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CHARACTERISTIC INVARIANTS OF FOLIATED BUNDLES*

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This paper gives a construction of characteristic invariants of foliated principal bundles in the category of smooth and complex manifolds or non-singular algebraic varieties. It contains a generalization of the Chern-Weil theory requiring no use of global connections. This construction leads for foliated bundles automatically to secondary characteristic invariants. The generalized Weil-homomorphism induces a homomorphism of spectral sequences. On the E_1-level this gives rise to further characteristic invariants (derived characteristic classes). The new invariants are geometrically interpreted and examples are discussed.

0. Introduction

In this paper we describe the construction of characteristic invariants for foliated bundles as announced in the preprints [32][33] and the notes [34][35].

A generalization of the Chern-Weil theory to foliated bundles is made which applies as well in the context of smooth and complex manifolds as for non-singular algebraic varieties and which requires no use of global connections. This construction leads for foliated bundles automatically to secondary characteristic invariants. The generalized Weil-homomorphism can be interpreted as a homomorphism of spectral sequences. On the E_1-level it leads to the con-

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struction of further characteristic classes. These derived characteristic classes give a generalization to foliated bundles of the characteristic invariants considered by Atiyah for holomorphic bundles [1] and which are interpreted by Grothendieck as invariants in the Hodge spectral sequence of De Rham cohomology [23][27]. The new invariants are geometrically interpreted and examples are discussed.

This work grew out of our extensive studies of foliated bundles, called ($\underline{L}$,$\underline{O}$)-modules in [29][30]. After seeing the Chern-Simons construction of secondary classes [11], we realized that Bott's vanishing theorem [5] interpreted for the Weil-homomorphism of a foliated bundle gave rise to new invariants in the sense of section 3, i.e. the contractible Weil algebra could be replaced by a cohomologically non-trivial algebra W/F. The first published announcement of our construction is [31].

We learned then about the Bott-Milnor construction [6] of characteristic invariants of foliations. The discovery of Godbillon-Vey [17] showed the interest of the Gelfand-Fuks cohomology of formal vectorfields [14][15]. Bott-Haefliger constructed in [8][25] invariants of Γ-foliations, generalizing the Godbillon-Vey classes. In this construction Γ denotes a transitive pseudogroup of diffeomorphisms on open sets of $\mathbb{R}^q$. If the construction here presented is applied to the transversal bundle of a Γ-foliation, it leads to the same invariants in the cases in which Γ is the pseudogroup of all diffeomorphisms of $\mathbb{R}^q$ or all holomorphic diffeomorphisms of $\mathbb{C}^q$. It is known on the other hand that this is not so in the symplectic case.

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1. Foliated bundles

We consider the categories of smooth and complex analytic manifolds ($\Lambda = \mathbb{R}$ or $\mathbb{C}$) or non-singular algebraic varieties over a field (alg. closed) Λ of characteristic zero. $\underline{O}=\underline{O}_M$ denotes the structure sheaf, $\Omega_M^{\bullet}$ the De Rham complex and $\underline{T}_M$ the tangent sheaf of M. To allow the discussion of singular foliations on M, we adopt the following point of view.

1.1 DEFINITION. A foliation on M is an integrable $\underline{O}_M$-module of 1-forms $\Omega \subset \Omega_M$, i.e. generating a differential ideal $\Omega.\Omega_M^{\bullet}$ in $\Omega_M^{\bullet}$. This means that for $\omega \in \Omega$ locally $d\omega = \sum_i \omega_i \wedge \alpha_i$ with $\omega_i \in \Omega$ and $\alpha_i \in \Omega_M^1$.

Denote by $\underline{L} \subset \underline{T}_M$ the annihilator sheaf of Ω, i.e. $\underline{L} = (\Omega_M^1/\Omega)^* = \underline{\mathrm{Hom}}_{\underline{O}}(\Omega_M^1/\Omega,\underline{O})$. The $\underline{O}$-submodule $\underline{L} \subset \underline{T}_M$ is then clearly a sheaf of Λ-Lie algebras. If Ω_M^1/Ω is a locally free $\underline{O}$-module of constant rank, so are Ω, $\underline{L}$ and the transversal sheaf $\underline{Q} = \underline{T}_M/\underline{L}$. This is the case of a non-singular foliation, which is usually described by the exact sequence

$$(1.2) \qquad 0 \to \underline{L} \to \underline{T}_M \to \underline{Q} \to 0 .$$

We do not wish to make this assumption on the foliation in this paper. The integer which plays a critical rôle for Ω throughout this paper is the following. Let for $x \in M$ be

$$V_x = \mathrm{im}\left(\Omega_x \otimes_{\underline{O}_x} \Lambda \;\to\; \Omega^1_{M,x} \otimes_{\underline{O}_x} \Lambda\right) \subset T^*_{M,x} \,.$$

The function $\dim_\Lambda V_x$ is lower semi-continuous on M. Define

$$q = \sup_{x \in M} \dim_\Lambda V_x \,, \qquad 0 \leqslant q \leqslant n \,. \tag{1.3}$$

Then any integer q' such that $q \leqslant q'$ will be an integer for which the construction of a generalized characteristic homomorphism holds in section 3. If e.g. Ω is locally generated over $\underline{O}_M$ by $\leqslant q'$ elements, then clearly $q \leqslant q'$ and q' will be an admissible integer. Note that for a non-singular foliation Ω we have $q = \mathrm{rank}_{\underline{O}}(\Omega)$ for the number q defined by (1.3).

Let now $P \xrightarrow{\pi} M$ be a G-principal bundle (in one of the three categories considered). We assume G connected and denote by $\underline{\underline{g}}$ its Lie algebra (over Λ). Let $\pi_*\Omega^\cdot_P$ be the direct image sheaf of $\Omega^\cdot_P$, on which G operates. $\pi^G_*\Omega^\cdot_P$ is the subsheaf of G-invariant forms on P and $\pi^G_*\Omega^\cdot_P = (\pi_*\Omega^\cdot_P)^{\underline{\underline{g}}}$, since G is connected. Note also that $\Omega^\cdot_M = (\pi_*\Omega^\cdot_P)_{\underline{\underline{g}}}$ (the $\underline{\underline{g}}$-basic elements in the sense of [9], see section 2). $P(\underline{\underline{g}}^*)$ denotes the bundle $P \times_G \underline{\underline{g}}^*$ with sheaf of sections $\underline{P}(\underline{\underline{g}}^*)$. Connections in P are then in bijective correspondence with splittings of the exact $\underline{O}$-module sequence (Atiyah-sequence [1])

$$\underline{A}(P): \qquad 0 \to \Omega^1_M \xrightarrow{\pi^*} \pi^G_*\Omega^1_P \xrightarrow{\rho} \underline{P}(\underline{\underline{g}}^*) \to 0 \,.$$

Consider for an integrable submodule $\Omega \subset \Omega^1_M$ the diagram of $\underline{O}_M$-homomorphisms

$$\begin{array}{ccccccccc}
 & & \Omega & = & \Omega & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\underline{A}(P): 0 & \longrightarrow & \Omega^1_M & \xrightarrow{\pi^*} & \pi^G_*\Omega^1_P & \xrightarrow{\rho} & \underline{P}(\underline{\underline{g}}^*) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\lambda} & \swarrow{\scriptstyle\sigma_0} & \downarrow{\scriptstyle\bar\lambda} & \swarrow{\scriptstyle\omega_0} & \| & & \\
\lambda_*\underline{A}(P): 0 & \longrightarrow & \Omega^1_M/\Omega & \xrightarrow[\bar\pi^*]{} & \pi^G_*\Omega^1_P/\Omega & \xrightarrow[\bar\rho]{} & \underline{P}(\underline{\underline{g}}^*) & \longrightarrow & 0
\end{array} \tag{1.4}$$

1.5 DEFINITION. A connection mod Ω in P is an $\underline{O}$-homomorphism $\omega_o: \underline{P}(\underline{\underline{g}}^*) \to \pi_*^G\Omega_P^1/\Omega$ which splits $\lambda_*\underline{A}(P)$. It corresponds to a unique $\underline{O}$-homomorphism $\sigma_o: \pi_*^G\Omega_P^1 \to \Omega_M^1/\Omega$ such that $\sigma_o\pi^* = \lambda$ (see diagram 1.4). The relation between ω_o and σ_o is given by

$$\bar{\pi}^*\sigma_o + \omega_o\rho = \bar{\lambda}: \ \pi_*^G\Omega_P^1 \to \pi_*^G\Omega_P^1/\Omega \ . \tag{1.6}$$

Dualizing (1.4) we get the diagram of $\underline{O}$-homomorphisms

$$\begin{array}{ccccccccc} & & & & & & \underline{Q} & & \\ & & & & & & \uparrow & & \\ 0 & \longrightarrow & \underline{P}(\underline{\underline{g}}) & \longrightarrow & \pi_*^G\underline{T}_P & \xrightarrow{\pi} & \underline{T}_M & \longrightarrow & 0 \\ & & & & & \nwarrow \sigma_o^* & \uparrow \lambda^* & & \\ & & & & & & \underline{L} = (\Omega_M^1/\Omega)^* & & \end{array} \tag{1.7}$$

The $\underline{O}$-homomorphism σ_o^* lifts vectorfields $\xi \in \underline{L}$ to G-invariant vectorfields $\sigma_o^*(\xi) = \tilde{\xi}$ on P and thus defines what one may call a partial connection in P along $\underline{L}$ (see [30] in the case of vectorbundles). For a non-singular foliation the latter viewpoint is equivalent to the point of view adopted here.

In practice a connection mod Ω in P is represented by an equivalence class of families of local connections as follows. First we need the notion of an admissible covering of M. This is an open covering $\mathcal{U} = (U_j)$ of M such that $H^q(U_\sigma,\underline{F}) = 0$, $q > 0$ for every coherent $\underline{O}$-module $\underline{F}$, where U_σ is a finite intersection of sets U_j. Admissible coverings exist in all categories considered. For a smooth manifold, a covering by normal convex neighborhoods (with respect to a Riemannian metric) is admissible. For a complex analytic manifold a Stein covering is admissible. For an algebraic variety an affine covering is admissible.

A connection mod Ω is then represented on $\mathcal{U}$ by a family $\omega = (\omega_j)$ of connections in $P|U_j$ such that on U_{ij} the difference $\omega_j - \omega_i \in \Gamma(U_{ij}, \underline{\mathrm{Hom}}_O(\underline{P}(\underline{\underline{g}}^*),\Omega))$. A connection mod Ω in P is called flat, if for a representing family $\omega=(\omega_j)$ the curvatures $K(\omega_j)$ are elements in $\Gamma(U_j,(\Omega.\Omega_M^{\bullet})^2 \otimes_O \underline{P}(\underline{\underline{g}}))$, where $\Omega.\Omega_M^{\bullet}$ denotes the ideal generated by Ω in $\Omega_M^{\bullet}$. The local connections ω_j are then called adapted (to the flat connection mod Ω in P). Our objects of study are then defined as follows.

1.8 DEFINITION. An Ω-foliated bundle (P,ω_o) is a principal bundle P equipped with a flat connection ω_o mod Ω.

This notion has been extensively used in [29],[30]. A similar notion has been used by Molino [42]. In the smooth or complex analytic case this means that the flow on M of a vectorfield $\xi \in \underline{L}$ lifts to a flow of G-bundle automorphisms of P generated by $\sigma_o^*(\xi) \in \pi_*^G \underline{T}_P$. If the sheaf $\underline{L}$ is defined by a finite-dimensional Lie algebra $\underline{\underline{\ell}}$ of vectorfields acting on M, then a lift of this action to P defines a foliation of P. See [29],[30] for more details. We describe now examples of foliated bundles.

1.9. $\Omega = \Omega_M^1$. In this case $\underline{L} = \{0\}$ and a foliated bundle is an ordinary principal bundle with no further data.

1.10. $\Omega = \{0\}$. In this case $\underline{L} = \underline{T}_M$ and a foliated bundle is a flat bundle equipped with a flat connection.

1.11. The transversal bundle of a non-singular foliation. In this case P is the frame-bundle of $Q = T_M/L$, equipped with the connection defined by Bott [5].

1.12. Submersions. Let $f: M \to X$ be a submersion and $\Omega = f^*\Omega_X^1$. In this case $\underline{L} = \underline{T}(f)$, the sheaf of tangent vectorfields along the fibers of f. The pullback $P = f^*P'$ of any principal G-bundle $P' \to X$ admits a canonical foliation with respect to Ω which is obtained as a special case of the following procedure.

