An Analog of the May-Milgram Model for Configurations with Multiplicities

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To Jim Milgram on his sixtieth birthday

Abstract. We point out a generalization of the May-Milgram and Segal models for iterated loop spaces (and their mapping space analog given by F. Cohen and C.F. Bodigheimer). We phrase our result in terms of labeled configuration spaces of "bounded multiplicity" and in so doing we answer a question of Carlsson and Milgram posed in the handbook. We relate this labeled construction to a theory of Lesh and as a result also obtain a generalization of a theorem of Quillen, Barratt and Priddy. Finally we point out that the stable splittings that occur in the classical case do not persist when higher multiplicities are allowed.

§1. Introduction

Ever since the work of Milgram on iterated loop spaces [M] and its generalization by May [Ma], the May-Milgram configuration space model has come to play a vital role in homotopy theory. Its applications are too numerous to list. Suffices to say that Nishida's nilpotence theorem, Mahowald's infinite family in stable homotopy groups of spheres and most recently Stolz's classification of certain families of constant curvature surfaces [St] make fundamental use of it.

We quickly recall what the model is: Let $F(\mathbb{R}^k, n) \subset (\mathbb{R}^k)^n$ be the subset of n-tuples of disjoint points of $\mathbb{R}^k$, and let X be a connected topological space with basepoint $*$. We assume X has the homotopy type of a CW complex. Observe that both spaces $F(\mathbb{R}^k, n)$ and X^n admit an action by the symmetric group Σ_n given by permuting coordinates. We can then consider the orbit quotients $F(\mathbb{R}^k, n) \times_{\Sigma_n} X^n$ for every $n \geq 1$ (the term

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corresponding to $n = 0$ is basepoint), and glue them together as follows

1.1 $$C(\mathbb{R}^k, X) = \coprod_{n\geq 0} F(\mathbb{R}^k, n) \times_{\Sigma_n} X^n / \sim$$

where $\sim$ is a (standard) basepoint identification

$$(m_1, \dots, m_n) \times_{\Sigma_n} (x_1, \dots, x_n) \sim$$
$$(m_1, \dots, \hat{m}_i, \dots, m_n) \times_{\Sigma_{n-1}} (x_1, \dots, \hat{x}_i, \dots, x_n) \text{ if } x_i = *.$$

("hat" here means deletion). The theorem of May and Milgram then states that

Theorem 1.2: *There is a map*

$$C(\mathbb{R}^k; X) \longrightarrow \Omega^k \Sigma^k X$$

which is a homotopy equivalence whenever X is connected.

In their paper in the *Handbook of algebraic topology*, G. Carlsson and J. Milgram ([CM], §7) consider the following space: Fix $d \geq 1$ and let $F^d(\mathbb{R}^k, n)$ be the space of ordered n-tuples of vectors in $\mathbb{R}^k$ so that no vector occurs more than d times in the n-tuple (when $d = 1$, $F^1(\mathbb{R}^k, n) = F(\mathbb{R}^k, n)$). Consider as before the "labeled" construction

1.3 $$C^d(\mathbb{R}^k, X) = \coprod_{n\geq 0} F^d(\mathbb{R}^k, n) \times_{\Sigma_n} X^n / \sim$$

Notice we have an inclusion of models $C(\mathbb{R}^k, X) \hookrightarrow C^d(\mathbb{R}^k, X)$. The authors in [CM] raise the question of determining the homotopy type of $C^d(\mathbb{R}^k, X)$ for connected X and $d > 1$. The case $d = 1$ being known (see above), an earlier attempt to answer their question for the case $d = 2$ was carried out by Karagueuzian [Kr]. In this paper we give a general answer to this question for all $d \geq 2$.

Let $SP^d(-)$ be the d-th symmetric product functor. This we recall is defined on spaces X as the quotient $SP^d(X) = X^d/\Sigma_d$ where Σ_d acts by permuting coordinates. We can of course replace $\mathbb{R}^k$ by any (ground) manifold M in 1.3 above and $F^d(\mathbb{R}^k, n)$ by $F^d(M, n)$ in the obvious way (see §2). Our first main result then takes the form.

Theorem 1.4: *Let M be a k-dimensional smooth and parallelizable manifold with non-empty boundary ∂M, and let X be a connected CW complex. Then there is a homotopy equivalence*

$$C^d(M; X) \xrightarrow{\simeq} Map(M, \partial M; SP^d(\Sigma^k X))$$

where the mapping space on the right corresponds to all maps of M into $SP^d(\Sigma^k X)$ sending ∂M to a fixed basepoint.

Remarks and Corollaries 1.5:

• The case $d = 1$ is due to F. Cohen and F. Bodigheimer and reduces to the May-Milgram model when $M = \mathbb{R}^k$ (viewed as the closed disc). There are indications that the full result has been known to them (although this isn't recorded in the literature) and in fact our proof is already largely in [B].

• An analog of theorem 1.4 for $M = \mathbb{R}^n$ based on little cubes of Boardman-Vogt has been obtained by F. Kato [Kt].
• In fact theorem 1.4 admits an extension to stably parallelizable M's. When M is closed, there is also an analog after substituting the space of maps with some appropriate section spaces.
• In the case $d = k = 1$, the space $C^1(\mathbb{R}; X) := C(\mathbb{R}; X)$ (X connected) is homotopy equivalent to the James construction $J(X)$ described as the free monoid on points of X with $*$ as the zero element. The equivalence 1.4 in this case is simply James' equivalence $J(X) \simeq \Omega\Sigma X$.

