ON FLAT BUNDLES

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A principal G-bundle ξ on X is *flat* if and only if it is induced from the universal covering bundle of X by a homomorphism $\pi_1 X \to G$ [6, Lemma 1]. First the holonomy map of a principal G-bundle is defined and flat bundles are characterized. Then the reduction problem with respect to a homomorphism $\tau: \Phi \to G$ of a finite abelian group Φ is discussed for $G = O(n)$, $SO(n)$ and $U(n)$.

1. **The holonomy map of a principal bundle.** For a differentiable principal G-bundle ξ on X a connection defines a holonomy map $\Omega X \to G$. The homotopy class of this map is an invariant of ξ, as shown e.g. in [2]. We first give a topological version of this invariant. Let G be a topological group, X a space and ξ a G-bundle with projection $p: T \to X$. EX denotes the space of paths starting from the basepoint of X. Choose a basepoint in T lying in the fiber over the basepoint of X. A section s of the principal EG-bundle $E(p): ET \to EX$ defines a map $h: \Omega X \to G$ as follows. For $\omega \in \Omega X$ there is a unique $h(\omega) \in G$ sending the basepoint of T to the endpoint of $s(\omega)$.

THEOREM 1.1.
(i) *$h: \Omega X \to G$ is an H-map (that is: h carries products into products, up to homotopy).*
(ii) *The equivalence class (under inner automorphisms of G) of the homotopy class of h is an invariant of ξ, called the holonomy map $h(\xi)$ of ξ.*
(iii) *$h(X, G): P(X, G) \to [\Omega X, G]$ defined by $h(X, G)(\xi) = h(\xi)$ is a natural transformation.*

Here $P(X, G)$ denotes the isomorphism classes of numerable G-bundles on X. No distinction is made between a G-bundle and its classifying map $X \to BG$. Then the classification theorem of [3] for numerable bundles over arbitrary spaces can be expressed by $P(X, G) = [X, BG]$.

PROPOSITION 1.2. *For the universal G-bundle η_G the holonomy map $h(\eta_G): \Omega BG \to G$ is a homotopy equivalence.*

2. **Flat bundles.** Let G_d be the underlying discrete group of G and

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$\iota: G_d \to G$ the canonical map. Observe that BG_d is an Eilenberg-MacLane space $K(G_d, 1)$.

THEOREM 2.1. *The following conditions for $\xi \in P(X, G)$ are equivalent.*

(i) *ξ is flat.*

(ii) *$\xi = \iota_* \eta$ for some $\eta \in P(X, G_d)$.*

(iii) *$h(\xi): \Omega X \to G$ factorizes through the natural projection $\Omega X \to \pi_1 X$, up to homotopy.*

For $SO(2)$-bundles one has the following result.

THEOREM 2.2. *$\xi \in P(X, SO(2))$ is flat if and only if the rational Euler class vanishes.*

The characteristic cohomology-homomorphisn of a flat bundle $\xi \in P(X, G)$ factorizes through $H^*(\pi_1 X)$. Thus one obtains necessary conditions for the characteristic classes of ξ.

3. **τ-flat bundles.** As a computational device we introduce an arbitrary discrete group Φ and a homomorphism $\tau: \Phi \to G$.

DEFINITION 3.1. $\xi: X \to BG$ is τ-flat if there is a map $\eta: X \to B\Phi$ with $B(\tau) \circ \eta \simeq \xi$.

PROPOSITION 3.2. *ξ is τ-flat if and only if there is a homomorphism $\gamma: \pi_1 X \to \Phi$, such that ξ is induced from the universal covering bundle ζ by $\tau \circ \gamma$. In particular, τ-flat implies flat.*

A homomorphism $\gamma: \pi_1 X \to G$ inducing a flat ξ can be thought of as the holonomy map and $\gamma(\pi_1 X) \subset G$ as the holonomy group of ξ. Then for injective $\tau: \Phi \to G$ a bundle is τ-flat if and only if it is flat with holonomy group contained in Φ.

We discuss τ-flat bundles for $G = O(n)$, $SO(n)$, $U(n)$ and Φ finite abelian. In order to simplify notations we restrict ourselves here to the case of a cyclic group $\boldsymbol{Z}_q$ of odd order. The case $\Phi = \boldsymbol{Z}_{2^\nu}$ can be treated similarly.

Let $\alpha: \boldsymbol{Z}_q \to SO(2)$ be defined by $\alpha(1) = \exp(1/q)$. A representation of $\boldsymbol{Z}_q$ is orientable and of the form $\tau = (\alpha^{\lambda_1}, \cdots, \alpha^{\lambda_m}): \boldsymbol{Z}_q \to SO(2)^m \hookrightarrow SO(n)$ with $\lambda_i = 1, \cdots, q$ and $m = [n/2]$.

