/2$ *be the quadratic form described above. Suppose that* $v_i(M) = 0$ *for* $i \le 2k$ *and that the pair* (n, k) *is not exceptional. Given*

$y \in K^{n-k}(f, b)$ *then*

$$q(Sq^k y) = \sum_{i=0}^{k-1} (Sq^{2k-i} y) \,.\, (Sq^i y).$$

Examples of normal maps are provided by framed manifolds and most of the applications of the above theorem discussed in this paper arise from these examples. Suppose that M^m is a compact smooth manifold and that νM is stably trivial. Let F be a stable trivialis .on, or a framing, of νM. If ∂M is empty, let $f\colon M^m \to S^m$ be a degree one map, then the framing F may be used to cover f by a bundle map $b_F\colon \nu M \to \epsilon$ where ϵ is a trivial bundle over S^m. Similarly if ∂M is non-empty there is a degree one normal map $(f, b_F)\colon (M, \partial M) \to (D^m, S^{m-1})$. If $m = 2n$ then $K^n(f, b_F) = H^n(M, \partial M)$ and the quadratic form is the quadratic form q_F associated to the framing.

COROLLARY 1.3. *Let* (M^{2n}, F) *be a framed manifold, and suppose* (n, k) *is not exceptional. Let* $q_F\colon H^n(M, \partial M) \to Z/2$ *be the quadratic form associated to the framing. Then if* $y \in H^{n-k}(M, \partial M)$

$$q_F(Sq^k y) = \sum_{i=0}^{k-1} (Sq^{2k-i} y) \,.\, (Sq^i y).$$

For all l the pair $(2^l - 1, k)$ is not exceptional provided $k < 2^l$. Then with (n, k) one of the pairs $(2^l - 1, 1)$, $(2^l - 1, 2)$, $(2^l - 1, 4)$ and ∂M empty this corollary is exactly the result 1.2 required in [8].

COROLLARY 1.4. *With the assumptions of* 1.3, *suppose* $y \in H^{n-k}(M, \partial M)$ *and* $Sq^k y = 0$. *Then*

$$\sum_{i=0}^{k-1} (Sq^{2k-i} y) \cdot (Sq^i y) = 0.$$

This corollary may be deduced from calculations with secondary cohomology operations, in the case $k = 1$ or 2 see [10, Theorem 2.2.1], in the general case it is necessary to use results contained in [6] and [7].

Corollary 1.4 has an application to immersions (compare Mahowald and Peterson's paper[10]). Suppose P is a p-dimensional closed manifold which immerses in R^{2p-2k}. Let η be the normal bundle of such an immersion. Then the disc bundle $D(\eta)$ is a $2p - 2k$ dimensional manifold, with boundary the sphere bundle $S(\eta)$, and $D(\eta)$ can be framed. In the statement of the next corollary $\bar{w}_i$ stands for the i-th normal Stiefel–Whitney class of P.

COROLLARY 1.5. *Suppose* P^p *immerses in* R^{2p-2k} *and that* $\bar{w}_k = 0$. *If* $(p - k, k)$ *is not exceptional then*

$$\bar{w}_{p-2k} \,.\, \left(\sum_{i=0}^{k-1} \bar{w}_{2k-i} \,.\, \bar{w}_i\right) = 0.$$

Proof. As above let η be the normal bundle of such an immersion and let $U \in H^{p-2k}(D(\eta), S(\eta))$ be the Thom class. Write the Thom isomorphism $H^i P \to H^{i+p-2k}(D(\eta), S(\eta))$ as $x \to Ux$. Note that $Sq^k U = U\bar{w}_k = 0$ by assumption and so applying 1.4 and writing $Sq^i U = U\bar{w}_i$ gives

$$\sum_{i=0}^{k-1} U\bar{w}_{2k-i} \,.\, U\bar{w}_i = 0.$$

Writing $U^2 = Sq^{p-2k} U = U\bar{w}_{p-2k}$ gives

$$U\bar{w}_{p-2k} \,.\, \left(\sum_{i=0}^{k-1} \bar{w}_{2k-i} \,.\, \bar{w}_i\right) = 0$$

and the corollary follows.