EXAMPLE: Suppose M is closed, almost parallelizable and $A \subset M$ a closed subset. If $T(A)$ is a tubular neighborhood of A, then $\partial(M - A) \simeq \partial T(A)$. In this case $C^d(M - A; X)$ gets identified with $\mathrm{Map}(M, A; SP^d(\Sigma^k X))$. Choose $A = Q_r$ to be a finite set of points $r \geq 1$. We can identify each point in Q_k with a small closed disk centered at that point. In this case $\partial M - Q_r$ is the boundary of all r disks. The quotient $M/\partial Q_r$ is obtained from M (up to homotopy) by attaching segments between a point $p_0 \in Q_r$ and the remaining $k - 1$ points. This means that $M/\partial Q_k \simeq M \vee \bigvee^{k-1} S^1$ and one gets the following corollary (compare [V])

Corollary 1.6 : *M^k is closed and Q_r a set of r distinct points in M. Then*

$$C^d(M - Q_r; X) \simeq Map^*(M, SP^d(\Sigma^k X)) \times \left(\Omega SP^d(\Sigma^k X)\right)^{r-1}$$

where Map^* refers to the space of based maps.

We next discuss the case when X is disconnected. We observe (after Segal) that $C^d(\mathbb{R}^k; X)$ is homotopy equivalent to an associative topological monoid $\bar{C}^d(\mathbb{R}^k; X)$ (cf. §2). This monoid admits a classifying space $B\bar{C}$ and we show (for $k \geq 2$, $d \geq 1$)

Theorem 1.7: $B\bar{C}^d(\mathbb{R}^k; X) \simeq \Omega^{k-1} SP^d(\Sigma^k X)$

Remarks 1.8:
(a) Notice that when X is connected, $\bar{C} = \bar{C}^d(\mathbb{R}^k; X)$ is also connected and hence $\Omega B\bar{C} \simeq \bar{C}$. This yields 1.4 as a corollary. The case $d = 1$ is a theorem of Segal [S].
(b) The case $X = S^0$ has been studied in [K1]. One has $C^d(\mathbb{R}^n, S^0) = \coprod C^d(\mathbb{R}^n, k)$ where $C^d(\mathbb{R}^n, k) = F^d(\mathbb{R}^n, k)/\Sigma_k$. Up to homotopy this is a disconnected topological monoid (see §2). If "+" denotes group completion $\Omega B(-)$, then 1.7 becomes

1.9
$$\left(\coprod_{k\geq 0} C^d(\mathbb{R}^n, k)\right)^+ \simeq \Omega^n SP^d(S^n)$$

For $n = 2$ we recover the equivalence $\left(\coprod_{k\geq 0} \mathrm{Pol}_d^k\right)^+ \simeq \Omega^2\mathbb{P}^d$ ([GKY], [K2]), where Pol_d^k is the space of complex (monic) degree k polynomials having roots of multiplicity not exceeding d.

In proving the theorems above we avoid entirely the theory of operads ([May]), iterated loop spaces ([CM]) or classifying spaces (as in [S]). Instead we use an interesting shortcut construction given in the form of the "scanning" map originally due to Segal (cf. [K1]). We refer mainly to [B], [K1] for the main ideas/arguments we use in the first part of this note.

PART 2: A CLASSIFYING FAMILY OF SUBGROUPS. The second part of this note ties the above results to work of K. Lesh [L1-2]. First recall that Σ_k acts freely on $F(\mathbb{R}^\infty, k)$ and a model for $B\Sigma_k$ is given by $C(\mathbb{R}^\infty, k)$. The space $\coprod_{k\geq 0} B\Sigma_k$ has the structure of a disconnected monoid with composition being induced from the pairings $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$. On the other hand we can "deform and add points marching to infinity" (see §2) and hence get stabilizing maps $C(\mathbb{R}^\infty, k) \longrightarrow C(\mathbb{R}^\infty, k+1)$ the limit of which is denoted by $C_\infty(\mathbb{R}^\infty) := B\Sigma_\infty$. The Baratt-Priddy-Quillen theorem (or a version of it) is the special case of 1.9 when $d = 1$ and $n = \infty$; namely it states that

$$\text{1.10} \qquad \left(\coprod B\Sigma_k\right)^+ := \Omega B\left(\coprod_{k\geq 0} B\Sigma_k\right) \simeq QS^0 \simeq_h \mathbb{Z} \times B\Sigma_\infty$$

where $\simeq_h$ means homology equivalence (here $B\Sigma_0$ is basepoint). From this point of view, the space $B\Sigma_\infty = C_\infty(\mathbb{R}^\infty)$ is closely associated with the group completion of the "family" of groups $\Sigma_k, k \geq 1$.

In the exact same way we can stabilize the spaces $C^d(\mathbb{R}^\infty, k)$ to a space $C^d_\infty(\mathbb{R}^\infty)$ and the question is then: what group completion is the space $C^d_\infty(\mathbb{R}^\infty)$ for $d > 1$ trying to describe?

In §4, we recall after T.tom Dieck the notion of a "family" of subgroups and of their classifying spaces. It turns out that given "compatible" families $\mathcal{F}_n$ (each consisting of a collection of subgroups of Σ_n), one can associate to them a topological monoid $\coprod B\mathcal{F}_n$ of which group completion is an infinite loop space ([L1]). As a special case, we consider the collection of subgroups

$$H_{i_1,\dots,i_k} = \Sigma_{i_1} \times \Sigma_{i_2} \times \cdots \times \Sigma_{i_k},\ i_1 + i_2 + \cdots + i_k = n$$

of Σ_n. We define a "family" $\mathcal{F}^d_n$ to consist of all $H_{i_1,\dots,i_k}$ with $i_j \leq d$ and $i_1 + i_2 + \cdots + i_k = n$, together with their subgroups. For every $n \geq 1$, the family $\mathcal{F}^d_n$ affords a classifying space construction $E\mathcal{F}^d_n$ (that is there is a Σ_n space $E\mathcal{F}^d_n$ such that the fixed point set under the action of $H_{i_1,\dots,i_k} \in \mathcal{F}^d_n$ is contractible, and otherwise it is empty). The quotient spaces $B\mathcal{F}^d_n = E\mathcal{F}^d_n/\Sigma_n$, $n \geq 1$, are compatible in the sense that the disjoint union $\coprod B\mathcal{F}^d_n$ has the structure of a topological monoid. We shall prove (cf. §4)

Proposition 1.11: *For all $d \geq 1$, we have the following homology equivalence*

$$\left(\coprod_{n\geq 0} B\mathcal{F}^d_n\right)^+ \simeq_h \mathbb{Z} \times C^d_\infty(\mathbb{R}^\infty)$$

When $d = 1$, $\mathcal{F}_n$ is the trivial family (consisting of the trivial subgroup in Σ_n), $B\mathcal{F}_n = B\Sigma_n$ and one recovers 1.10 this way. We note that theorem 1.11 is closely related to proposition 7.4 of [L2].

