Pseudohomology and Homology*

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Abstract

The notion of a *pseudocycle* is introduced in [13] to provide a framework for defining Gromov-Witten invariants and quantum cohomology. This paper studies the bordism groups of pseudocycles, called *pseudohomology groups*. These satisfy the Eilenberg-Steenrod axioms, and, for smooth compact manifold pairs, pseudohomology is naturally equivalent to homology. Easy examples show that this equivalence does not extend to the case of non-compact manifolds.

1 Introduction

Moduli spaces of pseudoholomorphic curves in a symplectic manifold X are generically smooth, finite-dimensional manifolds, but usually they are not compact. Thus, fundamental cycles are not available for extracting homology information or for defining invariants of X. One solution to this defect is to use a theorem of Gromov [8] to compactify the moduli spaces (e.g., see [19], [20]). An alternative approach, as presented in M$^{\text{C}}$Duff and Salamon [13], also applies Gromov's ideas by making use of certain dimension restrictions on the limit sets of evaluation maps associated with the moduli spaces. M$^{\text{C}}$Duff and Salamon call smooth X-valued maps satisfying these restrictions *pseudocycles* in X and use them to provide a framework for defining the Gromov-Witten invariants and quantum cohomology of X. This paper studies the bordism groups of pseudocycles in X, which we call the *pseudohomology* groups of X. To state our first result, let $\mathcal{C}$ denote the category of all smooth manifold pairs and smooth maps, as described in §3, let $\mathcal{C}^{prop}$ be the subcategory with the same objects and with morphisms the *proper* smooth maps, and let $\mathcal{A}$ denote the category of abelian groups and homorphisms. Pseudohomology groups are defined for all pairs in $\mathcal{C}$.

Theorem 1 *Pseudohomology groups and map-composition define functors $\Psi_k : \mathcal{C}^{prop} \to \mathcal{A}$, for all $k = 0, 1, 2, \ldots$. These functors satisfy the axioms of Eilenberg and Steenrod: homotopy-invariance, exact-sequence, excision, and dimension. In particular, for the dimension axiom, we have a canonical isomorphism $H_k(pt) \to \Psi_k(pt)$, for every k.*

The uniqueness theorem of Eilenberg and Steenrod, which they prove over the category of finite simplicial pairs, suggests that the isomorphism $H_*(pt) \to \Psi_*(pt)$ should extend purely formally to

*2000 *Mathematics Subject Classification* 55N35, 57R17, 57N65 a natural equivalence of functors $H_* \to \Psi_*$ over a suitable subcategory of $\mathcal{C}^{prop}$. In fact, this is the case, but we prefer to present a concrete realization of this equivalence.

In [13], Remark , M[C]Duff and Salamon outline a construction in the absolute case that provides a candidate for such a realization. Indeed, this paper was originally motivated by an effort to extend that construction and understand its properties, and to do some computations. Our next result shows that the construction extends to the case of pairs and does realize the equivalence that we want. In fact, we show not only that it gives an equivalence but that the equivalence is essentially unique.

Theorem 2 *There is a construction applying to all pairs (X, A) in $\mathcal{C}$,*

$$\{\textit{relative } k\textit{-cycles in } (X, A)\} \Longrightarrow \{\textit{relative } k\textit{-pseudocycles in } (X, A)\},$$

with the following properties:

a. *It induces well-defined homomorphisms $\psi : H_*(X, A) \to \Psi_*(X, A)$, which commute with boundary/connecting homomorphisms.*

b. *Restricted to $\mathcal{C}^{prop}$, the maps ψ form a natural transformation of functors $H_* \to \Psi_*$.*

c. *$\psi : H_*(pt) \to \Psi_*(pt)$ equals the isomorphism of Theorem 1.*

d. *Let $\mathcal{C}^{prop}_{hand}$ denote the full subcategory of $\mathcal{C}^{prop}$ with objects the pairs (X, A) admitting finite handle decompositions (cf. §3). Then the natural transformation ψ is an* equivalence *of functors on $\mathcal{C}^{prop}_{hand}$.*

   Furthermore, ψ is the only natural transformation $H_* \to \Psi_*$ on $\mathcal{C}^{prop}_{hand}$ satisfying a and c.

Remarks:

a. There are simple examples (Examples 1,2 §4) which show that although Ψ_* is defined for the objects of $\mathcal{C}$, it does not extend functorially to the maps in $\mathcal{C}$.

b. The notion of a finite handle decomposition of a pair (X, A) as described in §3 is a relative notion. Essentially, X is obtained from A by attaching finitely many handles. Thus, if $X = (X, \emptyset)$ is a non-compact manifold in $\mathcal{C}^{prop}$, it is not an object of $\mathcal{C}^{prop}_{hand}$.

   Theorem 2 d does not extend to $\mathcal{C}^{prop}$, which is to say that there is a substantial deviation between homology and pseudohomology for non-compact manifolds, as the following two results indicate:

   (i) **Proposition 1** *Let X be a non-compact, connected, orientable manifold with empty boundary. There is no surjective homomorphism $H_*(X) \to \Psi_*(X)$.*

   (ii) **Proposition 2** *Let X be a smooth, compact, orientable manifold. There is no natural transformation $H_* \to \Psi_*$ on $\mathcal{C}^{prop}$ that maps $H_*(X \times R) \to \Psi_*(X \times R)$ injectively.*

c. Unlike homology, Ψ_* does not commute with (infinite) direct limits. For if it did, neither of the above propositions would hold. Of course, this can be seen even more easily in the example of the identity map of a non-compact 0-manifold X. Such a map is a Ψ_0-cycle which clearly is not in the subgroup of $\Psi_0(X)$ generated by the pseudohomology of the compact submanifolds of X.

d. It is worth pointing out the connection between classical bordism and pseudocycle bordism:

   Let $\Omega_*(X, A)$ denote the classical, oriented bordism group of (X, A), which we may take to be defined via smooth maps, and let $\mu : \Omega_*(X, A) \to H_*(X, A)$ be the standard fundamental-class-evaluation homomorphism. Then, $\psi \circ \mu : \Omega_*(X, A) \to \Psi_*(X, A)$ is induced by the inclusion of bordism classes.

   This result is an immediate consequence of the construction and results of §§5,6.

e. In [13], M$^{\text{C}}$Duff and Salamon define an intersection pairing for pseudocycles in compact manifolds, which is analogous to the standard intersection pairing for homology. It is not hard to see that ψ respects intersection pairings: i.e., $\alpha \cdot \beta = \psi(\alpha) \cdot \psi(\beta)$, for all homology classes α, β of complementary dimensions. One way to see this is to use the relationship with classical bordism displayed in the preceding remark. Both μ and the inclusion-induced map $\Omega_* \to \Psi_*$ respect intersection pairing: for the former, this is a well-known fact (cf., [5]); for the latter, it follows from the definitions. Now use the classical fact that the index of the subgroup $\mu(\Omega_n(X))$ in $H_n(X)$ is finite for all n and all compact X.

   Given a closed, connected, oriented m-manifold X, the pairing on pseudocycles can be used to associate with every k-dimensional pseudocycle ϕ in X a homomorphism $\Psi_{m-k}(X) \to Z$, depending only on the bordism class of ϕ, which we continue to denote by the same letter. Composing this homomorphism with ψ, we obtain a homomorphism $H_{m-k}(X) \to Z$, which we call $\psi^*(\phi)$. M$^{\text{C}}$Duff and Salamon then say that a homology class $\alpha \in H_k(X)$ is *weakly represented* by ϕ if the intersection product with α determines the same homomorphism as $\psi^*(\phi)$. We prefer to say that such α and ϕ are *weakly related*. This notion is used in [13] both because the precise relationship between homology and pseudohomology was not clear and because the stated homomorphism was the main object of interest.

   In light of these comments and Theorem 2, one can see that the notion 'weakly related' has the following interpretation: Namely, the isomorphism ψ induces an isomorphism $\overline{\psi}$: $H_*(X)/tors \to \Psi_*(X)/tors$. The relation of weak relatedness is precisely the pullback of $\overline{\psi}$ to a relation $H_*(X) \leftrightarrow \Psi_*(X)$.

What is required for Theorem 2 is a well-structured, geometric notion of "k-cycle," as well as a relative version. Rourke and Sanderson present such a notion in [18], which they call a "cycle." We shall use the older term "circuit" [12] for this, to avoid too many uses of the word "cycle". A theorem of Rourke and Sanderson [18] asserts that bordism of singular circuits is naturally equivalent to singular homology (cf., Theorem in §5). Circuits are piecewise linear (PL) objects. Our procedure in §5 will be to show that singular circuits (both absolute and relative) can be smoothed (away from a codimension-two set) in an essentially unique way to produce pseudocycles (both absolute and relative, respectively), and similarly for bordisms of these.

Circuits have also been called pseudomanifolds (e.g., [22]). We point out that Ruan and Tian [19] also use the term "pseudomanifold" in the context of pseudoholomorphic curves: in particular, in their analysis of moduli spaces (i.e., the domains of the evaluation maps), which we have already alluded to. Their notion is not quite the same as that of Rourke and Sanderson (or the classical notion), since their definition combines the PL and smooth categories, effectively subsuming some smoothing results in their definition. Moreover, their codimension-two singular sets are always assumed to be skeleta of triangulations, which is too restrictive for our purposes in Theorem 2 (e.g., see §5).

We conclude this introduction by describing the organization of the paper.

Section 2 presents facts about limit sets of continuous maps and the limit dimension of these. These facts form the basis for deriving most of the properties of pseudocycles that appear in Theorem 1.

Section 3 describes the various categories of smooth manifolds that we use. In particular, it briefly describes the notion of manifold with corners, which turns out to require some special attention in our arguments. The Appendix discusses corners in more detail.

Section 4 presents proofs of the assertions in Theorems 1 and Propositions 1 and 2. Proposition 1 amounts to constructing a top-dimensional pseudocycle that cannot be realized by a homology class. An amusing variant of Hirsch's proof of the Brouwer Fixed Point Theorem accomplishes this. The most serious proof in this section is the proof of the excision property for pseudohomology. A key ingredient for this allows us to construct lower dimensional pseudocycles via transversality (cf., Corollary 7, §2)[1]

Section 5 begins with Rourke and Sanderson's definition of singular circuits and their theorem that these may be used to define singular homology. We show that singular circuits can be replaced by so-called "weakly piecewise smooth" circuits. This step is important, because singular circuits do not allow us to control limit dimension, whereas weakly piecewise smooth circuits do. Next we find certain codimension-two "singular" subpolyhedra in circuits whose complements are PL manifolds, and we show that these manifolds are smoothable in an essentially unique way. Similarly for relative circuits and bordisms. The main work here involves these last, which are essential for obtaining a transformation that is well defined, natural and commutes with the relevant connecting homomorphisms. Finally, we show how to smooth the weakly piecewise smooth maps away from the singular sets, so that away from the singular sets they are pseudocycles (resp., bordisms of pseudocycles). This constitutes the desired transition from cycles to pseudocycles.

In Section 6, we piece together these steps to define the transformation ψ and conclude the proof of Theorem 2.

The Appendix discusses manifolds and submanifolds with corners to the extent needed in this paper. Corners arise naturally in some of our constructions, and the usual technique of "straightening" or "rounding" them requires us to modify maps near the corners. This could potentially cause us to lose control of the limit dimension of these maps. Thus, the main goal of the appendix is to show how, after straightening corners, we may smooth maps while preserving their limit dimension.

2 Limit sets and limit dimension

All the topological spaces that we consider in this paper will be Hausdorff, second countable, and locally-compact. We remind the reader that a continuous map $f : X \to Y$ is *proper* if $f^{-1}(C)$ is compact for all compact C in Y. In particular, this holds if X is compact or f is a homeomorphism. A homotopy will be called proper if it is proper as a map.

[1] As this paper was being completed, the author became aware of the revision of [13], which is in preparation, and he thanks McDuff and Salamon for making a copy of some relevant chapters available. A result closely similar to Corollary 7 appears in the revision.

Definition 1 *Let $f : X \to Y$ be a continuous map. The* limit set *$L(f)$ of f consists of all $y \in Y$ for which there is a sequence $\{x_n\}$ in X with no limit points such that $f(x_n) \to y$.*

The following proposition and corollaries summarize the basic facts that we need about limit sets.

Proposition 3 *Let $f : X \to Y$, $g : Y \to Z$, and $f' : X' \to Y'$ be continuous. Then:*

a. *If $C \subseteq Y$ is compact and $C \cap L(f) = \emptyset$, then $f^{-1}(C)$ is compact.*

b. *f is proper iff $L(f) = \emptyset$.*

c. *If X_1 is closed in X, then $L(f|X_1) \subseteq L(f)$.*

d. *If $X = X_1 \cup X_2$, then $L(f) \subseteq L(f|X_1) \cup L(f|X_2)$, with equality if X_1 and X_2 are closed.*

e. *$L(f \times f') = L(f) \times \overline{f'(X')} \cup \overline{f(X)} \times L(f')$.*

f. *$g(L(f)) \subseteq L(gf) \subseteq g(L(f)) \cup L(g)$.*
Thus, if g is proper, $L(gf) = gL(f)$.

g. *If f is surjective, then $L(g) \subseteq L(gf)$. Thus, if f is surjective and proper, $L(g) = L(gf)$. As a special case, if K is compact and pr is the projection $X \times K \to X$, then $L(f \circ pr) = L(f)$.*

h. *If X is dense in a compact space W, and f extends to a continuous map $g : W \to Y$, then $L(f) = g(W \setminus X)$.*

i. *If A is a closed subset of Y, then $L(f \mid f^{-1}(A)) \subseteq L(f) \cap A$.*

These properties are straightforward consequences of the definition. We leave the proofs to the reader.