§2. THE QUADRATIC FORM

Let $(f, b)\colon (M, \partial M) \to (N, \partial N)$ be a degree one normal map of compact $2n$-manifolds. Then as described in Browder's book[3, p. 64] there is a *stable* map $g\colon N/\partial N \to M/\partial M$ such that

$$g^* f^* = 1\colon H^*(N, \partial N) \to H^*(N, \partial N).$$

The splitting α is taken to be g^*. Adjoint to g is a map $\bar{g}\colon N/\partial N \to Q(M/\partial M)$ where $QX = \lim_{n\to\infty} \Omega^n S^n X$.

Let K_n denote an Eilenberg–McLane space of type $(Z/2, n)$. To define the quadratic form some information concerning the space QK_n is required. Applying the functor Ω^L to the non-trivial map $S^L K_n \to K_{n+L}$ and letting L tend to infinity gives a map $r\colon QK_n \to K_n$. There is a canonical inclusion $i\colon K_n \to QK_n$ and ri is homotopic to the identity, and so r is referred to as the canonical retraction. Brown shows [4, lemma 2.2] that

(i) $\pi_{2n} QK_n = Z/2$

(ii) there is a short exact sequence

$$0 \longrightarrow [S^{2n}, QK_n] \xrightarrow[p^*]{} [N/\partial N, QK_n] \xrightarrow[r_*]{} H^n(N, \partial N) \longrightarrow 0 \tag{2.1}$$

where $p\colon N/\partial N \to S^n$ is a degree one map and $[X, Y]$ stands for homotopy classes of maps from X to Y.

Now choose any homomorphism $h\colon [N/\partial N, QK_n] \to Z/4$ such that hp^* is injective. Consider the function

$$q_h\colon H^n(N, \partial N) \to Z/4$$

defined to be the composite

$$H^n(N, \partial N) \cong [N/\partial N, K_n] \xrightarrow[i_*]{} [N/\partial N, QK_n] \xrightarrow[h]{} Z/4.$$

Brown shows in [4] that q_h is quadratic in the following sense. Let $2\colon Z/2 \to Z/4$ be the inclusion then

$$q_h(x + y) = q_h(x) + q_h(y) + 2(x \,.\, y).$$

To define the quadratic form q on an element $x \in K^n(f, b) \subset H^n(M, \partial M)$ regard x as a map of $M/\partial M$ into K_n. The functor Q gives a map $Q(x)\colon Q(M/\partial M) \to QK_n$ and so a function $Q\colon K^n(f, b) \to [Q(M/\partial M), QK_n]$. Consider the function

$$K^n(f, b) \xrightarrow[Q]{} [Q(M/\partial M), QK_n] \xrightarrow[\bar{g}^*]{} [N/\partial N, QK_n] \longrightarrow Z/4.$$

Referring to (2.1) note that

$$r_* \bar{g}^* Q(x) = g^* x = 0 \qquad \text{for } x \in K^n(f, b),$$

since by definition $K^n(f, b) = \ker(g^*\colon H^n(M, \partial M) \to H^n(N, \partial N))$. Thus $\bar{g}^* Q(x)$ is in the image of p^* and so from the definition of h it follows that $h(\bar{g}^* Q(x))$ does not depend on the choice of h and lies in $Z/2 \subset Z/4$. Define $q(x)$ by the formula

$$2q(x) = h(\bar{g}^* Q(x)).$$

§3. THE MAIN THEOREM

Suppose $y \in K^{n-k}(f, b)$ and $x = Sq^k y \in K^n(f, b)$, then the objective is to evaluate $q(x)$. Regard Sq^k as a map $K_{n-k} \to K_n$, then introduce the following notation:

$$\alpha = \bar{g}^* Q(y)\colon N/\partial N \to QK_{n-k}$$

$$\beta = Q(Sq^k)\colon QK_{n-k} \to QK_n.$$

Note that $\beta\alpha = \bar{g}^* Q(Sq^k y) = \bar{g}^* Q(x)$, and so, from §2, there exists a map $\gamma\colon S^{2n} \to QK_n$ such that $\beta\alpha = \gamma p$ where p is as in (2.1). Then the following diagram commutes

$$\begin{array}{ccc} N/\partial N & \xrightarrow{\alpha} & QK_{n-k} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \beta} \\ S^{2n} & \xrightarrow[\gamma]{} & QK_n \end{array}$$

Again from §2 it follows that

$$q(Sq^k y) = 0 \Leftrightarrow \gamma \text{ is null-homotopic.}$$

To get further we need to know how to detect the non-zero element of $\pi_{2n}QK_n$. It is known that this map actually induces a non-zero homomorphism in cohomology and so we need information about H^*QK_n.

