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FLAT BUNDLES AND CHARACTERISTIC CLASSES OF GROUP-REPRESENTATIONS.

By F. KAMBER and PH. TONDEUR.*

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0. Introduction. Let ξ be a principal *G*-bundle over a space *X*. ξ is flat if it is induced from the universal covering bundle of *X* by a homomorphism $\pi_1 X \to G$. In the differentiable case this is equivalent to the existence of a connection with curvature zero [15, Lemma 1]. In (2.5), various characterizations of this notion are given. It is shown in particular (6.1), that an SO(2)-bundle is flat if and only if its rational Euler class vanishes. Section 1 is devoted to the definition of a certain homotopy class of *H*-maps $\Omega X \to G$ (1.1), which in the case of a differentiable bundle reduces to the homotopy class of the holonomy map defined by an arbitrary connection on ξ . For the universal *G*-bundle, this class contains a homotopy equivalence $\Omega BG \to G$ (1.4). In Section 3, after some generalities, a characterization of flat U(n)-bundles over an *X* with finite $\pi_1 X$ is given (3.2) by a theorem of Brauer on induced representations [6].

From (2.5, iii) it is clear that the characteristic cohomology-homomorphism of a flat GL(n)-bundle ξ factorizes through $H^*(\pi_1 X)$. Thus one obtains necessary conditions for the characteristic classes of ξ . E. g. for finite $\pi_1 X$ the rational characteristic classes of ξ are trivial, and for $\pi_1 X$ finite of odd order (and $GL(n) = GL(n, \mathbf{R})$) the Stiefel-Whitney classes of ξ are trivial. But many examples of flat bundles with non-trivial characteristic

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classes are known, of which we list a few. The canonical \mathbf{R}^* -bundle on real projective space $P_n \mathbf{R}$ is flat, and $w_1 \in H^1(P_n \mathbf{R}, \mathbf{Z}_2)$ is the generator of $H^*(P_n R, \mathbf{Z}_2)$. In [10, p. 254] there is the example of the SO(2)-bundle on $P_2 \mathbf{R}$ induced from the universal covering bundle by the homomorphism $\mathbf{Z}_2 \to SO(2)$ sending the generator into the antipodal map. The Euler class is the generator of $H^*(P_2 \mathbf{R}, \mathbf{Z})$ and hence non-zero. The example of [15, p. 223, l. 5] shows the existence of a flat SO(2)-bundle on S^5/\mathbf{Z}_p with nonvanishing first Pontrjagin class. The main result of [15] proves the existence of flat $GL^+(2, \mathbf{R})$ -bundles with non-vanishing Euler class. In [2] there is given a five-dimensional flat Riemannian manifold with $w_2 \neq 0$. These examples suggest the study of characteristic classes of flat bundles.

In (3.2), the notion of a σ -flat *G*-bundle with respect to a homomorphism $\sigma: \Phi \to G$ is introduced, where Φ is an arbitrary discrete group. σ -flat bundles are flat (3.5) with a holonomy map $\pi_1 X \to G$ factorizing through σ . The introduction of σ serves as a computational device.

Next we turn to a detailed study of the characteristic classes of σ -flat bundles for G = O(n), SO(n), U(n) and Φ finite abelian. After recalling in Section 4 the relevant definitions and facts, in Section 5 the characteristic classes of $\sigma \in RG(\Phi)$ in the sense of [1] are computed (5.11) as polynomials in 1- and 2-dimensional classes of $H^*(\Phi, \mathbb{Z})$ and $H^*(\Phi, \mathbb{Z}_2)$. The main result is (5.13). For G-bundles which are classified by their characteristic classes, it gives necessary and sufficient conditions for the σ -flatness in terms of these classes. This is illustrated by (5.17) to (5.20). An explicitation for the case $\Phi = \mathbb{Z}_p$ is given in (5.21).

The last section 6 is concerned with the rational characteristic classes of flat bundles. As in all the paper, no differentiability assumptions are made. The triviality of the rational Chern classes for flat U(n)-bundles is proved by topological methods under some restrictions on the fundamental group.

Finally we would like to thank J. Milnor and P. E. Thomas for helpful conversations.¹

1. The holonomy map of a principal bundle. For general properties of principal bundles we refer to [3] [5] [13] and [22]. If ξ is a differentiable principal bundle over a connected space X with group G, a connection on ξ defines a holonomy map $h: \Omega X \to G$. The homotopy class of this map is an invariant of ξ , as shown e.g. in [9]. In this section, we construct a natural transformation $h(X,G): P(X,G) \to [\Omega X, G]$ such that in the case of a

¹Added in proof: The results of the paper were announced in the *Bulletin of the AMS*, vol. 72 (1966), pp. 846-849.

differentiable bundle ξ the class $h(X, G)(\xi)$ reduces to the homotopy class of the holonomy map.²

Let X be a space with basepoint *, EX the space of paths beginning at *, $\epsilon: EX \to X$ the endpoint map and ΩX the loop space. ΩX is an H-space. For a group G, EG is a group by functoriality. In the base space X of a G-bundle ξ we choose a fixed basepoint *. For any choice of a basepoint in the total space T mapped into * under $p: T \to X$, we obtain a principal EGbundle $E(\xi)$ with projection $E(p): ET \to EX$. A section $s: EX \to ET$ of E(p) is called a *path-lifting* in ξ . We think of s heuristically as a connection in ξ . s defines a map $h: \Omega X \to G$ as follows. If $\omega \in \Omega X$, the endpoint $(\epsilon \circ s)(\omega)$ of the lifted path lies over $* \in X$ and hence there is a unique $h(\omega) \in G$ with $* \cdot h(\omega) = (\epsilon \circ s)(\omega)$. Here $* \cdot h(\omega)$ denotes the right action of $h(\omega)$ on the chosen basepoint in T (over the fixed basepoint in X). It is clear that the maps $h: \Omega \to G$ defined for different choices of * in T differ by an inner automorphism of G.

THEOREM 1.1. (i) $h: \Omega X \to G$ is a map of H-spaces, that is: h carries products into products, up to homotopy.

(ii) The equivalence class (under inner automorphisms of G) of the homotopy class of h is an invariant of the bundle ξ . It is denoted $h(\xi)$ and called the holonomy map of ξ .

(iii) $h(\xi)$ is natural in X and G. More precisely, if P(X, G) denotes the equivalence classes of G-bundles on X, $[\Omega X, G]_{\mathfrak{A}}$ the equivalence classes (under inner automorphisms of G) of homotopy classes of H-maps $\Omega X \to G$, then $h(X,G)(\xi) = h(\xi)$ for $\xi \in P(X,G)$ defines a map h(X,G): P(X,G) $\to [\Omega X, G]_{\mathfrak{A}}$ which is natural in X and G.