1.13. Let $\mathfrak{U}$ be an open covering of M such that $P|\mathfrak{U}$ is trivial. Let $s_j: U_j \to P|U_j$ be trivializations and consider the corresponding flat connections ω_j in $P|U_j$ $(s_j^*\omega_j=0)$. With respect to a foliation Ω on M the family $\omega=(\omega_j)$ defines an Ω-foliation on P if and only if $(g_{ij}^{-1}\circ Dg_{ij})^*$: $\underline{\underline{g}}^* \to \Gamma(U_{ij},\Omega_M^1)$ has values in Ω, i.e. the coordinate functions g_{ij}: $U_{ij} \to G$ defined by $s_j = s_i \cdot g_{ij}$ are locally constant along the leaves of Ω. For a foliation defined by a Haefliger Γ-cocycle $\{f_j^i, \gamma_{ij}\}$ $(\Omega|U_j = \underline{O}|U_j(df_j^1 \ldots df_j^q))$ [24], this procedure defines a canonical Ω-foliation on the transversal frame bundle $F(\Omega)$.

Consider now the Weil-homomorphism of differential graded (DG)-algebras

$$(1.14) \qquad k(\omega):\ W(\underline{\underline{g}}) \ \to\ \Gamma(P,\Omega_P^\bullet)$$

defined by a connection ω in P [9]. Here $W(\underline{\underline{g}})$ denotes the Weil-algebra of the Lie algebra $\underline{\underline{g}}$ of the connected group G and $\Gamma(P,\Omega_P^\bullet)$ the algebra of global forms on P. This is the homomorphism inducing on the subalgebra of invariant polynomials $I(g) \subset W(g)$ the Chern-Weil homomorphism which assigns to $\Phi \in I(\underline{\underline{g}})$ the De Rham cohomology class $[k(\omega)\Phi] \in H_{DR}(M)$.

For a foliated bundle let now ω be a connection in P which is <u>adapted</u> to the foliation ω_o of P, i.e. a splitting of $\underline{A}(P)$ such that $\overline{\lambda}\circ\omega = \omega_o$ in diagram (1.4). We observe that the Weil-homomorphism (1.14) is then a <u>filtration-preserving</u> map in the following sense

$$(1.15) \qquad k(\omega):\ F^{2p}W(\underline{\underline{g}}) \ \to\ F^p\Gamma(P,\Omega_P^\bullet), \quad p \geqslant 0.$$

The filtration on $W(\underline{\underline{g}})$ is given by

$$(1.16) \quad F^{2p}W(\underline{\underline{g}}) = S^p(\underline{\underline{g}}^*).W(\underline{\underline{g}}) \ , \quad F^{2p-1}W \equiv F^{2p}W \ .$$

Further define [31]

$$(1.17) \qquad F^p\Gamma(P,\Omega_P^\bullet) = \Gamma\big(P,(\pi^*\Omega.\Omega_P^\bullet)^p\big) \ ,$$

where $(\pi^*\Omega.\Omega_P^1)^p$ denotes the p-th power of the ideal generated by $\pi^*\Omega$ in $\Omega_P^\bullet$. Both (1.16)(1.17) define decreasing ideal filtrations and these are preserved by the Weil-homomorphism. The fact that $F^p\Gamma(P,\Omega_P^\bullet) = 0$ for $p > q$, where q is the integer defined in (1.3), implies by (1.15) that $k(\omega)F^{2(q+1)} = 0$ and in particular $k(\omega)I(\underline{\underline{g}})^{2(q+1)} = 0$. This is Bott's vanishing theorem [5]. Moreover this fact gives rise to a homomorphism $W(\underline{\underline{g}})/F^{2(q+1)}W(\underline{\underline{g}}) \to \Gamma(P,\Omega_P^{\bullet})$, which in cohomology gives rise to secondary characteristic classes. Since the Weil-homomorphism is filtration-preserving it induces a morphism of the corresponding two spectral sequences. This will be studied in sections 6 to 8.

2. The semi-simplicial Weil algebras

The construction of the Weil-homomorphism $k(\omega)$ and its filtration properties for a foliated bundle depend on the existence of a global connection ω in P adapted to the foliation of P. We wish to generalize the construction of $k(\omega)$ so as to work also in the context of complex manifolds and non-singular algebraic varieties over a field of characteristic zero, where the existence of such connections in P cannot be generally assumed.

Consider an admissible covering $\mathfrak{U} = (U_j)$ of M and a family $\omega = (\omega_j)$ of local connections ω_j in $P|U_j$ adapted to the flat connection in P mod Ω. They always exist by (1.4) in view of the admissibility of $\mathfrak{U}$. Then $\omega = (\omega_j)$ is a connection

$$\Lambda^{\bullet}(\underline{\underline{g}}^*) \to \check{C}^{0}(\mathfrak{U},\pi_*\Omega_P^{\bullet}) \subset \check{C}^{\bullet}(\mathfrak{U},\pi_*\Omega_P^{\bullet})$$

in the (non-commutative) DG-algebra of Čech cochains $\check{C}^{\bullet}(\mathfrak{U},\pi_*\Omega_P^{\bullet})$ of the covering $\mathfrak{U}$ with coefficient-system defined by $\pi_*\Omega_P^{\bullet}$. $\check{C}$ is an algebra with respect to the assoc. Alexander-Whitney multiplication of cochains. As $W(\underline{\underline{g}})$ is universal only for connections in commutative DG-algebras [9], we wish to define an algebra $W_1(\underline{\underline{g}})$ which serves as domain of definition of a multiplicative generalized Weil-homomorphism with target $\check{C}$ and which has the same cohomological properties as $W(\underline{\underline{g}})$. A construction of the characteristic homomorphism $I(\underline{\underline{g}}) \to H_{DR}(M)$ using local connections has been indicated by Baum-Bott [2,p.34].

We need the notion of a $\underline{\underline{g}}$-DG-algebra A with respect to a Lie algebra $\underline{\underline{g}}$ (all algebras are over the groundfield Λ). This is a (not necessarily commutative) DG-algebra A equipped with Λ-derivations of $\theta(x)$ of degree zero, $i(x)$ of degree -1 for $x \in \underline{\underline{g}}$, $i(x)^2 = 0$ and satisfying formulas (1)(2)(3) of [9, exp. 19]. For any subalgebra $\underline{\underline{h}} \subset \underline{\underline{g}}$ we use the notations

$$A^{\underline{\underline{h}}} = \{a \in A \mid \theta(x)a = 0 \text{ for all } x \in \underline{\underline{h}}\} ,$$

$$A^{i(\underline{\underline{h}})} = \{a \in A \mid i(x)a = 0 \text{ for all } x \in \underline{\underline{h}}\} \text{ and}$$

$$A_{\underline{\underline{h}}} = A^{\underline{\underline{h}}} \cap A^{i(\underline{\underline{h}})} \quad (\underline{\underline{h}}\text{-basic elements in } A).$$

To explain the construction of $W_1(\underline{\underline{g}})$, we consider first a semi-simplicial object in the category of Lie algebras defined by $\underline{\underline{g}}$ as follows. Let $\underline{\underline{g}}^{\ell+1}$ denote for $\ell \geq 0$ the $(\ell+1)$-fold product of $\underline{\underline{g}}$ with itself. Define for $0 \leq i \leq \ell+1$, $0 \leq j \leq \ell$

$$\varepsilon_i^\ell : \underline{\underline{g}}^{\ell+2} \to \underline{\underline{g}}^{\ell+1}, \quad \varepsilon_i^\ell(x_0, \ldots, x_{\ell+1}) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{\ell+1})$$

$$\mu_j^\ell : \underline{\underline{g}}^{\ell+1} \to \underline{\underline{g}}^{\ell+2}, \quad \mu_j^\ell(x_0, \ldots, x_\ell) = (x_0, \ldots, x_j, x_j, x_{j+1}, \ldots, x_\ell).$$

Then ε and μ are the face and degeneracy maps for the semi-simplicial object in question and satisfy the usual relations (see e.g. [18,p.271] for the dual relations).

Next consider the Weil-algebra as a contravariant functor from Lie algebras to $\underline{\underline{g}}$-DG-algebras and apply it to the semi-simplicial object discussed. This gives rise to a cosemi-simplicial object $W_1(\underline{\underline{g}})$ in the category of g-DG-algebras. Note that

$$W_1^\ell(\underline{\underline{g}}) = W(\underline{\underline{g}}^{\ell+1}) \cong W(\underline{\underline{g}})^{\otimes \ell+1}$$

and the face and degeneracy maps $\varepsilon_i^\ell = W(\varepsilon_i^\ell) : W_1^\ell \to W_1^{\ell+1}$, $\mu_i^\ell = W(\eta_i^\ell) : W_1^{\ell+1} \to W_1^\ell$ are given by the inclusions omitting the i-th factors and multiplication of the i-th and (i+1)-th factors.

$W_1(\underline{\underline{g}})$ can in turn be given the structure of a (non-commutative) $\underline{\underline{g}}$-DG-algebra. For this purpose consider $W_1(\underline{\underline{g}})$ as the object

$$W_1(\underline{\underline{g}}) = \bigoplus_{\ell \geq 0} W_1^\ell(\underline{\underline{g}}).$$

Then W_1 can be interpreted as a cochain-complex on the semi-simplicial complex P (= point in the category of semi simplicial complexes) with one ℓ-simplex σ_ℓ for each $\ell \geqslant 0$ and with coefficients in the system assigning to every σ_ℓ the algebra $W_1^\ell = W^{\otimes \ell+1}$. As such it is equipped with the associative Alexander-Whitney multiplication.

The differential in W_1 is defined as follows. First let

$$(2.1) \qquad \delta_\ell = \sum_{i=o}^{\ell=1} (-1)^i \varepsilon_i^\ell : \quad W_1^\ell \to W_1^{\ell+1} .$$

If d denotes the differential on W_1^ℓ (induced from the differential on W), then the formula

$$(2.2) \qquad D = \delta_\ell + (-1)^\ell d \quad \text{on } W_1^\ell$$

defines a differential D on W_1 which turns it into a DG-algebra. It is a $\underline{\underline{g}}$-DG-algebra with respect to the $\underline{\underline{g}}$-operations on $W_1^\ell(\underline{\underline{g}}) = W(\underline{\underline{g}})^{\ell+1}$ obtained by restricting along the diagonal $\Delta: \underline{\underline{g}} \to \underline{\underline{g}}^{\ell+1}$. The construction performed with the functor W can now obviously be repeated with the functor W_1, which leads to a sequence of iterated cosemi-simplicial Weil-algebras $W_o(\underline{\underline{g}}) = W(\underline{\underline{g}}), W_1(\underline{\underline{g}}), W_2(\underline{\underline{g}}), \ldots$. The canonical projections

$$(2.3) \qquad \rho_s : \quad W_s(\underline{\underline{g}}) \to W_s^o(\underline{\underline{g}}) = W_{s-1}(\underline{\underline{g}}), \quad s > 0$$

are $\underline{\underline{g}}$-DG-algebra homomorphisms.

We proceed now to define inductively even filtrations $F_s^\bullet(\underline{\underline{g}})$ with respect to $\underline{\underline{g}}$ on $W_s(\underline{\underline{g}}^m)$ $(s \geqslant 0,\ m \geqslant 1)$ such that $F_o^\bullet(g)$ on $W_o(\underline{\underline{g}}) = W(\underline{\underline{g}})$ is given by (1.16):

$$(2.4) \qquad \begin{aligned} F_o^{2p}(\underline{\underline{g}})W(\underline{\underline{g}}^m) &= \mathrm{id}\big(W^+(\underline{\underline{g}}^m)^{i(\underline{\underline{g}})}\big)^p \\ F_s^\bullet(\underline{\underline{g}})W_s(\underline{\underline{g}}^m) &= \bigoplus_{\ell \geqslant o} F_s^\bullet(\underline{\underline{g}}) W_s^\ell(\underline{\underline{g}}^m) \\ &= \bigoplus_{\ell \geqslant o} F_{s-1}^\bullet(\underline{\underline{g}}) W_{s-1}\big((\underline{\underline{g}}^m)^{\ell+1}\big) , \quad s \geqslant 1 \end{aligned}$$

The odd filtrations are defined by $F_s^{2p-1} = F_s^{2p}$. The face and degeneracy operators of W_s are filtration-preserving. The filtration F_s is functorial for maps $W_s(\underline{\underline{g}}) \to W_s(\underline{\underline{g}}')$

induced by Lie homomorphisms $\underline{\underline{g}}' \to \underline{\underline{g}}$.