For any pointed space X let $X^{(n)}$ denote the n-fold smash product of X with itself. For any space Y let Y^+ denote the space obtained by adding a disjoint base point to Y. Let $E\Sigma_n$ be a contractible space on which the symmetric group Σ_n acts freely. Define D_nX by

$$D_nX = E\Sigma_n^+ \wedge_{\Sigma_n} X^{(n)}$$

where Σ_n acts on $X^{(n)}$ by permuting the factors. We will always identify D_1X with X without further comment. There is a natural isomorphism

$$\epsilon: \bigoplus_{n\geq 1} \tilde{H}^*D_nX \to \tilde{H}^*QX. \tag{3.1}$$

This isomorphism is an isomorphism of modules over the Steenrod algebra, however it does *not* preserve products. The existence of this isomorphism is proved in [1], indeed it arises from a *stable* homotopy equivalence $\vee_{n\geq 1} D_nX \to QX$.

Suppose $\phi: X \to Y$ is a *stable* map with adjoint $\bar{\phi}: X \to QY$. Then for all $y \in \tilde{H}^*D_1Y = \tilde{H}^*Y$

$$\bar{\phi}^*\epsilon y = \phi^* y. \tag{3.2}$$

We will not need more than $H^*X + H^*D_2X$ in the subsequent work and so we will only give more details concerning H^*D_2X. The space E_2X is defined as $E\Sigma_2 \times_{\Sigma_2} X \times X$. The structure of H^*E_2X is well-known, a summary of the details is given in [8, 4.6–4.10]. There is a collapsing map $E_2X \to D_2X$ which induces a monomorphism in cohomology. In the notation of [8] the image of H^*D_2X in H^*E_2X is the subalgebra generated by the classes $[x, y]$ and $e^k \otimes z \otimes z$ where x, y and z are in $\tilde{H}^*X$. We will always use the notation and results of [8, 4.6–4.10] when dealing with H^*D_2X.

Referring to the diagram at the beginning of this section note that γ represents the non-zero element of $\pi_{2n}QK_n$ if and only if $\gamma^*\epsilon(1 \otimes \iota_n \otimes \iota_n)$ is non-zero, where $\iota_n \in H^nK_n$ is the fundamental class and $1 \otimes \iota_n \otimes \iota_n \in H^{2n}D_2K_n$. However $p^*: H^{2n}S^{2n} \to H^{2n}(N, \partial N)$ is an isomorphism and so $\gamma^*\epsilon(1 \otimes \iota_n \otimes \iota_n)$ is non-zero if and only if $\alpha^*\beta^*\epsilon(1 \otimes \iota_n \otimes \iota_n)$ is non-zero. From the naturality of the isomorphism ϵ, it follows that $\beta^*\epsilon(1 \otimes \iota_n \otimes \iota_n) = \epsilon(1 \otimes Sq^k\iota_{n-k} \otimes Sq^k\iota_{n-k})$. Therefore we have shown:

(3.3) $q(Sq^k y) = 0$ if and only if $\alpha^*\epsilon(1 \otimes Sq^k\iota_{n-k} \otimes Sq^k\iota_{n-k}) = 0$.

To complete the proof of (1.2) the following two technical results are required.

PROPOSITION 3.4. *Suppose (n, k) is not an exceptional pair. Then in $H^{2n}D_2K_{n-k}$*

$$1 \otimes Sq^k\iota_{n-k} \otimes Sq^k\iota_{n-k} = \sum_{i=0}^{k-1} [Sq^{2k-i}\iota_{n-k}, Sq^i\iota_{n-k}] + A$$

where A is decomposable over the Steenrod algebra, that is $A = \sum_i a_i x_i$ where $x_i \in H^*D_2K_{n-k}$ and the a_i are stable primary cohomology operations of strictly positive degree.

The next proposition concerns the behaviour of products under the isomorphism $\epsilon: \bigoplus_{n\geq 0} \tilde{H}^*D_nX \to \tilde{H}^*QX$.

PROPOSITION 3.5. *Let a, b be elements of H^*X so that $[a, b]$ is in H^*D_2X. Then*

$$\epsilon a \cdot \epsilon b = \epsilon(a \cdot b + [a, b]).$$

We can now complete the proof of (1.2) using (3.4) and (3.5).