We shall call a homotopy $\Phi : Y \times [0,1] \to Z$ an *isotopy* if the associated map map Φ' defined by $(y,t) \mapsto (\Phi(y,t),t)$ is a homeomorphism $Y \times [0,1] \to Z \times [0,1]$.

Corollary 4 *Every isotopy is proper.*

Proof: Let $pr : Y \times [0,1] \to Y$ be the projection map, so that $\Phi = pr \circ \Phi'$. Then, by Proposition 3 b,f,together with the facts that Φ' and pr are proper, $L(\Phi) \subseteq pr(L(\Phi')) \cup L(pr) = \emptyset$. Now apply Proposition 3b. This completes the proof.

Corollary 5 *Let f be as in Proposition 3, and let $\Phi : Y \times [0,1] \to Z$ be a proper homotopy. Then*

$$L(\Phi \circ (f \times id_{[0,1]})) = \Phi(L(f) \times [0,1]).$$

Proof: Set $g = f \times id_{[0,1]}$. Then, by Proposition 3 b,g,

$$\Phi(L(g)) \subseteq L(\Phi \circ g) \subseteq \Phi(L(g)) \cup L(\Phi) = \Phi(L(g)).$$

It remains to observe that $L(g) = L(f) \times [0,1]$ (Prop. 3b,e). This completes the proof.

Definition 2 *Let $f, g : X \to Y$ be two maps, and assume Y is endowed with a metric d. We say that f and g are* equal at infinity, *written $f =_\infty g$, if, for every sequence $\{x_n\}$in X without limit points,* $\lim d(f(x_n), g(x_n)) = 0$.

For example, the functions $f, g : [1, \infty) \to [1, \infty)$ given by $f(x) = x$ and $g(x) = x + 1/x$ are equal at infinity when $[1, \infty)$ is given the standard metric. So are $f(x) = x$ and $g(x) = x + 1$ when $[1, \infty)$ is given the metric $d(x, y) = |x - y|/\max(x, y)$, but not when it has the standard metric. When necessary, we shall specify the metric on the space Y. The following fact is an immediate consequence of the definition.

Lemma 2.1 *If $f =_\infty g$, then $L(f) = L(g)$.*

In the next section, we shall discuss the various categories of smooth manifolds and smooth maps that we use in this paper, including some facts about manifolds with corners. The following definition is invariant under any any (reasonable) notion of smooth manifold and smooth map.

Definition 3 *Let Y be a smooth manifold, and let S be a non-empty subset of Y. The* smooth dimension *of S, denoted $sd(S)$, is the smallest integer m for which S is contained in the image of a smooth map $X \to Y$ with $dim X \leq m$. We define $sd(\emptyset) = -1$, in accord with the usual convention.*

If $f : X \to Y$ is a smooth map, then we define its limit dimension, *written $\ell d(f)$, by $\ell d(f) = sd(L(f))$.*

The following corollary to Proposition 3 follows immediately from this definition:

Corollary 6 *Let $f : X \to Y$ and $f' : X' \to Y'$ be smooth, with f' proper. Assume that X' has dimension n. Then:*

a. If X_1 is closed in X, then $\ell d(f|X_1) \leq \ell d(f)$.

b. If $X = X_1 \cup X_2$, then $\ell d(f) \leq max\{\ell d(f|X_1), \ell d(f|X_2)\}$, with equality when both X_1 and X_2 are closed.

c. $\ell d(f \times f') \leq \ell d(f) + n$.

d. If K is compact and pr is the projection $X \times K \to X$, then $\ell d(f \circ pr) = \ell d(f)$.

e. If $g : Y \to Z$ is smooth, then $\ell d(f) \leq \ell d(gf) \leq max(\ell d(f), \ell d(g))$.

f. If X is dense in a compact, smooth manifold W, and f extends to a smooth map $g : W \to Y$, then $\ell d(f) \leq sd(W \setminus X)$.

The final corollary to Proposition 3 plays an important role in the proof of the excision property. Suppose that U and V are smooth manifolds, which may have non-empty boundaries bU and bV but no corners, and suppose that (X, A) is a pair of smooth manifolds, no restrictions on boundary and corners.

Corollary 7 *Let $f : V \to X$ and $g : U \to X$ be smooth maps, both transversal to the submanifold $A \subseteq X$. We also assume that $f|bV$ and $g|bU$ are transversal to A. Finally, suppose that $L(f) \subseteq g(U)$. Then $\ell d(f|f^{-1}(A)) \leq dimU - codimA$.*

Proof: If A has codimension zero, then the transversality hypotheses are vacuous, and the conclusion reduces to the inequality $\ell d(f|f^{-1}(A)) \leq dimU$. In this case, the result follows immediately from Corollary 6 a and the fact that $f^{-1}(A)$ is closed in V. Now suppose that $codimA = n \geq 1$. Using Proposition 3 i, we calculate as follows:

$$L(f|f^{-1}(A)) \subseteq L(f) \cap A \subseteq g(U) \cap A = g(g^{-1}(A)).$$

By transversality, $g^{-1}(A)$ is a smooth submanifold of U of codimension equal to n, which yields the desired result.

3 Smooth manifolds

All the smooth manifolds in this paper are assumed to be finite-dimensional and second-countable. Beyond that, we need to impose certain restrictions depending on the context.

3.1 Various categories of smooth manifolds

The constructions and theory considered in this paper require that we distinguish among certain categories of smooth manifolds.

To begin with, pseudocycles are, by definition, not generally defined on compact manifolds. Indeed, the theory requires this. However, as described in the introduction, the theory also requires some sort of compactness or properness condition on the target manifolds (or pairs). So, we need to distinguish the class of manifolds allowed for the domains of pseudocycles from the class of manifolds (or maps) allowed for the targets.

Further, in this paper, corners arise occasionally both in the targets of pseudocycles and in some constructions involving the domains. Moreover, the standard procedures for dealing with corners can potentially affect the limit dimensions of the maps that we are studying. So corners cannot be disregarded completely. The reader is referred to the Appendix where we give some basic definitions and facts about corners. Here, we give only a brief description.

Smooth manifolds with corners are modeled locally on open subsets of the non-negative orthants of Euclidean spaces, by precise analogy with the usual definitions of smooth manifolds and smooth manifolds with boundary. A smooth submanifold of a smooth manifold with corners is a subset that inherits the structure of a smooth manifold with corners in the usual way via the restriction of charts. By a *smooth manifold pair*, we shall mean a smooth manifold with corners, together with a smooth submanifold that is *closed* as a subset of X. However, we shall occasionally restrict this generality. Henceforth, when we use the term "smooth manifold," we shall allow the possibility of corners unless we rule this out explicitly.

Definition 4 *Let $\mathcal{C}$ be the category with objects all smooth manifold pairs. and with morphisms all smooth maps of these pairs. Let $\mathcal{C}^c$ denote the full subcategory of $\mathcal{C}$ with objects the compact pairs and $\mathcal{C}^{prop}$ the subcategory with no restriction on objects but with all maps required to be proper.*

The category $\mathcal{S}$ is the full subcategory of $\mathcal{C}$ whose objects (X, A) consist of manifolds X and A with no corners. We observe that if A is connected, then either A is contained in the boundary bX or $bX \cap A = C$ is a union of components of bA with A transverse to bX at C. $\mathcal{S}^c$ will denote the full subcategory of $\mathcal{S}$ with objects the compact manifold pairs, and $\mathcal{S}^{prop}$ is the subcategory with the same objects as $\mathcal{S}$ but with all smooth maps required to be proper.

3.2 Handles and finite handle decompositions

Our proof of Theorem 2 parallels the uniqueness argument of Eilenberg and Steenrod [7], which takes place in the category of finite simplicial pairs. The smooth manifold structures that we take to parallel simplicial structures are finite handle decompositions. We alert the reader that our definition of these is slightly non-standard. As noted in the introduction, it is also possible by these methods to prove an existence result for the natural transformation ψ. However, this seems to require a more elaborate notion of handle decomposition, which we prefer to avoid in this paper.

Let D^n denote the unit n-ball and S^{n-1} its boundary. The corresponding ball and sphere of radius r will be denoted rD^n and rS^{n-1}, respectively. Let Y be a smooth m-manifold and $f : S^{i-1} \times 2D^{m-i} \to Y$ a smooth embedding with image a smooth submanifold of Y contained in bY. It follows easily from the definition of 'submanifold' that $f(S^{i-1} \times D^{m-i}) = B$ is contained in an open face of Y—see the Appendix for a definition of open face, an open submanifold of bY not meeting the corners of Y. Introduce a corner at bB and glue Y to $H = D^i \times D^{m-i}$ by identifying $S^{i-1} \times D^{m-i}$ with B via f. Again as indicated in the Appendix, the resulting manifold can be given a differentiable structure compatible with those of Y (with corner added) and H. We call this smooth manifold *the result of of attaching the i-handle H to Y* and denote it by $Y + H$. We use similar notation for any finite set of handles disjointly attached to Y.

Alternatively, if Y and H, as above, are smooth, codimension-zero submanifolds of Z, such that $Y \cap H = B$ and $H \subseteq intZ$, then we may 'round the corner' bB in Z, as indicated in the Appendix, obtaining a smooth, codimension-zero submanifold $r(Y \cup H)$ which is essentially the same as $Y + H$. We use the two constructions interchangeably.

Definition 5 *Let (X, A) be a smooth manifold pair in $\mathcal{C}^{prop}$, where $dimX = m$. By a* finite handle decomposition *of (X, A) we mean a nested sequence of codimension-zero submanifolds*

$$A \subset X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_{m-1} \subseteq X_m = X$$

satisfying

a. X_{-1} is a tubular neighborhood of A.

b. Each X_i , $i \geq 0$, admits a smooth, proper deformation retraction to

$$X_{i-1} + H_1 + H_2 + \ldots,$$

where $H_1, H_2, \ldots$ is a finite collection of i-handles.

We let $\mathcal{C}^{prop}_{hand}$ denote the full subcategory of $\mathcal{C}^{prop}$ whose objects are the pairs that admit finite handle decompositions. Similarly for $\mathcal{C}^{c}_{hand}$.

We note that the objects of $\mathcal{C}^{c}_{hand}$ include all the objects of $\mathcal{S}^c$. Indeed, if (X, A) is such a manifold pair with $A = \emptyset$, then this is a classical result that can be proved using Morse functions (as in, say, [16]). These techniques can be extended to the case of pairs.

4 Pseudohomology

In this section we extend to the relative case the definition of pseudocycle and bordism of pseudocycles presented in [13] , and we derive the basic properties of the resulting bordism (i.e., *pseudohomology*) groups. In effect, these results constitute a proof of Theorem 1 of the introduction.

Definition 6 *Let (X, A) be a smooth manifold pair in $\mathcal{C}$ and k a non-negative integer. A k-dimensional* relative pseudocycle *in (X, A) (or* relative Ψ_k-cycle *in (X, A)) is a pair (V, f) such that V is an oriented manifold in $\mathcal{S}$, f is smooth, and the following dimension conditions are satisfied:*

a. $dim V = k$;

b. $\ell d(f) \leq \max(-1, k-2)$;

c. $\ell d(f|bV) \leq \max(-1, k-3)$.

We use the following standard conventions. Namely, when $bV = \emptyset$, we call (V, f) an *absolute* pseudocycle (or absolute Ψ_k-cycle) in (X, A). And when A is empty, we write X instead of $(X, \emptyset)$ and omit the word "absolute."

If (V, f) is a relative Ψ_k-cycle in (X, A), then clearly $(bV, f|bV)$ is a Ψ_{k-1}-cycle in A.

We now remind the reader of the usual notion of *oriented, relative smooth bordism*, but without the usual compactness condition (cf.,e.g., [4]). Let $f_i : (V_i, \partial V_i) \to (X, A), i = 0, 1$, be smooth maps of (oriented) k-manifolds. By a relative smooth bordism in (X, A) between f_0 and f_1 we mean a pair (W, F), such that:

- W is an oriented $(k+1)$-manifold in $\mathcal{S}$;
- bW contains the disjoint union $V_0 \sqcup V_1$ as a codimension-0 submanifold, closed as a subset of bW;
- $F : W \to X$ is a smooth map extending $f_0 \sqcup f_1$;
- $F(\overline{bW \setminus (V_0 \sqcup V_1)}) \subseteq A$.

Definition 7 *A relative Ψ_k-bordism between relative Ψ_k-cycles (V_i, f_i) in $(X, A), i = 0, 1$ is a relative smooth bordism (W, F) in (X, A) between (V_0, f_0) and (V_1, f_1) as above, such that:*

a. $\ell d(F) \leq \max(-1, k-1)$;

b. $\ell d(F|\overline{bW \setminus (V_0 \sqcup V_1)}) \leq \max(-1, k-2)$.

When the pseudocycles (V_i, f_i) are absolute and $V_0 \sqcup V_1 = bW$, we call (W, F) an *absolute bordism.* When $A = \emptyset$, forcing pseudocycles and bordisms to be absolute, we omit this word.

If (W, F) is a relative Ψ_k-bordism between (V_0, f_0) and (V_1, f_1), then the restriction

$$(\overline{bW \setminus (V_0 \sqcup V_1)}, F|\overline{bW \setminus (V_0 \sqcup V_1)})$$

is a Ψ_{k-1}-bordism between $(bV_0, f_0|bV_0)$ and $(bV_1, f_1|bV_1)$.

We now form bordism classes of relative Ψ_k-cycles in (X, A) and bordism classes of Ψ_k-cycles in X and in A. The usual argument that bordism gives an equivalence relation works because of Corollary 6b, in §2, together with the techniques of gluing and smoothing maps described in the Appendix (to verify transitivity). Thus, we obtain bordism or pseudohomology groups $\Psi_k(X, A), \Psi_k(X)$, and $\Psi_k(A)$. As usual, by restricting a relative pseudocycle to the boundary, we obtain a homomorphism $\Psi_k(X, A) \to \Psi_{k-1}(A)$, which we denote by ∂.