2.5 LEMMA. $F_s^{\cdot}W_s$ *is an even, bihomogeneous and multiplicative filtration by* $\underline{\underline{g}}$-*DG-ideals.*

The split exact sequence

$$0 \to \underline{\underline{g}} \xrightarrow{\Delta} \underline{\underline{g}}^{\ell+1} \to V_\ell \to 0$$

defines the $\underline{\underline{g}}$-module V_ℓ, whose dual is given by $V_\ell^* = \ker \Delta^* = \{(\alpha_o,\dots,\alpha_\ell) \mid \sum_{i=o}^{\ell} \alpha_i = 0\}$. The filtration $F_1^{2p}W_1^\ell(\underline{\underline{g}}) = F_o^{2p}W(\underline{\underline{g}}^{\ell+1})$ is then given by

$$(2.6) \qquad F_1^{2p}W_1^\ell(\underline{\underline{g}}) \cong \bigoplus_{|r| \geq p} \Lambda^{\cdot}\underline{\underline{g}}^* \otimes \left(\Lambda^{\cdot}V_\ell^* \otimes S^{\cdot}(\underline{\underline{g}}^{*\ell+1})\right)^{|r|}$$

where the reduced degree $|r|$ is determined by $\deg \Lambda^1 V_\ell^* = \deg S^1(\underline{\underline{g}}^{*\ell+1}) = 1$. For the graded object we have therefore

$$G_1^{2p}W_1^\ell(\underline{\underline{g}}) = \Lambda\underline{\underline{g}}^* \otimes \left(\Lambda V_\ell^* \otimes S(\underline{\underline{g}}^{*\ell+1})\right)^{|p|}.$$

For every subalgebra $\underline{\underline{h}} \subset \underline{\underline{g}}$ the filtrations $F_s^{\cdot}$ induce filtrations on the relative algebras

$$(2.7) \qquad W_s(\underline{\underline{g}},\underline{\underline{h}}) = \left(W_s(\underline{\underline{g}})\right)_{\underline{\underline{h}}}, \quad s \geq 0 .$$

It is immediate that the canonical projections $\rho_s\colon W_s \to W_{s-1}$ are filtration-preserving. Define for $s \geq 0$

$$(2.8) \qquad W_s(\underline{\underline{g}},\underline{\underline{h}})_k = W_s(\underline{\underline{g}},\underline{\underline{h}})/F_s^{2(k+1)}W_s(\underline{\underline{g}},\underline{\underline{h}}), \quad k \geq 0 .$$

For $k = \infty$ we set $F^\infty = \bigcap_{p \geq o} F^{2p} = 0$, so that

$$(2.9) \qquad W_s(\underline{\underline{g}},\underline{\underline{h}})_\infty = W_s(\underline{\underline{g}},\underline{\underline{h}}) .$$

The main result concerning the relationship between the W_s is as follows. The proof will appear elsewhere.

2.10 THEOREM. *Let* $(\underline{\underline{g}},\underline{\underline{h}})$ *be a reductive pair of Lie algebras. The homomorphisms of spectral sequences induced by the filtration-preserving canonical projections* $\rho_s\colon W_s(\underline{\underline{g}},\underline{\underline{h}}) \to W_{s-1}(\underline{\underline{g}},\underline{\underline{h}})$ *induce isomorphisms on the* E_1*-level and hence* isomorphisms for every $0 \leqslant k \leqslant \infty$

$$H(\rho_s): \ H(W_s(\underline{\underline{g}},\underline{\underline{h}})_k) \xrightarrow{\cong} H(W_{s-1}(\underline{\underline{g}},\underline{\underline{h}})_k) \ , \quad s > 0.$$

The E_1-term can be computed as follows:

2.11 THEOREM [34]. *Let* $(\underline{\underline{g}},h)$ *be a reductive pair of Lie algebras*, $0 \leqslant k \leqslant \infty$.

(i) $E_1^{2p,q}(W(\underline{\underline{g}},\underline{\underline{h}})_k) \cong H^q(\underline{\underline{g}},\underline{\underline{h}}) \otimes I^{2p}(\underline{\underline{g}})_k$;

(ii) $d_{2r+1} = 0$ *and* d_{2r} *is induced by a transgression* $\tau_{\underline{\underline{g}}}: P_{\underline{\underline{g}}} \to I(\underline{\underline{g}})_k$;

(iii) *the terms* E_{2r}, E_∞ *and* $H^\bullet(W(\underline{\underline{g}},\underline{\underline{h}})_k)$ *can be computed under a mild condition on* $(\underline{\underline{g}},\underline{\underline{h}})$.

Here $H(\underline{\underline{g}},\underline{\underline{h}})$ denotes $H(\Lambda^\bullet(\underline{\underline{g}}/\underline{\underline{h}})^{*\underline{\underline{h}}})$, which can be computed [9] [19] as

(2.12) $$H(\underline{\underline{g}},\underline{\underline{h}}) \cong \Lambda\hat{P} \otimes I(\underline{\underline{h}})/I(\underline{\underline{g}})^+.I(\underline{\underline{h}})$$

for pairs $(\underline{\underline{g}},\underline{\underline{h}})$ satisfying the condition

(2.13) $$\dim \hat{P} = \operatorname{rank} \underline{\underline{g}} - \operatorname{rank} \underline{\underline{h}}$$

for a Samelson space $\hat{P} \subset P_{\underline{\underline{g}}}$ of primitive elements of $\underline{\underline{g}}$. The condition mentioned under (iii) is the following [34]:

(2.14) There exists a transgression $\tau_{\underline{\underline{g}}}$ for $\underline{\underline{g}}$ such that

$$\ker(\iota^*: I(\underline{\underline{g}}) \to I(\underline{\underline{h}})) = \operatorname{ideal}(\tau_{\underline{\underline{g}}}\hat{P}) \subset I(\underline{\underline{g}}) \cong S(\tau_{\underline{\underline{g}}}P_{\underline{\underline{g}}}).$$

This condition is satisfied for all *symmetric* pairs and many interesting examples. Condition (2.14) implies (2.13) and has been used for the general computation in [34]. For the pairs $(\underline{\underline{g\ell}}(n), \underline{\underline{so}}(n))$, $(\underline{\underline{g\ell}}(n), O(n))$ and $k = n$ the algebras $H(W(\underline{\underline{g}},\underline{\underline{h}})_k)$ have been computed by Vey [16].

3. The generalized characteristic homomorphism of a foliated bundle

We return to the geometric situation considered before, i.e. a foliated bundle $P \xrightarrow{\pi} M$ equipped with a family $\omega=(\omega_j)$

of adapted connections on $P|U_j$ with respect to an admissible covering $\mathfrak{U} = (U_j)$ of M. We define then a homomorphism

$$(3.1) \qquad k_1(\omega): \quad W_1(\underline{\underline{g}}) \to \check{C}(\mathfrak{U}, \pi_* \Omega_P)$$

as follows. For $\ell \geq 0$, let $\sigma = (i_o, \dots, i_\ell)$ be an ℓ-simplex of the nerve $N(\mathfrak{U})$. Consider the compositions
$\omega_{i_j}: \quad \Lambda(\underline{\underline{g}}^*) \to \Gamma(U_{i_j}, \pi_*\Omega_P^\bullet) \to \Gamma(U_\sigma, \pi_*\Omega_P^\bullet)$ for $j = 0, \dots, \ell$.
This defines

$$(3.2) \qquad k(\omega_\sigma): \quad W(\underline{\underline{g}}^{\ell+1}) \to \Gamma(U_\sigma, \pi_*\Omega_P^\bullet)$$

as the universal $\underline{\underline{g}}$-DG-algebra homomorphism extending

$$(3.3) \qquad \Lambda(\omega_\sigma): \quad \Lambda(\underline{\underline{g}}^{*\,\ell+1}) \to \Gamma(U_\sigma, \pi_*\Omega_P^\bullet)$$

given on the factor j by ω_{i_j}. We get therefore a homomorphism $k_1(\omega): W_1^\ell(\underline{\underline{g}}) \to C^\ell(\mathfrak{U}, \pi_*\Omega_P^\bullet)$ by setting $k_1(\omega)_\sigma = k(\omega_\sigma)$.

(3.1) is a homomorphism of $\underline{\underline{g}}$-DG-algebras, where the $\underline{\underline{g}}$-operations on $\check{C}(\mathfrak{U}, \pi_*\Omega_P^\bullet)$ are defined simplex-wise by $(\theta(x)\varphi)_\sigma = \theta(x)\varphi_\sigma$ and $(i(x)\varphi)_\sigma = (-1)^\ell i(x)\varphi_\sigma$ for $\varphi \in \check{C}^\ell(\mathfrak{U}, \pi_*\Omega_P^\bullet)$ and $\sigma \in N(\mathfrak{U})_\ell$. (3.1) is the <u>generalized Weil-homomorphism</u> of P.

The crucial result for our construction is the following:

3.4 <u>PROPOSITION</u>. $k_1(\omega)$ <u>is filtration-preserving in the sense that</u>

$$k_1(\omega): \quad F_1^{2p} W_1 \to F^p \check{C}(\mathfrak{U}, \pi_*\Omega_P^\bullet), \quad p \geq 0.$$

The filtration on the image complex is defined by

$$(3.5) \qquad F^p \check{C}(\mathfrak{U}, \pi_*\Omega_P^\bullet) = \check{C}(\mathfrak{U}, F^p\Omega_P^\bullet) = \check{C}(\mathfrak{U}, (\Omega \cdot \pi_*\Omega_P^\bullet)^p) .$$

Similarly $\check{C}(\mathfrak{U}, \Omega_M^\bullet)$ is filtered by

$$(3.5') \qquad F^p \check{C}(\mathfrak{U}, \Omega_M^\bullet) = \check{C}(\mathfrak{U}, F^p\Omega_M^\bullet) = \check{C}(\mathfrak{U}, (\Omega . \Omega_M)^p) .$$

Proposition 3.4 follows by the multiplicativity of $k_1(\omega)$ and (2.6) from

$$k(\omega_\sigma)\tilde{\alpha} \in \Gamma(U_\sigma, F^1\pi_*\Omega_P^2),\ k(\omega_\sigma)\alpha \in \Gamma(U_\sigma, F^1\pi_*\Omega_P^1)$$

for $\tilde{\alpha} \in S^1(\underline{\underline{g}}^{*\ell+1})$, $\alpha \in \Lambda^1 V_\ell^*$.

For our construction it is essential to observe that this filtration is zero for $p > q$, where q is the integer as defined in (1.3). It follows from (3.4) that $k_1(\omega)$ induces a homomorphism $k_1(\omega)\colon W_1(\underline{\underline{g}})_q \to \check{C}$, which in cohomology gives rise to the generalized characteristic homomorphism.

More generally for a (connected) closed subgroup $H \subset G$ with Lie algebra $\underline{\underline{h}} \subset \underline{\underline{g}}$ we have an induced map between the $\underline{\underline{h}}$-basic algebras of (3.1). If $\hat{\pi}\colon P/H \to M$ denotes the projection induced from $\pi\colon P \to M$, then $(\pi_*\Omega_P^\bullet)_{\underline{\underline{h}}} = \hat{\pi}_*\Omega_{P/H}^\bullet$ and hence

$$k_1(\omega)\colon\ W_1(\underline{\underline{g}},\underline{\underline{h}}) \to \check{C}(\mathfrak{U}, \hat{\pi}_*\Omega_{P/H})$$

Since this map is still filtration-preserving, and the filtration on the RHS is zero for degrees exceeding q, we get an induced homomorphism, also denoted by $k_1(\omega)$:

$$(3.6)\quad k_1(\omega)\colon\ W_1(\underline{\underline{g}},\underline{\underline{h}})_q \to \check{C}(\mathfrak{U}, \hat{\pi}_*\Omega_{P/H})\ .$$

To define invariants in the base manifold M, we need an H-reduction of P given by a section $s\colon M \to P/H$ of $\hat{\pi}\colon P/H \to M$ as the pull-back $P' = s^*P$. Before we formulate the result, observe that $H^\bullet(\check{C}(\mathfrak{U}, \hat{\pi}_*\Omega_{P/H}^\bullet))$ maps canonically into the hypercohomology $\mathbb{H}^\bullet(M, \hat{\pi}_*\Omega_{P/H}^\bullet)$, which maps under $\hat{\pi}^*$ into $\mathbb{H}^\bullet(P/H, \Omega_{P/H}^\bullet)$, the De Rham cohomology $H_{DR}(P/H)$ [22]. The map (3.6) gives then under observation of Theorem 2.10 rise to the homomorphism in the following theorem:

3.7 THEOREM. *Let (P,ω_o) be an Ω-foliated principal G-bundle, $H \subset G$ a (connected) closed subgroup such that $(\underline{\underline{g}},\underline{\underline{h}})$ is a reductive pair of Lie algebras, and q the number defined by* (1.3).

(i) *There exists a homomorphism depending only on* (P,ω_o)

$$(3.8)\quad k_*\colon\ H^\bullet(W^\bullet(\underline{\underline{g}},\underline{\underline{h}})_q) \to \mathbb{H}^\bullet(M, \hat{\pi}_*\Omega_{P/H}^\bullet) \to H_{DR}^\bullet(P/H).$$

(ii) *If* P *admits an* H-*reduction* $P' = s^*P$ *given by a section* s *of* $\hat{\pi}$, *there exists a homomorphism*

$$(3.9) \qquad \Delta_* \equiv s^* \circ k_*: \; H\big(W^\bullet(\mathfrak{g},\mathfrak{h})_q\big) \to H^\bullet_{DR}(M).$$

This is the generalized characteristic homomorphism of P *(depending on* P'*).*

To establish the independence of k_* for two choices $\omega^0 = (\omega^0_j)$ and $\omega^1 = (\omega^1_j)$ of adapted connections on a covering $\mathfrak{U} = (U_j)$ we consider the commutative diagram

$$(3.10) \qquad \begin{array}{ccc} W_2^\bullet(\mathfrak{g},\mathfrak{h})_q & \xrightarrow{\;k_2(\omega^0,\omega^1)\;} & C^\bullet(\Delta_1,\check{C}^\bullet) \\ \rho_2 \big\downarrow & & \big\downarrow j_i \\ W_1^\bullet(\mathfrak{g},\mathfrak{h})_q & \xrightarrow{\;k_1(\omega^i)\;} & \check{C}^\bullet \end{array} \qquad (i=0,1)$$

where $C(\Delta_1,\check{C})$ is the cochain-complex on the standard 1-simplex Δ_1 with coefficients in the constant system $\check{C}$, j_i is the restriction to the i-th vertex (i=0,1) and $k_2(\omega^0,\omega^1)$ is defined analogously to k_1. As the vertical maps induce isomorphisms in cohomology (2.10), and $H(j_i)$ is independent of i, it follows that $H\big(k_1(\omega^0)\big) = H\big(k_1(\omega^1)\big)$.