Proof. (i) is proved in Lemma 1.2. To prove (ii) it is sufficient to show that for a choice of $* \in T$ over the basepoint of X the homotopy class of h is an invariant of the bundle ξ . We first observe that any two path liftings $s_0, s_1: EX \to ET$ are homotopic as path liftings, i.e. there is a family $s_t, t \in I$ of path liftings beginning with s_0 and ending with s_1 .

Let $\mu \in EX$. By the principality of $E(\xi)$ there is a unique $\gamma(\mu) \in EG$ with $s_1(\mu) = s_0(\mu) \cdot \gamma(\mu)$. Hence s_0, s_1 define $\gamma: EX \to EG$. Now let $\Phi: EG \times I \to EG$ be the contraction of EG along paths, i.e. $(\Phi_i\gamma)(s) = \gamma(ts)$ for $\gamma \in EG$ and $s, t \in I$. Note that $\Phi_0\gamma = e \in G, \ \Phi_1\gamma = \gamma$. Hence

² Added in proof: The holonomy-map of a fiber space has also been considered by E. H. Brown, "Twisted tensor products, I," *Annals of Mathematics*, vol. 69 (1959), pp. 223-246 and J. D. Stasheff, "Parallel transport in fiber spaces," to appear.

$$s_t(\mu) = s_0(\mu) \cdot (\Phi_t \gamma)(\mu)$$

for $\mu \in EX$ defines a family $s_t \colon EX \to ET$ of path liftings as required.

Now each path lifting s_t defines as before a map $h_t: \Omega X \to G$, characterized by $* \cdot h_t(\omega) = \epsilon(s_t(\omega))$ for $\omega \in \Omega X$. We claim that

$$h_t(\omega) = h_0(\omega) \cdot \gamma(\omega)(t) \text{ for } \omega \in \Omega X$$

Namely

$$* \cdot h_t(\omega) = \epsilon(s_t(\omega)) = \epsilon(s_0(\omega) \cdot (\Phi_t \gamma)(\omega)) = s_0(\omega)(1) \cdot \gamma(\omega)(t)$$

and

$$* \cdot h_0(\omega) = s_0(\omega)(1) \cdot \gamma(\omega)(0) = s_0(\omega)(1),$$

as $\gamma(\omega)(0) = e \in G$. This equation shows that $h_0, h_1: \Omega X \to G$ are homotopic and hence (ii) is proved. (iii) follows from the construction of h. We observe that strictly speaking the functor $[\Omega X, G]_{\mathscr{H}}$ has to be defined with respect to a basepoint $* \in X$ and basepoint preserving maps $X \to Y$. It can be shown however that the sets $[\Omega X, G]_{\mathscr{H}}$ (equivalence classes—under inner automorphisms of G—of homotopy classes of H-maps $\Omega X \to G$) for different choices of $* \in X$ are canonically isomorphic and that arbitrary maps $X \to Y$ induce well defined maps $[\Omega Y, G]_{\mathscr{H}} \to [\Omega X, G]_{\mathscr{H}}$.

LEMMA 1.2. Let s be a path lifting in ξ and $h: \Omega X \rightarrow G$ the corresponding map. Then the following diagram is homotopy-commutative



Proof. Let $\mu \lor \omega$ denote the composition of $\mu \in EX$ with $\omega \in \Omega X$ (first ω , then μ). Then both $s(\mu \lor \omega)$ and $s(\mu) \cdot h(\omega) \lor s(\omega)$ project under E(p) on $\mu \lor \omega \in EX$. By the principality of $E(\xi)$ there is a unique $\eta: EX \times \Omega X \to EG$ with $s(\mu \lor \omega) = \{s(\mu) \cdot h(\omega) \lor s(\omega)\} \cdot \eta(\mu, \omega)$. From this follows that for $\omega_1, \omega_2 \in \Omega X$

$$h(\omega_2 \vee \omega_1) = h(\omega_2)h(\omega_1)\eta(\omega_2,\omega_1)(1).$$

With the definition

$$k_t(\omega_2 \vee \omega_1) = h(\omega_2)h(\omega_1)\eta(\omega_2,\omega_1)(t)$$

we get $k_0(\omega_2 \vee \omega_1) = h(\omega_2)h(\omega_1)$, $k_1(\omega_2 \vee \omega_1) = h(\omega_2 \vee \omega_1)$ and the lemma is proved.

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We adopt the point of view of Dold [10] and restrict our attention to numerable G-bundles (trivial over a numerable covering of the base). Then by [10], [16] for any topological group G there exist a contractible universal bundle η_G on the classifying space BG of G and the map $[X, BG] \rightarrow P(X, G)$ sending a (not necessarily basepoint preserving) homotopy class $f \in [X, BG]$ into the bundle $f^*\eta_G$ is bijective without any restriction on X. We will denote the classifying map of ξ with the same letter. Put $h_G = h(\eta_G)$. Then from (1.1, iii) follows

COROLLARY 1.3. For a bundle $\xi: X \to BG$

 $h(\xi) = h_G \circ \Omega(\xi).$

It is known that ΩBG and G are homotopy equivalent. We prove now PROPOSITION 1.4. The H-map $h_G: \Omega BG \rightarrow G$ is a homotopy equivalence.

Proof. Let s be a path lifting in the universal bundle η_G , $p_G: TG \to BG$, i.e. $s: EBG \to ETG$ with $E(p_G) \circ s = 1$. Consider the map $k = \epsilon \circ s: EGB \to TG$. The diagram



is commutative (if h_G is represented by the map induced from *s*, otherwise homotopy commutative, which is all we need). This says that *k* is h_G -equivariant and certainly a map of the two fiber spaces.

Now EBG and TG are contractible. By comparing the homotopy sequences of these two fibrations we obtain for any space X a bijection

$$(h_G)_*: [X, \Omega BG] \xrightarrow{\cong} [X, G].$$

Hence $h_G: \Omega GB \to G$ is a homotopy equivalence.

Remark 1.5. That the map h(X, G) of (1.1, iii) need not be injective is seen as follows. Let $\gamma_0, \gamma_1: G \to H$ be homomorphisms of the groups G, Hand $B(\gamma_0), B(\gamma_1): BG \to BH$ the induced maps of classifying spaces. Then by $(1.3) h(BG, H)(B(\gamma_i)) = h_H \circ \Omega B(\gamma_i)$ for i = 1, 2. Consider the homotopy commutative diagram



where h_G , h_H are equivalences by (1.4). Assume $\gamma_1 \simeq \gamma_2$. Then $\Omega B(\gamma_1) \simeq \Omega B(\gamma_i)$ and hence h(BG, H) has the same value on $B(\gamma_1)$, $B(\gamma_2)$. But in the example in [10], p. 254, this situation is realised and moreover $B(\gamma_1) \not\simeq B(\gamma_2)$ (γ_1 and γ_2 are not homotopic as homomorphisms, but only as maps). Hence h(BG, H) is not injective in this example.