When V (resp.,W) is compact in the definitions above we call the pseudocycle (resp., bordism) compact. If we restrict to compact pseudocycles and bordisms, then we obtain the classical (oriented) bordism groups $\Omega_k(X, A)$, which may be defined by smooth maps, since our targets (X, A) are smooth. Thus, we have natural "inclusion" homomorphisms $\Omega_k(X, A) \to \Psi_k(X, A)$, etc.

After the following preliminary lemma, we present some of the basic properties of pseudohomology.

Lemma 4.1 *Let (X, A) and (Y, B) be smooth manifold pairs, and let $h : (X, A) \to (Y, B)$ be a proper, smooth map. If (V, f) is a relative Ψ_k-cycle in (X,A), then (V, hf) is a relative Ψ_k-cycle in (Y, B). If (W, g) is a relative Ψ_k-bordism in (X, A), then (W, hg) is a relative Ψ_k-bordism in (Y, B).*

Proof: Clearly (V, hf) represents a smooth, relative bordism class in (Y, B) and (W, hg) represents a smooth relative bordism. The required dimension conditions follow immediately from Corollary 6e (§2).

Thus, for proper h, the rule $(V, f) \mapsto (V, hf)$ defines a homomorphism

$$\Psi_k(h) : \Psi_k(X, A) \to \Psi_k(Y, B).$$

As one example, given the smooth manifold pair (X, A) in $\mathcal{C}^{prop}$, the inclusion $i : A \hookrightarrow X$ gives the homomorphism $\Psi_k(i) : \Psi_k(A) \to \Psi_k(X)$. Usually, to simplify notation, we replace $\Psi_k(h)$ by h_*.

Note that without the properness condition on h, the lemma would be false (see Examples 1 and 2 below in this section).

Lemma 4.1 immediately gives the following corollary.

Proposition 8 (functoriality) *Ψ_k is a functor from $\mathcal{C}^{prop}$ to the category of abelian goups and homomorphisms.*

It will be convenient from time to time to make use of the corresponding *reduced* functor $\tilde{\Psi}_*$, which is defined in the usual way for pairs of the form $(X, \emptyset)$. That is, $\tilde{\Psi}_*(X) = ker(\Psi_*(X) \to \Psi_*(pt))$. Of course, the collapse map $X \to pt$ must be proper, i.e., X must be compact.

Proposition 9 (homotopy invariance) *Ψ_k is a homotopy functor on $\mathcal{C}^{prop}$.*

Proof: Choose any relative Ψ_k-cycle (V, f) in (X, A), and let

$$H : (X, A) \times [0, 1] \to (Y, B)$$

be a smooth, proper homotopy. Then $H \circ (f \times id_{[0,1]})$ is a smooth map in $\mathcal{C}$ but not in $\mathcal{S}$. So, we smooth the corners of $V \times [0, 1]$ and adjust the map $H(f \times id_{[0,1]})$ as in the Appendix, obtaining a smooth map $G : (V \times [0, 1])_\alpha \to Y$ which is a smooth-bordism between $(V, H_0 f)$ and $(V, H_1 f)$. It remains to verify that the smooth dimension conditions of Definition 6 are satisfied. The construction in the Appendix insures that the limit sets of G and of $G|bV \times [0, 1]$ are the same as those of $H(f \times id)$ and $H((f|bdV) \times id)$, respectively, so we may compute with the latter. But, by Corollary 5 (§2),

$$L(H \circ (f \times id)) = H(L(f \times id)) = H(L(f) \times [0, 1]),$$

from which the inequality $\ell d(H \circ (f \times id)) \leq k - 1$ follows immediately. Similarly for the inequality $\ell d(H \circ ((f|bV) \times id)) \leq k - 2$. This completes the proof of the proposition.

Of course, the results above all specialize to the absolute case.

Now let (X, A) be a pair in $\mathcal{C}^{prop}$, and let i and j be the (proper) inclusions $A \hookrightarrow X$ and $X \hookrightarrow (X, A)$, respectively. These maps, together with restriction to the boundary already described above, induce a sequence

$$\cdots \xrightarrow{\partial} \Psi_k(A) \xrightarrow{i_*} \Psi_k(X) \xrightarrow{j_*} \Psi_k(X, A) \xrightarrow{\partial} \Psi_{k-1}(A) \xrightarrow{i_*} \cdots,$$

which we call the Ψ_*-sequence of the pair (X, A).

Proposition 10 (exact sequence) *The Ψ_*-sequence of the pair (X, A) is exact.*

The proof follows closely the standard proof of exactness for classical bordism. However, here there are two added complications. First, we need to check various smooth dimension conditions to make sure the constructions remain within the class of pseudocycles or bordisms of pseudocycles. These conditions are not hard to check with the aid of Proposition 3 and Corollary 6 of §2. Details for this are left to the reader. Secondly, in one part of the proof of exactness at $\Psi_k(X, A)$, we must glue together along their boundaries a relative Ψ_k-cycle (U, f) in (X, A) and a Ψ_{k-1}-nullbordism (V, g) of $(bU, f|bU)$ in A, smoothing both $U \cup V$ and $f \cup g$. Calling the result (W, h), we construct $(W \times [0, 1], h \circ pr)$, which is to serve as a relative bordism between (W, h) and the original (V, f). However, first corners must be rounded and $h \circ pr$ correspondingly smoothed, and this all must be done without violating the requisite limit-dimension conditions. These steps all follow from results and techniques of the Appendix, to which we refer the reader.

Proposition 11 (excision) *Let (X, A) be a smooth manifold pair in $\mathcal{C}^{prop}$, and let U be an open subset of X such that $\overline{U} \subseteq int(A)$ and such that the inclusion $inc : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ belongs to $\mathcal{C}^{prop}$. Then*

$$\Psi_*(inc) : \Psi_*(X \setminus U, A \setminus U) \to \Psi_*(X, A)$$

is an isomorphism.

The proof makes use of the following general fact about smooth manifold pairs.

Lemma 4.2 *Let (X, A) be a smooth manifold pair such that $dimA = dimX$, and let K be a closed subset of the topological interior A° of A in X. Then there exists a smooth manifold pair (X, B) such that $K \subseteq B^\circ \subseteq B \subseteq A^\circ$ and (X, B) is a (strong) smooth, proper deformation retract of (X, A).*

Proof: This fact follows quickly from results and methods of J. Cerf [3], in particular, Chapitre II, §§2,3. We refer the reader to that paper.

Proof of Proposition 11: (1) *Surjectivity.* Setting $K = \overline{U}$, we make use of the smooth manifold pair (X, B) of Lemma 4.2, which is a smooth, proper deformation retract of (X, A) and satisfies $\overline{U} \subseteq B^\circ \subseteq B \subseteq A^\circ$. It follows from the homotopy property of Ψ_* that, given a class $z \in \Psi_k$(X,A), we may represent it by a Ψ_k-cycle (V, f) such that $f(bV) \subseteq B$. By definition, there exist smooth maps $F : M \to X$ and $G : N \to A$ such that $dimM = k - 2$, $dimN = k - 3$, $F(M) \supseteq L(f)$, and $G(N) \supseteq L(f|bV)$.

We now use a theorem of Whitney (e.g., see [1]) to construct a smooth function $\lambda : X \to [0, 1]$ such that $\lambda(B) = 1$ and $\lambda(X \setminus A^\circ) = 0$. Whitney's theorem and proof hold just as well in the presence of corners as in their absence. By Sard's Theorem, there exists a $c \in (0, 1)$ that is a regular value simultaneously for the maps λf, $\lambda(f|bV)$, λF, and λG. Let $W = (\lambda f)^{-1}([0, c])$, and set $g = f|W$. We claim that (W, g) is a Ψ_k-cycle in $(X \setminus U, A \setminus U)$ which is Ψ_k-bordant to (V, f) in (X, A).

To verify the claim, note first that since $\lambda f(W) \subseteq [0, c]$, W is a smooth submanifold of $intV$ with boundary $(\lambda f)^{-1}(c)$ and no corners. Further, g maps (W, bW) smoothly into $(X \setminus U, A \setminus U)$, and $(V \times [0, 1], f \circ pr)$ is a smooth relative bordism in (X, A) between (V, f) and (W, g). It remains only to verify the requisite dimension conditions on limit sets.

First we verify the dimension conditions for (W, g). Since W is a closed subset of V, we have $L(g) \subseteq L(f)$ (Proposition 3, §2), and so $\ell d(g) \leq \ell d(f) \leq k - 2$, as required. Further, since $bW = \lambda f^{-1}(c)$, we have $\ell d(g|bW) = \ell d(g|(\lambda f)^{-1}(c)) = \ell d(f|f^{-1}(\lambda^{-1}(c)))$. By construction, both f and F are transverse to $\lambda^{-1}(c)$. Thus, applying Corollary 7 of §2, we get $\ell d(g|bdW) \leq dimM - 1 \leq k - 3$, as required.

The dimension conditions for $(V \times [0, 1], f \circ pr)$ are verified more easily. Thus, $\ell d(f \circ pr) = \ell d(f) \leq k - 2 < k - 1$, (Corollary 6, §2), as required. Also, again using Corollary 6,

$$\begin{aligned} \ell d(f \circ pr|\overline{b(V \times [0,1]) \setminus ((V \times 0) \sqcup (W \times 1))}) &= \ell d(f \circ pr|(bV \times [0,1]) \cup \overline{V \setminus W} \times 1) \\ &= \max\{\ell d(f|bV), \ell d(f|\overline{V \setminus W})\} \\ &\leq k - 2. \end{aligned}$$

This completes the proof of surjectivity.

(2) *Injectivity.* Let B and λ be as in the proof of surjectivity. The smooth, proper deformation retraction $(X, A) \to (X, B)$ of that proof restricts to a smooth proper deformation retraction $(X \setminus U, A \setminus U) \to (X \setminus U, B \setminus U)$. It follows that any class $w \in \Psi_k(X \setminus U, A \setminus U)$ that maps to $0 \in \Psi_k(X, A)$ can be represented by a Ψ_k-cycle in $(X \setminus U, B \setminus U)$ that is Ψ_k-nullbordant in (X, B). Accordingly, let (V, f) be such a Ψ_k-cycle in $(X \setminus U, B \setminus U)$ representing w, and let (W, g) a Ψ_k-nullbordism of (V, f) in (X, B). Recall that we have $\lambda(B) = 1$ and $\lambda(X \setminus A^\circ) = 0$.

Now we again make use of smooth maps provided by the definitions, $F : M \to X \setminus U$, $F' : M' \to A \setminus U$, $G : N \to X$, and $G' : N' \to A$, which satisfy the following: $dimM = k-2$ and $F(M) \supseteq L(f)$; $dimM' = k-3$ and $F'(M') \supseteq L(f|bV)$; $dimN = k-1$ and $G(N) \supseteq L(g)$; $dimN' = k-2$ and $G'(N') \supseteq L(g|\overline{bW \setminus V})$. Using the function λ, we choose a $d \in (0,1)$ that is simultaneously a regular value for λf, $\lambda f|bV$, λg, $\lambda g|bW$, λF, $\lambda F'$, λG, and $\lambda G'$.

Note that $(\lambda g)^{-1}(d)$ separates W(resp.,bW) into two submanifolds $W_\pm$ (resp., $(bW)_\pm$), with the notation chosen so that $W_+ = (\lambda g)^{-1}([0,d])$, etc. Thus, $(bW)_+ \subseteq int(V)$. Set $h = g|W_+$. The remainder of the proof now consists of verifying the following two claims:

a. $((bW)_+, h|(bW)_+)$ *is* Ψ_k*-bordant to* (V, f) *in* $(X \setminus U, A \setminus U)$.

b. (W_+, h) *is a* Ψ_k*-nullbordism of* $((bW)_+, h|(bW)_+)$ *in* $(X \setminus U, A \setminus U)$.

For the first claim, we make use of the smooth bordism $(V \times [0,1], f \circ pr)$ between (V, f) and $((bW)_+, h|(\partial W)_+)$ in $(X \setminus U, A \setminus U)$. Of course, corners must be straightened and the map smoothed so as to preserve limit dimension. Again, this is done by the techniques of Appendix A. The verification of dimension conditions proceeds similarly to that in the proof of surjectivity. The only slightly subtle point involves $\ell d(h|f^{-1}(d))$, which is evaluated with the aid of Corollary 7, §2, as before. Similar arguments apply to the second claim.

This concludes the proof of the excision property.

Corollary 12 Ψ_* *takes finite disjoint unions to finite direct sums.*

The proof is a well-known induction on the number of terms in the union, which works for any functor satisfying the Eilenberg-Steenrod axioms. For the induction step, one looks at the long exact sequence for the pair (full union, union with one term deleted). The excision property allows one to remove the subspace, etc.

Suppose now that $\{X_i\}$ is a handle decomposition for the pair (X, A) (cf., §3). Combining the foregoing results, we compute just as for homology:

$$\Psi_k(X_i, X_{i-1}) \approx \bigoplus_j \Psi_k(D^i_j, S^{i-1}_j),$$

where the subscript j indexes the i-handles attached to X_{i-1}. To complete this calculation, we need to compute the groups $\Psi_k(D^i, S^{i-1})$. This can be done with the aid of some of the computational facts that we now present. Let $\iota : \Omega_*(X, A) \to \Psi_*(X, A)$ denote the homomorphism induced by the inclusion of bordism classes. There is also a corresponding homomorphism $\tilde{\iota}$ of the reduced theories. Recall that there is a natural transformation of functors $\mu : \Omega_* \to H_*$ defined by evaluating fundamental classes.