The construction of Δ_* is functorial in P. It is also functorial in $(\mathfrak{g},\mathfrak{h})$ in an obvious sense.

For $\mathfrak{h} = \mathfrak{g}$ we take $s = id: M \to P/G = M$. Then $H\big(W(\mathfrak{g},\mathfrak{g})_q\big) = I(\mathfrak{g})_q$ and

$$(3.12) \qquad \Delta_* = k_*: \; I(\mathfrak{g})_q \to H_{DR}(M) .$$

This is the Chern-Weil homomorphism of P, but constructed *without the use of a global connection* on P. Note that on the cochain-level it is realized with the help of a family $\omega = (\omega_j)$ of adapted connections ω_j on $P|U_j$ ($\mathfrak{U} = (U_j)$ an admissible covering of M) as a homomorphism

$$(3.13) \qquad k_1(\omega): \; W_1^\bullet(\mathfrak{g},\mathfrak{g})_q \to \check{C}^\bullet(\mathfrak{U},\Omega^\bullet_M) .$$

By theorem 2.17 we have

$$H(W_1(\underline{\underline{g}},\underline{\underline{g}})_q) \cong H(W(\underline{\underline{g}},\underline{\underline{g}})_q) \cong I(\underline{\underline{g}})_q \cong I(\underline{\underline{g}})/F^{2(q+1)}I(\underline{\underline{g}}).$$

For H = {e} we have by Theorem 3.7, (i) a well-defined homomorphism

$$(3.14) \qquad k_*:\ H(W(\underline{g})_q) \to \mathbb{H}^{\cdot}(M,\pi_*\Omega^{\cdot}_P) \to H_{DR}(P).$$

Thus for every $\omega \in W(\underline{\underline{g}})^+$ such that $d_W(\omega) \in F^{2(q+1)}W(\underline{\underline{g}})$ there is a well-defined De Rham class $k_*(\omega) \in H_{DR}(P)$. This is a construction of the type considered by Chern and Simons [11] [12], where they consider more particularly $\Phi \in I(g)$ such that $k(\omega)\Phi = 0 \in \Gamma(M,\Omega^{\cdot}_M)$. As mentioned in the introduction, this observation was one of the motivations for our construction.

For a non-singular foliation Ω with oriented transversal bundle Q let $P = F(\Omega) = F(Q^*)$ be the canonically foliated $GL^+(q)$-frame bundle of $\Omega = \underline{Q}^*$ (1.11). For H=SO(q) the bundle P/H has the contractible fibre $GL^+(q)/SO(q)$ and hence there exists up to homotopy a unique section s of $\hat{\pi}$. The generalized characteristic homomorphism Δ_* defines then invariants of the <u>foliation</u> Ω in $H^{\cdot}_{DR}(M)$. Using the Gelfand-Fuks cohomology of formal vectorfields [15], Bott-Haefliger construct in [8] [25] invariants of Γ-foliations, generalizing the classes discovered by Godbillon-Vey [17]. Here Γ denotes a transitive pseudogroup of diffeomorphisms on open sets of $\mathbb{R}^q$, and a Γ-foliation on M is defined by a family of submersions $f_U: U \to f_U(U) \subset \mathbb{R}^q$, $\mathcal{U} = \{U\}$ an open covering of M, these submersions differing on $U \cap V$ by an element of Γ. For Γ the pseudogroup of all diffeomorphisms of $\mathbb{R}^q$ or all holomorphisms of C^q the two constructions give the same invariants, but this is known not to be so in the symplectic case.

The following result gives a more detailed description of the generalized characteristic homomorphism.

3.15 <u>THEOREM</u>. <u>Let</u> P <u>be</u> <u>a</u> <u>foliated</u> <u>bundle</u> <u>as</u> <u>in</u> <u>Theorem</u> 3.7

and $P' = s^*P$ an H-reduction of P.

(i) There is a split exact sequence of algebras

$$(3.16)\quad 0 \to H(K_q^{\cdot}) \to H(W(\mathfrak{g},\mathfrak{h})_q) \overset{\varkappa}{\underset{\leftarrow}{\to}} I(\mathfrak{h}) \otimes_{I(\mathfrak{g})} I(\mathfrak{g})_q \to 0$$

and the composition $\Delta_* \circ \varkappa$ is induced by the characteristic homomorphism: $I(h) \to H_{DR}(M)$ of P'.

(ii) If the foliation of P is induced by a foliation of P', then $\Delta_* | H(K_q^{\cdot}) = 0$.

The ideal $H(K_q^{\cdot}) \subset H(W(\mathfrak{g},\mathfrak{h})_q)$ is the algebra of universal secondary characteristic invariants. By part (ii) in Theorem 3.15 the secondary invariants $\Delta_*(H(K_q^{\cdot}))$ for a foliated bundle (P,ω_o) are a measure for the non-compatibility of the foliation ω_o of P with the H-reduction P'. The proof of this fact is an immediate consequence of the functoriality of Δ_*. Namely under the assumption of (ii) in 3.15 Δ_* factorizes as follows:

$$\begin{array}{ccc} H(W(\mathfrak{g},\mathfrak{h})_q) & \xrightarrow{\quad\Delta_*\quad} & \\ \downarrow & & H_{DR}(M) \\ H(W(\mathfrak{h},\mathfrak{h})_q) \cong I(\mathfrak{h})_q & \nearrow & \end{array}$$

But the vertical homomorphism is the composition

$$H(W(\mathfrak{g},\mathfrak{h})_q) \to I(\mathfrak{h}) \otimes_{I(\mathfrak{g})} I(\mathfrak{g})_q \to I(\mathfrak{h})_q$$

which implies that $\Delta_* | H(K_q) = 0$. More generally under the assumption of (ii) in 3.15 the vanishing of $k(\omega)$ on $F^{2(\ell+1)}$ for some $\ell \geqslant 0$ implies $\Delta_* | H(K_\ell^{\cdot}) = 0$.

4. Interpretation and examples of secondary characteristic classes.

Before we turn to the discussion of examples of secondary characteristic classes, we comment again on the computation of $H(W(\mathfrak{g},\mathfrak{h})_k)$ for $k \geqslant 0$. (See also the end of section 2.) For reductive pairs $H(W(\mathfrak{g},\mathfrak{h})_k)$ can be computed [34] as the cohomology of the complex

(4.1) $$A = \Lambda P_{\underline{\underline{g}}} \otimes I(\underline{\underline{g}})_k \otimes I(\underline{\underline{h}})$$

where $P_{\underline{\underline{g}}}$ denotes the primitive elements of $\underline{\underline{g}}$. The differential d_A is a derivation of degree 1, which is zero in the last two factors and is given in $P_{\underline{\underline{g}}}$ by

(4.2) $$d_A(x) = 1 \otimes \tau_g(x) \otimes 1 - 1 \otimes 1 \otimes \iota^*\tau_g(x),$$

where $\tau_{\underline{\underline{g}}}: P_{\underline{\underline{g}}} \to I(\underline{\underline{g}})$ is a transgression for $\underline{\underline{g}}$ and $i: \underline{\underline{h}} \subset \underline{\underline{g}}$. This realization of $H^{\cdot}(W(\underline{\underline{g}},\underline{\underline{h}})_k)$ allows its computation for reductive pairs satisfying condition (2.14) [34].

The spectral sequence

$$E_1^{2p,q}(W(\underline{\underline{g}},\underline{\underline{h}})_k) \cong H^q(\underline{\underline{g}},\underline{\underline{h}}) \otimes I^{2p}(\underline{\underline{g}})_k \Rightarrow H^{2p+q}(W(\underline{\underline{g}},\underline{\underline{h}})_k)$$

discussed at the end of section 2 arises from the filtration of the A-complex (4.1) by $I(\underline{\underline{g}})$. One may approximate $H(W(\underline{\underline{g}},\underline{\underline{h}})_k)$ by another spectral sequence (involving a graded Koszul complex) also deduced from A:

$$'E_1^{r,s} \cong \mathrm{Tor}_{r-s}^{I(\underline{\underline{g}})}(I(\underline{\underline{h}}),I(\underline{\underline{g}})_k)^{2r} \Longrightarrow H^{r+s}(W(\underline{\underline{g}},\underline{\underline{h}})_k).$$

For k=0 we have $I(\underline{\underline{g}})_o \cong \Lambda$ (ground field) and

$$'E_1^{r,s} \cong \mathrm{Tor}_{r-s}^{I(\underline{\underline{g}})}(I(\underline{\underline{h}}),\Lambda)^{2r} \Longrightarrow H^{r+s}(\underline{\underline{g}},\underline{\underline{h}}) .$$

For $k = \infty$ we have $I(\underline{\underline{g}})_\infty = I(\underline{\underline{g}})$, $'E_1^{r,s} = 0$ for $r \neq s$ and since $d_1 = 0$

$$'E_1^{r,s} = I(\underline{\underline{h}})^{2r} \xrightarrow{\cong} H^{2r}(W(\underline{\underline{g}},\underline{\underline{h}}))$$

whereas $H^{2r+1}(W(\underline{\underline{g}},\underline{\underline{h}})) = 0$.

4.3 Flat bundles. A flat G-bundle is a G-bundle $P \xrightarrow{\pi} M$ foliated with respect to $\Omega = \{0\} \subset \Omega_M^1$, i.e. $\underline{L} = \underline{T}_M$ and $q = 0$ (P is equipped with a curvature free global connection). The generalized characteristic homomorphism is now a map

(4.4) $$\Delta_*: \; H^{\cdot}(\underline{\underline{g}},\underline{\underline{h}}) \cong H(W(\underline{\underline{g}},\underline{\underline{h}})_o) \to H_{DR}^{\cdot}(M) .$$

It will be shown in section 8 that Δ_* may be injective in certain cases and that Δ_* is rigid in degrees > 1.

For a flat smooth M^m the tangent principal bundle F(M) is a flat GL(m)-bundle. For H = O(m) there is hence a well-defined homomorphism $\Delta_*: H(\mathfrak{gl}(m), O(m)) \to H_{DR}(M)$, defining invariants of the flat structure of M. If the primitive elements $P_{\mathfrak{gl}(m)}$ transgressing to the Chern classes $c_i \in I(\mathfrak{gl}(m))$ are denoted x_i, then $H^{\cdot}(\mathfrak{gl}(m), O(m)) \cong \Lambda^{\cdot}(x_1, x_3, \ldots, x_{m'})$, $m' = 2[\frac{m+1}{2}] - 1$, and we get the following result:

4.5 <u>THEOREM</u>. *Let* M^m *be a flat smooth manifold. There are well-defined secondary invariants*

$$\Delta_*(x_i) \in H_{DR}^{2i-1}(M) \qquad (i = 1, 3, \ldots, m').$$

For a Riemannian flat manifold these invariants are zero (by 3.15, (ii)). *Moreover, if* $h: \pi_1(M) \to GL(m)$ *denotes the holonomy of the frame bundle* $F(M)$, *we have*

$$\int_\gamma \Delta(\omega) x_1 = \int_\gamma s^* tr(\omega) = -\log|\det h(\gamma)|$$

for $\gamma \in \pi_1(M)$, *and* $s: M \to F(M)/O(m)$ *a Riemannian metric on* M.

Let M^m be a compact affine hyperbolic manifold, i.e. equipped with a flat and torsionfree connection and such that the universal covering is isomorphic to an open convex subset of $\mathbb{R}^m$ containing no complete line. The hyperbolicity of the affine structure on M is then characterized according to Koszul [37] by the existence of a closed 1-form with positive definite covariant derivative. The De Rham class of this 1-form is precisely the affine invariant $\Delta_*(x_1)$ of Theorem 4.5.

4.6 *The transversal bundle Q of a foliation*. This case has been discussed already in section 3 and it has been explained in which cases our construction furnishes the same invariants as the Bott-Haefliger construction [8][25]. If the foliation of Q is induced from a foliation of an H-reduction $(H \subset GL(q))$, this is called a *transverse* H-*structure*, Colon [13]. The secondary invariants are then trivial by Theorem 3.15, (ii).