The assumptions on Wu-classes ensure that, in the notation of (3.4), $\alpha^*\epsilon A = 0$. Therefore using (3.4) and (3.5)

$$\alpha^*\epsilon(1 \otimes Sq^k\iota_{n-k} \otimes Sq^k\iota_{n-k}) = \alpha^*\epsilon \sum_{i=0}^{k-1} [Sq^{2k-i}\iota_{n-k}, Sq^i\iota_{n-k}]$$

$$= \alpha^* \sum_{i=0}^{k-1} (\epsilon Sq^{2k-i}\iota_{n-k} \cdot \epsilon Sq^i\iota_{n-k} + \epsilon(Sq^{2k-i}\iota_{n-k} \cdot Sq^i\iota_{n-k})).$$

From the definition of α, the naturality of ϵ and (3.2) we deduce

$$\begin{aligned} \alpha^*\epsilon Sq^i\iota_{n-k} &= \bar{g}^*Q(y)^*\epsilon Sq^i\iota_{n-k} \\ &= \bar{g}^*\epsilon Sq^i y \\ &= Sq^i g^* y. \end{aligned}$$

However $y \in K^{n-k}(f, b)$ and so $g^*y = 0$. Therefore

$$\alpha^*\epsilon(1 \otimes Sq^k\iota_{n-k} \otimes Sq^k\iota_{n-k}) = \alpha^*\epsilon \sum_{i=0}^{k-1} Sq^{2k-i}\iota_{n-k} \cdot Sq^i\iota_{n-k}$$

$$= g^* \sum_{i=0}^{k-1} Sq^{2k-i}y \cdot Sq^i y$$

where this equality follows from the definition of α, the naturality of ϵ and (3.2). However g^* is an isomorphism in $2n$-dimensional cohomology and so using (3.3) the proof is completed.

§4. DEFERRED PROOFS

This section is devoted to proving (3.4) and (3.5) and therefore finishing the proof of (1.2). We begin with (3.4).

In any graded vector space V let V_t be the subspace of elements of degree t. Let A denote the mod 2 Steenrod algebra.

If $p \leq 4(n-k)$ there is a homomorphism

$$\delta: H^pD_2K_{n-k} \to A_{p+k-n+1}$$

defined by

$$\delta[x, y] = 0 \quad \text{for } x, y \in H^*K_{n-k}$$

$$\delta(e^t \otimes Sq^I\iota_{n-k} \otimes Sq^I\iota_{n-k}) = Sq^{t+|I|+n-k+1}Sq^I$$

where I is an admissible sequence, so that Sq^I is an element of the Cartan–Serre basis for the Steenrod algebra (see [13]) and $|I|$ stands for the degree of I so that $Sq^I \in A_{|I|}$. The reason for the assumption that $p \leq 4(n-k)$ is that we have not defined $\delta(e^t \otimes a \otimes a)$ where a is a product $S_q^I\iota_{n-k}$, $S_q^J\iota_{n-k}$ with I or J non-trivial. It is straightforward to check from 4.10 of [8] and the Adem relations that $\delta(Sq^i x) = Sq^i\delta x$.

Let $B(q)$ be the left ideal of A consisting of those cohomology operations which vanish on all cohomology classes of dimension $\leq q$. As a $Z/2$ vector space $B(q)$ has a basis consisting of those elements Sq^I in the Cartan–Serre basis for A with $e(I) > q$ where $e(I)$ is the excess of I (see [13]).

Let $C \subset H^*D_2K_{n-k}$ be the subvector space generated by those classes of the form $e^t \otimes x \otimes x$ where $t > 0$. Note that from 4.10 of [8] C is actually a sub-A-module of $H^*D_2K_{n-k}$.

LEMMA 4.1. *δ maps C_t isomorphically onto $B(n-k+1)_{t+k-n+1}$ for $t \leq 4(n-k)$.*

Proof. This is clear from the definitions.

Although these facts have been presented algebraically there is a topological result underlying them. This result, due to Milgram, is that if we denote by F_L the fibre of the non-trivial map $S^LK_{n-k} \to K_{n+L-k}$ where L is large, then in dimensions $\leq 3(n-k)$ –

$2+L$, F_L is homotopy equivalent to $S^L D_2 K_{n-k}$, see [12]. The homomorphism δ is in fact the boundary map in the Serre exact cohomology sequence of this fibration.