Remark 1.7. Proposition 1.4 also shows immediately that a homomorphism $\gamma: G \to H$ inducing a homotopy equivalence $B(\gamma): BG \to BH$ must be an (ordinary) homotopy equivalence ([10], one half of Theorem 9.1). This follows from (1.6), as h_G , h_H and $\Omega B(\gamma)$ are homotopy equivalences.

2. Flat bundles. We turn now to a description of flat bundles. Let G_d be the underlying discrete group of a topological group G, $\iota: G_d \to G$ the canonical map. Taking induced bundles, ι defines a natural map $\iota_*: P(X, G_d) \to P(X, G)$ from discrete G-bundles to G-bundles. ι also induces a map $B(\iota): BG_d \to BG$ and the diagram

$$P(X, G_d) \xrightarrow{\iota_*} P(X, G)$$
$$\uparrow \cong \qquad \uparrow \cong \\ [X, BG_d] \xrightarrow{B(\iota)_*} [X, BG]$$

is commutative for any space X (and natural in X and G).

LEMMA 2.1. BG_d is an Eilenberg MacLane space $K(G_d, 1)$.

Proof. BG_d is the base of the universal G_d -bundle π_{G_d} with $p_{G_d}: TG_d \rightarrow BG_d$, where TG_d is contractible [10]. The homotopy sequence of this fibration shows $\pi_n(BG_d) = 0$ for n > 1 and $\pi_1(BG_d) \cong \pi_0(G_d) \cong G_d$, which proves the lemma.

PROPOSITION 2.2. There is a natural map

 $\pi_1: [X, BG_d] \to Hom(\pi_1 X, G_d) \Longrightarrow H^1(\pi_1 X, G_d)$

where the right hand side denotes the set of equivalence classes (under inner automorphisms of G) of homomorphisms $\pi_1 X \to G_d$.

Proof. This follows from Lemma 2.1, as $\pi_1 BG_d \cong G_d$. Note that

 $[X, BG_d]$ are ordinary homotopy classes (not necessarily basepoint preserving). See also the comment at the end of the proof of 1.1.

PROPOSITION 2.3. The canonical map $q: \Omega X \to \pi_1 X$ induces a bijection $q^*: Hom(\pi_1 X, G_d) \to [\Omega X, G_d]_{\mathscr{H}}.$

Proof. As G_d is discrete, any homotopy class of maps $\Omega X \to G_d$ contains a single element. Hence $[\Omega X, G_d]_{\mathcal{H}}$ is the set of equivalence classes (under inner automorphisms of G_d) of continuous maps $\Omega X \to G_d$. Now a continuous map $\Omega X \to G_d$ is constant on any homotopy class of loops. But $\pi_1 X \cong \pi_0 \Omega X$, from which the bijectivity of q^* follows.

With these notations we can state

THEOREM 2.4. The following diagram is commutative (and natural in X and G)



All vertical maps are bijective.

Proof. The only fact which remains to prove is the commutativity of the front square, which follows from a simple argument.

Observe that the map $s = h(X, G_d)^{-1} \circ q^*$ is the map of [22, thm. 13.9]. According to this theorem, s is bijective (and hence also $h(X, G_d)$ and π_1) if X is arcwise connected, arc-wise locally connected, and semi locally 1connected. From now on we assume that X satisfies these conditions. Then X has a universal covering space $\tilde{X} \to X$, which is a principal $\pi_1 X$ -bundle, the universal covering bundle ζ . Its classifying map is also denoted by $\zeta: X \to B\pi_1 X$. Any homomorphism $\gamma: \pi_1 X \to G$ induces a G-bundle $\gamma_* \zeta$ over X with classifying map $B(\gamma) \circ \zeta$. Observe that the "identification" $\iota_*: \underline{Hom}(\pi_1 X, G_d) \to \underline{Hom}(\pi_1 X, G)$ is bijective. Then it is clear that the map sending γ into $\gamma_* \zeta$ is just the bijection s. COROLLARY 2.5. For $\xi \in P(X, G)$ the following conditions are equivalent.

(i) $\xi = \iota_* \eta$ for some $\eta \in P(X, G_d)$.

(ii) $\exists \eta: X \rightarrow BG_d$, such that the following diagram is homotopy commutative



(iii) \exists a homomorphism $\gamma: \pi_1 X \to G$, such that the following diagram is homotopy commutative

$$B\pi_1 X \xrightarrow{B(\gamma)} BG$$
$$\zeta \searrow \checkmark \xi$$

(iv) $\xi = \gamma_* \zeta$ for some homomorphism $\gamma : \pi_1 X \to G$.

(v) $h(\xi): \Omega X \to G$ factorizes through $q: \Omega X \to \pi_1 X$, up to homotopy.

Proof. (ii) \Rightarrow (iii). Define $\overline{\gamma} = \pi_1(\eta)$. Then $B(\overline{\gamma})\zeta \simeq \eta$ by the mentioned interpretation of the map s. Hence with $\gamma = \iota \circ \overline{\gamma}$ we have $B(\gamma)\zeta \simeq B(\iota)\eta = \xi$. The rest of (2.5) follows from (2.4).

Definition 2.6. $\xi \in P(X, G)$ is called flat if one (and hence any) of the conditions of (2.5) is satisfied.

Remark 2.7. If ξ is a differentiable *G*-bundle, Lemma 1 of [15] shows that ξ is flat if and only if it possesses a connection with curvature zero.

The main point of (2.5) is the following. We want to decide if a given $\xi \in P(X, G)$ is flat. According to (iii) we have to fill in the diagram with a map $B(\gamma)$ which is induced by a homomorphism $\gamma: \pi_1 X \to G$. This problem has only been solved in very special cases [15] (The well-known description of complete flat Riemannian or affine manifolds can also be considered as an answer in these cases). Much simpler is (ii), where we have to fill in the diagram with a map η , i.e. one has to solve a lifting problem for the map ξ . A new difficulty arises from the generality of BG_d . We introduce at the end of section 3 an intermediate discrete group Φ , which in the applications will be finite. We emphasize that the introduction of Φ serves as a computational device.

As an immediate consequence of the definition we obtain

PROPOSITION 2.8. Let $\xi \in P(X, G)$ be flat and $\pi: \tilde{X} \to X$ the universal covering of X. The lifted bundle $\pi^*\xi$ on \tilde{X} is trivial.