PART 3: STABLE SPLITTINGS. In this last part we point out that in general there can be no stable splittings for the labeled configuration space constructions in 1.3 whenever the labels have multiplicity d at least 2. This is in stark contrast with the $d = 1$ case where these splittings (due to Snaith and Kahn) are almost a trademark of the May-Milgram construction.

We owe this section to Fred Cohen who first informed the author of this non-splitting result and to the referee who suggested the line of proof adopted here.

Given the labeled construction $C^d(M,X) = \coprod_{n\geq 0} F^d(M,n)\times_{\Sigma_n} X^n/\sim$, we can consider the successive quotients

$$D^d_k(X) = C^d_k/C^d_{k-1}$$

where $C^d_k = \coprod_{n=0}^k F^d(M,n)\times_{\Sigma_n} X^n/\sim$. We define $D^d_0(X)$ as the basepoint. A standard argument due originally to Dold and Steenrod shows that the space $C^d(M,X)$ splits in homology with direct summands $H_*(D^d_k(X))$. More precisely we shall give in §5 a short proof of the following proposition

Proposition 1.12: *For all $d \geq 1$, there is a Steenrod splitting*

$$H_*(C^d(M,X)) \cong \bigoplus_{k\geq 0} H_*(D^d_k(X))$$

where homology is taken with untwisted coefficients.

The next step is then to check whether such a homology splitting is actually induced from a stable splitting. When X is a connected sphere, this turns out to be the case only when $d = 1$.

Proposition 1.13: *Let $M = \mathbb{R}^k$, $k \geq 1$ and $X = S^j$ a sphere with $j > 1$. Then $C^d(\mathbb{R}^k, S^j)$ stably splits as a bouquet $\bigvee_{k\geq 0} D^d_k$ if and only if $d = 1$.*

Of course the arguments we provide can be applied to other choices of M and X. In fact, this non-stable splitting should be clear in light of the following example. As $d \to \infty$, $C^d(M,X)$ is in homotopy more and more like $SP^\infty(M\ltimes X)$ (where $M\ltimes X$ is the half-smash $M\times X/M\times *$). However SP^∞ is a functor modeled over Eilenberg-MacLane spaces and these abound in cohomology operations which prevent them from splitting. An obvious example: $SP^n(S^2) = \mathbb{P}^n$ the n-th complex projective space, $SP^\infty(S^2) = \mathbb{P}^\infty$ and $H_*(\mathbb{P}^\infty) = \bigoplus H_*(\mathbb{P}^n, \mathbb{P}^{n-1})$ while $\mathbb{P}^\infty$ certainly doesn't split as $\bigvee(\mathbb{P}^n/\mathbb{P}^{n-1})$.

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§2. Constructions and Notation

NOTATION: A configuration in M is a formal finite sum $\sum n_i x_i$, $n_i \in \mathbf{N}$, $x_i \in M$ ($x_i \neq x_j, i \neq j$). Such a configuration represents by definition a point of $SP^n(M), n = n_1 + \cdots + n_k$.

Assume M to be a connected manifold (in this paper, M will be either open or compact with non-empty boundary) and define

$$F^d(M,n) = \bigcup \{(\underbrace{x_1,\dots,x_1}_{i_1},\underbrace{x_2,\dots,x_2}_{i_2},\cdots,\underbrace{x_k,\dots,x_k}_{i_k}) \mid x_i \in M, x_i \neq x_j, i \neq j, \text{ and } i_j \leq d,\ i_1 + \cdots + i_k = n\}.$$

We write $C^d(M,n) = F^d(M,n)/\Sigma_n$ for the unoriented configuration space. We now consider the labeled construction 1.3. When $d = 1$ and $M = \mathbb{R}^\infty$, it is customary to write $C^1(\mathbb{R}^\infty; X) = QX$. A stable version of the theorem of May and Milgram asserts that

$$Q(X) \simeq \Omega^\infty \Sigma^\infty X \text{ for connected } X$$

and this of course implies that $\pi_i(Q(X)) \cong \pi_i^s(X)$.

Remark 2.1: One notices that singular homology and stable homotopy sit at opposite ends of the labeled configuration space construction given in the form

$$\coprod_{n\geq 1} F(n) \times_{\Sigma_n} X^n/\sim$$

where $\sim$ is the usual basepoint identification. When $F(n) = *$ and Σ_n acts trivially, we get $SP^\infty(X) = \coprod_n * \times_{\Sigma_n} X^n/_\sim$. The functor $SP^\infty(-)$ gives singular homology by virtue of the well-known theorem of Dold and Thom to the effect that $\pi_*(SP^\infty(X)) \cong \tilde{H}_*(X;\mathbb{Z})$. On the other hand and when $M = \mathbb{R}^\infty$, we get the functor $Q(-)$ and hence $\pi_*^s(X)$. The labeled constructs $C^d(M,-)$ for $d > 1$ provide intermediate functors between stable homotopy and integral homology and this circle of ideas is discussed in [L1-2].

The labeled construction 2.1 defines a bifunctor $C^d(-;-)$ which is a homotopy functor in X and an isotopy functor in M. When $X = S^0$, we write $C^d(M) := C^d(M,S^0) = \coprod_{k\geq 0} C^d(M,k)$.