4.7 Characteristic numbers of a foliated bundle. Let $\Omega \subset \Omega^1_M$ be a foliation on a complex manifold M and assume that Ω is locally free of rank n-1 off the disjoint union N of < n-dimensional closed submanifolds of M, $n = \dim_{\mathbb{C}} M$. The number q defined in (1.3) for Ω is then necessarily $q = n-1$, since $\dim_\Lambda V_x$ is lower semi-continuous. It follows from Theorem 3.7 that for a bundle $P \to M$ foliated with respect to Ω, the characteristic numbers necessarily vanish. Consider on the other hand the annihilator sheaf $\underline{L} = (\Omega^1_M/\Omega)^*$. Since P is foliated with respect to Ω, P carries in particular an action of $\underline{L}$ by infinitesimal bundle automorphisms. If $\underline{L}$ is of rank 1, i.e. the sheaf of sections of a holomorphic line bundle, the characteristic numbers of P can be evaluated by Bott [4] as the sum of residua attached to the singularities of $\underline{T}_M/\underline{L}$. In the situation described above this sum is necessarily zero.

4.8 Pfaffian systems. (Martinet [40]). Let the submodule $\underline{E} \subset \Omega^1_M$ be a Pfaffian system of rank p on M, i.e. the sheaf of sections of a subbundle $E \subset T^*_M$ of dimension p. Then $\underline{E}$ and $\Omega^1_M/\underline{E}$ are locally free of rank p, n-p respectively ($n = \dim M$). The characteristic system Ω of $\underline{E}$ is a foliation in the sense of section 1, i.e. generates a differential ideal in $\Omega^\cdot_M$. Martinet's result in [40] can be interpreted as showing that the frame bundle F(E) of E is foliated with respect to Ω and hence gives rise to a homomorphism

$$H(W(\underline{\underline{g\ell}}(p))_q) \to H_{DR}(F(E))$$

where q is the number defined in (1.3), the class of the system $\underline{E}$. Note that $p \leq q$ and $p = q$ if and only if the original Pfaffian system $\underline{E}$ is already involutive. One of the features of our localized construction of the characteristic homomorphism is that this example can be generalized to the holomorphic case. The same comment applies to the characteristic invariants defined recently by Malgrange for systems of smooth partial differential equations.

5. The spectral sequence associated to a foliation

From this section on we assume that a non-singular foliation

(5.1) $$0 \to \Omega \to \Omega^1_M \to \Omega^1_M/\Omega \to 0$$

is given on M, i.e. Ω^1_M/Ω -- and hence Ω -- is supposed to be locally free ($q = \mathrm{rk}_{\underline{O}}\Omega$). The finite ideal-filtration $F^p\Omega^\cdot_M = \Lambda^p\Omega\cdot\Omega^\cdot_M$ used in (3.5') determines then a multiplicative spectral sequence with respect to the hypercohomology functor $\mathbb{H}^\cdot(M;-) = R^\cdot\mathbf{\Gamma}_M$ [21, 0_{III}, 13.6.4]:

(5.2) $$E_1^{p,q}(\Omega) = \mathbb{H}^{p+q}(M;G^p\Omega^\cdot_M) \Longrightarrow \mathbb{H}^{p+q}(M;\Omega^\cdot_M) = H^{p+q}_{DR}(M).$$

Here $G^p\Omega^\cdot_M = F^p/F^{p+1}$ and the final term is equipped with the filtration $F^pH^\cdot_{DR}(M) = \mathrm{im}(\mathbb{H}^\cdot(M;F^p) \to \mathbb{H}(M;\Omega^\cdot_M))$. We shall determine the E_1- and E_2-terms of this spectral sequence. To do so we have to make extensive use of the cohomology theory of the twisted sheaf of Lie algebras $\underline{L}=(\Omega^\cdot_M/\Omega)^* \subset \underline{T}_M$ with coefficients in a $(\underline{L},\underline{O})$-module. This theory was developed in [30] and we refer to this work for details. A $(\underline{L},\underline{O})$-module is an $\underline{O}$-module $\underline{E}$ equipped with a partial curvature-free connection along $\underline{L}$

(5.3) $$\mu: \ \underline{E} \to \underline{L}^* \otimes_{\underline{O}} \underline{E}\ .$$

Equivalently $(\underline{L},\underline{O})$-modules can be described as $\underline{U}(\underline{L},\underline{O})$-modules, where $\underline{U}(\underline{L},\underline{O})$ is the universal envelope of the twisted Lie algebra $\underline{L}$ [30,§3]. If $\underline{E}$ is locally free of finite rank, a $(\underline{L},\underline{O}_M)$-module structure on $\underline{E}$ is the same as an Ω-foliation in the frame bundle $F(\underline{E})$.

5.4 EXAMPLE. Ω is a $(\underline{L},\underline{O})$-module by the Lie derivative $\theta(\xi)\alpha = i(\xi)d\alpha$, $\alpha \in \Omega$, $\xi \in \underline{L}$, $(i(\xi)\alpha = 0)$. This is the Bott-connection on the dual of the transversal bundle of $\underline{L}$. $\Lambda^p\Omega$, $p \geqslant 0$ carry then also $(\underline{L},\underline{O})$-structures in an obvious way.

For a $(\underline{L},\underline{O})$-module $\underline{E}$ there is a Chevalley-Eilenberg-type differential d_L on $\underline{T}^\cdot_{\underline{L}}(\underline{E}) \equiv \underline{\mathrm{Hom}}_{\underline{O}}(\Lambda^\cdot_{\underline{L}},\underline{E})$ [30,4.21] whereby $\underline{T}^\cdot_{\underline{L}}(\underline{E})$ becomes a complex. It is now easy to verify that

(5.5) $$G^p\Omega^{\cdot}_M \cong \underline{T}^{\cdot -p}_{\underline{L}}(\Lambda^p\Omega) ,$$

and that under this isomorphism $G(d) = d_L$ for the exterior differential d in the De Rham complex $\Omega^{\cdot}_M$. $\Lambda^p\Omega$ has the $(\underline{L},\underline{O})$ structure described in 5.4. In fact for a local splitting of (5.1) we obtain a local decomposition $\Omega^{\cdot}_M \cong \Lambda^{\cdot}\Omega \otimes_{\underline{O}} \Lambda^{\cdot}\underline{L}^*$. As Ω is integrable, the differential d decomposes into $d = d' + d'' + \tilde{d}$ of bidegrees (1,0),(0,1),(2,-1) respectively. $d^2 = 0$ is equivalent to the relations $d''^2 = 0$, $\tilde{d}^2 = 0$, $d'd'' + d''d' = 0$, $d'\tilde{d} + \tilde{d}\,d' = 0$ and $d''\tilde{d} + \tilde{d}\,d'' + d'^2 = 0$. (5.5) is now immediate and also $G(d) = G(d'') = \pm d_L$. Observe that the isomorphism (5.5) is independent of the local splitting and hence globally defined. Similarly d' induces a globally defined morphism of sheaf complexes of degree 0

(5.6) $$d' : \underline{T}^{\cdot}_{\underline{L}}(\Lambda^p\Omega) \to \underline{T}^{\cdot}_{\underline{L}}(\Lambda^{p+1}\Omega),$$

satisfying $d'^2\alpha = -d''\tilde{d}\alpha$ for $\alpha \in \underline{T}^{\cdot}_{\underline{L}}(\Lambda^p\Omega)$ such that $d''\alpha = 0$. We finally mention that $\underline{T}^{\cdot}_{\underline{L}}(\underline{E})$ is a resolvent functor for the functor $\underline{t}(\underline{E}) = \underline{\mathrm{Hom}}_{\underline{U}}(\underline{O},\underline{E}) \cong \underline{E}^{\underline{L}}$ ($\underline{L}$-invariant elements) from $(\underline{L},\underline{O})$-modules to abelian sheaves [30,4.22] and hence by Grothendiecks general theory [20] there are natural equivalences

(5.7) $$\mathbb{H}^{\cdot}(M;\underline{T}^{\cdot}_{\underline{L}}(\underline{E})) \cong \mathrm{Ext}^{\cdot}_{\underline{U}(\underline{L},\underline{O})}(M;\underline{O},\underline{E}),$$

(5.8) $$\underline{H}^{\cdot}(\underline{T}^{\cdot}_{\underline{L}}(\underline{E})) \cong \underline{\mathrm{Ext}}^{\cdot}_{\underline{U}}(\underline{O},\underline{E}).$$

(5.7) and (5.8) are analogous to the cohomology of a Lie algebra. However, the groups $H^{\cdot}(M,\underline{L};\underline{E}) \equiv \mathrm{Ext}^{\cdot}_{\underline{U}}(M;\underline{O},\underline{E})$ are of global nature and involve also the cohomology of M (cf. Examples 5.11-5.14).

5.9 <u>THEOREM</u>. <u>The</u> E_1-<u>term of the multiplicative spectral sequence</u> (5.2) <u>is given by</u>

$$E_1^{p,q}(\Omega) \cong \mathbb{H}^q(M;\underline{T}^{\cdot}_{\underline{L}}(\Lambda^p\Omega)) \cong H^q(M,\underline{L};\Lambda^p\Omega) .$$

The differential d_1 is induced by the homomorphism d' in (5.6) and hence

$$(5.10) \qquad E_2^{p,q}(\Omega) \cong H_{d'}^p H^q(M,\underline{L};\Lambda^{\cdot}\Omega) \Longrightarrow H_{DR}^{p+q}(M) \;.$$

The edge maps of (5.2) are given by

$$E_2^{p,o} = H_{d'}^p(\Gamma(M,\Lambda^{\cdot}\Omega^{\underline{L}}) \to H_{DR}^p(M) \to E_2^{o,p} = H_{d'}^o(H^p(M,\underline{L};\underline{O})).$$

The $E_2^{p,o}$-terms are the cohomology groups of $\underline{L}$-basic forms on M, i.e. forms α annihilated by $i(\xi)$, $\theta(\xi)$, $\xi \in \underline{L}$. The fibre-terms $E_2^{o,p}$ contain information about the De Rham cohomology-groups along the leaves of the foliation. See example 5.14 where $E_2^{o,p}$ has an explicit geometric interpretation.

This spectral sequence is of a very general nature as will be seen from the discussion of a few special cases.

5.11. Let $\Omega = 0$: Then $\underline{L} = \underline{T}_M$ and $\underline{U}(\underline{L},\underline{O})$ is the sheaf $\underline{D}_M$ of differential operators on M. In this case (5.2) collapses and we obtain the isomorphism

$$E_1^{o,q} = \operatorname{Ext}^q_{\underline{D}_M}(M;\underline{O},\underline{O}) \cong \mathbb{H}^q(M;\Omega_M) = H_{DR}^q(M).$$

5.12. Let $\Omega = \Omega_M^1$: Then $\underline{L} = 0$, the filtration F^p is the Hodge filtration $F^p\Omega_M^{\cdot} = \Omega_M^p \cdot \Omega_M^{\cdot}$ on $\Omega_M^{\cdot}$ and (5.2) is the Hodge spectral sequence [22]

$$E_1^{p,q} = H^q(M,\Omega_M^p) \Longrightarrow H_{DR}^{\cdot}(M).$$

In the complex-analytic and algebraic categories this spectral sequence need not be trivial.

5.13. Assume that locally free $\underline{O}$-Modules $\underline{E}$ of finite type are Γ-acyclic on M and that $0 \to \underline{E}^{\underline{L}} \to \underline{T}_{\underline{L}}^{\cdot}(\underline{E})$ is a resolution of $\underline{E}^{\underline{L}}$, $\underline{E}$ a $(\underline{L},\underline{O})$-module of above type. Then the hypercohomology spectral sequences for $\mathbb{H}^{\cdot}(M;\underline{T}_{\underline{L}}(\underline{E}))$ collapse to isomorphisms [20]

$${}^{I}E_2^{p,O} = H^p(\Gamma(M,\underline{T}_{\underline{L}}^{\cdot}(\underline{E}))) \xrightarrow[{}^{I}\varkappa]{\cong} \mathbb{H}^p(M;\underline{T}_{\underline{L}}^{\cdot}(\underline{E})) \xleftarrow[{}^{II}\varkappa]{\cong} H^p(M,\underline{E}^{\underline{L}}) = {}^{II}E_2^{p,O}$$

The spectral sequence now takes the form

$$E_1^{p,q}(\Omega) \cong H^q(M,\Lambda^p\Omega^{\underline{L}}) \Longrightarrow H_{DR}^{p+q}(M)$$

and

$$E_2^{p,q}(\Omega) \cong H_{d'}^p H^q(M,\Lambda^{\cdot}\Omega^{\underline{L}}) \Longrightarrow H_{DR}^{p+q}(M),$$

where d' in (5.6) induces a $\underline{O}^{\underline{L}}$ - linear differential

$$d': \quad \Lambda^{\cdot}\Omega^{\underline{L}} \to \Lambda^{\cdot+1}\Omega^{\underline{L}}$$

in the sheaf $\Lambda^{\cdot}\Omega^{\underline{L}}$ of $\underline{L}$ - basic forms. This is in particular the case for the C^{∞} - category where one has a Poincaré-Lemma with parameters [46]. The groups $E_1^{p,q}$ coincide in this case with the groups $H_F^{p,q}(M)$ and $E_2^{p,o} = H_{d'}^p(\Gamma(M,\Lambda^{\cdot}\Omega^{\underline{L}}))$ are the cohomology groups of $\underline{L}$-basic forms (see Reinhart [46] and Molino [42], Vaisman [48],[49]).