Proposition 13 *a. If* X *is compact, then* $\iota : \Omega_0(X) \to \Psi_0(X)$ *is an isomorphism. Therefore, the composite* $\iota \circ (\mu)^{-1} : H_0(X) \to \Psi_0(X)$ *is an isomorphism.*

b. If X *is* n*-dimensional, then* $\Psi_k(X) = 0$*, for all* $k > n$*.*

Proof: 1. If (V, f) (resp., (W, g)) is a Ψ_0-cycle (resp., Ψ_0-bordism) in X, then V (resp., W) is compact (Proposition 3a). It follows that ι is just the identity homomorphism.

2. By definition, when $k > n$, every smooth bordism in X is a Ψ_k-bordism. Therefore, given any Ψ_k-cycle (V, f) in X, we can find a Ψ_k-nullbordism of (V, f): namely, $(V \times [0, \infty), f \circ pr)$.

This completes the proof.

In the special case $X = pt$, Proposition 13 implies the dimension axiom asserted in Theorem 1. Thus, combining this proposition with Propositions 8—11, we have completed the proof of Theorem 1.

Both assertions of Proposition 13 carry over to the reduced groups. Note that the first assertion immediately implies the following: *if $f : X \to Y$ establishes a bijection between the components of the compact manifolds X and Y, then $f_* : \Psi_0(X) \to \Psi_0(Y)$ is an isomorphism.*

Proposition 14 *a. The inclusion $pt = D^0 \hookrightarrow D^n$ induces an isomorphism $\Psi_*(pt) \approx \Psi_*(D^n)$.*

b. There is an isomorphism $\Psi_*(D^n, S^{n-1}) \approx \tilde{\Psi}_{*-n}(S^0)$, which is a composition of boundary maps of long exact sequences and excision isomorphisms.

The proof is the same as for ordinary homology.

The remaining computational observations imply items a and b, listed in the Remark in §1.

Example 1: Let $U = V$ denote the open unit interval $(0, 1)$, and define $f : U \to V$ and $g : V \to S^1$ by $f(x) = x$ and $g(y) = \exp(2\pi iy)$. Then $L(f) = \emptyset$, so (U, f) defines a Ψ_0-cycle in V. On the other hand, $L(gf) \neq \emptyset$, so (U, gf) is not a Ψ_0-cycle in S^1.

This example shows that composition with maps that are not proper does not define a homomorphism of pseudohomology.

Example 2: Let $U = W = S^1$, the unit sphere, and let $V = U \times R$. Let i denote the inclusion map $U \hookrightarrow V$ given by $x \mapsto (x, 0)$, and let $p : V \to W$ be the projection. As we shall see below, the pair (U, id_U) represents a non-zero class in Ψ_1(U), and so $(p \circ i)_*[U, id_U] \neq 0$ in $\Psi_1(W)$. However, (U, i) is Ψ_0-nullbordant in V (with null bordism the inclusion $S^1 \times [0, \infty) \hookrightarrow S^1 \times R$), so that,if defined, the homomorphism p_* satisfies $p_* i_*[U, id_U] = 0$.

Therefore, in the absence of properness, Ψ_* does not define a functor.

Proposition 15 *Let X be any smooth k-manifold with empty boundary, let $V = X$, and let $f : V \to X$ be any diffeomorphism. Then (V, f) is a Ψ_k-cycle in X that does not represent* $0 \in \Psi_k(X)$.

Thus, $\Psi_k(X) \neq 0$, whereas $H_k(X) = 0$. It follows that when X has non-compact components, there is no natural surjection $H_k(X) \to \Psi_k(X)$.

Proof of Proposition 15: Since f is a diffeomorphism, $L(f) = \emptyset$, so (V, f) is a Ψ_k-cycle. The remainder of the proof is similar to Hirsch's proof of the Brower Fixed Point Theorem, as presented in [17]. Suppose then that, counter to our proposed conclusion, we have a Ψ_k-nullbordism of (V, f) in X, say (W, g). Since $g|V = f$, g is surjective. We observe that $sdL(g) \leq k-1$, so that $L(g)$ has Lebesgue measure 0 in X. Similarly, by Sard's Theorem, the set of critical values of g has measure 0. It follows that there exists a regular value p of g, that is not in $L(g)$. By the definition of g, p is also a regular value of $g|bW = g|V = f$. Set $g^{-1}(p) = A$. Then A is a non-empty, compact (Proposition 3a, §2), 1-dimensional submanifold of W such that $bA = A \cap bW = \{f^{-1}(p)\}$, a singleton subset of A. However, there is no such manifold, because every compact 1-manifold has an *even* number of boundary points. This contradiction proves the claim.

Proposition 16 *Let X be any smooth manifold, and let $j : X \to X \times R$ denote the (proper) inclusion given by $x \mapsto (x, 0)$. Then $j_* : \Psi_*(X) \to \Psi_*(X \times R)$ is trivial.*

Proof: If (V, f) is a Ψ_k-cycle in X, then $(V \times [0, \infty), f \times inc)$ is a Ψ_k-nullbordism of (V, f) in $X \times R$.

Corollary 17 *Suppose that X is a compact, orientable n-manifold with empty boundary, and suppose that $\psi : H_* \to \Psi_*$ is a natural tranformation of functors on $\mathcal{C}^{prop}$. Then $\psi : H_*(X \times R) \to \Psi_*(X \times R)$ is non-injective.*

Proof: The homomorphism $j_* : H_n(X) \to H_n(X \times R)$ is a non-zero isomorphism. The result is now immediate from the foregoing proposition.

5 Smoothing circuits

We describe the representation of homology by oriented, singular k-circuits due to Rourke and Sanderson [18] (see also [2]) and then pass to weakly piecewise smooth (WPS) circuits . These can be used to represent the homology of smooth manifold pairs, and they have the virtue of allowing us some control over smooth dimension. We conclude by showing how to smooth WPS circuits so as to produce pseudocycles.

5.1 Circuits

We begin by reminding the reader of some basic terminology from piecewise linear (PL) topology.

Definition 8 *All of the simplicial complexes that we consider will be locally finite collections of simplexes contained in some finite dimensional Euclidean space. If K is a simplicial complex and S is a subset of K, then we let $|S|$ denote the union of all open simplexes $\mathring{\sigma}$ for $\sigma \in S$. We call $|K|$ a* polyhedron *and K a* triangulation *of $|K|$. The set $St(S, K)$ consists of all $\sigma \in K$ having some face in S. The subset $|St(S, K)| \subseteq |K|$ is always open; we sometimes call it the (open) star of $|S|$ in $|K|$.*

We let $S(i)$ denote the collection of i-simplexes in S, and S^i the union of all $S(j)$, $j \leq i$. If S is a subcomplex of K, then so is S^i.

A polyhedral pair (Q, P) is a pair of topological spaces for which there exist a simplicial complex L and subcomplex K with $Q = |L|$ and $P = |K|$. The pair (L, K) is said to triangulate (Q, P). The dimension of (Q, P) is the usual dimension of L. This is independent of the choice of triangulating L.

Definition 9 *a. A* relative k-circuit *is a* compact *polyhedral pair $(Q, \delta Q)$ such that there is a $(k-2)$-dimensional subpolyhedron $S(Q) \subseteq Q$ satisfying:*

(i) $Q = \overline{Q \setminus S(Q)}$.

(ii) $Q \setminus S(Q)$ *is a piecewise-linear k-manifold with boundary $\delta Q \setminus S(Q)$.*

(iii) δQ *is a relative $(k-1)$-circuit with $\delta(\delta Q) = \emptyset$ and $S(\delta Q) = S(Q) \cap \delta Q$.*

If $\delta Q = \emptyset$, we say that Q is a closed or absolute k-circuit (or simply a k-circuit). Then (c) may be rephrased as

(iv) δQ *is a $(k-1)$-circuit with $S(\delta Q) = S(Q) \cap \delta Q$.*

b. A null bordism *of a relative k-circuit Q is a relative $(k+1)$-circuit R such that $Q \subset \delta R$ and $S(R) \cap Q = S(Q)$. It follows that $S(R) \cap \delta Q = S(\delta R) \cap \delta Q = S(\delta Q)$ and that $Q \setminus S(Q)$ and $\delta Q \setminus S(\delta Q)$ are properly embedded PL submanifolds of $\delta R \setminus S(\delta R)$. Absolute bordisms are the obvious specializations. For example, if R is a null bordism of the relative k-circuit Q, then $\delta R \setminus (Q \setminus \delta Q)$ is a null bordism of the absolute circuit δQ.*

c. A singular k-circuit in a space Z *is a pair (P, a), where P is a k-circuit and $a : P \to Z$ is a continuous map. Similarly for bordisms of singular k-circuits. The same terminology applies to the relative case.*

d. We say that a circuit or bordism of circuits T is oriented if the manifold $T \setminus S(T)$ is oriented; then it induces an orientation on the manifold $\delta T \setminus S(T)$, so that δT inherits an orientation from T, etc. In this paper we restrict attention entirely to oriented circuits and bordisms although we may not mention this further explicitly.

With these definitions, one can now define groups of bordism classes of relative (and absolute) k-circuits. If (X, A) is a pair of spaces, then we denote its circuit bordism group by $\mathcal{H}_k(X, A)$, and similarly in the absolute case.

Let (Q, a) be a singular, relative k-circuit in (X, A). As stated in Definition 9d, $Q \setminus \delta Q$ is oriented. It follows that Q carries a fundamental (singular) homology class. (To see this in the absolute case, triangulate Q such that $S(Q)$ is a subcomplex, and orient every k-simplex compatibly with Q. The formal sum of these is a simplicial k-chain on Q, which one quickly computes to be a k-cycle. This cycle can now be used to define a singular k-cycle that represents $[Q]$. The relative case is similar.) The class $[Q]$ allows us to define an evaluation map $e : \mathcal{H}_k(X, A) \to H_k(X, A)$ by the usual rule $e[Q, a] = a_*([Q])$.

Theorem (Rourke and Sanderson [18]): *The evaluation map $e : \mathcal{H}_k(X, A) \to H_k(X, A)$ is a natural isomorphism for all k.*

This theorem provides us with almost the right context in which to compare homology with pseudohomology. However, notice that circuit homology uses continuous maps, whereas pseudohomology uses smooth maps with constraints on their limit dimensions. Therefore, it is convenient to modify circuit homology slightly to adapt to this. Our first step is to introduce the notion of a *weakly piecewise-smooth* map (which we henceforth call a WPS map).

Definition 10 *Let S be a subset of a locally-finite simplicial complex K, let Y be smooth, and let $f : |S| \to Y$ be a continuous map. We say that f is weakly piecewise-smooth with respect to S (WPS w.r.t.S) if $f| \overset{\circ}{\sigma} : \overset{\circ}{\sigma} \to Y$ is smooth for every simplex σ in S.*[2]

If P is a polyhedron, then we say that f is WPS if f is WPS w.r.t. K for some K triangulating P.

The following properties of this definition are easily checked:

- If f is WPS w.r.t. K, and K' is a subdivision of K, then f is WPS w.r.t. K'.
- If $f : P \to Y$ is WPS, then for every L triangulating P, there is a subdivision L' of L such that f is WPS w.r.t. L'.

At this point, we observe that we can repeat Definition 9c, with the topological space Z replaced by a smooth manifold and continuous maps replaced by WPS maps; similarly for the relative case. This results in bordism groups $\mathcal{H}_k^{WPS}(X, A)$ and an "inclusion" homomorphism

$$\jmath : \mathcal{H}_k^{WPS}(X, A) \to \mathcal{H}_k(X, A),$$

for each smooth manifold pair (X, A).

Proposition 18 *The homomorphism $\jmath$ is an isomorphism.*

Proof (sketch): For simplicity, start with the absolute case. An induction argument reduces the proof to showing that if (Q, P) is a compact, polyhedral pair and $f : Q \to X$ a continuous map such that $f|P$ is WPS, then there is a homotopy $rel P$ (i.e., pointwise stationary on P) between f and a WPS map. In turn, another induction argument reduces this to the special case in which Q is a simplex σ and P its boundary $\dot{\sigma}$. So suppose that we are in this situation, choose a topological metric on X, and then choose a smooth ϵ-approximation to $f| \overset{\circ}{\sigma}$, say g'. Here ϵ is a continuous function on σ which is positive on $\overset{\circ}{\sigma}$ and 0 on $\dot{\sigma}$. We extend g' to a map g on σ by setting $g = f$ on $\dot{\sigma}$, and we check that g is WPS. Moreover, by choosing ϵ suitably small, it is easy to arrange that g be homotopic to $f\ rel \dot{\sigma}$. This concludes the sketch in the absolute case.

The relative case can be derived from the absolute case. We start with Q, P, and f as before, only now we have a smooth manifold pair (X, A) and subpolyhedra $Q' \subseteq Q$ and $P' = P \cap Q'$, with f a map of pairs $(Q, Q') \to (X, A)$. We want to conclude that there is a homotopy $rel\ P$ of maps of pairs between f and a WPS map. To obtain this, first apply the absolute case to Q', P', and $f|Q'$. Then apply the homotopy extension property to obtain a homotopy of pairs $rel\ P$ between f and a continuous map which is WPS on $P \cup Q'$. Then apply the absolute case again to the pair $(Q, P \cup Q')$.

[2] The usual notion of piecewise smooth map, which we *do not* want here, requires that f be smooth on each σ.

This concludes our sketch.

It follows that we may use WPS circuits to represent the homology of smooth manifold pairs.

We conclude this subsection with a result which shows that smooth dimension can be defined in terms of WPS maps.