When M has an open end or a boundary, we observe that we can put a monoidal structure on $C^d(M,X)$ up to homotopy. We explain this for $M = \mathbb{R}^k$: Let $\mathbb{R}^k_t$ be given as follows

2.2 $$\mathbb{R}^k_t = \{(x_1,\dots,x_n) \in \mathbb{R}^k,\ \mid 0 < x_n < t\}$$

and define as in [S] the space

$$\bar{C}^d(\mathbb{R}^k;X) = \{(\zeta,t) \in C^d(\mathbb{R}^k_t;X) \times \mathbb{R}^+\}$$

Again we have that $\bar{C}^d(\mathbb{R}^k,X) \simeq C^d(\mathbb{R}^k,X)$ and this new modified space has now the structure of an associative (*homotopy commutative for* $k \geq 2$) topological monoid with a composition law given by juxtaposition

$$C(\mathbb{R}^k_t;X) \times C(\mathbb{R}^k_{t'};X) \longrightarrow C(\mathbb{R}^k_{t+t'};X),\ (\zeta,\zeta') \mapsto (\zeta + T_t\zeta')$$

where T_t is translation $(0,t'))\longrightarrow(t,t+t')$. In the case $d=k=1$ this is the same up to homotopy as the well-known James construction $J(X)$.

GROUP COMPLETION: Since $\bar{C}^d=\bar{C}^d(\mathbb{R}^k;X)$ is a monoid (possibly disconnected), it admits a "group completion" $\Omega B\bar{C}^d$ (i.e. $\pi_0(\Omega BM)$ is a group completion of $\pi_0(M)$ for M a monoid). A handy description of the homology of this space is given as follows. We shall suppose that X has finitely many components (the countable case follows from a direct limit argument). We then have that $\bar{C}^d(\mathbb{R}^k;X)$ has $\mathbb{N}^m$ components, with $m=|\pi_0(X)|-1$. For each $i\leq|\pi_0(X)|$ choose a point p_i in the i-th component of X (the zero component is the component containing basepoint $*=p_0$). Let $\mathbb{R}^k_t$ be as in 2.2 and choose a point $z_i\in\mathbb{R}^k_{i,i+1}=\mathbb{R}^k_{i+1}-\mathbb{R}^k_i$, for all $i\geq 1$. We can then consider the inclusion

2.3
$$\begin{array}{ccc}\bar{C}^d(\mathbb{R}^k;X) & \xrightarrow{\tau_i} & \bar{C}^d(\mathbb{R}^k;X)\\ (\sum(m_r,x_r),i) & \mapsto & (\sum(m_r,x_r)+(z_i,p_i),i+1)\end{array}$$

The direct limit over these maps is denoted by $\hat{C}^d(\mathbb{R}^k;X)$. When X is connected, we have the equivalence $\hat{C}^d(\mathbb{R}^k;X)\simeq C^d(\mathbb{R}^k;X)$.

EXAMPLE 2.4: We can define maps $C^d(\mathbb{R}^k,i)\xrightarrow{\cong}C^d(\mathbb{R}^k_i,i)\xrightarrow{+z_i}C^d(\mathbb{R}^k,i+1)$ the direct limit of which we write $C^d_\infty(\mathbb{R}^k)$. It is now easy to see that $\hat{C}^d(\mathbb{R}^k;S^0)\simeq\mathbb{Z}\times\lim_i C^d(\mathbb{R}^k;i):=\mathbb{Z}\times C^d_\infty(\mathbb{R}^k)$.

Lemma 2.5: *Let X CW, $k\geq 2$. Then $H_*(\hat{C}^d(\mathbb{R}^k;X))\cong H_*(\Omega B\bar{C}^d(\mathbb{R}^k;X))$.*

PROOF: If we let $\pi=\pi_0(\bar{C}^d)$, then a theorem of Kahn and Priddy (cf. [MS]) states that

2.6
$$H_*(\bar{C}^d)[\pi^{-1}]\cong H_*(\Omega B\bar{C}^d)$$

where the left hand side means localization with respect to the multiplicative set π (provided that π is in the center of $H_*(\bar{C}^d)$ which is the case since $\bar{C}^d$ is homotopy abelian as pointed out earlier). The idea of 2.6 is of course that by inverting π, we are "turning" multiplication by elements of π into isomorphisms (that this is necessary is clear since $\Omega B\bar{C}^d$ is a group and hence the image of π under $M\to\Omega B\bar{C}^d$ must consist of units.) Now notice that the point $(z_i,p_i)\in\bar{C}^d(\mathbb{R}^k,X)$ constructed above represents a point $e_i\in\pi_0(\bar{C}^d)$. The stabilization maps in 2.3 correspond therefore to maps

$$\hat{C}^d(\mathbb{R}^k;X)\simeq\varinjlim_{e_i\in\pi}\left(\bar{C}^d(\mathbb{R}^k;X)\xrightarrow{\cdot e_i}\bar{C}^d(\mathbb{R}^k;X)\right).$$

and this direct limit (by construction) must satisfy $H_*(\hat{C}^d(\mathbb{R}^k;X))\cong H_*(\bar{C}^d(\mathbb{R}^k;X))[\pi^{-1}]$. The claim follows from 2.6. ∎

§3. Proof of Theorem 1.4

The correspondence (known as scanning) which associates to configurations on a parallelizable manifold a mapping space has been described in a few places (cf. [K1], [GKY]). This scanning procedure extends to labeled configuration spaces without modification (and so we refer to [K1] for the details). Let M be parallelizable with non-empty boundary ∂M and $\dim M = k$. Given $x \in M$ we can canonically identify a neighborhood D^k_x of it with the closed disc D^k and hence to every configuration $\zeta \in C^d(M;X)$ we have a map

$$\zeta \times M \longrightarrow C^d(D^k;X)$$

which associates to $x \in M$ part of the configuration ζ lying in D^k_x. We compose with the map $C^d(D^k;X)\longrightarrow C^d(D^k,\partial D^k;X)$ to make this association continuous. Here $C^d(D^k,\partial D^k;X)$ is the quotient of $C^d(D^k;X)$ with the additional identification

$$\begin{aligned}&(m_1,\dots,m_n)\times_{\Sigma_n}(x_1,\dots,x_n)\sim\\ &\qquad (m_1,\dots,\hat{m}_i,\dots,m_n)\times_{\Sigma_n}(x_1,\dots,\hat{x}_i,\dots,x_n),\ \text{if}\ \ m_i\in\partial D^k\end{aligned}$$

