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Jordan normal form projections

By

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Let K be a field, and let $T\colon V \to V$ be an endomorphism of a finite-dimensional vector space V over K such that K contains the roots $\lambda_1, \lambda_2, \ldots, \lambda_m$ of the characteristic polynomial

$$\chi(t) = \det(tI - T) = (t - \lambda_1)^{n_1}(t - \lambda_2)^{n_2} \cdots (t - \lambda_m)^{n_m}.$$

V is a direct sum of T-invariant subspaces $V_j = \ker(T - \lambda_j I)^{n_j}$, one for each eigenvalue λ_j, such that $T - \lambda_j I\colon V_k \to V_k$ is nilpotent for $j = k$ and an automorphism for $j \neq k$. We obtain in this paper an explicit formula for the projection $p_j(T) = p_j(T)^2\colon V \to V$ onto the subspace V_j, as a polynomial in T.

A *near-projection* in a ring A is an element $p \in A$ such that $q = p(1-p) \in A$ is nilpotent, that is $q^n = 0$ for some exponent $n \geqq 1$. We refer to Lück and Ranicki [3] for the general theory of near-projections, and for the construction of the unique projection $p_\omega = (p_\omega)^2 \in A$ such that $p_\omega - p$ is nilpotent and $p p_\omega = p_\omega p$, namely

$$\begin{aligned} p_\omega &= (p^n + (1-p)^n)^{-1} p^n = p + (1/2)(2p - 1)((1 - 4q)^{-1/2} - 1) \\ &= p + (2p-1)(q + 3q^2 + 10q^3 + 35q^4 + 126q^5 + 462q^6 + 1716q^7 \\ &\quad + 6435q^8 + 24310q^9 + 92378q^{10} + 352716q^{11} \\ &\quad + 1352078q^{12} + \cdots \in A. \end{aligned}$$

(In the special case when A is of characteristic 2 this simplifies to

$$p_\omega = p + q + q^2 + q^4 + q^8 + q^{16} + \cdots \in A).$$

If R is any ring and $p\colon V \to V$ is a near-projection in the endomorphism ring of an R-module V then $p_\omega\colon V \to V$ is the projection onto the p-invariant submodule

$$P = \operatorname{im}(p_\omega) = \operatorname{im}(p^n) = \ker((1-p)^n) \subseteqq V,$$

such that $V = P \oplus Q$ with Q the p-invariant submodule

$$Q = \operatorname{im}(1 - p_\omega) = \operatorname{im}((1-p)^n) = \ker(p^n) \subseteqq V.$$

$p\colon P \to P$ is an isomorphism and $p\colon Q \to Q$ is nilpotent. An endomorphism $p\colon V \to V$ of a finite-dimensional vector space V over a field K is a near-projection if and only if the characteristic polynomial is of the type $\chi(t) = t^m(1-t)^n$, in which case $(p(1-p))^{\max(m,n)} = 0\colon V \to V$.

Let R be a commutative ring with 1, with polynomial extension ring $R[t]$. A polynomial $\varphi(t) \in R[t]$ is *completely reducible* if it can be expressed as a product of linear factors

$$\varphi(t) = (t - \lambda_1)^{n_1}(t - \lambda_2)^{n_2} \cdots (t - \lambda_m)^{n_m} \in R[t].$$

Let $T: V \to V$ be an endomorphism of an R-module V. A polynomial $\varphi(t) \in R[t]$ *annihilates* T if

$$\varphi(T) = 0: V \to V,$$

in which case V is an $R[t]/(\varphi(t))$-module with t acting on V by T. Any expression of $R[t]/(\varphi(t))$ as a product of rings

$$R[t]/(\varphi(t)) = A_1 \times A_2 \times \cdots \times A_m$$

determines a decomposition of V as a direct sum of T-invariant submodules

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m,$$

with $V_j = A_j V$ an A_j-module.