Recall that the indecomposable quotient of an A module M is the $Z/2$ vector space $M/\bar{A}M$ where $\bar{A}$ is the ideal of elements of strictly positive degree. An element of M is called A-decomposable if it maps to zero in the indecomposable quotient of M. This notion of decomposability agrees with that used in (3.4). The proof of (3.4) uses a result due to Harper on the indecomposable quotient of $B(q)$.

THEOREM 4.2. (See theorem A of [7]). *Suppose the pair (n, k) is not exceptional and that $n+k+1 \neq 2^l$ for any l. Then every element of $B(n-k+1)_{n+k+1}$ is A-decomposable. If $n+k+1 = 2^l$ or (n, k) is exceptional then the indecomposable quotient of $B(n-k+1)_{n+k+1}$ is $Z/2$. Finally if $n+k+1 = 2^l$, then an element of $B(n-k+1)_{n+k+1}$ is A-decomposable if and only if it is an A-decomposable element of A.*

We now give the proof of (3.4).

Proof of 3.4. Consider $Sq^{2k}(1 \otimes \iota_{n-k} \otimes \iota_{n-k})$ in $H^{2n}D_2K_{n-k}$. By 4.10 of [8]

$$\begin{aligned} Sq^{2k}(1 \otimes \iota_{n-k} \otimes \iota_{n-k}) &= 1 \otimes Sq^k\iota_{n-k} \otimes Sq^k\iota_{n-k} \\ &+ \sum_{i=0}^{k-1} \binom{n-k-i}{2k-2i} e^{2k-2i} \otimes Sq^i\iota_{n-k} \otimes Sq^i\iota_{n-k} \\ &+ \sum_{i=0}^{k-1} [Sq^{2k-i}\iota_{n-k}, Sq^i\iota_{n-k}]. \end{aligned}$$

Write Z for the class $\sum_{i=0}^{k-1} \binom{n-k-i}{2k-2i} e^{2k-2i} \otimes Sq^i\iota_{n-k} \otimes Sq^i\iota_{n-k}$, then $Z \in C_{2n}$ and so $\delta Z \in B(n-k+1)_{n+k+1}$. Suppose $n+k+1 \neq 2^l$ then the assumption that (n, k) is not exceptional gives us, from (4.2), that δZ is A-decomposable. If $n+k+1 = 2^l$ then we see from the definition of δ and (4.2) that δZ is A-decomposable if and only if

$$\binom{n-k}{2k} \equiv 0 \bmod 2.$$

However $n-k = 2^l - 1 - 2k$ and one checks that

$$\binom{2^l-1-2k}{2k} \equiv 0 \bmod 2.$$

So we have shown that δZ is A-decomposable in $B(n-k+1)$ if (n, k) is not exceptional and in view of (4.1) we have completed the proof of (3.4).

We come now to the proof of (3.5). The strategy is obvious: we use known information about the homology of QX and the homomorphism induced by the diagonal map to deduce the result.

Using the Pontryagin product H_*QX becomes an algebra; given $x, y \in H_*QX$ denote their product by $x \circ y$. The map $i: X \to QX$ induces an injection in homology and H_*QX is generated, as an algebra, by iterated Dyer–Lashof operations evaluated on i_*H_*X, see [5] and [11, pp. 40–42]. Let $\epsilon^*: \tilde{H}_*QX \to \bigoplus_{n\geq 1} \tilde{H}_*D_nX$ be the dual to (3.1). We require the following facts concerning ϵ^*, see [1].

(4.3) $\epsilon^* i_*: \tilde{H}_*X \to \bigoplus_{n\geq 1} \tilde{H}_*D_nX$ is the embedding of the direct summand $\tilde{H}_*X$.

Let $\pi: \tilde{H}_*QX \to \tilde{H}_*X$ be the composite of ϵ^* and the projection onto the direct summand $\tilde{H}_*X$.

(4.4) For $x \in H_*QX$, then $\pi Q^i x = 0$, here Q^i is the Dyer–Lashof operation.

The usual maps $D_nX \times D_mX \to D_{n+m}X$ provide $\bigoplus_{n\geq 1} \tilde{H}_*D_nX$ with an algebra structure.

(4.5) ϵ^* is an isomorphism of algebras.