Proof. The classifying map $\xi \circ \pi : \tilde{X} \to BG$ of $\pi^*\xi$ factorizes through the contractible total space of the universal $\pi_1 X$ -bundle on $B\pi_1 X$ and is hence $\simeq 0$.

3. The transformation α . Let G be the infinite orthogonal or unitary group, KX the ring of G-vectorbundles on X and $RG(\Phi)$ the ring of Grepresentations of a finite group Φ . $\sigma \in RG(\Phi)$ induces a map $\sigma_* : P(X, \Phi) \rightarrow P(X, G)$ sending ξ to $\sigma_*\xi$. Its associated G-vectorbundle is still denoted $\sigma_*\xi$ and $\alpha(\xi)(\sigma) = \sigma_*\xi$ defines a ringhomomorphism $\alpha(\xi) : RG(\Phi) \rightarrow KX$. For all this see [1]. Let $[RG(\Phi), KX]$ be the ringhomomorphisms $RG(\Phi) \rightarrow KX$. Then

$$\alpha \colon P(X, \Phi) \to [RG(\Phi), KX]$$

is natural in X and Φ . In the sequel we shall use this and the following fact without comment.

Let X be a space with finite $\pi_1 X = \Phi$ and $f: X' \to X$ the finite covering corresponding to a subgroup $\iota: \Phi' \subset \Phi$. The direct image $f_{!\eta}$ of a vectorbundle η on X' is defined by $(f_{!\eta})_x = \bigoplus_y \eta_y$, where $y \in f^{-1}(x)$. Inducing representations by ι gives a module homomorphism $\iota_1: RG(\Phi') \to RG(\Phi)$. With the universal coverings $\zeta, \pi: \tilde{X} \to X$ and $\zeta', \pi': \tilde{X} \to X'$ the following diagram is commutative

$$(3.1) \qquad \begin{array}{c} RG(\Phi') & \xrightarrow{\alpha(\zeta')} KX' \\ \downarrow \iota_1 & \downarrow f_1 \\ RG(\Phi) & \xrightarrow{\alpha(\zeta)} KX. \end{array}$$

A vectorbundle is called flat, if its associated principal bundle is flat. A theorem of Brauer [6] on characters of finite groups gives the following description of flat U(n)-vectorbundles.

PROPOSITION 3.2. Let X be a CW-complex with $\pi_1 X$ finite, and ξ a U(n)-vector bundle on X. ξ is flat if and only if the following holds. There exists a finite number of coverings $f_i: X_i \to X$ and C-line bundles η_i on X_i $(i = 1, \dots, m)$ such that

(i)
$$c_1(\eta_i) \boldsymbol{\varrho} = 0$$
 ($c_1(\eta_i) \boldsymbol{\varrho}$ = rational Chern class).

(ii)
$$\xi = \sum_{i=1}^m f_{i_1} \eta_i.$$

Proof. We first observe that by (6.3) condition (i) is equivalent to the existence of a unique family of 1-dimensionl **C**-representations $\rho_i \in RU(\pi_1 X_i)$ inducing the η_i from the coverings $\zeta_i, \pi_i : \tilde{X} \to X_i; \ \eta_i = \rho_{i*} \zeta_i$.

To prove sufficiency, consider $j_i = \pi_1(f_i) : \pi_1 X_i \to \pi_1 X$ and define

$$\rho = \sum_{i=1}^{m} j_{i} \rho_i \in RU(\pi_1 X)$$

We claim that ξ is induced from the universal covering bundle ζ by ρ and hence flat. Namely

$$\rho_{*}\zeta = \alpha(\zeta)\rho = \sum_{i=1}^{m} \alpha(\zeta) j_{i_{1}}\rho_{i} = \sum_{i=1}^{m} (j_{i_{1}}\rho_{i})_{*}\zeta$$
$$= \sum_{i=1}^{m} f_{i_{1}}(\rho_{i*}\zeta_{i}) = \sum_{i=1}^{m} f_{i_{1}}\eta_{i} = \xi.$$

Conversely assume $\xi = \rho_* \zeta$ for some $\rho \in RU(\pi_1 X)$. By the theorem of Brauer [6] there exists a finite number of subgroups $j_i \colon \Phi_i \to \pi_1 X$ and 1-dimensional **C**-representations $\rho_i \in RU(\Phi_i)$ $(i = 1, \dots, m)$ with $\rho = \sum_{i=1}^m j_{i_1} \rho_i$. Let $f_i \colon X_i \to X$ be the coverings with $\pi_1 X_i = \Phi_i$ and put ζ_i for the covering $\pi_i \colon X_i \to X$. For $\eta_i = \rho_{i*} \zeta_i$ we have $c_1(\eta_i)_{\mathbf{Q}} = 0$ by (6.1). (ii) follows from

$$\sum_{i=1}^{m} f_{i_1} \eta_i = \sum_{i=1}^{m} f_{i_1}(\rho_{i*}\zeta_i) = \sum_{i=1}^{m} (j_{i_1}\rho_i)_* \zeta = \rho_* \zeta = \xi \text{ and the proposition is proved. } \blacksquare$$

For the rest of this section, G is again an arbitrary topological group and Φ a discrete group. Let $\sigma: \Phi \to G$ be a homomorphism. If we replace ι by σ and G_d by Φ in Theorem 2.4, it still holds. Let $B(\sigma): B\Phi \to BG$ be the induced map of classifying spaces.

Definition 3.3. $\xi \in P(X, G)$ is called σ -flat if $\exists \eta \colon X \to B\Phi$, such that the following diagram is homotopy commutative

$$(3.4) \qquad \qquad \begin{array}{c} B\Phi \xrightarrow{B(\sigma)} BG \\ \eta \swarrow \xrightarrow{\chi} \checkmark \xi \end{array}$$

A map η with this property is called a σ -structure on ξ . ξ is σ -flat if there exists a σ -structure on ξ .

THEOREM 3.5. $\xi \in P(X, G)$ is σ -flat if and only if \exists a homomorphism $\gamma' : \pi_1 X \to \Phi$, such that ξ is induced from the universal covering bundle ζ of X by $\gamma = \sigma \circ \gamma' : \xi = \gamma_* \zeta$. In particular, σ -flat implies flat.

Proof. As Corollary (2.5).

Comments on Definition 3.3 (see also end of Section 2). For $\Phi = G_d$ and $\sigma = \iota \colon G_d \to G$ a $\xi \in P(X, G)$ is flat if and only if it is ι -flat, i.e. a

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discrete G-bundle. For a flat $\xi \neq \gamma : \pi_1 X \to G$ with $\xi = \gamma_* \zeta$ can be thought of as the holonomy map and $\gamma(\pi_1 X) \subset G$ as the holonomy group of ξ . Then for injective σ a bundle ξ is σ -flat if and only if it is flat with holonomy group contained in Φ .