Equivalently, when points of the ground space D^k tend to the boundary they are discarded together with their labels. To every configuration in $\zeta \in C^d(M;X)$ we then have a map $\zeta \times M\longrightarrow C^d(D^k,\partial D^k;X)$. We can demand that the points making up ζ live away from the boundary (or end) of M and in fact with a bit of care we get a correspondence

$$S' : C^d(M;X)\longrightarrow \mathrm{Map}(M,\partial M;C^d(D^k,\partial D^k;X))$$

where ∂M is sent to basepoint in $C^d(D^k,\partial D^k;X)$. It is not hard to see (by a standard radial retraction argument, cf. [K1], [McD]) that

3.1 $$C^d(D^k,\partial D^k;X)\simeq SP^d((D^k/\partial D^k)\wedge X) = SP^d(\Sigma^k X)$$

and hence S gives rise to the map

$$S : C^d(M;X)\longrightarrow \mathrm{Map}(M,\partial M;SP^d(\Sigma^k X))$$

which extends to the stabilized space $\hat{C}^d(\mathbb{R}^k;X)$ (2.3).

Proposition 3.2: *For X a topological space, scanning induces an (integral) homology equivalence*

$$S_* : H_*(\hat{C}^d(\mathbb{R}^n;X))\xrightarrow{\ \cong\ } H_*(\Omega^n SP^d(\Sigma^n X))$$

S is a homotopy equivalence whenever X is connected.

The arguments that go into verifying 3.2 are by now standard. The idea is to induct on an (ingenious) handle decomposition of the closed unit cube D^n (given for instance by Bodigheimer in [B]) and to use properties of the functor C^d. Define $C^d(M,A)$ for a closed ANR $A\subset M$ to be the quotient of $C^d(M)$ by the identification which requires that points be discarded when they are in A (exactly as in the case $A=\partial D$ above). The functor $C^d(M,A;X)$ is an isotopy functor in M and A. The following easy extension of results in [B], [K1] can be shown

Lemma 3.3: *Let M and N be connected manifolds, $N\subset M$, $M_0\subset M$ and consider the cofibration $(N,N\cap M_0)\longrightarrow(M,M_0)\longrightarrow(M,N\cup M_0)$. Then*

(a) $C^d(N, N\cap M_0; X)\longrightarrow C^d(M, M_0; X)\longrightarrow C^d(M, N\cup M_0; X)$ *is a quasifibration if* $N \cap M_0 \neq \emptyset$ *or* X *connected.*
(b) Assume $N\cap M_0 = \emptyset$ *and* N *has an end or a boundary, then*

$$\hat{C}^d(N; X)\longrightarrow \hat{C}^d(M, M_0; X)\longrightarrow C^d(M, N\cup M_0; X)$$

is a quasifibration if X *connected and a homology fibration otherwise.*

SKETCH OF PROOF: Let's consider the case $X = S^0$ and $N\cap M_0 = \emptyset$. The point in showing that the sequence of spaces in (b) above is a homology fibration (resp. a quasifibration) boils down to showing that maps

$$\hat{C}^d(N)\xrightarrow{\ +\ }\hat{C}^d(N)$$

given by adjointing a given set of configurations is a homology equivalence (resp. a weak homotopy equivalence). Because of the very construction of $\hat{C}$, adding configurations simply switches components and since these components are the same, "addition" induces a homology equivalence. This is not (necessarily) a homotopy equivalence since there is no obvious map backwards ("subtraction") which when composed with addition induces the identity on components. When either X is connected or $N\cap M_0\neq\emptyset$ it is possible to "subtract" by moving labels to $*\in X$ or points to $N\cap M_0$ where they get discarded. ∎

SKETCH OF PROOF OF THEOREM 3.2: We let $A^k = S^{k-1}\times D^{n-k}$ denote part of the boundary of the unit cube $D^n = [0,1]^n$ (note that $A^0 = \emptyset$). Now retracting then scanning gives a map $C^d(D^n, A^k; X)\longrightarrow\Omega^{n-k}SP^d(\Sigma^n X)$ (cf. [K2]) which in the case $k = n$ (i.e. $A^n = \partial D^n$) is a homotopy equivalence according to 3.1.

Let $I_k\subset D^n$ denote the subset of $(y^1,\dots,y^n)$ such that $y^i = 0$ or $y^i = 1$ for some $i = k+1,\dots,n$, or $y^k = 1$ (that is I_k consists of $D^k\times S^{n-k-1}\subset\partial(D^n = D^k\times D^{n-k})$, for $1\leq k<n$, together with one face of D^k; cf. [B]). Now let $H_k = [0,1]^{k-1}\times[0,\frac{1}{2}]\times[0,1]^{n-k}$. Then there is a cofibration sequence

$$(H_k, H_k\cap I_k)\longrightarrow(D^n, I_k)\longrightarrow(D^n, H_k\cup I_k)$$

The pair $(H_k, H_k\cap I_k)$ can be identified with the pair (D^n, A^{k-1}), while $(D^n, H_k\cup I_k) = (D^n, S^{k-1}\times D^{n-k})$ represents a "handle" (D^n, A^k). Applying the functor $C^d(-; X)$ and then scanning yields the commutative diagram for all $k>1$

$$\begin{array}{ccc}
C^d(D^n, A^{k-1}; X) & \longrightarrow & \Omega^{n-k+1}SP^d(\Sigma^n X)\\
\downarrow & & \downarrow\\
C^d(D^n, I_k; X) & \longrightarrow & \Gamma\\
\downarrow & & \downarrow\\
C^d(D^n, A^k; X) & \longrightarrow & \Omega^{n-k}SP^d(\Sigma^n X)
\end{array}$$

where Γ is an appropriate section space. When $k>1$, $A^{k-1}\neq\emptyset$ and 3.3 asserts that the left vertical sequence is a quasifibration. Note that for $k>0$, D^n retracts onto I_k implying that $C^d(D^n, I_k, X)$ is contractible (similarly so is Γ) and so inductively $C^d(D^n, A^k; X)\simeq\Omega^{n-k}SP^d(\Sigma^n X)$ for $k>1$. When