The *characteristic polynomial* of an endomorphism $T: V \to V$ of a finitely generated free R-module V is defined by

$$\chi(t) = \det((tI - T): V[t] \to V[t]) \in R[t],$$

as usual. $\chi(t)$ annihilates T by the Cayley-Hamilton theorem. More generally, Almkvist [1] defined the characteristic polynomial of an endomorphism $T: V \to V$ of a finitely generated projective R-module V by

$$\chi(t) = t^n \lambda(-1/t) \in R[t],$$

with n the degree of the polynomial

$$\lambda(t) = \det(I + t(T \oplus 0): (V \oplus W)[t] \to (V \oplus W)[t]) \in R[t]$$

defined for any finitely generated projective R-module W such that $V \oplus W$ is a finitely generated free R-module. This agrees with the previous definition, and is also such that $\chi(T) = 0: V \to V$.

Theorem. *Let $T: V \to V$ be an endomorphism of an R-module V such that there exists a completely reducible annihilating polynomial of degree n*

$$\varphi(t) = (t - \lambda_1)^{n_1}(t - \lambda_2)^{n_2} \cdots (t - \lambda_m)^{n_m} \in R[t] \quad \left(n = \sum_{j=1}^{m} n_j\right),$$

with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m \in R$ such that each $\lambda_j - \lambda_k \in R$ $(j \neq k)$ is a unit. For $j = 1, 2, \ldots, m$ define the polynomial of degree $(n - n_j)$

$$g_j(t) = \prod_{k \neq j} (t - \lambda_k)^{n_k} \in R[t],$$

so that $\varphi(t) = (t - \lambda_j)^{n_j} g_j(t)$. Then

$$g_j(\lambda_j)^{-1} g_j(T): V \to V$$

is a near-projection in the endomorphism ring of V, and its associated projection

$$p_j(T) = (g_j(\lambda_j)^{-1} g_j(T))_\omega : V \to V$$

is the projection onto the direct summand

$$V_j = \operatorname{im}(g_j(T): V \to V) = \ker((T - \lambda_j I)^{n_j}: V \to V) \subseteqq V,$$

with

$$\sum_{j=1}^{m} p_j(T) = 1: V \to V, \quad p_j(T) p_k(T) = 0 \quad \text{for} \quad j \neq k.$$

V is the direct sum of the T-invariant submodules V_j $(j = 1, 2, \ldots, m)$, such that $T - \lambda_j I: V_k \to V_k$ is nilpotent for $j = k$ and an automorphism for $j \neq k$.

Proof. For $j = 1, 2, \ldots, m$ define

$$a_j(t) = \sum_{k=0}^{n_j - 1} \frac{(t - \lambda_j)^k}{k!} \left(\frac{1}{g_j(t)}\right)^{(k)}_{t = \lambda_j} \in R[t],$$

the first n_j terms in the Taylor expansion of $1/g_j(t)$ around $t = \lambda_j$, such that

$$a_1(t) g_1(t) + a_2(t) g_2(t) + \cdots + a_m(t) g_m(t) = 1 \in R[t].$$

The left hand side is the degree $(n_j - 1)$ Lagrange-Sylvester interpolation polynomial $f(t)$ with $f(\lambda_j) = 1$, $f^{(k)}(\lambda_j) = 0$ $(1 \leqq j \leqq m, 1 \leqq k \leqq n_j - 1)$ (Gantmacher [2, V.1]). The coefficients of $a_j(t)$ are defined in R, since the coefficients of the formal power series

$$\begin{aligned}(t - \lambda_i)^{-n_i} &= (\lambda_j - \lambda_i)^{-n_i} (1 + (t - \lambda_j)/(\lambda_j - \lambda_i))^{-n_i} \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{n_i + k - 1}{k} (t - \lambda_j)^k / (\lambda_j - \lambda_i)^{k - n_i} \in R[[t - \lambda_j]]\end{aligned}$$

are defined in R, for $i \neq j$, $\varphi(t)$ is a factor of $g_j(t) g_k(t)$ for $j \neq k$, so that there are defined projections

$$p_j(t) = a_j(t) g_j(t) \in R[t]/(\varphi(t)) \quad (1 \leqq j \leqq m)$$

such that

$$\sum_{j=1}^{m} p_j(t) = 1, \quad p_j(t) p_k(t) = 0 \quad \text{for} \quad j \neq k.$$