The following facts concerning the Kronecker pairing between H_*QX and H^*QX can be deduced from (4.3)–(4.5).

(4.6) Let $a \in H^*X$ and $x \in H_*QX$, then $\langle \epsilon a, x\rangle = \langle a, \pi x\rangle$.

(4.7) Let $a, b \in H^*X$ and $x, y \in H_*QX$, then

$$\langle \epsilon[a, b], x \circ y\rangle = \langle a, \pi x\rangle\langle b, \pi y\rangle + \langle a, \pi y\rangle\langle b, \pi x\rangle.$$

Next we show:

(4.8) If $a, b \in H^*X$ and $x, y \in \tilde{H}_*QX$, then

$$\langle \epsilon a \cdot \epsilon b, x \circ y\rangle = \langle a, \pi x\rangle\langle b, \pi y\rangle + \langle a, \pi y\rangle\langle b, \pi x\rangle.$$

This follows since $\langle \epsilon a \cdot \epsilon b, x \circ y\rangle = \langle \epsilon a \otimes \epsilon b, \Delta_*(x \circ y)\rangle$ where $\Delta: QX \to QX \times QX$ is the diagonal map. However

$$\Delta_*(x \circ y) = x \otimes y + y \otimes x + \sum_i u_i \otimes v_i$$

where at least one of u_i, $v_i \in H_*QX$ is decomposable under the Pontryagin product. From (4.5) we conclude that either πu_i or $\pi v_i = 0$ and (4.8) follows from (4.6).

Now suppose $x\ u \in H_*QX$ and $\Delta_* x = \sum_i x_i' \otimes x_i''$, then (see [11, p. 6])

$$\Delta_* Q^t x = \sum_{i,j} Q^{t-j}x_i' \otimes Q^j x_i''.$$

From this formula and (4.4) we deduce:

(4.9) For $a, b \in H^*X$ and $x \in H_*QX$,

$$\langle \epsilon a \cdot \epsilon b, Q^t x\rangle = 0.$$

One more result is required.

(4.10) For $a, b \in H^*X$ and $x \in H_*X$,

$$\langle \epsilon a \cdot \epsilon b, i_* x\rangle = \langle a \cdot b, x\rangle.$$

This completes the proof of (3.5) for (4.7)–(4.10) show that

$$\langle \epsilon a \cdot \epsilon b, x\rangle = \langle \epsilon[a, b], x\rangle + \langle \epsilon(ab), x\rangle$$

for all $x \in H_*QX$.

§5. TWO EXAMPLES

The purpose of this section is to show how to apply the main theorem of this paper to the study of the connectivity of Arf-changeable manifolds. For the terminology and the context in which to set these examples the reader is referred to [9], particularly §3 of that paper.

Suppose $k = 2^r - 1$ and that M^{2k} is a stably parallelizable closed manifold. Let $y_r \in H^{2^r-1}SO$ be the class defined by $y_r = \Omega w_{2^r}$ where $w_{2^r} \in H^{2^r}BSO$ is the universal Stiefel–Whitney class. The manifold M^{2k} is called Arf-changeable if there exists a map $f: M \to SO$ and a framing F of M such that $q_F(f^*y_r) = 1$. If M is Arf changeable then it has two framings F_1 and F_2 such that the quadratic forms q_{F_1} and q_{F_2} have different Arf invariants, see [9].

Suppose M is s-connected and $f: M \to SO$, then a result due to Stong[14] shows that

(i) If $\phi(s+1) \geq r+1$ then $f^*y_r = 0$.

(ii) If $\phi(s+1) = r$ then there exists a stable primary cohomology operation a_r such that $f^*y_r = a_r x$ for some $x \in H^{s+1}M$.

Here $\phi(m)$ is the number of integers l in the range $0 < l \leq m$ such that $l \equiv 0, 1, 2, 4 \bmod 8$.

The main result of this paper can be used to give evidence for the following conjecture:

(5.1) Suppose $k = 2^r - 1$ with $r \geq 4$, and M^{2k} is stably parallelizable and s-connected where $\phi(s+1) \geq r$. Then M is not Arf-changeable.

If the condition $r \geq 4$ is relaxed then (5.1) is trivially false as the examples $S^1 \times S^1$, $S^3 \times S^3$, and $S^7 \times S^7$ show. Here (5.1) will be verified for $r = 4$ and 5.