$k=1$, $A^{k-1}=A^0=\emptyset$ and we need to pass to group completed spaces $\hat{C}^d$. In this case we have the diagram

$$\begin{array}{ccc} \hat{C}^d(D^n;X) & \longrightarrow & \Omega^nSP^d(\Sigma^nX) \\ \downarrow & & \downarrow \\ \hat{C}^d(D^n,*;X) & \xrightarrow{\simeq} & \Gamma \\ \downarrow & & \downarrow \\ C^d(D^n,A^1;X) & \xrightarrow{\simeq} & \Omega^{n-1}SP^d(\Sigma^nX) \end{array}$$

where the left hand side is now a homology fibration according to 3.3 (or quasifibration if X connected). Since $\Omega^nSP^d(\Sigma^nX)$ must be the homotopy fiber of $\hat{C}^d(D^n,*;X)\rightarrow C^d(D^n,A^1;X)$ it follows that $H_*(\hat{C}^d(D^n;X))\cong H_*(\Omega^nSP^d(\Sigma^nX))$ as asserted. ∎

Proof of theorem 1.4: A manifold M with non-empty boundary can be obtained from $\mathbb{R}^k$ by a sequence of attachments of handles of index k, $0\leq k<n$. Theorem 1.2 now follows by repeated use of lemma 3.3 and an inductive argument on a handle decomposition for M (cf. [K1], [McD]). ∎

Proposition 3.4: *For X a CW complex, there is a homotopy equivalence*

$$B\bar{C}^d(\mathbb{R}^k;X)\simeq\Omega^{k-1}SP^d(\Sigma^kX).$$

Proof: First of all there is a homotopy commutative diagram

$$\begin{array}{ccc} & \hat{C}^d(\mathbb{R}^k;X) & \\ \swarrow & & \searrow \\ \Omega B\bar{C}^d & \longrightarrow & \Omega^kSP^d(\Sigma^kX) \end{array}$$

where the right hand map is the "natural fiber" inclusion (see [MS]) and the bottom map is induced by scanning $\bar{C}^d\longrightarrow\Omega^kSP^d(\Sigma^kX)$. By combining 2.5 and 3.2 we see that all maps are homology equivalences. The bottom map is a homology equivalence between two simple spaces (being H-spaces) and also induces an isomorphism at the level of $\pi_1(=H_1$ being abelian). This is then a homotopy equivalence and the proof is complete (note that X being a CW complex, $\Omega^kSP^d(\Sigma^kX)$ has the homotopy type of a CW complex as well). ∎

§4. Classifying Families of Subgroups of Σ_n and $B\bar{C}^d$

In this section we describe a construction of K. Lesh (cf. [L1-2]) which associates to a (compatible) family of groups an infinite loop space. We then describe how our labeled construction fits in and prove proposition 1.11 of the introduction.

Let G be a group and let $\mathcal{F}$ be a collection of subgroups of G which is closed under conjugation; meaning that

- If $H\in\mathcal{F}$ and $g\in G$, then $g^{-1}Hg\in\mathcal{F}$

• If $H \in \mathcal{F}$ and K a subgroup of H, then $K \in \mathcal{F}$.
Such a collection is called a *family*.

The prototypical example of a family would be to take all subgroups of a group G (a variant is to consider only the finite subgroups). A less trivial example would be to consider the family of elementary abelian p-subgroups of Σ_n which are generated by disjoint p-cycles together with their subgroups. This family is studied in [L1].

It turns out that to a family $\mathcal{F}$ of subgroups of a group G there is associated a *classifying* space $B\mathcal{F}$ by work of T. tom Dieck. More precisely, tom Dieck constructs a G-space $E\mathcal{F}$ with the property that the fixed point set $E\mathcal{F}^H$ of $H \subset G$ is such that

$$E\mathcal{F}^H \simeq * \text{ for } H \in \mathcal{F}, \text{ and } E\mathcal{F}^H = \emptyset \text{ for } H \notin \mathcal{F}$$

Note that $E\mathcal{F}$ is always contractible since $* \in \mathcal{F}$ for any family. Naturally one then defines the classifying space $B\mathcal{F}$ to be the orbit space of the G action on $E\mathcal{F}$.

EXAMPLE 4.1: Let $\mathcal{F}$ consists only of the trivial subgroup in G. Then $E\mathcal{F} = EG$.

We now specialize to the symmetric groups Σ_n and we suppose that for each n we are given a family $\mathcal{F}_n$ of subgroups for $G = \Sigma_n$. We recall that given two subgroups $H \in \Sigma_n$ and $K \in \Sigma_m$, we have a group $H \times K \in \Sigma_{n+m}$ obtained as the image of the composite

$$H \times K \hookrightarrow \Sigma_n \times \Sigma_m \hookrightarrow \Sigma_{n+m}.$$

DEFINITION 4.2: The families $\{\mathcal{F}_n\}_{n\in\mathbf{Z}^+}$ are *compatible* if whenever $H \in \mathcal{F}_n$ and $K \in \mathcal{F}_m$, then $H \times K \in \mathcal{F}_{n+m}$.