Proposition 19 *Let Y be smooth, and let S be a subset of Y. Then sdS is the smallest integer m for which S is contained in the image of a WPS map $P \to Y$ with $dimP \leq m$.*

Proof: The proposition is clearly equivalent to the statement,

$sdS \leq m$ iff S is in the image of a WPS map $P \to Y$ with $dimP \leq m$,

for which we now give a proof. If $sdS \leq m$, then S is in the image of a smooth map $X \to Y$ with $dimX \leq m$. Use a smooth triangulation of X to transform this into a WPS map. On the other hand, if there is a WPS map $f : P \to Y$ with image containing S and with $dimP \leq m$, form a smooth map as follows. Let $dimP = k \leq m$, and choose a triangulation K of P such that f is WPS w.r.t. K. Choose any $\sigma \in K(i)$. The smooth k-manifold $\overset{\circ}{\sigma} \times R^{k-i}$ maps smoothly to Y by first projecting to $\overset{\circ}{\sigma}$ in P and then mapping to Y by f. The disjoint union of all these maps is a smooth map to Y from a k-manifold with image containing S. Thus, $sdS \leq m$. This completes the proof.

5.2 Manifolds in circuits

The Rourke-Sanderson definition of a k-circuit Q is independent of the choice of simplicial complex L triangulating Q. Thus, for example, the singular subset $S(Q) \subseteq Q$ is simply defined to be a $(k-2)$-dimension subpolyhedron of Q and *not* necessarily the $(k-2)$-skeleton of a triangulation. While this invariance is undoubtedly a useful feature of their theory, it poses some obstacles for us from the viewpoint of smoothing the complement of the singular set. We deal with these by choosing arbitrary polyhedral triangulations of Q (i.e., not necessarily respecting $S(Q)$) and choosing $(k-2)$-dimensional subcomplexes that both contain the "bad points" in $S(Q)$ and have smoothable complements. By "bad points" we mean non-manifold points, which we now describe.

Definition 11 *Let K be a k-dimensional simplicial complex. We call a point p in $|K|$ a* manifold-point *if it has a neighborhood in $|K|$ that is PL-homeomorphic to R^k or R^k_+ (closed, Euclidean half-space). Otherwise we say that p is a* non-manifold point*.*

Clearly the non-manifold points form a closed subset of $|K|$, but we can say more than this. The following lemma presents a standard fact from PL topology; we provide a proof for the reader's convenience.

Lemma 5.1 *The non-manifold points of $|K|$ form a subpolyhedron triangulated by a subcomplex of K.*

Proof: If $p, q \in |K|$, both belonging to the same open simplex, then it is easy to construct a PL homeomorphism $f : |K| \to |K|$ such that $f(p) = q$. It follows that both p and q are manifold points or non-manifold points. Therefore, the non-manifold points of $|K|$ consist of a union of open simplexes. Since this set is closed, it must be a union of closed simplexes, completing the proof.

In the remainder of this subsection, we shall be consistently dealing with the following separate three cases:

a. P is an absolute $(k-1)$-circuit triangulated by a finite simplicial complex K.

b. (Q, P) is a relative k-circuit triangulated by (L, K).

c. R is a relative $(k+1)$-nullbordism of (Q, P) with $(R, \delta R)$ triangulated by (N, M) and L and K subcomplexes that triangulate Q and P respectively.

In each of these cases, we define a subpolyhedron Σ of codimension two, and we show that it contains all the bad points.

Definition of Σ:

a. $\Sigma = |K^{k-3}|$.

b. $\Sigma = |L^{k-2} \setminus K(k-2)|$.

c. $\Sigma = |N^{k-1} \setminus (M(k-1) \cup St(K(k-2), N))|$.

The reader can easily convince herself/himself that Σ is a compact subpolyhedron of codimension two.

Proposition 20 *a. $P \setminus \Sigma$ is a PL $(k-1)$-manifold with empty boundary.*

b. $Q \setminus \Sigma$ is a PL k-manifold with boundary equal to $P \setminus \Sigma$.

c. $R \setminus \Sigma$ is a PL $(k+1)$-manifold with boundary $\delta R \setminus \Sigma$. Moreover, $Q \setminus \Sigma$ is a properly embedded, codimension-0 submanifold of $\delta R \setminus \Sigma$ with boundary $P \setminus \Sigma$.

Proof: We show first that each of the stated complements of Σ is a PL manifold. Since each complement is an open set, it suffices to show that it contains only manifold points.

By definition, in each case the non-manifold points are contained in a codimension-two subpolyhedron ($S(P)$, $S(Q)$, $S(R)$, as the case may be). Since they form a subcomplex, they must be contained in K^{k-3} (Case a), or L^{k-2} (Case b), or N^{k-1} (Case c). Already this gives the desired result in Case a. In Case b, the only potentially bad points that Σ misses are in $|K^{k-2}| \subseteq P$. However, since $S(Q) \cap P = S(P)$, which has codimension two in P, the points in $|K^{k-2}|$ are all manifold points, disposing of Case b. This argument also shows that in Case c, all points except possibly those in $St(K(k-2), N)$ are manifold points, so it remains to deal with these. First note that all points in $K(k-2)$ are manifold points, since $S(R)$ meets P in a codimension two subpolyhedron. Next, consider any $\sigma \in St(K(k-2), N)$. We claim that $\mathring{\sigma}$ consists entirely of manifold points of R. For by Lemma 5.1, $\mathring{\sigma}$ is either disjoint from or contained in the set of non-manifold points. In the latter case, each of its $(k-2)$-faces would also consist of non-manifold points, which we just showed does not happen. This verifies the claim and concludes the proof that all the complements of Σ are PL manifolds. It remains to verify the remaining statements involving boundaries of these manifolds.

Case a. Choose any $p \in P \setminus \Sigma$, and let $\sigma \in K$ be its carrier, i.e., $p \in \overset{\circ}{\sigma}$. Since $dim\sigma \geq k-2$, there is a $q \in \overset{\circ}{\sigma}$ that is not in $S(P)$, i.e., q is a non-boundary point in the PL manifold $P \setminus S(P)$. Since, as in the argument for Lemma 5.1, there is a PL homeomorphism $P \to P$ that takes p to q, p cannot be a boundary point of $P \setminus \Sigma$.

Case b. If p belongs to $Q \setminus \Sigma$ but not to P, then it belongs to an open simplex $\overset{\circ}{\sigma}$ of dimension k or $(k-1)$ that does not meet P. As before, we find a PL homeomorphism f of Q such that $f(p)$ is in $\overset{\circ}{\sigma}$ and not in $S(Q)$. Therefore, $f(p)$ is in the manifold interior of $Q \setminus S(Q)$, and so p is in the manifold interior of $Q \setminus \Sigma$. This shows that the boundary of $Q \setminus \Sigma$ is contained in $P \setminus \Sigma$. For the reverse inclusion, choose any $p \in P \setminus \Sigma$ and argue as before to find a PL homeomorphism $f : Q \to Q$ keeping p in its carrier in P and mapping it to $q \in P \setminus S(Q)$. By hypothesis, q is a boundary point in $Q \setminus S(Q)$. Hence, p is a boundary point in $Q \setminus \Sigma$.

Case c. The arguments here are similar to the foregoing and will be left to the reader.

This completes our proof.

Remark: We note for future reference that each of the manifolds $P \setminus \Sigma$, $Q \setminus \Sigma$, $R \setminus \Sigma$ is contained in the complement of a skeleton:

- $P \setminus \Sigma \subseteq |K \setminus K^{k-3}|$.
- $Q \setminus \Sigma \subseteq |L \setminus L^{k-3}|$.
- $R \setminus \Sigma \subseteq |N \setminus N^{k-3}|$.

5.3 Smoothing circuits

Smoothing a singular circuit has two parts: smooth the domain (away from a codimension-two subcomplex), and then smooth the map. Similarly with singular circuit bordisms. In this subsection, we show how to smooth the domains.

We continue to deal with the three cases listed in the previous subsection and use the notation introduced there. By a *smoothing* of a PL manifold T we mean a differentiable structure on T, say α, such that the identity map $T \to T_\alpha$ restricts to a smooth embedding of each closed top-dimensional simplex of a triangulation of T. (Here T_α denotes the differentiable manifold with underlying space T and differentiable structure α.) By a *concordance* of smoothings α, β of T, we mean a smoothing of $T \times R$ that coincides with the product smoothing $\alpha \times R$ on $T \times (-\infty, \epsilon)$ and $\beta \times R$ on $T \times (1-\epsilon, \infty)$, for some suitable, small, positive ϵ. We usually abbreviate this by saying that it is a smoothing of $T \times [0,1]$ that coincides with α on $T \times 0$ and β on $T \times 1$. Concordances of smoothings define an equivalence relation on the smoothings of T. This notion has obvious refinements to the relative case.

Proposition 21 *Consider the $(k-1)$-circuit P, the relative k-circuit Q, and the relative nullbordism R of Q discussed in 5.2 above, and also let Σ denote the singular subpolyhedra defined there. Then:*

a. $P \setminus \Sigma$ admits a smoothing, which is unique up to concordance.

b. Every smoothing of $P \setminus \Sigma$ extends to a smoothing of $Q \setminus \Sigma$, which is unique up to relative concordance.

c. Every smoothing of $Q \setminus \Sigma$ extends to a smoothing of $R \setminus \Sigma$, which is unique up to relative concordance.

Proof: The proof in all three cases follows the same pattern.[3] Let (T, S) be one of the three pairs $(P \setminus \Sigma, \emptyset)$, $(Q \setminus \Sigma, P \setminus \Sigma)$, $(R \setminus \Sigma, Q \setminus \Sigma)$. We use the obstruction theory of Hirsch [9] to prove the existence of the stated smoothing. The obstructions in this theory reside in local cohomology groups $H^i(T, S; \Gamma_{i-1})$, with coefficents defined as follows: Γ_n is obtained from the group of all diffeomorphisms $Diff(S^{n-1})$ by factoring out the subgroup of diffeomorphisms that extend to D^n. These groups are known to be finite and abelian for all n, [10], and moreover,

$$\Gamma_n = 0, \text{for all}\, n \leq 6.$$

The proof consists in showing that each pair (T, S) has the homotopy type of a CW pair of small dimension (in particular, at most one, two, and three in the three respective cases). Thus the obstruction groups all vanish identically, and so the desired smoothings exist.

The uniqueness assertions are proved similarly. Here obstructions lie in the groups $H^i(T, S; \Gamma_i)$, which vanish identically.

It remains, then, to check that (T, S) has the stated homotopy type. We do this via three lemmas.

Lemma 5.2 *Suppose that K is a finite simplicial complex of dimension k and S is a subset of K containing the r-skeleton K^r, for some r. Then $|K \setminus S|$ has the homotopy type of a finite CW complex of dimension $\leq k - r - 1$.*

Proof: We begin by proving the result when $S = K^r$. Let K' denote the barycentric subdivision of K, and, for every $\sigma \in K$, let $\hat{\sigma}$ denote the barycenter of σ. For simplexes s and t, write $s < t$ to indicate that s is a proper face of t. Now, for any $\tau \in K$, let

$$\dot{D}(\tau) = \{\hat{\tau}_1 \hat{\tau}_2 \cdots \hat{\tau}_s | \tau < \tau_1 < \tau_2 < \cdots < \tau_s\}.$$

Clearly, $\dot{D}(\tau)$ is a subcomplex of K'. If τ has dimension d, then every simplex of $\dot{D}(\tau)$ has dimension $\leq k - d - 1$. Thus, setting

$$L = \bigcup_{dim \tau = r} \dot{D}(\tau),$$

L is a subcomplex of K' of dimension $\leq k - r - 1$. We show that $|K \setminus K^r| \simeq |L|$ (where " $\simeq''$ denotes homotopy equivalence).

Every simplex of K' decomposes uniquely as a join $\lambda * \mu$, with $\lambda \in (K^r)'$ and $\mu \in L$. (We allow either λ or μ to be empty, so that, for example, if $\lambda = \emptyset$, we get $\mu \in L$, etc.) To see this, write the simplex uniquely as $\hat{\tau_1} \cdots \hat{\tau_s}$, where $\tau_1 < \cdots < \tau_s$. Then $\lambda = \tau_1 \cdots \tau_\ell$ and $\mu = \tau_{\ell+1} \cdots \tau_s$, where ℓ is the maximal subscript i such that $dim \tau_i \leq r$. It follows that we may deform $|K \setminus K^r|$ to $|L|$ by sliding points uniformly along join lines.

[3] An easy direct construction gives a smoothing in Case a (cf., [13], pp.90–91). However, existence in the other cases, as well as the uniqueness assertions, is less transparent. So we give auniform proof for all cases.

We now deal with the simplexes in $S \setminus K^r$. Since each simplex σ of K of dimension $> r$ is a union of join lines beginning in $\sigma \cap |K^r|$ and ending in $\sigma \cap |L|$ and meeting at most at endpoints, and each open join line in the decompositions $\lambda * \mu$ is contained in a unique open simplex of K (indeed of K'), it follows that the deformation described in the last paragraph leaves $|K \setminus S|$ invariant and deforms it to $|L \setminus S|$. Observe that this last is a union of open simplexes of L. The proof is now completed with the following lemma.

Lemma 5.3 *Let B be a finite simplicial complex and $A \subset B$ a non-empty subset consisting of simplexes of dimension $\leq m$. Then $|A|$ has the homotopy type of a finite CW complex of dimension $\leq m$.*

Proof: Let σ be a top-dimensional simplex in A, and let C denote $\{\tau \in A | \tau \leq \sigma\}$. Clearly $|C|$ contracts to the barycenter $\hat{\sigma}$ by sliding points along "radial" lines. So, if $C = A$, we are done. Otherwise, set $A' = A \setminus \{\sigma\}$ and $C' = C \setminus \{\sigma\}$. Note that $|A'|$, $|C|$, and $|C'|$ are closed subsets of $|A|$ satisfying $|A'| \cup |C| = |A|$ and $|A'| \cap |C| = |C'|$. Moreover, by using the join structure coming from barycentric subdivision, it is not hard to show that $|C'|$ is a neigborhood deformation retract (NDR) in both $|A'|$ and $|C|$ so that the inclusion maps $|C'| \hookrightarrow |A'|$ and $|C'| \hookrightarrow |C|$ are cofibrations, (cf. [ ], p.). By induction on the number of simplexes, both $|A'|$ and $|C'|$ are homotopy-equivalent to finite CW complexes of dimension $\leq m$ and $m-1$, respectively, and we have shown that this also holds trivially for $|C|$. Standard homotopy-theoretic arguments now yield the desired result for $|A| = |A'| \cup |C|$. This completes our proof.