4. Characteristic classes of group representations. In this section we list some facts on characteristic classes of G-bundles, G = U(n), O(n), SO(n), which will be needed in Sections 5 and 6. We recall the definition of characteristic classes of group-representations [1, Appendix] and give some interpretations of low-dimensional classes. Our general references here are [1], [3], [4], [5], [13] and [17]. Let X be a paracompact, connected, locally connected, semi-locally contractible space, e.g. a CW-complex. If G is a topological group, we denote by G_X (or G if no confusion is possible) the group-valued sheaf of germs of continuous functions $X \to G$. Then we have the following natural isomorphisms:

(4.1)
$$\dot{H}^1(X,\underline{G}) \cong P(X,G) \cong [X,BG],$$

where $H^1(X, \underline{G})$ is the Čech cohomology set of X with coefficients in the (non-abelian) sheaf \underline{G} .

G = U(n), n > 0: By [3] one has $H^*(BU(n), \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n],$ deg $c_i = 2i$ and the inclusion $\iota: T(n) = U(1)^n \subset U(n)$ induces a map

$$B(\iota)^* \colon H^*(BU(n), \mathbb{Z}) \to H^*(BT(n), \mathbb{Z}) \cong \mathbb{Z}[x_1, \cdots, x_n], \ \deg x_i = 2$$

given by

(4.2)
$$B(\iota) * c_i = \sigma_i(x_1, \cdots, x_n)$$
, *i*-th elementary symmetric function.

For a U(n)-bundle $\xi: X \to BU(n)$, the Chern-classes are given by $c_i(\xi) = \xi^* c_i$. G = O(n): By [4] one has $H^*(BO(n), Z_2) \cong \mathbb{Z}_2[w_1, \cdots, w_n]$, deg w_i = i and the inclusion $\mu: Q(n) = \mathbb{Z}_2^n \subset O(n)$ sends w_i to

$$B(\mu)^* w_i = \sigma_i(y_1, \cdots, y_n) \in H^*(Q(n), \mathbb{Z}_2) \simeq \mathbb{Z}_2[y_1, \cdots, y_n], \ \deg y_i = 1.$$

For an O(n)-bundle $\xi: X \to BO(n)$ the Stiefel-Whitney classes are given by $w_i(\xi) = \xi^* w_i$.

Recall the formula [4], [25]

(4.4)
$$\beta_2 w_j = Sq^1 w_j = \begin{cases} 0, & j \text{ odd} \\ w_{j+1} + w_1 w_j, & j \text{ even} \end{cases}$$

where β_2 is the \mathbb{Z}_2 -Bockstein-operation.

In the case G = SO(n), one just has to put $w_1 = 0$ and all formulas remain valid.

The universal Pontrjagin classes p_i are defined by the inclusion

 $\kappa: O(n) \subset U(n): \check{p}_{j} = B(\kappa)^{*}c_{j}, \ p_{i} = (-1)^{i}\check{p}_{2i}, \ i = 1, \cdots, k = \lfloor n/2 \rfloor.$ Then one has $\lfloor 5 \rfloor$:

(4.5)
$$\begin{aligned} H^*(BO(n), \mathbf{Z}) &\cong \mathbf{Z}[p_1, \cdots, p_k] \oplus T_2 \\ T_2 &\cong \operatorname{im}(\rho_2 \mid T_2) \cong \operatorname{im} Sq^1, \ 2 \cdot T_2 = 0, \end{aligned}$$

where $\rho_p: H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z}_p), p \ge 0$, means reduction mod p $(p=0: \mathbb{Z}_0 = \mathbb{Q})$. Moreover the classes \check{p}_{2i+1} are redundant, since we have

(4.6)
$$\check{p}_{2i+1} = (\delta_2 w_{2i})^2 + (\delta_2 w_1) p_i \in T_2$$

In Section 5 we will need some reduction formulae for Pontrjagin classes which we list here without comment. They can be found in [5]. The inclusions $T(k) \stackrel{\iota}{\subset} U(k) \stackrel{r}{\subset} O(n) \stackrel{\kappa}{\subset} U(n), \ k = \lfloor n/2 \rfloor$, induce cohomology maps given by

(4.7)
$$B(r)^* \check{p}_i = \sum_{j+l=i}^{\infty} (-1)^{l} c_j c_l$$

and therefore

(4.8)
$$B(r \circ \iota)^* p_i = \sum_{j+l=2i} (-1)^{l+i} \sigma_i (x_1, \cdots, x_k) | \sigma_l (x_1, \cdots, x_k) = \sigma_i (x_1^2, \cdots, x_k^2) \in H^*(BT(k), \mathbb{Z}) \cong \mathbb{Z}[x_1, \cdots, x_k],$$

(4.9)
$$B(r \circ \iota) * \check{p}_{2i+1} = 0.$$

The inclusion $Q(n) \stackrel{\mu}{\subset} O(n)$ induces a cohomology map given by

(4.10)
$$B(\mu)^* \mathring{p}_j = \sigma_j(\delta_2 y_1, \cdots, \delta_2 y_n) \in H^*(Q(n), Z),$$

 $(\delta_2 = integral Bockstein-operation)$ and therefore

(4.11)
$$B(\mu) * \rho_2 \check{p}_j = \sigma_j(y_1^2, \cdots, y_n^2) = \sigma_j(y_1, \cdots, y_n)^2 = B(\mu) * w_j^2$$

which implies

(4.12)
$$\rho_2 \tilde{p}_j = (w_j)^2$$

For G = SO(2k+1), (4.5) remains valid, the universal Euler-class being given by $\chi_{2k+1} = \delta_2 w_{2k}$.

For G = SO(2k) one has

(4.13)
$$H^*(BSO(2k), \mathbb{Z}) \cong \mathbb{Z}[p_1, \cdots, p_{k-1}, \chi_{2k}] \oplus T_2',$$

 $\chi^2 = p_k, \ 2 \cdot T_2' = 0$

and

(4.14)
$$\rho_2 \chi_{2k} = w_{2k}, \ B(r \circ \iota)^* \chi_{2k} = \prod_{i=1}^k x_i.$$

The formulae (4.6) \div (4.12) remain true after obvious modifications ($w_1 = 0$, $\sum_{i=1}^{n} y_i = 0$).