5.14. Submersions. Let $(M,\underline{O}_M) \xrightarrow{f} (X,\underline{O}_X)$ be a morphism such that

(5.15) $$0 \to \underline{T}(f) \to \underline{T}_M \to f^*\underline{T}_X \to 0$$

is exact, i.e. f is a submersion (f smooth in the algebraic case). The tangentbundle along the fibres $\underline{L} = \underline{T}(f)$ is a Lie algebra sheaf, the annihilator of the integrable sheaf $\Omega = f^*\Omega_X^1$ of rank $q = \dim X$:

$$0 \to \Omega \to \Omega_M^1 \to \Omega_{M/X}^{\cdot} \cong \underline{L}^* \to 0.$$

Here $\Omega_{M/X}^{\cdot}$ denotes the relative cotangent complex of forms along $\underline{T}(f)$. In this case we have $G^p\Omega_M^{\cdot} \cong f^*\Omega_X^p \otimes \Omega_{M/X}^{\cdot -p}$.

Assume now that (quasi-) coherent $\underline{O}_X$-modules are Γ_X-acyclic. Following [36](in the algebraic case) we may then compute the E_1-term as

(5.16) $$E_1^{p,q}(\Omega) \cong \Gamma(X,\Omega_X^p \otimes \mathbb{R}^q f_*(\Omega_{M/X}^{\cdot})),$$

where $\mathbb{R}^q f_*$ is the hyperderived functor of $\mathbb{R}^o f_* = \underline{H}^o \circ f_* = f_* \circ \underline{H}^o$. The differential d_1 (resp. d') is now induced by the flat Gauss-Manin connection ∇ in the relative De Rham sheaves $\mathcal{H}_{DR}^q(M/X) = \mathbb{R}^q f_*(\Omega_{M/X}^{\cdot})$:

$$d_1 = \nabla : \quad \Omega_X^p \otimes_{\underline{O}_X} \mathcal{H}_{DR}^q(M/X) \to \Omega_X^{p+1} \otimes_{\underline{O}_X} \mathcal{H}_{DR}^q(M/X),$$

with $\nabla^2 = 0$. Using the acyclicity condition on $\Omega_X^p \otimes_{\underline{O}_X} \mathcal{H}_{DR}^q$ we obtain

5.17 PROPOSITION. For a submersion $f: M \to X$ the spectral sequence (5.10) is isomorphic to the Leray spectral sequence for De Rham cohomology

$$E_2^{p,q}(\Omega) = H_\nabla^p(\Gamma(X, \Omega_X^\cdot \otimes_{\underline{O}_X} \mathcal{H}_{DR}^q(M/X) \cong \mathbb{H}^p(X; \Omega_X^\cdot \otimes_{\underline{O}_X} \mathcal{H}_{DR}^q(M/X)) \Rightarrow H_{DR}^{p+q}(M).$$

The acyclicity condition on X is satisfied e.g. for affine algebraic varieties X, Stein manifolds X in the complex analytic case (Theorem B for coherent modules) and paracompact C^∞-manifolds X (all $\underline{O}_X$-modules are fine and hence Γ_X-acyclic).

Proposition 5.17 shows that the spectral sequence (5.10) is a proper substitute for the Leray-spectral seqence in the case where the foliation is not globally given by a submersion.

5.18. It would be interesting to know criteria for the degeneracy of the spectral sequence $E(\Omega)$ in (5.9) either at the E_1- or E_2-level ($d_r = 0$, $r \geq 1$ or $r \geq 2$). Thus in example (5.12) the spectral sequence stops at E_1 if M is a Kähler manifold (see also Deligne, IHES, Publ. Math., No. 35, 1968). For two complementary foliations Ω_i, $\Omega_M^1 = \Omega_1 \otimes \Omega_2$ the differentials $\tilde{d}_i$ of degree (2,-1) are zero, i=1,2. Together with the theory of harmonic forms on a foliated manifold [47],[48] this might well lead to degeneracy results.

6. Derived characteristic classes

In this section we relate the constructions of sections 3 and 5. We want to show that the construction of the characteristic homomorphism Δ_* in section 3 determines a multiplicative map of spectral sequences:

(6.1) $$\Delta_r\colon\ E_{2r}^{2p,n-2p}\big(W(\underline{\underline{g}},\underline{\underline{h}})_q\big) \to E_r^{p,n-p}(\Omega)\ ,\ r \geqslant 1.$$

To do this we need the following remarks. Let $\underline{A}^\cdot$ be a complex of $\underline{O}$-modules on M and $\underline{C}(M;\underline{A})^\cdot$ the canonical resolution of $\underline{A}^\cdot$ equipped with the total differential and degree. For an open covering $\mathcal{U}$ of M there are canonical chain maps

(6.2) $$\check{C}(\mathcal{U};\underline{A}^\cdot)^\cdot \xrightarrow{j'} K^\cdot(\underline{A}^\cdot) = \check{C}\big(\mathcal{U};\underline{C}^\cdot(M,\underline{A}^\cdot)\big)^{tot} \xleftarrow{j''} \Gamma\underline{C}(M;\underline{A})^\cdot$$

which induce edge maps for the two spectral sequences associated to K. As the second spectral sequence collapses for every $\underline{A}^\cdot$ we obtain a natural homomorphism [18,Ch.II,5.5.4]:

(6.3) $$j = (j''_*)^{-1} \circ j'_*\colon\ \mathbb{H}^\cdot(\mathcal{U};\underline{A}^\cdot) \to \mathbb{H}^\cdot(M;\underline{A}^\cdot)$$

$K(\underline{A}^\cdot)$ and $\Gamma\underline{C}(M;A^\cdot)$ are exact in $\underline{A}^\cdot$ and so is $\check{C}(\mathcal{U};\underline{A}^\cdot)$ for an admissible $\mathcal{U}$. For a filtered $\underline{A}^\cdot$ it follows that (6.2) defines a mapping of spectral sequences associated to the filtration which on the E_1-level is given by

$$j_1 = (j''_1)^{-1} \circ j'_1\colon\ \mathbb{H}^\cdot(\mathcal{U};G^p\underline{A}^\cdot) \to \mathbb{H}^\cdot(M;G^p\underline{A}^\cdot)$$

Let now (P,ω_o) be an Ω-foliated G-bundle with an H-reduction $s\colon M \to P/H$. The characteristic homomorphism Δ_* in (3.7) is defined as follows by the chain homomorphism

(6.4) $$\Delta(\omega) \equiv s^* \circ k_1(\omega)\colon\ W_1(\underline{\underline{g}},\underline{\underline{h}})_q \to \check{C}(\mathcal{U};\Omega_M^\cdot)$$

which is filtration-preserving in the sense of (3.4). Consider the diagram of filtration-preserving chain maps

(6.5) $$\begin{array}{ccccc} W_1(\underline{\underline{g}},\underline{\underline{h}})_q & \xrightarrow{\Delta(\omega)} & \check{C}(\mathcal{U};\Omega_M^\cdot) & \xrightarrow{j'} & K^\cdot(\Omega_M^\cdot) \\ \Big\downarrow{\scriptstyle \rho_1} & & & & \Big\uparrow{\scriptstyle j''} \\ W(\underline{\underline{g}},\underline{\underline{h}})_q & & & & \Gamma\underline{C}^\cdot(M;\Omega_M^\cdot) \end{array}$$

As the vertical maps are isomorphisms on the E_1-level (2.10), there exist unique homomorphisms Δ_r as in (6.1) and

(6.6) $\Delta_*: \ H^\bullet(W(\underline{\underline{g}},\underline{\underline{h}})_q) \rightarrow \mathbb{H}^\bullet(M;\Omega^\bullet_M) = H^\bullet_{DR}(M)$

making the diagrams corresponding to (6.5) commutative. The homomorphisms Δ_r in (6.1) for $r \geq 1$ are called the <u>derived characteristic homomorphisms</u> of (P,ω_o). As the spectral sequence of $W(\underline{\underline{g}},\underline{\underline{h}})_q$ is defined by an even filtration, we have $d_{2r-1} = 0$ for $r > 0$. This, together with the property $\Delta(F^{2p}W_1) \subseteq F^p\check{C}$, explains the indices in (6.1). Thus a foliated G-bundle (P,ω_o) with H-structure determines a sequence of characteristic homomorphisms $\{\Delta_*,\Delta_r\}_{r\geq 1}$, with Δ_r approximating Δ_*.

By (2.11), (5.9) we have for r=1:

(6.8) $\Delta_1^{s,t}: \ E_2^{2s,t}(W) \cong H^t(\underline{\underline{g}},\underline{\underline{h}}) \otimes I(\underline{\underline{g}})_q^{2s} \rightarrow E_1^{s,s+t} \cong H^{s+t}(M,\underline{L};\Lambda^t\Omega)$.

As Δ_1 is multiplicative, it is completely determined by the maps $\Delta_1^{s,o}$ and $\Delta_1^{o,t}$. These will be computed in the next two sections.

7. Atiyah classes

In this section we will give an interpretation of the derived characteristic classes of <u>basis-type</u>

(7.1) $\Delta_1^{p,o} \ : \ I(\underline{\underline{g}})_q^{2p} \rightarrow H^p(M,\underline{L};\Lambda^p\Omega) \qquad 0 \leq p \leq q.$

It turns out that these classes depend only on the splitting obstruction of a certain short exact sequence of $\underline{U}(\underline{L},\underline{Q})$-modules associated to a Ω-foliation ω_o in the G-bundle $P \xrightarrow{\pi} M$. Let $\tilde{\Omega} \subset \Omega^1$ denote the dual transversal bundle of the foliation lifted to P (1.7):

$\tilde{\Omega} = \{\varphi \in \Omega^1_P / i(\tilde{\xi})\varphi = 0\}$, $\tilde{\xi} = \sigma_o^*(\xi)$, $\xi \in \underline{L}$. As the foliation Ω on M is non-singular, we have $\pi_*^G\tilde{\Omega} \cong \ker(\sigma_o)$ in diagram (1.4). Thus we may complete (1.4) in the following way

$$
\begin{array}{llccccccccc}
& & & & 0 & & & & & & \\
& & & & \downarrow & & & \overset{\omega}{\dashleftarrow} & & & \\
& \underline{A}(P,\omega_o): & 0 & \longrightarrow & \Omega & \longrightarrow & \pi_*^G\tilde{\Omega} & \longrightarrow & \underline{P}(\underline{\underline{g}}^*) & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \| & & \\
(7.2) & \underline{A}(P): & 0 & \longrightarrow & \Omega_M^1 & \xrightarrow{\pi^*} & \pi_*^G\Omega_P^1 & \xrightarrow{\rho} & \underline{P}(\underline{\underline{g}}^*) & \longrightarrow & 0 \\
& & & & \lambda\downarrow & \swarrow\sigma_o & \downarrow\bar{\lambda} & \swarrow\omega_o & \| & & \\
& & 0 & \longrightarrow & \Omega_M^1/\Omega & \rightarrow & \pi_*^G\Omega_P^1/\Omega & \longrightarrow & \underline{P}(\underline{\underline{g}}^*) & \longrightarrow & 0 \\
& & & & \downarrow & & & & & & \\
& & & & 0 & & & & & &
\end{array}
$$

In the case of example (1.12), $\underline{A}(P,\omega_o)$ is the pull-back by f of the Atiyah sequence $\underline{A}(P')$ of P' on X: $\underline{A}(P,\omega_o) = f^*\underline{A}(P')$.

The local sections ω of $\underline{A}(P,\omega_o)$ are exactly the local connections in P, adapted to ω_o in the sense of section 1: $\bar{\lambda}\omega = \omega_o$. Globally there is an obstruction to the existence of an adapted connection which is represented by an element in $H^1(M,\Omega\otimes_{\underline{O}} \underline{P}(\underline{\underline{g}}))$.

The sequence $\underline{A}(P,\omega_o)$ has an intrinsic additional structure: it is naturally a sequence of $\underline{U}(\underline{L},\underline{O})$-modules.

7.3 LEMMA. The operation $\xi\cdot\varphi = \theta(\tilde{\xi})\varphi = i(\tilde{\xi})d\varphi$, $\xi\in\underline{L}$, $\varphi\in\pi_*\tilde{\Omega}$ and the canonical map $\underline{O}_M \to \pi_*\underline{O}_P$ define an $\underline{U}(\underline{L},\underline{O})$-module structure on $\pi_*\tilde{\Omega}$ such that $\Omega\subset\pi_*^G\tilde{\Omega}$ is a submodule and the structure induced in Ω coincides with the one defined in (5.4).