In any case Stong's result shows that (5.1) is true if $\phi(s+1) \geq r+1$. Consider the case $\phi(s+1) = r$, in the first instance when $r = 4$ and so $s = 7$. Let $f: M^{30} \to SO$ be a map of a 7-connected manifold into SO, then

$$f^* y_4 = (Sq^7 + Sq^4 Sq^2 Sq^1)x$$

for some $x \in H^8 M$ with $Sq^2 x = 0$, see [14, p. 526, p. 543].

Let F be a framing of M, then

$$(5.2)\quad q_F(Sq^7 x + Sq^4 Sq^2 Sq^1 x) = q_F(Sq^7 x) + q_F(Sq^4 Sq^2 Sq^1 x) + Sq^7 x \cdot Sq^4 Sq^2 Sq^1 x.$$

Note that (15, 7) and (15, 4) are not exceptional pairs and so using (1.3) we deduce

$$(5.3)\quad \begin{cases} q_F(Sq^7 x) = \sum_{i=0}^{6} Sq^{14-i} x \cdot Sq^i x \\ q_F(Sq^4 Sq^2 Sq^1 x) = \sum_{i=0}^{3} Sq^{8-i} Sq^2 Sq^1 x \cdot Sq^i Sq^2 Sq^1 x. \end{cases}$$

Note that x is an 8-dimensional class so the first formula in (5.3) reduces to

$$(5.4)\quad q_F(Sq^7 x) = Sq^8 x \cdot Sq^6 x.$$

From the Adem relations we can conclude that

$$Sq^2 Sq^2 Sq^1 = 0 \qquad Sq^3 Sq^2 Sq^1 = 0$$

and so the second formula in (5.3) reduces to

$$(5.5)\quad q_F(Sq^4 Sq^2 Sq^1 x) = Sq^8 Sq^2 Sq^1 x \cdot Sq^2 Sq^1 x + Sq^7 Sq^2 Sq^1 x \cdot Sq^1 Sq^2 Sq^1 x.$$

We now use the following fact concerning products and cohomology operations in the cohomology of a stably parallelizable manifold N^n. Let $x \in H^{n-k-l} N$, $y \in H^k N$ and $a \in A_l$. Then $x \cdot ay = (\chi a)(x) \cdot y$ where $\chi: A \to A$ is the canonical antiautomorphism of A. Therefore, referring to (5.5), we see that

$$Sq^7 Sq^2 Sq^1 x \cdot Sq^1(Sq^2 Sq^1 x) = (\chi Sq^1) Sq^7 Sq^2 Sq^1 x \cdot Sq^2 Sq^1 x = 0$$

since $\chi Sq^1 = Sq^1$ and $Sq^1 Sq^7 = 0$. So (5.5) reduces to

$$(5.6)\quad q_F(Sq^4 Sq^2 Sq^1 x) = Sq^8 Sq^2 Sq^1 x \cdot Sq^2 Sq^1 x.$$

Putting (5.2), (5.4) and (5.6) together and using the above remark about products and cohomology operations gives

$$(5.7)\quad q_F(Sq^7 x + Sq^4 Sq^2 Sq^1 x) = x \cdot ((\chi Sq^6) Sq^8 x + \chi(Sq^2 Sq^1) Sq^8 Sq^2 Sq^1 x + (\chi Sq^7) Sq^4 Sq^2 Sq^1 x).$$

After a straightforward calculation with χ the Adem relations, and using the fact that $Sq^2 x = 0$ we deduce the right hand side of (5.7) is zero. This verifies (5.1) in this case $r = 4$.

Now consider the case $\phi(s+1) = r$ when $r = 5$ and so $s = 8$. Let $f: M^{62} \to SO$ be a map of an 8-connected manifold into SO, then

$$f^* y_5 = (Sq^{12} Sq^6 Sq^3 Sq^1 + Sq^{14} Sq^6 Sq^2 + Sq^{15} Sq^4 Sq^2 Sq^1 + Sq^{15} Sq^7 + Sq^{16} Sq^4 Sq^2)x$$

for some x in $H^9 M$ with $Sq^3 x = 0$, see [14, p. 526, p. 543].

A similar argument to the one given above, but one involving a much longer calculation with χ and the Adem relations will show that

$$q_F(f^* y_5) = 0$$

for any framing F of M and so will verify (5.1) when $r = 5$.

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