Let Φ be a finite group, η_{Φ} the universal Φ -bundle with base space $B\Phi = K(\Phi, 1)$. The transformation α of Section 3 allows us to define the Chern-classes $c_i(\sigma)$ of a representation $\sigma \in RU(\Phi)$ by [1, Appendix], [12]

(4.15)
$$c_i(\sigma) = c_i(\alpha(\eta_{\Phi})(\sigma) = c_i(\sigma_*\eta_{\Phi}) \in H^{2i}(\Phi, \mathbb{Z}).$$

Stiefel-Whitney classes $w_i(\sigma)$, Pontrjagin classes $p_i(\sigma)$, $\sigma \in RO(\Phi)$ and the Euler-class $\chi(\sigma)$, $\sigma \in RSO(\Phi)$ are similarly defined. The Pontrjagin classes are again determined by the Chern-classes:

$$p_i(\sigma) = p_i(\alpha(\eta_{\Phi})(\sigma) = p_i(\sigma_*\eta_{\Phi}) = c_{2i}(\kappa_*\sigma_*\eta_{\Phi}) = c_{2i}(\kappa_*(\sigma)),$$

for $\sigma \in RO(\Phi)$, $k_*: RO(\Phi) \to RU(\Phi)$ induced by $\kappa: 0 \to U$.

Since $\alpha(\eta_{\Phi})$ is a natural ring-homomorphism, the characteristic classes of representations inherit the formal properties of the characteristic classes of bundles.

Characteristic classes can also be defined for non-finite discrete groups Φ . $R(\Phi)$ has then to be interpreted as the Grothendieck-ring finite-dimensional representations of Φ .

The low dimensional characteristic classes are described by the following propositions.

PROPOSITION 4.16.

(i) Let $RU(\Phi) \xrightarrow{\det} Hom(\Phi, U(1))$ be the determinant-homomorphism. Then

$$c_1(\sigma) = c_1(\det \sigma), \sigma \in RU(\Phi).$$

(ii) Let $RO(\Phi) \xrightarrow{\det} Hom(\Phi, O(1) = \mathbb{Z}_2)$ be the determinant-homomorphism. Then

$$w_1(\sigma) = w_1(\det \sigma), \sigma \in RO(\Phi).$$

Proof. (i) The result is known for U(n)-bundles. Exterior powers λ^i make KU and RU into λ -rings and $\alpha(\eta_{\Phi})$ becomes a λ -map [1, §12]. But for dim $\sigma = n$ we have $\lambda^n \sigma = \det \sigma$ and therefore

$$c_1(\det \sigma) = c_1(\alpha(\eta_{\Phi})\lambda^n \sigma) = c_1(\lambda^n \alpha(\eta_{\Phi})\sigma)$$
$$= c_1(\det \alpha(\eta_{\Phi})\sigma) = c_1(\alpha(\eta_{\Phi})\sigma) = c_1(\sigma), \text{ qed.}$$

(ii) is proved by the same argument.

The following proposition gives a representation of the fundamental characteristic class (primary obstruction) of an arbitrary *G*-bundle as a coboundary map in a (non-abelian) cohomology sequence (cf. [13, 4.3.1] for $G = C^*$). Assume that *G* is path-connected and that *BG* has a *CW*-structure Let ξ a *G*-bundle on *X*, $\tau_{\xi} \colon H^1(G, \pi_1 G) \to H^2(X, \pi_1 G)$ the transgression in ξ and $\iota \in H^1(G, \pi_1 G) \cong \operatorname{Hom}(\pi_1 G, \pi_1 G)$ the fundamental class of *G* with respect to the natural orientation of $S^1 \subset C$.

PROPOSITION 4.17. (i) Consider the exact sequence of group-valued sheaves on X

$$0 \to \pi_1 G \to \tilde{G} \to G \to 1$$

defined by the universal covering of G. Since π_1G is abelian, we have a (non-abelian) exact sequence in Čech-cohomology

 $* \to H^1(X, \pi_1 G) \to \check{H}^1(X, \tilde{G}) \to \check{H}^1(X, \bar{G}) \xrightarrow{\delta} H^2(X, \pi_1 G).$

Then $\delta(\xi) = -\tau_{\xi}(\iota)$ for $\xi \in \check{H}^1(X, \check{G})$.

(ii) Taking $X = B\Phi$ (Φ discrete), we have a (non-abelian) grouptheoretic exact cohomology sequence

$$* \to H^1(\Phi, \pi_1 G) \to H^1(\Phi, \tilde{G}_d) \to H^1(\Phi, G_d) \xrightarrow{\delta'} H^2(\Phi, \pi_1 G)$$

and again

$$\delta'(\sigma) = -\tau_{\sigma_*\eta_\Phi}(\iota) = -(B\sigma)^*\tau_{\eta_G}(\iota),$$

where η_G resp. η_{Φ} are universal G- resp. Φ -bundles, and

$$\sigma \in H^1(\Phi, G_d) \cong Hom(\Phi, G_d).$$

(iii) Let $\sigma \in H^1(\Phi, G_d)$; the fundamental group $\Gamma = \pi_1(E\Phi \times_{\Phi} G)$ of the total space $E\Phi \times_{\Phi} G$ of the bundle $\sigma_{*\eta\Phi}$ is an extension of the form

$$0 \to \pi_1 G \to \Gamma \to \Phi \to 1$$

whose class in $H^2(\Phi, \pi_1 G)$ is equal to $\delta'(\sigma)$.

Remark 4.18. The cohomology sequences in (i), (ii) are exact in the category of sets with basepoint (see e.g. [21*] for the group-theoretic case).

COROLLARY 4.19. $\xi \in \check{H}^1(X, G)$ can be lifted to a \tilde{G} -bundle if and only if $\tau_{\xi}(\iota) = 0$. Moreover, if X is a CW-complex, $-\tau_{\xi}(\iota) = o(\xi)$ where $o(\xi)$ is the primary obstruction of ξ [22, p. 189] and therefore: $\xi \in \check{H}^1(X, G)$ has a \tilde{G} -structure if and only if $o(\xi) = 0$.