The obstruction for a global $\underline{U}(\underline{L},\underline{O})$-splitting of $\underline{A}(P,\omega_o)$ is as usual defined as a coboundary:

(7.4) $\to \mathrm{Hom}_{\underline{U}}(\underline{P}(\underline{\underline{g}}^*),\pi_*^G\tilde{\Omega}) \to \mathrm{Hom}_{\underline{U}}(\underline{P}(\underline{\underline{g}}^*),\underline{P}(\underline{\underline{g}}^*)) \xrightarrow{\partial} \mathrm{Ext}^1_{\underline{U}}(M;\underline{P}(\underline{\underline{g}}^*),\Omega) \to \dots$

(7.5) $\tilde{\zeta}(\underline{P},\omega_o) \overset{\text{Def.}}{=} -\partial(\mathrm{id}_{\underline{P}(\underline{\underline{g}}^*)}) \in \mathrm{Ext}^1_{\underline{U}}(M;\underline{P}(\underline{\underline{g}}^*),\Omega) \cong H^1(M,\underline{L};\Omega\otimes_{\underline{O}}\underline{P}(\underline{\underline{g}}))$.

To describe $\tilde{\zeta}(P,\omega_o)$ on the cochain level, observe that for a connection ω in $P|U$ adapted to ω_o we have

(7.6) $i(\tilde{\xi})K(\omega)(\varphi) = \theta(\tilde{\xi})\omega(\varphi) - \omega(\theta_o(\tilde{\xi})\varphi) \in \Gamma(U,\Omega)$, $\xi \in \underline{L}$, $\varphi \in \underline{P}(\underline{\underline{g}}^*)$,

where θ_o denotes the $\underline{L}$-action induced on $\underline{P}(\underline{\underline{g}}^*)$ by (7.3), and $K(\omega) \in \Gamma(U,(\Omega\cdot\Omega_M^{\cdot})^2 \otimes_{\underline{O}} \underline{P}(\underline{\underline{g}}))$ is the curvature of ω. Let $\mathfrak{U}$ be an admissible covering and $\omega=(\omega_j)$ a family of connections in $P|\mathfrak{U}$ adapted to ω_o. We define then a cochain $\zeta'=(\zeta^o,\zeta^1) \in \check{C}^{\cdot}(\mathfrak{U},\underline{T}^{\cdot}_{\underline{L}}(\Omega \otimes_{\underline{O}} \underline{P}(\underline{\underline{g}})))^1$ of total degree 1 by $\zeta^o(j)(\xi) = i(\tilde{\xi})\ K(\omega_j)$, $\xi \in \underline{L}|U_j$, $\zeta^1(i,j) = \omega_j - \omega_i \in \Gamma(U_{ij},\Omega \otimes_{\underline{O}} \underline{P}(\underline{\underline{g}}))$. Using (7.6) one shows that it is closed under the total differential $D=\delta\pm d_L$ in $\check{C}$ and hence ζ' defines a cohomology class in $H^1(\check{C}^{\cdot})$. Using (5.7) we obtain

7.7 <u>LEMMA</u>. <u>Under the canonical homomorphism</u> (6.3)

$$j:\quad H^1(\check{C}(\mathfrak{U},\underline{T}^{\cdot}_{\underline{L}}(\Omega \underset{\underline{O}}{\otimes} \underline{P}(\underline{\underline{g}}))) \to H^1(M,\underline{L};\Omega \underset{\underline{O}}{\otimes} \underline{P}(\underline{\underline{g}}))$$

<u>we have</u> $j(\zeta') = \pm\ \tilde{\zeta}(P,\omega_o)$.

We define a $\underline{L}$-<u>basic</u> connection in (P,ω_o) as a connection satisfying

$$(7.8) \qquad i(\tilde{\xi})\ \omega(\varphi) = 0$$

$$(7.9) \quad \theta(\tilde{\xi})\omega(\varphi) - \omega(\theta_o(\tilde{\xi})\varphi) = i(\tilde{\xi})K(\omega)(\varphi) = 0,\ \forall \xi \in \underline{L},\ \varphi \in \underline{P}(\underline{\underline{g}}^*).$$

It is then clear that $\underline{L}$-basic connections exist if and only if $\tilde{\zeta}(P,\omega_o) = 0$, that they are in a 1-1-correspondence with $\underline{U}$-splittings of $\underline{A}(P,\omega_o)$ if $\tilde{\zeta}(P,\omega_o) = 0$ and that they form a convex set: ω,ω' $\underline{L}$-basic $\Longrightarrow \omega'-\omega \in \Gamma(M,(\Omega \otimes_{\underline{O}} \underline{P}(\underline{\underline{g}}))^{\underline{L}})$. Furthermore by (7.8),(7.9) ω is $\underline{L}$-basic if and only if it is adapted to ω_o in P and $i(\tilde{\xi})K(\omega) = 0$, $\xi \in \underline{L}$, i.e.

$$(7.10) \qquad K(\omega) \in \Gamma(M,(\Lambda^2\Omega \otimes_{\underline{O}} \underline{P}(\underline{\underline{g}}))^{\underline{L}}).$$

There is also a <u>local</u> obstruction for a $\underline{U}(\underline{L},\underline{O})$-splitting of $\underline{A}(\underline{P},\omega_o)$ [30;§4]. It is a section $\zeta(P,\omega_o) \in \Gamma(M,\underline{Ext}^1_{\underline{U}}(\underline{P}(\underline{\underline{g}}^*),\Omega)) \cong \Gamma(M,\underline{H}^1(\underline{T}^{\cdot}_{\underline{L}}(\Omega \otimes_{\underline{O}} \underline{P}(\underline{\underline{g}}))))$. If $\zeta(P,\omega_o) = 0$, there exist $\underline{L}$-basic connections in P <u>locally</u> on a sufficiently fine covering of M. This is notably so in the C^∞ case (see example 5.13).

To describe $\Delta_1^{2p,\cdot}$ let now $\Phi \in I(\underline{\underline{g}})^{2p}$ be an invariant polynomial on $\underline{\underline{g}}$ of degree p and consider the mapping

(7.11) $\alpha_\Phi\colon H^1(M,\underline{L};\Omega \otimes_{\underline{O}} \underline{P}(\underline{\underline{g}}))^{\otimes p} \to H^p(M,\underline{L};\Lambda^p\Omega \otimes_{\underline{O}} \underline{P}(S^p\underline{\underline{g}})) \xrightarrow{\Phi_*} H^p(M,\underline{L};\Lambda^p\Omega)$.

This defines in turn

(7.12) $a^p\colon \; I(\underline{\underline{g}})^{2p}_q \to H^p(M,\underline{L};\Lambda^p\Omega)$

by $a^p(\Phi) = \alpha_\Phi(\tilde{\zeta}(P,\omega_o)^{\otimes p})$. The classes $a^p(\Phi)$ are called the <u>Atiyah classes</u> of the Ω-foliated bundle (P,ω_o).

7.13 <u>THEOREM</u>. <u>The derived characteristic classes of basis-type coincide with the Atiyah classes of</u> (P,ω_o): $\Delta_1^{p,o} = a^p$. <u>In particular, if</u> $\tilde{\zeta}(P,\omega_o)=0$, <u>i.e. if there exists a L-basic connection in</u> P, <u>then</u> $\Delta_1^{p,o} = a^p = 0$, $p > 0$.

In general the derived characteristic classes in $E_1(\Omega)$ are not d_1-cocycles. But as the classes $I(\underline{\underline{g}})_q \to W(\underline{\underline{g}},\underline{\underline{h}})_q$ are mapped into the cocycles of $W(\underline{\underline{g}},\underline{\underline{h}})_q$, we obtain as a consequence of 7.13 and the definition of Δ_1:

7.14 <u>COROLLARY</u>. <u>For</u> $\Phi \in I(\underline{\underline{g}})^{2p}$ <u>the Atiyah class</u> $a^p(\Phi)$ <u>satisfies</u> $a^p(\Phi) \in Z_\infty(E_1)^{p,p}$ <u>and the De Rham class</u> $k_*(\Phi)$ (3.12) <u>satisfies</u> $k_*(\Phi) \in F^pH^{2p}_{DR}(M)$. <u>Moreover the two classes correspond to each other via the canonical homomorphisms</u>

$$Z_\infty(E_1)^{p,p} \twoheadrightarrow E_\infty^{p,p} \cong F^pH^{2p}_{DR}(M)/F^{p+1} \twoheadleftarrow F^pH^{2p}_{DR}(M).$$

<u>Thus if</u> $a^p(\Phi)=0$, <u>we have</u> $k_*(\Phi) \in F^{p+1}H^{2p}_{DR}(M)$.

The following examples show that our construction of the classes $\tilde{\zeta}(P,\omega_o)$ and $a = \Delta_1^{\cdot,o}$ generalizes and unifies some known constructions:

7.15. $\Omega = \Omega^1_M$ (see 5.12): In this case the obstruction class $\tilde{\zeta}(P) \in H^1(M,\Omega^1_M \otimes \underline{P}(\underline{\underline{g}}))$ coincides with the obstruction defined by Atiyah [1] for the existence of a global holomorphic connection in a holomorphic principal bundle. The classes $a^p(\Phi) \in H^p(M,\Omega^p_M)$ coincide then by construction with the characteristic classes in Hodge cohomology defined in [1]; see also Illusie [27,I].

7.16. In the C^∞-case (5.13) the obstruction $\tilde{\zeta}(P,\omega_o)$ is an element of $H^1(M,(\Omega \otimes_{\underline{O}} \underline{P}(\underline{\underline{g}}))^{\underline{L}}) = H_F^{1,1}(M,\underline{P}(\underline{\underline{g}}))$ and can be shown to coincide with the class defined by Molino [42]. The derived characteristic classes $\Delta_1^{p,o} = a^p$ induce by (7.14) a homomorphism

$$\bar{a}^p\colon I(\underline{\underline{g}})_q^{2p} \to E_2^{p,p} = H_{d'}^p(H^p(M,\Lambda^\cdot\Omega^{\underline{L}})) = H_{d'}^p(H_F^{\cdot,p}(M)).$$

7.17. In the case of a submersion $f\colon M \to X$ and $\Omega = f^*\Omega_X^1$ (5.14), the Atiyah classes of a Ω-foliated bundle $P \to M$ induce by (7.14) a homomorphism into the E_2-term of the Leray spectral sequence of f, $\bar{a}^p\colon I(\underline{\underline{g}})_q^{2p} \to H_\nabla^p(\Gamma(X,\Omega_X^\cdot \otimes_{\underline{O}} \mathcal{H}_{DR}^p(M/X)))$. $\tilde{\zeta}(P,\omega_o)$ and a^p, $p>0$ are zero if $P = f^*P'$ for a G-principal bundle $P' \to X$, since the canonical foliation on P (1.12) is obtained by pull-back $\omega=f^*\omega'$ of a connection ω' in P'. In fact $\tilde{\zeta}(P,\omega_o) = f^*\tilde{\zeta}(P')$ and $\tilde{\zeta}(P') = 0$ by acyclicity. Connections which locally are of this form are the CTP of Molino [42].

7.18. By Cor. 7.14 the Atiyah classes $a^p(\Phi)$ may be considered as a first approximation to the De Rham classes $k_*(\Phi) \in H_{DR}^{2p}(M)$ relative to the given foliation Ω on M. In some cases they actually determine the De Rham classes (e.g. for Kähler manifolds, $\Omega = \Omega_M^1$; compare [1]). In general the question of determinacy of $k_*(\Phi)$ by $a^p(\Phi)$ is related to the degeneracy (5.18) of the spectral sequence $E(\Omega)$.

Consider now the special case where $\tilde{\zeta}(P,\omega_o) = 0$, i.e. P admits a <u>global</u> <u>L</u>-<u>basic</u> <u>connection</u>. It follows from (7.10) that $\Delta(\omega)$ in (6.4) preserves filtrations in the strict sense: $\Delta(\omega)F^{2p}W_1 \subseteq F^{2p}\check{C}$. We therefore obtain for $q = rk_{\underline{O}}(\Omega)$

7.19 <u>THEOREM</u>. <u>If</u> $\tilde{\zeta}(P,\omega_o) = 0$ <u>there is a factorization of the characteristic homomorphism</u> Δ_*:

$$(7.20)\qquad \begin{array}{ccc} H(W(\underline{\underline{g}},\underline{\underline{h}})_q) & \longrightarrow & H(W(\underline{\underline{g}},\underline{\underline{h}})_{q_o}) \\ & \searrow^{\Delta_*} \quad {}^{\Delta_{o,*}}\swarrow & \\ & H^{\cdot}_{DR}(M) & \end{array}$$

where $q_o = \left[\frac{q}{2}\right]$ *and the horizontal homomorphism is induced by the canonical projection* $W(\underline{\underline{g}},\underline{\underline{h}})_q \to W(\underline{\underline{g}},\underline{\underline{h}})_{q_o}$. *Moreover* $\Delta_{o,*}$ *factorizes by* (7.8),(7.9) *as indicated in the diagram below*

$$(7.21)\qquad \begin{array}{cccccc} \Delta_{o,*}: & H(W(\underline{\underline{g}},\underline{\underline{h}})_{q_o}) & \longrightarrow & H(\Gamma(P,\Lambda^{\cdot}\tilde{\Omega}^{\underline{L}})_H) & \xrightarrow{s^*} & H^{\cdot}_{DR}(M) \\ & \uparrow & & \uparrow & & \| \\ & I(\underline{\underline{g}})_{q_o} & \longrightarrow & H(\Gamma(M,\Lambda^{\cdot}\Omega^{\underline{L}})) & \longrightarrow & H^{\cdot}_{DR}(M). \end{array}$$

Hence in the presence of an $\underline{L}$-basic connection on P the homomorphism $\Delta_{o,*}$ should be considered the characteristic homomorphism of (P,ω_o).