THEOREM 3.3. *Let $\xi: X \to BO(n)$ be a bundle and $\tau = (\alpha^{\lambda_1}, \cdots, \alpha^{\lambda_m}): \boldsymbol{Z}_q \to SO(n)$ a representation. There exists a τ-flat bundle $\xi': X \to BSO(n)$ with the same Pontrjagin classes as ξ if and only if there is an $u \in H^2(X, \boldsymbol{Z})$ with $q \cdot u = 0$ and $p_i(\xi) = \sigma_i(\lambda_1^2, \cdots, \lambda_m^2) \cdot u^{2i} \in H^{4i}(X, \boldsymbol{Z})$, $i = 1, \cdots, n$, where σ_i is the ith elementary symmetric function. If ξ is moreover oriented, ξ' and ξ have the same Euler class if and only if $\chi(\xi) = \lambda_1 \cdots \lambda_m$*

$\cdot u^m$ *for* $n=2m$ *and* $\chi(\xi)=0$ *for* $n=2m+1$.

Note that the Stiefel-Whitney classes of ξ' are trivial, as the characteristic cohomology map factorizes through $H^*(\boldsymbol{Z}_q, \boldsymbol{Z}_2)\cong\boldsymbol{Z}_2$.

The proof of 3.3 is based on the computation of the characteristic classes of τ in the sense of [1] and a method due to Massey-Szczarba [5].

If the bundles $\xi: X\to BG$ are classified by their characteristic classes, 3.3 gives necessary and sufficient conditions for the τ-flatness of ξ. E.g. [11, Theorems 4.2, 4.3] for $G=O(n)$ and [4], [8] for $G=SO(n)$ prove the following.

COROLLARY 3.4. *Let X be a* CW-*complex*, $\xi: X\to BO(n)$.

(i) *Assume* $\dim X\leqq \min(8, n-1)$; $H^4(X, \boldsymbol{Z})$ *without* 2-*torsion and* $H^8(X, \boldsymbol{Z})$ *without* 6-*torsion. Then ξ is τ-flat if and only if* $w_1(\xi)=0$, $w_2(\xi)=0$ *and* $\exists u\in H^2(X, \boldsymbol{Z})$ *with* $q\cdot u=0$ *and* $p_i(\xi)=\sigma_i(\lambda_1^2, \cdots, \lambda_m^2)\cdot u^{2i}$, $i=1, 2$.

(ii) *Assume* $\dim X\leqq 4$, $H^4(X, \boldsymbol{Z})$ *without* 2-*torsion and ξ oriented. Then ξ is τ-flat if and only if* $w_2(\xi)=0$, $w_4(\xi)=0$ ($\chi(\xi)=0$ *if* $n=4$) *and* $p_1(\xi)=\sum_{i=1}^n \lambda_i^2\cdot u^2$ *with* $u\in H^2(X, \boldsymbol{Z})$, $q\cdot u=0$.

Consider a unitary representation $\tau: \boldsymbol{Z}_q\to U(n)$. It factorizes through the maximal torus of $U(n)$ and hence is of the form $\tau=(\alpha^{\lambda_1}, \cdots, \alpha^{\lambda_n})$: $\boldsymbol{Z}_q\to U(1) \hookrightarrow U(n)$. A result similar to 3.3 implies via the classification theorems of [7], [11].

COROLLARY 3.5. *Let X be a* CW-*complex with* $\dim X\leqq 2n$ *and* $H^{2j}(X, \boldsymbol{Z})$ *without* $(j-1)!$-*torsion. Then* $\xi: X\to BU(n)$ *is τ-flat if and only if there is an* $u\in H^2(X, \boldsymbol{Z})$ *with* $q\cdot u=0$ *and* $c_i(\xi)=\sigma_i(\lambda_1, \cdots, \lambda_n)\cdot u^i$, $i=1, \cdots, n$, *where $c_i(\xi)$ is the ith Chern class of ξ.*

These results can be extended to a representation $\tau: \Phi\to G$ of a finite abelian group Φ, $G=O(n)$, $SO(n)$, $U(n)$ as follows. Let $G=O(n)$, m the number of irreducible 2-dimensional components of τ and $k=n-2m$. Then one has the following factorization of τ[2]

$$\Phi \to F_m = SO(2)^m \times Q(k) \overset{\rho}{\hookrightarrow} O(n)$$

where $Q(k)=(\boldsymbol{Z}_2)^k$ and ρ is the standard inclusion. First we compute the characteristic classes of ρ. The characteristic classes of the 1- and 2-dimensional representations of Φ are then computed by the additivity of ω_1: $\mathrm{Hom}(\Phi, O(1))\to H^1(\Phi, \boldsymbol{Z}_2)$ and χ: $\mathrm{Hom}(\Phi, SO(2))\to H^2(\Phi, \boldsymbol{Z})$. A detailed exposition will appear in the American Journal of Math.

[2] We are indebted to J. M. G. Fell for pointing out this fact.

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