In order to apply the foregoing to the case of pairs, we use the following result, whose proof is an easy exercise in homotopy theory and will be omitted.

Lemma 5.4 *Suppose that a space B has the homotopy type of an ℓ-dimensional CW complex, and a space A has the homotopy type of a k-dimensional CW complex. Let $f : A \to B$ be a map with mapping cylinder $M(f)$. Then the pair $(M(f), A)$ has the homotopy type of a CW pair (L, K) such that $dim L = max(k+1, \ell)$.*

Completing the proof of Proposition 21:

Case a. $(T, S) = (P \setminus \Sigma, \emptyset)$. In this case, $P = |K| = |K^{k-1}|$, and $\Sigma = |K^{k-3}|$. By Lemma 5.2, then, $P \setminus \Sigma$ has the homotopy type of a graph, as desired.

Case b. $(T, S) = (Q \setminus \Sigma, P \setminus \Sigma)$. In this case, $Q = |L| = |L^k|$ and $\Sigma \supseteq |L^{k-3}|$ (see the Remark at the end of subsection 5.2). Thus, Lemma 5.2 implies that Q has the homotopy type of a 2-dimensional CW complex. Since $P \setminus \Sigma$ is the boundary of $Q \setminus \Sigma$, it posseses a collar neighborhood. This implies that the pair (Q, P) is homeomorphic to the pair $(M(i), P)$, where i is the inclusion $P \hookrightarrow Q$. Now apply Case 1 (to P) and Lemma 5.4, to conclude that (Q, P) has the homotopy type of a 2-dimensional CW pair.

Case c. $(T, S) = (R \setminus \Sigma, Q \setminus \Sigma)$. In this case $R = |N| = |N^{k+1}|$ and $\Sigma \supseteq |N^{k-3}|$. Therefore, $R \setminus \Sigma$ has the homotopy type of a 3-dimensional CW complex. Arguing as in Case 2, we see that (R, Q) has the homotopy type of a 3-dimensional CW pair.

This completes the proof of Proposition 21.

5.4 Smoothing singular and WPS circuits

In this section we show how to smooth singular circuits and bordisms away from subcomplexes of codimension two. We continue with the notation of the preceding subsections.

Proposition 22 *Let (Q, a) be a relative singular k-circuit in (X, A), and suppose that α is a smoothing of $Q \setminus \Sigma$. Suppose that $a|(P \setminus \Sigma)_\alpha$ is smooth, and choose any $\epsilon > 0$. Then there exists (Q, b), a relative singular k-circuit in (X, A), such that*

a. $b|(Q \setminus \Sigma)_\alpha$ *is smooth.*

b. $b|\Sigma \cup P = a|\Sigma \cup P$.

c. $d(a(x), b(x)) < \epsilon$, *for all* $x \in Q$.

d. $L(b|Q \setminus \Sigma) = b(\Sigma)$ *and* $L(b|P \setminus \Sigma) = a(P \cap \Sigma)$.

Proof: Let $\delta : Q \to [0, \epsilon)$ be a continuous function such that $\Sigma \cup P = \delta^{-1}(0)$. Choose a map $b_1|(Q \setminus \Sigma)_\alpha \to X$ which is a smooth extension of $a|(P \setminus \Sigma)_\alpha$ and a δ-approximation of $a|(Q \setminus \Sigma)_\alpha$. Define $b = (a|\Sigma) \cup b_1$. It is not hard to check that b is continuous. Properties (a) - (c) are true by construction. Assertion (d) is an immediate application of Proposition 3i. This completes the proof.

Corollary 23 *Suppose that the relative singular k-circuit (Q, a) in Proposition 22 is WPS. Then the resulting k-circuit (Q, b) is WPS, and*

a. $\ell d(b|(Q \setminus \Sigma)_\alpha) \leq k - 2$.

b. $\ell d(b|(P \setminus \Sigma)_\alpha) \leq k - 3$.

Proof: That (Q, b) is WPS when (Q, a) is follows immediately from the construction. The assertions concerning limit dimension follow immediately from assertion d of the proposition, together with the definition of the singular set Σ given above Proposition 20. This completes the proof.

The following variation will be useful later.

Corollary 24 *Let (Q, a) be a relative singular k-circuit in (X, A), and let Σ be as before. Then there exists a smoothing α of $Q \setminus \Sigma$ and a relative WPS k-circuit (Q, b) in (X, A) such that*

a. $b|(Q \setminus \Sigma)_\alpha$ *is smooth.*

b. (Q, b) *is bordant to* (Q, a)

c. $L(b|Q \setminus \Sigma) = b(\Sigma)$ *and* $L(b|P \setminus \Sigma) = b(P \cap \Sigma)$.

Proof: First smooth Q by applying Proposition 21 a,b. Denote the smoothing by α. Set $a' = a|P$ and apply Proposition 22 to (P, a') to get a map b' that ϵ-approximates a', and is smooth on $(P \setminus \Sigma)_\alpha$. For ϵ small enough, we can obtain b' homotopic to a'. Consequently, by the homotopy extension property, b' extends to a continuous map b'' on Q, which is homotopic to a as a map of pairs. Now apply Proposition 22 and Corollary 23 to (Q, b'') to obtain a WPS map $b : (Q, P) \to (X, A)$ that extends a', approximates b'', and is smooth on $(Q\setminus\Sigma)_\alpha$. We choose the approximation good enough to insure that b is homotopic to b'', hence homotopic to a. The first assertion of the corollary is now true by construction, and the third is true by the argument previously given. Finally, the second assertion holds by Rourke and Sanderson's theorem (Theorem 4), because a and b are homotopic, implying $a_*([Q]) = b_*([Q])$. This completes the proof.

Closely similar arguments prove the next two results. We leave details to the reader.

Proposition 25 *Let a constant $\epsilon > 0$ be given, and let (R, c) be a singular circuit null bordism in (X, A) of the singular relative k-circuit (Q, b). Let Σ be the exceptional or singular set for R as before. Finally, assume that α is a smoothing of $Q \setminus \Sigma$ such that $b|(Q \setminus \Sigma)_\alpha$ is smooth. Then there is a smoothing β of $R \setminus \Sigma$ extending α and a continuous map $d : R \to X$ such that*

a. $d|(R \setminus \Sigma)_\beta$ is smooth.

b. $d|\Sigma \cup Q = c|\Sigma \cup Q$.

c. d is an ϵ-approximation of c.

d. Any such d necessarily satisfies

$$L(d|R \setminus \Sigma) = d(\Sigma)$$

and

$$L(d|\overline{(\delta R \setminus \Sigma) \setminus (Q \setminus \Sigma)}) = c(\Sigma \cap \overline{\delta R \setminus Q}),$$

where the first closure is relative to $\delta R \setminus \Sigma$ and the second relative to δR.

Corollary 26 *Suppose that the nullbordism (R, c) of the preceding proposition is WPS. Then the resulting nullbordism (R, d) is WPS, and*

a. $\ell d(d|(R \setminus \Sigma)_\beta) \leq k - 1$

b. $\ell d(d|\overline{(\delta R \setminus \Sigma)_\beta \setminus (Q \setminus \Sigma)_\beta}) \leq k - 2$.

It is clear that the maps obtained in Corollary 24 and Corollary 26 are, respectively a pseudocycle in (X, A) and a nullbordism of pseudocycles in (X, A). We are finally prepared, therefore, to construct the desired homomorphism $\psi : H_*(X, A) \to \Psi_*(X, A)$.

6 The natural equivalence $\psi : H_* \to \Psi_*$

Let (X, A) be a smooth manifold pair in $\mathcal{C}^{prop}$, and choose any class $z \in H_k(X, A)$. According to the theorem of Rourke and Sanderson (Theorem 4, §5), we may represent z by a unique bordism class of singular k-circuits in (X, A), say (Q, a). Choose any triangulation (L, K) of $(Q, P) = (Q, \delta Q)$ and define the subpolyhedron Σ of Q, all as in the preceding subsections. Corollary 24 produces a WPS k-circuit (Q, b) bordant to (Q, a), and a smoothing α of $Q \setminus \Sigma$ such that $b|(Q \setminus \Sigma)_\alpha$ is smooth. Further, Corollary 23 shows that $((Q \setminus \Sigma)_\alpha, b|)$ satisfies the limit dimension constraints required of a pseudocycle.

Definition 12 *We define $\psi(z)$ to be the Ψ_k-bordism class of this pseudocycle.*

Proposition 27 *ψ is a well-defined homomorphism. Furthermore, ψ is natural with respect to smooth proper maps of manifold pairs, and it commutes with boundary maps of long exact sequences.*

Proof: This follows from the results of §5. Thus, given z as above, different choices leading to the definition of $\psi(z)$ would yield, say, a WPS relative k-circuit $(\tilde{Q}, \tilde{b})$ representing z and a corresponding smooth relative Ψ_k-cycle $((\tilde{Q} \setminus \tilde{\Sigma})_{\tilde{\alpha}}, \tilde{b}|(\tilde{Q} \setminus \tilde{\Sigma})_{\tilde{\alpha}})$. This k-circuit is bordant to the original (as circuits). Hence, we may use Proposition 26 to obtain a smooth Ψ_k-bordism between $((Q \setminus \Sigma)_\alpha, b|(Q \setminus \Sigma)_\alpha)$ and $((\tilde{Q} \setminus \tilde{\Sigma})_{\tilde{\alpha}}, \tilde{b}|(\tilde{Q} \setminus \tilde{\Sigma})_{\tilde{\alpha}})$, showing that $\psi(z)$ is well-defined. We leave details to the reader. That ψ is a homomorphism follows immediately from the definitions.

To see that ψ is natural with respect to smooth maps of compact manifold pairs, suppose that $f : (X, A) \to (Y, B)$ is such a map, and let z be as before. Using the same notation as above, we have seen that z may be represented by a WPS relative k-circuit (Q, b) such that the restriction of b to the smooth manifold $(Q \setminus \Sigma)_\alpha$ is a smooth map of manifold pairs $((Q \setminus \Sigma)_\alpha, (P \setminus \Sigma)_\alpha) \to (X, A)$. This restriction represents $\psi(z)$. By definition, $((Q \setminus \Sigma)_\alpha, f \circ (b|(Q \setminus \Sigma)_\alpha))$ represents $f_*(\psi(z))$. Also by definition $(Q, f \circ b)$ represents $f_*(z)$. But $f \circ b|(Q \setminus \Sigma)_\alpha$ is a smooth map of pairs, which, by the discussion above and the first part of this proof, may be taken to represent $\psi(f_*(z))$. Therefore, $f_*(\psi(z)) = \psi(f_*(z))$, as required.

The proof that ψ commutes with boundary maps of long exact sequences follows along similar lines. Start with z and (Q, b) as in the previous paragraph, obtaining $((Q \setminus \Sigma)_\alpha, b|(Q \setminus \Sigma)_\alpha)$ representing $\psi(z)$. Then $(P, b|P) = (\delta Q, b|\delta Q)$ represents $\partial(z)$, and, by the first paragraph of the proof, $((P \setminus \Sigma)_\alpha, b|(P \setminus \Sigma)_\alpha)$ represents $\psi(\partial(z))$. On the other hand, this same pair clearly represents $\partial(\psi(z))$.

This completes the proof of the proposition.

The following corollary summarizes some of what we have shown here and in Section 5.

Corollary 28 *$\psi : H_* \to \Psi_*$ is a natural transformation of functors on $\mathcal{C}^{prop}$, and it induces maps of long exact sequences.*

Proposition 29 *$\psi : H_*(pt) \to \Psi_*(pt)$ is the canonical isomorphism described in Proposition 13.*

We leave this easy calculation to the reader.

Proposition 30 *ψ is a natural equivalence of functors on $\mathcal{C}^{prop}_{hand}$.*

Proof: Begin by proving that ψ is an isomorphism for the following pairs: $(D^k, \emptyset)$, (D^k, S^{k-1}), $(S^k, \emptyset)$, $(D^k \times D^\ell, S^{k-1} \times D^\ell)$. This parallels the usual inductive homology computation for these pairs (cf. Proposition 14), using the Five Lemma at each stage. Since both homology and pseudohomology map finite disjoint unions to direct sums, ψ is an isomorphism for all pairs that can be obtained from those listed by taking finite disjoint unions.

Now choose (X, A) in $\mathcal{C}^{prop}_{hand}$, where X has dimension n. By definition, there is a nested sequence of codimension-0 submanifolds of X, $\{X_i\}$, $-1 \leq i \leq n$, such that, X_{-1} is a tubular neighborhood of A, $X = X_n$, and each X_i smoothly deformation retracts to X_{i-1} together with finite number of disjointly attached i-handles. The results of the previous paragraph, together with the excision and homotopy properties for homology and pseudohomology, imply that ψ is an isomorphism for each pair (X_i, X_{i-1}). (Cf. the comment after Corollary 12 in §4.) The desired result then follows inductively by applying the Five Lemma to the long-exact homology and pseudohomology sequences of the triples (X_i, X_{i-1}, A).