For $I(\underline{\underline{g}})$ the improvement of the Bott vanishing theorem contained in (7.21) was observed by Molino [43] and Pasternack [45]. Diagram (7.20) gives a non-trivial result even in the case when $\Omega = \Omega^1_M$. Let $P \to M$ be a holomorphic principal bundle which admits a holomorphic connection and a holomorphic H-reduction P'. As in this case $q=n=\dim_{\mathbb{C}}M$, we obtain a characteristic homomorphism

$$\Delta_{o,*}: H^{\cdot}(W(\underline{\underline{g}},\underline{\underline{h}})_{n_o}) \to H^{\cdot}(M,\mathbb{C}), \qquad n_o = \left[\frac{n}{2}\right]$$

In particular for H = G this means that the ordinary characteristic homomorphism $k_*: I(\underline{\underline{g}}) \to H^{\cdot}(M,\mathbb{C})$ breaks off in degrees >n. Of course this example is interesting only if M is not Stein.

8. Derived classes of fibre-type

We will now describe the derived classes of *fibre-type*:

$$(8.1)\qquad \Delta_1^{o,n}: H^n(\underline{\underline{g}},h) \to H^n(M,\underline{L};\underline{O}) = \mathrm{Ext}^n_{\underline{U}}(M;\underline{O},\underline{O}), \quad n\geq 0.$$

First we remark that these classes are always given by global forms on the cochain level, even if they are constructed with respect to a family $\omega=(\omega_j)$ of adapted connections in (P,ω_o). Using the notation of section 3 we consider the mapping

$$(8.2)\quad b: (\Lambda^{\cdot}\underline{\underline{g}}^*)_{\underline{\underline{h}}} \xrightarrow{s^*\circ\omega} \check{C}^o(\mathcal{U},\Omega^{\cdot}_M) \xrightarrow{(\Lambda\lambda)} \check{C}^o(\mathcal{U},\underline{T}^{\cdot}_{\underline{L}}(\underline{O})) .$$

Since $\omega_j-\omega_i \in \Gamma(U_{ij},\Omega \otimes_{\underline{O}} P(\underline{\underline{g}}))$ we have $b(\Phi)_j-b(\Phi)_i =$

$$= \sum_{k=1}^{r} \lambda^* s^*\Phi(\omega_j, \quad ,\omega_j,\overset{k}{\widehat{\omega_j-\omega_i}},\omega_i,\ldots,\omega_i) = 0, \Phi \in (\Lambda^r\underline{\underline{g}}^*)_{\underline{\underline{h}}},$$ and

hence $(b(\Phi)_j)$ defines a global form in $\Gamma(\underline{T}^r_{\underline{L}}(\underline{O}))$. It now follows from $\underline{T}^{\cdot}_{\underline{L}}(\underline{O}) \cong \Omega^{\cdot}_M/F^1\Omega^{\cdot}_M$ that b is a chain-map

$$(8.3)\quad b: (\Lambda^{\cdot}\underline{\underline{g}}^*)_{\underline{\underline{h}}} \longrightarrow \Gamma(M,\underline{T}^{\cdot}_{\underline{L}}(\underline{O})) = \Gamma(M,\Lambda^{\cdot}\underline{L}^*) .$$

Using (2.11) one proves

8.4 PROPOSITION. The derived classes of fibre-type are given by the composition

$$\Delta_1^{o,\cdot}: H^{\cdot}(\underline{\underline{g}},\underline{\underline{h}}) = H^{\cdot}((\Lambda\underline{\underline{g}}^*)_{\underline{\underline{h}}}) \xrightarrow{b_*} H^{\cdot}(\Gamma(M,\Lambda\underline{L}^*)) \xrightarrow{'\varkappa} H^{\cdot}(M,\underline{L};\underline{O})$$

where the second homomorphism $'\varkappa$ is the edge-map in the hypercohomology spectral sequence (compare 5.12).

We emphasize the importance of the fibre-type classes by giving a few examples and applications.

8.5. $\Omega=0$ (see 4.3, 5.11): In this case P is a flat G-bundle and the characteristic homomorphisms Δ_* and Δ_1 coincide. To exhibit examples of flat bundles with non-trivial $\Delta_*=\Delta_1$ we return to the examples of flat G-bundles with non-trivial (topological) characteristic homomorphism which we constructed in [28;4.14].Let G be a connected semisimple Lie group with finite center which contains no compact factor, $K\subset G$ a maximal compact subgroup and (U,K) the compact symmetric pair dual to the pair (G,K). By [3] there exist discrete uniform torsionfree subgroups $\Gamma_\alpha \subset G$. The flat G-bundle $P=(K\backslash G)\times_{\Gamma} G \xrightarrow{\pi} M_\alpha=(K\backslash G)/\Gamma$ has a canonical K-reduction given by the isomorphism $P \overset{\varphi}{\cong} (G/\Gamma) \underset{K}{\times} G$ induced by $\varphi(g,g') = \varphi(g,gg')$. Then $B_\alpha : M_\alpha = B_\Gamma \to B_G$ classifies P and if we denote by $\xi_\alpha : M_\alpha \to B_K$, $\eta : U/K \to B_K$ the classifying maps of the K-bundles $G/\Gamma \to M_\alpha$ resp. $U \to U/K$, we have

8.6 PROPOSITION. There is a commutative diagram

$$\begin{array}{ccccc} & H^\bullet(B_G, \mathbb{R}) & \xrightarrow{\cong} & H^\bullet(B_K, \mathbb{R}) \cong I(\underline{\underline{k}}) & \\ & \downarrow B_\alpha^* & \swarrow \xi_\alpha^* & \downarrow \eta^* & \\ H^\bullet(\Gamma, \mathbb{R}) \cong & H^\bullet(M_\alpha, \mathbb{R}) & \underset{b_*}{\longleftarrow\!\supset} & H^\bullet(U/K, \mathbb{R}) \cong H^\bullet(\underline{\underline{g}}, \underline{\underline{k}}) & \end{array}$$

where $b_* = \Delta_*$ is the homomorphism in (8.4) for $\underline{L} = \underline{T}_M$; b_* is injective.

In fact it follows easily from our construction that b_* is injective in top-dimension and hence injective by Poincaré-duality for $H(M_\alpha, \mathbb{R})$. In this case the map b_* can be identified with the map constructed by Matsushima [40] using harmonic forms and it also coincides with Hirzebruchs proportionality map which transforms the characteristic classes of the K-bundle $U \to U/K$ into those of the K-bundle $G/\Gamma \to M_\alpha$.

8.7. Deformations. Let $f: M \to X$ be a submersion as in (5.14). A Lie-algebra subsheaf $\underline{L} \subset \underline{T}(f)$ may then be considered as a deformation of foliations $\underline{L}_x$ on the fibers $M_x = f^{-1}(x)$, $x \in X$. Similarly a foliated G-bundle (with respect to $\Omega = (\underline{T}_M/\underline{L})^*$) defines a deformation of foliated bundles $P_x \to M_x$ and an H-structure on P defines a deformation of H-structures on P_x, $x \in X$. We obtain then a commutative diagram:

(8.8)
$$\begin{array}{ccc}
H^n(W(\underline{\underline{g}},h)_{q+m}) & \xrightarrow{\Delta_*} & H^n_{DR}(M) \\
\Big\downarrow \text{can} & & \Big\downarrow \\
 & & E_2^{o,n}(f^*\Omega^1_X) = \Gamma(X, \mathcal{H}^n_{DR}(M/X)^{\nabla}) \\
 & & \Big\downarrow \\
H^n(W(\underline{\underline{g}},\underline{\underline{h}})_q) & \xrightarrow{\tilde{\Delta}_*} & \Gamma(X, \mathcal{H}^n_{DR}(M/X)) \\
 & \searrow^{\Delta_*(x)} & \Big\downarrow ev_x \\
 & & H^n_{DR}(M_x)
\end{array}$$

where $q = rk_{\underline{O}}(\underline{T}(f)/\underline{L})$, $m = rk_{\underline{O}_X}(\Omega^1_X) = \dim X$, and $\tilde{\Delta}_*$ is a homomorphism defined like Δ_* but with respect to the relative De Rham complex $\Omega^{\bullet}_{M/X}$. $\tilde{\Delta}_*$ represents the family of characteristic homomorphisms $\Delta_*(x)$ of the foliated bundles $P_x \to M_x$, $x \in X$.

The commutativity of this diagram implies

8.9. <u>THEOREM</u>. <u>The classes</u> $\tilde{\Delta}_*(u)$ <u>for</u> $u \in \operatorname{im}(H^{\bullet}(W(\underline{\underline{g}},\underline{\underline{h}})_{q+m}) \to H^{\bullet}(W(\underline{\underline{g}},\underline{\underline{h}})_q))$ <u>are rigid, namely they are invariant in</u> $\mathcal{H}^{\bullet}_{DR}(M/X)$ <u>under parallel transport by the Gauss-Manin connection</u> ∇.

For a product family $M = N \times \mathbb{R} \to \mathbb{R}$ where $\nabla = \frac{d}{dt}$, $m=1$, this means the independence of the classes $\Delta_*(t)u$ from the parameter $t \in \mathbb{R}$. This implies in particular the result of Heitsch in [26] on the rigidity of characteristic classes of a foliation under one-parameter deformations.

In the case $\underline{L} = \underline{T}(f)$, $\Omega = f^*\Omega^1_X$ the above situation defines a deformation of <u>flat</u> bundles $P_x \to M_x$, $x \in X$. As we have $\Delta_* = \Delta_1$ for flat bundles (8.5), we may compare $\tilde{\Delta}_*$ with the <u>derived</u> characteristic homomorphisms Δ_1 and Δ_2 for Ω on M. Using (5.17) we obtain a commutative diagram:

$$
\begin{array}{ccc}
E_4^{0,n}(W(\underline{\underline{g}},\underline{\underline{h}})_m) & \xrightarrow{\Delta_2} & E_2^{0,n}(\Omega) = \Gamma(X, \mathcal{H}^n_{DR}(M/X)^\nabla) \\
\cap & & \cap \\
E_2^{0,n}(W(\underline{\underline{g}},\underline{\underline{h}})_m) \cong H^n(\underline{\underline{g}},\underline{\underline{h}}) & \xrightarrow[\tilde{\Delta}_*=\Delta_1]{} & E_1(\Omega) = \Gamma(X, \mathcal{H}^n_{DR}(M/X))
\end{array} \tag{8.10}
$$

From (2.11),(2.12) it follows that $E^{0,\cdot}_{2(s+1)} \cong \Lambda^\cdot \hat{P}^{(2s)} \otimes I(\underline{\underline{h}})^\cdot / I(\underline{\underline{g}})^+ \cdot I(\underline{\underline{h}}) \subset E_2^{0,\cdot} = H^\cdot(\underline{\underline{g}},\underline{\underline{h}})$, where $P^{(2s)} = \{x \in \hat{P} / \deg x > 2s\}$. This gives

8.11. <u>THEOREM</u>. <u>For a deformation of flat bundles</u> $(\underline{L}=\underline{T}(f))$ <u>we have</u> $\tilde{\Delta}_* = \Delta_1$ <u>on</u> $E_1^{0,\cdot}(W(\underline{\underline{g}},\underline{\underline{h}})_m) \cong H^\cdot(\underline{\underline{g}},\underline{\underline{h}})$. <u>Moreover on</u> $E_4^{0,\cdot} \cong \Lambda^\cdot \hat{P}^{(2)} \otimes I(\underline{\underline{h}})/I(\underline{\underline{g}})^+ \cdot I(\underline{\underline{h}}) \subset H^\cdot(\underline{\underline{g}},\underline{\underline{h}})$ <u>the homomorphism</u> $\tilde{\Delta}_*$ <u>is rigid, i.e. maps into the sections of</u> $\mathcal{H}^\cdot_{DR}(M/X)$ <u>which are parallel under the Gauss-Manin connection</u> ∇.

It follows in particular that the classes $\Delta_*(x_i) \in H^{2i-1}_{DR}(M)$, $i>1$, in Theorem 4.5 are rigid under deformation of the flat structure on M.

We finally want to point out that similar results hold for the rigidity of derived characteristic classes in the general case $(\underline{L} \subseteq \underline{T}(f))$.

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