The isomorphism of rings defined by projection on each factor

$$R[t]/(\varphi(t)) \to \prod_{j=1}^{m} R[t]/((t - \lambda_j)^{n_j}) \ : \ x \to (x, x, \ldots, x)$$

has inverse

$$\prod_{j=1}^{m} R[t]/((t - \lambda_j)^{n_j}) \to R[t]/(\varphi(t));$$

$$(x_1, x_2, \ldots, x_m) \to x_1 p_1(t) + x_2 p_2(t) + \cdots + x_m p_m(t).$$

The elements $g_j(t), (t-\lambda_j)^{n_j} \in R[t]/(\varphi(t))$ are coprime and have product 0, so that

$$\operatorname{im}(g_j(T)\colon V \to V) = \ker((T-\lambda_j I)^{n_j}\colon V \to V).$$

Let $V_j \subseteqq V$ be the image of the projection

$$p_j(T) = a_j(T)\, g_j(T)\colon V \to V,$$

an $R[t]/((t-\lambda_j)^{n_j})$-module with t acting by $T\colon V_j \to V_j$, and with $g_j(T), a_j(T)\colon V_j \to V_j$ inverse automorphisms. Thus

$$V_j = \operatorname{im}(p_j(T)\colon V \to V) = \operatorname{im}(g_j(T)\colon V \to V) = \ker((T-\lambda_j I)^{n_j}\colon V \to V).$$

As λ_j is a root of $g_j(t) - g_j(\lambda_j)$ there exists a polynomial $h_j(t)$ such that

$$g_j(t) - g_j(\lambda_j) = (t-\lambda_j)\, h_j(t) \in R[t].$$

Similarly, there exists a polynomial $b_j(t)$ such that

$$a_j(t) - a_j(\lambda_j) = (t-\lambda_j)\, b_j(t) \in R[t].$$

Now

$$a_j(\lambda_j)\, g_j(t) = g_j(\lambda_j)^{-1} g_j(t) \in R[t]/(\varphi(t))$$

is a near-projection, since

$$g_j(\lambda_j)^{-1} g_j(t)\,(1 - g_j(\lambda_j)^{-1} g_j(t)) = -\, g_j(\lambda_j)^{-2} g_j(t)\,(t-\lambda_j)\, h_j(t) \in R[t]/(\varphi(t))$$

is nilpotent (of exponent at most n_j). The near-projection differs from the projection $p_j(t)$ by a nilpotent

$$p_j(t) - g_j(\lambda_j)^{-1} g_j(t) = g_j(t)\,(t-\lambda_j)\, b_j(t) \in R[t]/(\varphi(t)),$$

so that

$$p_j(t) = (g_j(\lambda_j)^{-1} g_j(t))_\omega \in R[t]/(\varphi(t)). \quad \square$$

Remark 1. The hypotheses of the Theorem are satisfied for any endomorphism $T\colon V \to V$ of a finite-dimensional vector space V over a field K which contains the roots of the minimal polynomial $\mu(t)$, with $\varphi(t)$ either $\mu(t)$ or the characteristic polynomial $\chi(t)$. The near-projections $g_j(\lambda_j)^{-1} g_j(t)\colon V \to V$ for the two cases need not be the same, but they differ by nilpotents so that the projections $p_j(T) = (g_j(\lambda_j)^{-1} g_j(T))_\omega\colon V \to V$ are the same. $\square$

Remark 2. The projections $p_j(T)$ can also be expressed as

$$p_j(T) = ((T-\lambda_j I)^{n_j} + g_j(T))^{-1} g_j(T)\colon V \to V. \quad \square$$

Remark 3. The proof of the Theorem does not actually use the Jordan normal form, and could be used to simplify the standard proofs. We are indepted to S. J. Patterson for describing to us Fitting's proof, making use of the projections $p_j(T)$ but without an explicit formula. $\square$

References

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