Proposition 31 *Any natural transformation $H_* \to \Psi_*$ on $\mathcal{C}^{prop}$ that induces maps of long exact sequences and equals the canonical isomorphism at pt coincides with ψ on $\mathcal{C}^{prop}_{hand}$.*

Proof: Let (X, A) and $\{X_i\}$ be as in the preceding proof. Define chain complexes $\mathbf{C}^{\Psi}_*$ and $\mathbf{C}^{H}_*$ as follows: $\mathbf{C}^{\Psi}_i(X, A) = \Psi_i(X_i, X_{i-1})$; the boundary map ∂_i is given by the appropriate connecting homomorphism in the long exact Ψ_*-sequence for the triple (X_i, X_{i-1}, X_{i-2}). Similarly for $\mathbf{C}^{H}_*$. Let $\mathbf{H}^{\Psi}_*$ and $\mathbf{H}_*$ denote the corresponding homology groups. It is well known that $\mathbf{H}_*$ is naturally equivalent to ordinary (singular) homology. The same relation holds between $\mathbf{H}^{\Psi}_*$ and Ψ_*. It will be convenient for us to make this explicit, which we do in the case of Ψ_*. A precise analog holds for H_*. So, consider the following sequence of homomorphisms:

$$\Psi_i(X, A) \xrightarrow{j_*} \Psi_i(X, X_{-1}) \xleftarrow{k_*} \Psi_i(X_i, X_{-1}) \xrightarrow{\ell_*} \Psi_i(X_i, X_{i-1}) \xrightarrow{\partial} \Psi_{i-1}(X_{i-1}, X_{i-2}),$$

and

$$ker(\partial) \xrightarrow{\pi} \mathbf{H}^{\Psi}{}_i(X, A) \to 0,$$

where j_*, k_* and ℓ_* are induced by inclusion, and π is the obvious projection. Note that j_* is an isomorphism, k_* is onto, and $im(\ell_*) \subseteq ker(\partial)$. In fact $\partial = m_*\partial'$, where m_* is the injection $\Psi_{i-1}(X_{i-1}, X_{-1}) \to \Psi_{i-1}(X_{i-1}, X_{i-2})$ induced by inclusion, and ∂' is the boundary homomorphism in the long exact sequence of the triple (X_i, X_{i-1}, X_{-1}). It follows that $im(\ell_*) = ker(\partial)$. Further, k_* factors through the inclusion-induced isomorphism $\Psi_i(X_{i+1}, X_{-1}) \to \Psi_i(X, X_{-1}))$, so that

$$\ell_*(ker(k_*) = \partial(\Psi_{i+1}(X_{i+1}, X_i)) = B_i(\mathbf{C}^{\Psi}_*(X, A)) \subseteq ker(\partial).$$

Together, these observations give a proof that the relation $\pi\ell_*(k_*)^{-1}j_*$ defines an isomorphism

$$\Psi_i(X, A) \to \mathbf{H}^{\Psi}{}_i(X, A).$$

To deal with naturality, we introduce the notion of a *handle-adapted* map. Suppose that (X, A) and (Y, B) are smooth manifold pairs equipped with handle decompositions $\{X_i\}$ and $\{Y_i\}$, respectively. If $i > dimX = m$, we set $X_i = X_m$, and similarly for Y_i. We then say that a smooth map of pairs $f : (X, A) \to (Y, B)$ is *handle-adapted* (with respect to the given decompositions) if $f(X_i) \subseteq Y_i$, for all i. A similar definition applies to homotopies. We make use of the following lemma, which is proved in virtually the same way as the analogue for cellular maps of CW complexes (e.g., see [21], p. 404).

Lemma 6.1 *Suppose that f is a smooth, proper map of smooth manifold pairs equipped with handle decompositions. Then f is properly and smoothly homotopic to a handle-adapted map. Moreover, if two handle-adapted maps are properly and smoothly homotopic, then the homotopy may be chosen to be handle-adapted.*

With the aid of this lemma, it is now easy to see that the chain complexes we have defined extend to functors on $\mathcal{C}_{hand}^{prop}$ and that the isomorphism $\Psi_i(X,A) \to \mathbf{H}^{\Psi}{}_i(X,A)$ is natural with respect to smooth maps. Of course this is clear in the case of homology.

Now any natural transformation $H_* \to \Psi_*$ that induces maps of long exact sequences as hypothesized will induce a map of chain functors $\mathbf{C}_*^H \to \mathbf{C}_*^{\Psi}$. Moreover, if such a transformation equals the canonical isomorphism at *pt*, it is easy to see that it must induce the same map of chain functors as ψ. Hence, it induces the same isomorphism of corresponding homology functors $\mathbf{H}_* \to \mathbf{H}_*^{\Psi}$ as ψ.

The desired conclusion now follows from the fact that both the transformation in question and ψ commute with the homomorphisms $j_*,\ k_*, \ell_*,\ \partial$, and π used above to define the isomorphisms

$$H_*(X,A) \to \mathbf{H}_*(X,A).$$

and

$$\Psi_*(X,A) \to \mathbf{H}_*^{\Psi}(X,A).$$

This completes the proof.

Appendix on Corners

Corners appear naturally in differential topology when, for example, one takes the product of manifolds with non-empty boundaries, or, for another example, when a manifold with non-empty boundary is separated by a codimension-one submanifold that meets the boundary transversally. Most of the time, such corners can be "straightened" or "rounded" on a case by case basis without great inconvenience to the construction at hand. (See [15], [6], or the discussion below for a description of angle straightening. Note, as described later, we distinguish between straightening and rounding.) As a result, corners are rarely considered in the literature (e.g., see [11], p.3). However, there are contexts in which a more systematic approach to corners is desirable, as for example when dealing with spaces of embeddings. J.Cerf gave the first such account in [3] for exactly this purpose. Some aspects of Cerf's treatment were amplified and extended in the Cartan Seminar lectures of A. Douady [6]. Since that time, the only other systematic treatment that I have been able to find is in the unpublished e-manuscript of R. Melrose [14].

A Smooth manifold pairs

We take as known all the usual definitions involving smooth manifolds with boundary. For all integers $m \geq 0$ and all finite sequences $I = (i_1, \ldots, i_k)$ such that $0 \leq k \leq m$ and $i_1 < i_2 < \ldots < i_k$— we allow the empty sequence $\emptyset$, in which case $k = 0$— define $R^m_I = \{x = (x_1, \ldots, x_m) \in R^m | x_i \geq 0$, for all $i \in I\}$. We may use the notation $|I|$ to denote k. These are our model spaces. Note that $R^m = R^m_\emptyset$. If U is open in R^m_I and $h = (h^1, \ldots, h^n) : U \to R^n$ is a map, then we say that it is smooth if each coordinate function h^i extends to a C^∞ function on an open subset of R^m. This leads to the definitions of diffeomorphisms, smooth charts, smooth atlases, and differentiable structures in the usual way. Note that if V is open in R^n_J and $h : U \to V$ is a diffeomorphism, then $m = n$, but $|I|$ is not necessarily equal to $|J|$ unless $0 \in U$ and $h(0) = 0$.

Definition 13 *a. A smooth m-manifold with corners (or smooth m-manifold for short) is a second-countable Hausdorff space X together with a differentiable structure $\{(U_\alpha, h_\alpha)\}$ for which the image of each chart is an open subset of some R^m_I, m fixed, I variable. Topologically, X is an m-manifold with boundary.*

b. Let X be a smooth m-manifold and n a non-negative integer $\leq m$. A subset $A \subseteq X$ is called a smooth n-dimensional submanifold of X if every $a \in A$ is contained in a smooth chart (U, h) for X such that $h(U \cap A) = h(U) \cap R^n_J$, for some J. Such an A inherits a topology and a smooth atlas—hence a differentiable structure—from X, making it into a smooth n-manifold.

c. A smooth manifold pair (X, A) is a topological pair such that A is a closed subset of X, X and A are smooth manifolds, and the differentiable structure of A is inherited from that of X. Note that in this case the inclusion $A \hookrightarrow X$ is a proper, smooth embedding.

B Corner indices and corner sets

Fix the integer m, and choose any $p \in R^m_I$. The *corner index* of p is the smallest integer ℓ such that p has a neighborhood U in R^m_I diffeomorphic to an open subset of R^m_J with $\ell = |J|$. Thus, for example, the origin of R^m_I has corner index $|I|$ and $(1, 1, \ldots, 1)$ has corner index 0. If U and V are open subsets of R^m_I and R^m_J, respectively, and $h : U \to V$ is a diffeomorphism, it is easy to show that h preserves the corner indices of points. It follows that we can well-define the corner index of a point in a smooth manifold. We call the maximal corner index of all points in X the *corner index* of X. When there is no possibility of confusion, we shall omit the adjective "corner." It is easy to see that the corner index of a point p equals $|I|$ for any smooth coordinate chart (U, h) around p such that $h(p) = 0$ and $h(U)$ open in R^m_I.

In general, the points of index ≥ 1 in X comprise the topological boundary of X, a closed set denoted bX. The complement $X \setminus bX$ is called the *manifold interior of* X and is denoted $intX$. The points of index ≥ 2 comprise a closed subset $cX \subseteq bX$, called the corner set of X, (or the set of corners of X). $X \setminus cX$ and $bX \setminus cX$ are smooth submanifolds of X of index ≤ 1, and clearly $b(X \setminus cX) = bX \setminus cX$. The components of $bX \setminus cX$ are called *open faces* of X. If F is an open face, it may happen that its closure $\overline{F}$ is a smooth codimension-one submanifold of X, in which case we call $\overline{F}$ a *closed face* of X. However, the example of one lobe of a lemniscate [3] (including its interior), shows that not every open face is the interior of a closed face.

When X has index 0, then X is a standard smooth manifold with empty boundary. When X has index 1, then it is a standard smooth manifold with non-empty boundary. When X has index 2, then cX is a codimension-one submanifold of bX with empty boundary and is a codimension-two smooth submanifold of X.

Let A be a connected, codimension-one submanifold of X of corner index ≤ 1. It is not hard to show that the intersection $intA \cap bX$ is both closed and open in $intA$ and does not meet cX. Thus, either $intA$ is contained in an open face of X or $intA \subseteq intX$. In the latter case, either $A \subseteq intX$ or A meets $bX \setminus cX$ transversally in a union of components of bA.

C Straightening angles, introducing corners, and gluing

The simplest way in which corners appear is in the product of two manifolds with boundary $X \times Y$, in which case $b(X \times Y) = bX \times Y \cup X \times bY$, and $c(X \times Y) = bX \times bY$. In this case, the closure of every open face is a closed face. For example, when X, Y, bX, and bY are connected, the closed faces of $X \times Y$ are $bX \times Y$ and $X \times bY$. Using smooth collars $bX \times [0, 1) \to X$ and $bY \times [0, 1) \to Y$, we see that $c(X \times Y)$ has a neighborhood N diffeomorphic to $c(X \times Y) \times R^2_2$. We identify N with this product. Milnor's construction of *straightening angles (or corners)*, [15], [6], sends N homeomorphically to $c(X \times Y) \times R^2_1$ via the rule $(x, \rho, \theta) \mapsto (x, \rho, 2\theta)$, in which polar coordinates are used for the second and third entries. This homeomorphism h is then used to endow N with a diferentiable structure, whereas $X \times Y \setminus c(X \times Y)$ keeps the structure induced from that of $X \times Y$. These are compatible where they are jointly defined, so they determine a differentiable structure, say α, on $X \times Y$. We denote $X \times Y$ with this structure $(X \times Y)_\alpha$ and say that $(X \times Y)_\alpha$ is obtained from $X \times Y$ by straightening angles (or corners).

Note that $h|\{p \in N|\theta(p) = 0\}$ is a smooth embedding. Similarly, for $h|\{p \in N|\theta(p) = \pi/2\}$. Therefore, the closed faces of $X \times Y$ inherit the same differentiable structures from $X \times Y$ as from $(X \times Y)_\alpha$.

This procedure may be extended to all X of index ≤ 2 by making use of the relevant tubular neighborhood theory (see [6]). The important point is that the original inherited structures on

$X \setminus cX$, $bX \setminus cX$, and cX remain unchanged, and similarly for the structures on the closed faces of X. All that changes is the structure for X at points on cX.

The procedure of straightening corners can be reversed. Thus, suppose that X is a smooth manifold with non-empty boundary and (perhaps) corners. Choose a connected codimension-one, properly embedded submanifold B of X contained in an open face of bX and let C be a component of bB. Then C has a tubular neighborhood in X with fibre R^2_1. As shown in [6], its differentiable structure can be replaced by that of a corresponding tube with fibre R^2_2 by "halving" angles. This *introduces* a corner at C. This procedure is useful when gluing two manifolds together along a codimension-0 submanifold of their boundaries, as we now describe in a special case.

Suppose that A and B are two smooth m-manifolds and that E and F are two diffeomorphic $(m-1)$-manifolds of index 1 which are smooth, properly embedded submanifolds of A and B, respectively, with $E \subseteq bA$ and $F \subseteq bB$. E meets the corner set cA in a disjoint union of components of bE, in which all points have index 2. Introduce corners at every component not meeting cA,calling the resulting manifold A again. Do the same for F and B. bE and bF become a union of components of cA and cB with each point having index 2 in A and B, respectively. E and F become closed faces of A and B, respectively.

Now in $A \sqcup B$, identify E and F via the given diffeomorphism, calling the result $A +_{E=F} B$, or $A + B$, for short. Let C denote the common image of E and F in $A + B$, and let D denote the common image of bE and bF. It remains to give $A + B$ the structure of a smooth manifold compatible with the structures of A and B. For this, we make use of collar neighborhoods of E and F in A and B and quarter-disc tubular neighborhoods of bE and bF. Restricting the collars to $intE$ and $intF$, these fit together in the standard way to form a product neighborhood of $intC$ in $A + B$, which is given the product smoothing. The quarter-disc tubular neighborhoods fit together to form a half-disc tubular neighborhood, which we use to smooth a neighborhood of D. Finally, use the structures on $A \setminus E$ and $B \setminus F$ to smooth the complement $A + B \setminus C$. These all fit together to give a smooth atlas for $A + B$. The resulting smooth manifold contains A and B as smooth submanifolds.