Proof. (i) By naturality of the transgression and δ , it is sufficient to prove the statement for the universal bundle $\eta_G \in \check{H}^1(BG, \underline{G})$. We reduce the problem further to bundles on S^2 by the following trick. Orient $S^2 = \Sigma S^1$ coherently with the canonical orientation of S^1 and let $[f] \in \pi_2 BG$. Consider the diagram, in which all isomorphisms are either canonical or given by the orientations:

$$[BG, BG] \cong \check{H}^{1}(BG, \underline{G}) \xrightarrow{\delta} \check{H}^{2}(BG, \pi_{1}G) \cong H^{2}(BG, \pi_{1}G) \cong \operatorname{Hom}(\pi_{2}BG, \pi_{1}G)$$

$$(4.20) \quad f^{*} \bigvee_{[f]} \overset{\operatorname{id}}{\longrightarrow} \eta_{G} \bigvee_{\delta} f^{*} \bigvee_{f^{*}} f^{*} \bigvee_{f^{*}} f^{*} \bigvee_{f^{*}} f^{*} \bigvee_{f^{*}} f^{*} \bigvee_{[f]} f^{*} \bigvee_{\delta} f^{*} \int_{\delta} f^{$$

By naturality of δ and the isomorphisms involved, the diagram is commutative and the f^* at the right is essentially an evaluation map. Therefore it is sufficient to show that $f^*(-\tau_{\eta_G}(\iota)) = f^*(\delta(\eta_G))$ for all $[f] \in \pi_2 BG$ or equivalently by $(4.20): -\tau_{f^*\eta_G}(\iota) = \delta(f^*\eta_G)$. But $-\tau_{f^*\eta_G}(\iota) \in \operatorname{Hom}(\pi_2 S^2, \pi_1 G)$ $\cong \pi_1 G$ is given by $\gamma([f]) \in \pi_1 G$ where $\gamma: \pi_2 BG \cong \pi_1 G$ is the canonical isomorphism and hence we are done if we show: The composition of the maps in the bottom line of (4.20) is equal to γ . This follows by an argument similar to that used in [13, 4.3.1] if one observes that the bundle $f^*\eta_G$ on $S^2 = \Sigma S^1$, $[f] \in [S^2, BG]$ is obtained by clutching the trivial bundles on the 2 hemispheres of S^2 along the common equator S^1 by a map $S^1 \to G$ representing the class $\gamma([f]) \in \pi_1 G$.

(ii) follows from (i) and the commutative diagram

$$\begin{array}{ccc}
\delta' \\
H^{1}(\Phi, G_{d}) & \longrightarrow & H^{2}(\Phi, \pi_{1}G) \\
& & & \downarrow & \alpha(\eta_{\Phi}) & & & \\
\check{H}^{1}(B\Phi, \underline{G}) & \longrightarrow & H^{2}(B\Phi, \pi_{1}G)
\end{array}$$

where $\alpha(\eta_{\Phi})(\sigma) = \sigma_*\eta_{\Phi}, \sigma \in H^1(\Phi, G_d)$.

(iii) The extension corresponding to $\delta'(\sigma)$ is given by a pull-back operation

(4.21)
$$\begin{array}{c} 0 \to \pi_1 G \to \tilde{G} \xrightarrow{p} G \to 1 \\ & \left\| \begin{array}{c} \uparrow \\ j \downarrow \\ i \\ f \end{array} \right\| \sigma \\ \delta'(\sigma) : 0 \to \pi_1 G \to \Gamma \xrightarrow{p} \sigma \to 1 \\ q \end{array}$$

where $\Gamma = \{(s, \tilde{g}) \in \Phi \times \tilde{G}/\sigma s = p\tilde{g}\}$ and j, q are the natural projections. Define a free Γ -action on $E_{\Phi} \times \tilde{G}$ by $(e, \tilde{g}')^{(s, \tilde{g})} = (e^s, \tilde{g}^{-1} \cdot \tilde{g}'), (s, \tilde{g}) \in \Gamma$. The $\pi_1 G$ -action on $E_{\Phi} \times \tilde{G}$ is then given by $(e, \tilde{g}')^{\alpha} = (e, \alpha^{-1} \cdot \tilde{g}'), \alpha \in \pi_1 G \subset \tilde{G}$ and thus $E_{\Phi} \times \tilde{G}/\pi_1 G \cong E_{\Phi} \times G$. The quotient group Φ acts now on $E_{\Phi} \times G$ by $(e, g)^s = (e^s, \sigma s^{-1} \cdot g), s \in \Phi$ and therefore $E_{\Phi} \times G/\Phi = E_{\Phi} \times_{\Phi} G$. Since $E_{\Phi} \times \tilde{G}$ is 1-connected, we have a universal covering

$$E_{\Phi} \times \tilde{G} \xrightarrow{\pi_1 G} E_{\Phi} \times G \xrightarrow{\Phi} E_{\Phi} \times_{\Phi} G$$

with decktransformation groups given by the bottom line of (4.21) and the result follows by a standard argument.

Examples.

4.22.
$$G = U(n) : \delta(\xi) = c_1(\xi) \in H^2(X, \mathbb{Z}), \xi \in \check{H}^1(X, U(n)),$$

 $\delta'(\sigma) = c_1(\sigma) \in H^2(\Phi, \mathbb{Z}), \sigma \in H^1(\Phi, U(n)_d),$
[13, 4.3.1].
4.23. $G = GL^+(2, \mathbb{R}) : \delta(\xi) = \chi(\xi) \in H^2(X, \mathbb{Z}), \xi \in \check{H}^1(X, GL^+(2, \mathbb{R}))$
 $\delta'(\sigma) = \chi(\sigma) \in H^2(\Phi, \mathbb{Z}), \sigma \in H^1(\Phi, GL^+(2, \mathbb{R})_d)$

4.24.
$$G = SO(n), n > 2: \delta(\xi) = w_2(\xi) \in H^2(X, \mathbb{Z}_2), \xi \in \check{H}^1(X, \underline{SO(n)}),$$
$$\delta'(\sigma) = w_2(\sigma) \in H^2(\Phi, \mathbb{Z}_2), \sigma \in H^1(\Phi, \underline{SO(n)}_d).$$

This includes the result of [5, p. 350]: $w_2(\xi) = 0$ if and only if ξ can be lifted to a SO(n) = Spin(n)-bundle.

4.25. By (4.17, ii) and (4.20) one can compute [1, p.62] the Chern class c_1 of the irreducible representations

$$\rho^{\lambda} \colon \mathbb{Z}_n \to U(1), \ \rho^{\lambda}(1) = \exp(\lambda/n), \ \lambda = 0, \cdots, n-1:$$

$$c_1(\rho^{\lambda}) = \lambda \cdot u \in H^*(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}[u]/(n \cdot u), \deg u = 2.$$

4.26. Milnor's characterisation of flat $GL^+(2, \mathbf{R})$ -bundles on compact oriented 2-manifolds M_g of genus $g \ge 1$ [15]: Proposition 4.17 allows to give a formal description of Milnor's construction as follows: The fundamental group $\pi = \pi_1 M_g$ is generated by elements $\alpha_1 \cdots \alpha_{2g}$ with the single relation $\phi(\alpha_1 \cdots \alpha_{2g}) = \prod_{i=1}^g \alpha_{2i-1} \alpha_{2i} \alpha_{2i-1}^{-1} \alpha_{2i}^{-1} = 1$. Since M_g is covered by the open 2-disc (Uniformisation Theorem), it follows that $M_g = B\pi$ and that $H^*(M_g, A)$ $\cong H^*(\pi, A)$, for any π -module A of coefficients. Thus π is of cohomological dimension 2 and moreover $H^1(\pi, \mathbf{Z}) \cong \mathbf{Z}^{2g}$, $H^2(\pi, \mathbf{Z}) \cong \mathbf{Z}$.