Theorem 4.3 (Lesh): *Let $\{\mathcal{F}_n\}_{n\in\mathbf{Z}^+}$ be a compatible choice of families, then $\coprod B\mathcal{F}_n$ has a monoid structure whose group completion is an infinite loop space $L\mathcal{F}$. Such a space comes equipped with (natural) maps*

$$QS^0 \longrightarrow L\mathcal{F} \longrightarrow \mathbb{Z}.$$

EXAMPLE 4.4: Let $\mathcal{F}_n$ be the family consisting of the trivial subgroup in Σ_n. Then $B\mathcal{F}_n = B\Sigma_n$ and so $L\mathcal{F}$ in this case is the group completion of $\coprod B\Sigma_n$ which is known to correspond by a theorem of Barratt-Priddy and Quillen to the infinite loop space QS^0.

We now relate the above constructions to the spaces $C^d(\mathbb{R}^\infty, n)$ and their stable version $C^d(\mathbb{R}^\infty)$ constructed in §3. Given $n \geq 1$ we consider the following subgroups of Σ_n;

$$H_{i_1,\dots,i_k} = \Sigma_{i_1} \times \Sigma_{i_2} \times \cdots \times \Sigma_{i_k} \subset \Sigma_n,\ i_j \leq n \text{ and } i_1 + i_2 + \cdots + i_k = n.$$

Each such subgroup $H_{i_1,\dots,i_k}$ acts on $F^d(\mathbb{R}^\infty, n)$ by permuting points. Let

$$\mathcal{F}_n^d = \{H_{i_1,\dots,i_k} \mid i_j \leq d,\ i_1+i_2+\cdots+i_k = n, \text{ together with their subgroups}\}.$$

It is not hard to see that $\mathcal{F}^d_n$ satisfies the conditions of a family, and that the newly obtained families $\{\mathcal{F}^d_n\}_{n\in\mathbf{Z}^+}$ form a compatible collection.

Lemma 4.5: $E\mathcal{F}^d_n \simeq F^d(\mathbb{R}^\infty, n)$.

PROOF: Pick $H = H_{i_1,\dots,i_k} \in \mathcal{F}^d_n$ ($i_j \leq d$ and $\sum i_j = n$.) Then

$$(E\mathcal{F}^d_n)^{H_{i_1,\dots,i_k}} = \{(\underbrace{x_1,\dots,x_1}_{i_1},\underbrace{x_2,\dots,x_2}_{i_2},\cdots,\underbrace{x_k,\dots,x_k}_{i_k}) \mid x_i \in \mathbb{R}^\infty\}$$

where the x_i need not be distinct. If we let $X(\mathbb{R}^j)$ be the subset of $(E\mathcal{F}^d_n)^H$ consisting of the $x_i \in \mathbb{R}^j \subset \mathbb{R}^\infty$, then we see that $X(\mathbb{R}^j)$ is open in $\mathbb{R}^{jn}$ and is the complement of hyperplanes of codimension at least j (implying in particular that it is $j-2$ connected). Since $(E\mathcal{F}^d_n)^H$ is the direct limit of $X(\mathbb{R}^j) \hookrightarrow X(\mathbb{R}^{j+1})$ it must be contractible and $(E\mathcal{F}^d_n)^H \simeq *$ as desired. What is left to show is that the fixed point set of $H \notin \mathcal{F}^d_n$ is empty. Observe that any such H must contain a cycle on (at least) $d+1$ letters. The fixed points of such a cycle consists of configurations containing a $d+1$ (or maybe more) repeated point. Such a configuration cannot exist in $F^d(\mathbb{R}^\infty, n)$ (by definition) and $(E\mathcal{F}^d)^H = \emptyset$. ∎

Proposition 4.6: *For all $d \geq 1$, we have the following equivalence*

$$B\left(\coprod_{n\geq 0} B\mathcal{F}^d_n\right) \simeq \Omega^{\infty-1}SP^d(S^\infty).$$

PROOF: By lemma 4.5, $\coprod_{n\geq 0} B\mathcal{F}^d_n = \coprod_{n\geq 0} C^d(\mathbb{R}^\infty, n) = C^d(\mathbb{R}^\infty; S^0) \simeq \bar{C}^d(\mathbb{R}^\infty; S^0)$. On the other hand, it is easy to see that theorem 1.7 still holds true for $k=\infty$ and the claim follows. ∎

Combining now 3.2 with the above proposition yields 1.11 immediately. Note that by construction $\hat{C}^d(\mathbb{R}^\infty, \infty) = \mathbb{Z} \times C^d_\infty(\mathbb{R}^\infty)$ (cf. 2.4).

§5. Stable Splittings

As discussed in the introduction we are concerned with the existence of stable splittings for $C^d(M,X)$ and more generally for the (*filtration n*) subspace

$$C^d_n(M,X) = \coprod_{n=0}^{k} F^d(M,n) \times_{\Sigma_n} X^n/\sim$$

in terms of successive quotients as in

$$C^d_n(M,X) \simeq_s \bigvee_{k=0}^{n} D^d_k(M,X)$$

where $D^d_0(M,X) = *$ and $D^d_k(M,X) = C^d_k/C^d_{k-1}$ for $k \geq 1$. As far as homology is concerned, this splitting always occurs; i.e.

Proposition 5.1: $H_*(C^d_n(M,X)) \cong \bigoplus_{k=0}^n H_*(D^d_k(M,X))$

PROOF:(compare [C]) There are transfer maps

$$\iota_j : C_n^d(M,X)\longrightarrow SP^\infty(C_j^d(M,X))\longrightarrow SP^\infty(D_j^d)$$

for all $j\leq n$ given in the standard way: the first map associates to each n-tuple in $C_n^d(M,X)$ all choices of j-subtuples and then concatenates them in SP^∞, while the second map is collapsing C_{j-1}^d in C_j^d. The maps ι_j combine to give a map

$$C_n^d(M,X)\longrightarrow \prod_{0\leq j\leq k} SP^\infty(D_j^d) = SP^\infty(\bigvee_{0\leq j\leq k} D_j^d)$$

which in turn can be extended multiplicatively to a map

$$\iota : SP^\infty(C_n^d(M,X))\longrightarrow SP^\infty(\bigvee_{0\leq j\leq k} D_j^d(M,X))$$

We need then show that ι is a weak equivalence (for then the proposition will follow from the known correspondence: $\pi_*(SP^\infty(X))\cong \tilde H(X;\mathbb{Z})$).