D Rounding corners

The process of straightening angles does not change the underlying topological manifold, but when we are dealing with *submanifolds* with corners, it might be necessary to do so. We describe one method for doing this. Assume that X is compact for simplicity. Let A be a submanifold of X of codimension zero and having corner index ≤ 2. Let C be an open, connected subset of cA over which the normal, orthogonal R^2_2-bundle in A is trivial. Choose a fixed trivialization, and use it to identify the normal bundle with $C \times R^2_2$. Choose any $\epsilon > 0$, and let $\phi : [0, \infty) \to [0, \infty)$ be a smooth function satisfying the following conditions:

- $\phi'(t) < 0$, for $t \in (0, \epsilon)$.
- $\phi^{(n)}(\epsilon) = 0$, for all $n \geq 0$.
- $\phi(t) = 0$, for all $t \geq \epsilon$.
- $\phi(t) = \phi^{-1}(t)$, for $0 \leq t \leq \epsilon$.

Now define $W_\epsilon = \{(x,s,t) \in C \times R_2^2$ such that $t < \phi(s)\}$, regarded as a subset of A via the previously mentioned trivialization. *A priori* this set would appear to depend on the trivialization. However, as shown in [6], the group of the orthogonal normal bundle of cX is Z_2, generated by the automorphism of R_2^2 that exchanges the factors. This induces a self-bundle-map of $C \times R_2^2$ that leaves W_ϵ invariant. Thus, given a component K of cA, we can cover its normal bundle by trivializations and then piece together copies of the corresponding W_ϵ's that fit together smoothly to obtain a subset of the normal bundle that we call Y_ϵ. Set $r(A) = A \setminus Y_\epsilon$. This eliminates the corner of A at K and is called the 'rounding' of A at K. Sometimes we delete reference to ϵ or K, and sometimes we take K to be a union of components of cA. It is not hard to show, as in Lemma A.1, that there is a smooth deformation retraction $(X, A) \to (X, r(A))$. Moreover $r(A)$ is diffeomorphic to A with angles straightened at K.

This procedure can be applied in the following situation. Let f and g be smooth, real-valued functions on X such that:

- 0 is a regular value of $f, g, f|bX$, and $g|bX$;
- $(0,0)$ is a regular value of $F = (f,g)$;
- $F^{-1}(0,0) \cap bX = \emptyset$.

Set $A = f^{-1}[0,\infty), B = f^{-1}(-\infty,0], C = g^{-1}[0,\infty), D = g^{-1}(-\infty,0]$. Then both $F^{-1}(R_2^2) = A \cap C$ and $F^{-1}(-R_2^2) = B \cap D$ are smooth, codimension-0 submanifolds with corners at $F^{-1}(0,0)$ in $intX$. We can round these by the above procedure. Furthermore, $\overline{X \setminus r(B \cap D)}$ is a smooth submanifold of X coinciding with $A \cup C$ outside a neighborhood of $F^{-1}(0,0)$ and deformation retracting onto $A \cup C$. By abuse of notation we denote it by $r(A \cup C)$.

Now consider a handle H attached to a submanifold W in X, supposing that $cW \subseteq bX$ and $H \subseteq intX$. Let g be a smooth real-valued function that has regular value 0 and $W = g^{-1}[0,\infty)$. Extend the handle H slightly into W and round the result, obtaining a smooth codimension-0 manifold K in $intX$ that equals H outside of W and has boundary transverse to bW. Let f be a smooth real-valued function with regular value 0 and $K = f^{-1}[0,\infty)$. Then f and g have the properties above and $W \cup K = W \cup H$. The manifold $r(W \cup K)$ is what we mean when we speak of attaching the handle H to W (in X) and rounding the result.

If A is a submanifold of X, and B is a handle attached to A in X, it is straightforward to show that $r(A \cup B)$ is diffeomorphic to $A + B$.

E Smoothing maps

In this paper, corners are encountered in two settings: in domains of smooth maps and in their targets. We deal first with corners in the domain.

Lemma E.1 *Suppose that we are given smooth manifolds X and Y, a closed subset A of X, a continuous map $f : X \to Y$ that is smooth in a neighborhood of A, and a continuous function $\epsilon : X \to (0,1]$. Suppose further that Y is endowed with a metric. Then there exists a smooth map $g : X \to Y$ that is an ϵ-approximation of f such that $g = f$ in a neighborhood of A and $g =_\infty f$.*

Proof: Leaving aside the last condition asserted, the result is completely standard in the case of smooth manifolds without corners. The standard proof carries over without essential change to the general case. It remains to show that the last condition can be satisfied. For this, we suppose that X is non-compact—since otherwise the assertion is vacuously true—and we construct a continuous function $\delta : X \to (0, 1]$ satisfying $\delta =_\infty 0$ and $\delta \leq \epsilon$. Then, we use the earlier statements in the lemma to find a smooth δ-approximation g of f that equals f in a neighborhood of A. It is easy to check that $g =_\infty f$. This completes the proof.

It is useful to have a slight variant of this result.

Corollary E.2 *Let X, Y, A, and ϵ be as in the lemma above, and suppose additionally that A is a codimension-zero submanifold of X. If $f : X \to Y$ is a continuous map such that $f|A$ is smooth, then there exists a smooth map $g : X \to Y$ that is an ϵ-approximation of f such that $g|A = f|A$ and $g =_\infty f$.*

Proof: Results of [3] imply that there is a smooth self-map h of X that is arbitrarily close to the identity and retracts a neighborhood of A onto A. The approximation may be chosen so close that fh is a $\delta/2$-approximation of f, where δ is as in the proof of the above lemma. Clearly fh is smooth in a neighborhood of A and $fh|A = f|A$. Apply the lemma to find an $\delta/2$-approximation g of fh that equals fh in a neighborhood of A. It is easy to check that g has the desired properties. This completes the proof.

Corollary E.3 *Let X and Y be a smooth manifolds endowed with metrics, and suppose that X has corner index 2. Let α be a differentiable structure on X obtained by smoothing corners as described in the previous subsection. Let $f : X \to Y$ be smooth and $\epsilon : X \to (0, 1]$ any continuous function. Then there is a smooth $g : X_\alpha \to Y$ such that*

a. g is an ϵ-approximation of f.

b. $g = f$ outside any prescribed neighborhood of cX.

c. $g =_\infty f$.

Proof: Let N be a prescribed neighborhood of cX, and let $T \subseteq N$ be a closed tubular neighborhood of cX in X_α. Set $A = \overline{X \setminus T}$, and apply the previous lemma.

Remark: Note that item (c) above implies that $L(g) = L(f)$ (Lemma 2.1).

It takes a bit more work to prove the following relative version of this corollary.

Lemma E.4 *We continue with the notation and hypotheses of Corollary E.3 and suppose additionally that W is a disjoint union of closed faces of bX. Then the conclusion of Corollary E.3 holds, and furthermore, g may be chosen to coincide with f on W.*

Proof (sketch): Since Corollary E.3 already gives the desired result outside a neighborhood of cX and the only possible problem occurs near bW, we may assume WLOG that X is a neighborhood of bW, indeed, after corners are smoothed, that $X = bW \times R^2_1$, with the notation chosen so that W is identified with $bW \times [0, \infty) \times 0$. Let T be a tubular neighborhood of cX as in the proof of Corollary E.3 above, and let T' be the union of those components of T that meet bW, rounding corners so that the fibres of the bundle projection $T' \to bW$ are smooth discs which intersect bX in closed line segments forming a 1-disc bundle over bW. Let $\delta : X \to (0,1)$ be an arbitrary, continuous function. Finally, set $B = \overline{X \setminus T'} \cup W$.

We claim that there is a smooth map $h : X \to X$ with the following properties:

- $h = id_X$ on B.
- h is a δ-approximation of id_X.
- $h^{-1}(B)$ is a neighborhood of B.

We leave the construction of such an h to the reader. The function δ may be chosen so small that $f =_\infty fh$ and that fh is an $\epsilon/2$-approximation of f. Moreover, fh is smooth in a neighborhood of B and coincides with f on B. We then apply Lemma E.1 to fh to obtain a smooth $\epsilon/2$-approximation g that equals fh at infinity and coincides with fh—hence with f—on a neighborhood of B. g is the desired ϵ-approximation of f.

This completes the sketch of the proof.

The following result is an immediate consequence of the foregoing.

Corollary E.5 *Let X be a smooth manifold of index ≤ 1, let A be a codimension-zero submanifold of $X \setminus bX$, closed as a subset of X, and let $H : X \times [0,1] \to Y$ be a smooth map. Smooth the corners of $X \times [0,1]$ via a differentiable structure β, and endow $X \times [0,1]$ and Y with metrics. Let $\epsilon : X \times [0,1] \to (0,1]$ be a continuous function. Then, there exists a smooth ϵ-approximation of H, $G : (X \times [0,1])_\beta \to Y$, which is equal to H at infinity and equals H on $(X \times 0) \cup A \times [0,1] \cup (X \times 1)$.*

We now return to Corollary E.2, and we continue to use the notation of that result and its proof. Let β be the differentiable structure on $X_\alpha \times [0,1]$ obtained by straightening the angles at the corner $b(X_\alpha) \times \{0,1\}$. As already remarked, the closed faces $X_\alpha \times \{0,1\}$ and $bX_\alpha \times [0,1]$ keep their original differentiable structures in $(X_\alpha \times [0,1])_\beta$.

We are interested in the relationship between two close, smooth approximations of the map f. It is a standard fact in algebraic topology that when the approximation is close enough, the two maps are homotopic, and this translates without difficulty to differential topology. Here, we take into account the smoothing of corners, as well as what happens at infinity.

Lemma E.6 *Let X, Y, A, f, ϵ be as in Corollary E.2. Then, we may choose $\delta \leq \epsilon$ such that if $g_0, g_1 : X_\alpha \to Y$ are smooth δ-approximations of f extending $f|A$, there is a smooth map $G : (X_\alpha \times [0,1])_\beta \to Y$ which satisfies*

a. $G(x,i) = g_i(x)$, for all x and for $i = 0, 1$.

b. G is a $2\delta \circ pr$-approximation of $f \circ pr$.

c. $G|A \times [0,1] = f \circ pr|A \times [0,1]$.

d. $G =_\infty f \circ pr$.

Proof (Sketch): Corollary E.5 shows that we need only find a smooth homotopy rel$A$, $X_\alpha \times [0,1] \to Y$, between g_0 and g_1 that is suitably close to $f \circ pr$. There are two well-known proofs of the classical analogue. One is inductive, proceeding stepwise over a countable union of coordinate neighborhoods. The other uses differential geometry. Both methods can be extended to the case of manifolds with corners. In particular, as is shown in [6], connections can be defined for manifolds with corners in such a way as to respect the corner structure. Thus, geodesics are defined in the usual way, as well as convex, normal geodesic neighborhoods. Thus, two close approximations to f can be deformed to one another by sliding along uniquely determined geodesic arcs. Wherever the maps g_0 and g_1 coincide, the homotopy is stationary. This completes the sketch.

References

[1] Th. Bröcker, *Differentiable germs and catastrophes*, LMS Lect. Note Series 17, Camb. Univ. Press (1975).

[2] S. Buoncristiano, C. Rourke, and B. Sanderson, *A geometric approach to homology theory*, LMS Lect. Note Series 18, Camb. Univ. Press (1976)

[3] J. Cerf, Topologie de certains espaces de plongments, *Bull. Soc. math. France* **89** (1961), 227–380.

[4] P. Conner and E. Floyd, *Differentiable periodic maps*, Springer (1964).

[5] A. Dold, *Lectures on algebraic topology*, Springer (1972).

[6] A Douady, Variétés à bord anguleux et voisinages tubulaires, *Séminaire Henri Cartan* **14** (1961/62).

[7] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton University Press (1952).

[8] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985), 307–347.

[9] M. Hirsch, Obstruction theories for smoothing manifolds and maps, *Bull. Amer. math. Soc.* **69** (1963), 352–356.

[10] M. Kervaire and J. Milnor, Groups of homotopy spheres I, *Ann. Math.* **77** (1963), 504-537.

[11] R. Kirby, *The topology of 4-manifolds*, Springer (Lecture Notes in Math. No. 1374) (1989).

[12] S. Lefschetz, *Topology*, Chelsea, New York (1956).

[13] D. McDuff and D. Salamon, *J-holomorphic Curves and Quantum Cohomology*, University Lecture Series, Vol. 6, AMS (1994).

[14] R. Melrose, *Differential analysis on manifolds with corners*, e-preprint MIT (1996).

[15] J. Milnor, *Differentiable manifolds which are homotopy spheres*, preprint, Princeton U. (1959)

[16] J. Milnor, *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann and J. Sondow, Princeton University Press (1965).

[17] J. Milnor, *Topology from the differentiable viewpoint*, Notes by D. Weaver, The University Press of Virginia (1965)

[18] C. Rourke and B. Sanderson, *Introduction to piecewise-linear topology*, Springer (1972).

[19] Y. Ruan and G. Tian, A mathematical theory of quantum cohomology, *J. Diff. Geom.* **42**, No. 2 (1995), 259–367.

[20] D. Salamon, Lectures on Floer homology, in *Symplectic geometry and topology*, ed. Y. Eliashberg and L. Traynor, IAS/Park Cit mathematical Series Vol. 7, Amer. Math. Soc./Inst. for Adv. Study (1999).

[21] E. Spanier, *Algebraic topology*, McGraw-Hill (1966).

[22] H. Seifert and WS. Threlfall, *A textbook of topology*, Academic Press (1980).

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