To see that ι is a weak equivalence, we use again the fact that SP^∞ converts cofibrations to fibrations. As in the proof of 3.2 we get the following diagram of fibrations

$$\begin{array}{ccc}
SP^\infty(C_k^d(M,X)) & \longrightarrow & SP^\infty(\bigvee_{j=0}^k D_j^d)\\
\downarrow & & \downarrow\\
SP^\infty(C_{k+1}^d(M,X)) & \longrightarrow & SP^\infty(\bigvee_{j=0}^{k+1} D_j^d)\\
\downarrow & & \downarrow\\
SP^\infty(D_{k+1}^d) & \xrightarrow{id} & SP^\infty(D_{k+1}^d)
\end{array}$$

The upper part of the diagram commutes because the transfer map is compatible with the inclusions $C_k^d(M,X)\hookrightarrow C_{k+1}^d(M,X)$ in a natural way. The proof now proceeds by easy induction. ∎

Remark 5.2: A direct limit argument shows that the homology splitting extends to $H_*(C^d(M,X))\cong\bigoplus_{k\geq 0} H_*(D_k^d(M,X))$.

Remark 5.3: One can also consider the directed system $\iota_d : C^d(M,X)\longrightarrow C^{d+1}(M,X)$ with direct limit $SP^\infty(M\ltimes X)$ where $M\ltimes X$ be the half smash $M\times X/M\times *$ (which can also be written as $M_+\wedge X$ where M_+ is M with a disjoint point added). The homology splitting for $C^d(M,X)$ is compatible (as is easy to see) with the corresponding homology splitting for $SP^\infty(M\ltimes X)$ (a fact we use in the proof of 5.4 below).

Consider the inclusion $C^d(M,X)\hookrightarrow SP^\infty(M\ltimes X)$. Again X has the homotopy type of CW-complex. We claim that this inclusion is a homotopy equivalence through a range depending on d. To see this, first write

$$D_k^d(M,X) = F^d(M,k)_+\wedge_{\Sigma_k} X^{(k)}$$

where $X^{(k)}$ is the smash product of $X\wedge\cdots\wedge X$ (k-times) and $F^d(M,k)_+$ is $F^d(M,k)$ with a disjoint basepoint adjointed. From this it follows directly that if M is r-connected, then $D^d_k(M,k)$ is kr-connected (which is the connectivity of $X^{(k)}$).

Proposition 5.4: *Suppose X is r-connected with $r\geq 1$, then the inclusion*

$$\alpha : C^d(M,X)\longrightarrow SP^\infty(M\ltimes X)$$

is a homotopy equivalence up to dimension $r(d+1)$.

PROOF: We first show that α induces an isomorphism in homology up to dimension $r(d+1)$. Remark 5.3 implies that the inclusion $C^d(M,X)\xrightarrow{\alpha} SP^\infty(M\ltimes X)$ is stably equivalent to the map

$$\bigvee D^d_k\longrightarrow \bigvee SP^k(M\ltimes X)/SP^{k-1}(M\ltimes X)$$

Clearly $D^d_k = SP^k/SP^{k-1}$ for $k\leq d$ and the map above is the identity on those summands. Homology may then differ starting with the term D^d_{d+1} which has non-trivial homology (possibly) in $r(d+1)+1$ but not before. This implies right away that α_* is an isomorphism in homology up to dimension $r(d+1)$.

To see that this is a homotopy equivalence through that range, it is enough to show $C^d(M,X)$ is simply-connected if X itself is simply connected. To see this, we can apply theorem 1.4 and get that

$$\begin{aligned}\pi_1(C^d(M,X)) &= \pi_1\left(\mathrm{Map}(M,\partial M; SP^d(\Sigma^kX)\right)\\ &= \pi_1\left(\mathrm{Map}^*(M/\partial M, SP^d(\Sigma^kX)\right)\\ &= \left[\Sigma(M/\partial M), SP^d(\Sigma^kX)\right]_*\end{aligned}$$

where $k=\dim M$ and $[--]_*$ means based homotopy classes of maps. Now $\Sigma(M/\partial M)$ is a CW complex of dimension $k+1$ whereas $SP^d(\Sigma^kX)$ is $k+1$ connected (which is the connectivity of Σ^kX). The set of based homotopy classes is therefore trivial and the proof follows. ∎

We are now in a position to prove proposition 1.13 of the introduction

Proposition 5.5: *Let $M=\mathbb{R}^k$ and $X=S^j$ with $j>1$. Then $C^d(\mathbb{R}^k,S^j)$ stably splits as a bouquet $\bigvee_{k\geq 0}D^d_k$ if and only if $d=1$.*

PROOF: $C^d(\mathbb{R}^k,S^j)$ is $SP^\infty(\mathbb{R}^k\ltimes S^j)=SP^\infty(S^j)=K(\mathbb{Z},j)$ up to homotopy and through dimension $j(d+1)>2j$ (provided $d>1$). If $\iota_j\in H^j(SP^\infty(S^j))$ denotes the fundamental class, then $\mathrm{Sq}^2(\iota_j)\neq 0\in H^{2j}(SP^2(S^j)) = H^{2j}(SP^\infty(S^j))$ which means that $SP^2(S^j)/S^j = D^d_2(S^j)$ (when $d>1$) cannot split off $SP^\infty(S^j)$ and hence off $C^d(\mathbb{R}^k,S^j)$. ∎

Remark 5.6: When X is not connected, stable splittings can occur for $d>1$. For instance [GKY] verify that $C^d(\mathbb{R}^k;S^0)$ (cf. 1.9) stably splits for all d.

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