The diagram

now shows that $\xi \in \check{H}^1(M_g, \underline{GL^+(2, \mathbf{R})})$ is flat if and only if $\chi(\xi) \in \operatorname{im}(\delta')$. In fact, the proofs on pp. $2\overline{17}$ - $2\overline{20}$ of [15] consist essentially in a determination of $\operatorname{im}(\delta')$:

$$\operatorname{im}(\delta') = \{ \alpha \in \mathbb{Z} / | \alpha | < g \}.$$

5. The lifting problem for finite abelian Φ . In this section the lifting problem 3.4 for $\xi \in [X, BG]$ is discussed for Φ finite abelian, G = O(n), SO(n), U(n). First we compute the characteristic classes of representations $\tau: \Phi \to G$ as polynomials in 1- and 2-dimensional classes of $H^*(\Phi, \mathbb{Z})$ and $H^*(\Phi, \mathbb{Z}_2)$. This yields necessary conditions for the characteristic classes of τ -flat bundles. These conditions turn out to be sufficient in certain cases.

Let $\tau: \Phi \to O(n)$ be a representation of Φ , *m* the number of irreducible 2-dimensional components of τ and k = n - 2m. Then we have the following factorization of $\tau:^3$

(5.1)
$$\Phi \longrightarrow F_m = SO(2)^m \times Q(k) \stackrel{\rho}{\subseteq} O(n),$$

 ρ being the standard inclusion.

The characteristic classes for the reduction ρ are given by

Proposition 5.2.

³We are indebted to J. M. G. Fell for pointing out this fact.

(i)
$$B(\rho)^* w_i = \sum_{2j+l=i} \sigma_j(\rho_2 x_1, \cdots, \rho_2 x_m) \cdot \sigma_l(y_1, \cdots, y_k)$$
.
 $\in H^*(BF_m, \mathbb{Z}_2) \cong \mathbb{Z}_2[\rho_2 x_1, \cdots, \rho_2 x_m] \otimes \mathbb{Z}_2[y_1, \cdots, y_k]$
(deg $x_i = 2$, deg $y_i = 1$).
(ii) $B(\rho)^* p_i = \sum_{i=1}^{n} \sigma_j(x_1^2, \cdots, x_m^2) \sigma_{2l}(\delta_2 y_1, \cdots, \delta_2 y_k)$

(iii) If τ is oriented, τ factors through $SF_m = SO(2)^m \times SQ(k)$ and we have

$$B(\rho)^* \chi_n = \delta_2 B(\rho)^* w_{n-1} \qquad if \ n \ odd, \ k > 1$$
$$= 0 \qquad if \ n \ odd, \ k = 1$$
$$2 \cdot B(\rho)^* \chi_n = \delta_2 B(\rho)^* w_{n-1} = 0 \qquad if \ n \ even, \ k > 0$$
$$B(\rho)^* \chi_n = x_1 \cdot \cdots \cdot x_m \qquad if \ n \ even, \ k = 0.$$

Proof. (i) is a simple consequence of the Whitney formula.
(ii) Factor ρ in the following form

$$F_{m} = SO(2)^{m} \times Q(k) \xrightarrow{\rho} O(n)$$

$$\vec{\rho} \times \mu \searrow \qquad \nearrow \iota$$

$$O(2m) \times O(k)$$

Then by the Whitney formula

$$(-1)^{i}B(\iota)^{*}p_{i} = \sum_{j+l=2i} \tilde{p}_{j}\tilde{p}_{l} = \sum_{j+l=i} [(-1)^{i}p_{j}p_{l} + \tilde{p}_{2j+1}\tilde{p}_{2l-1}]$$

$$(-1)^{i}B(\rho)^{*}p_{i} = \sum_{j+l=i} [(-1)^{i}B(\rho)^{*}p_{j} \cdot B(\mu)^{*}p_{l} + B(\rho)^{*}\tilde{p}_{2j+1}B(\mu)^{*}\tilde{p}_{2l-1}].$$

But now

$$B(\tilde{\rho}) * p_{j} = \sigma_{j}(x_{1}^{2}, \cdots, x_{m}^{2}) \text{ by } (4.8),$$

$$B(\mu) * p_{l} = \sigma_{2l}(\delta_{2}y_{1}, \cdots, \delta_{2}y_{k}) \text{ by } (4.10),$$

$$B(\tilde{\rho}) * \tilde{\rho}_{2j+1} = 0 \text{ by } (4.9),$$

and the formula follows.

(iii) The first two formulae are obvious. The last follows from (4.14). If n is even, k > 0 we have $\delta_2 w_{n-1} = 0$ by (4.4), $\rho_2(2 \cdot \chi_n) = 2 \cdot w_n = 0$ and $\rho_0(B(\rho)^*\chi_n) = B(\rho)^*(\rho_0\chi_n) = 0$ since k > 0. The third formula now follows by a torsion argument [18, p. 112].

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Remark 5.3. If we look only at the reduction

$$O(m) \times Q(k) \underset{i\delta \times \mu}{\subset} O(m) \times O(k) \underset{\iota}{\subset} O(m+k)$$

we obtain in the same manner, applying (4.6)

$$B(\iota \circ (i\delta \times \mu))^* p_i = \sum_{j+l=i} [p_j \cdot \sigma_{2l}(\delta_2 y_1, \cdots, \delta_2 y_k) + \{(\delta_2 w_{2j})^2 + (\delta_2 w_1) \cdot p_j\} \sigma_{2l-1}(\delta_2 y_1, \cdots, \delta_2 y_k)].$$

This is formula (2.1) of [14], which follows here simply from the productformula for \check{p}_i , (4.6) and (4.10). (5.2, iii) remains true in this case except when *n* even and k = 0.

Remark 5.3'. The Euler-class satisfies a product-formula and therefore $B(\rho)^*\chi_n = x_1 \cdots x_m \cdot B(\mu)^*\chi_k$. In the case *n* even, k > 0 we show that the class $B(\mu)^*\chi_k \in H^*(SQ(k), \mathbb{Z})$ cannot be an element of the subring $\mathbb{Z}[\delta_2 y_1, \cdots, \delta_2 y_k]/(2\delta_2 y_1, \cdots, 2\delta_2 y_k, \sum_{i=1}^m \delta_2 y_i)$. If it were so then $B(\mu)^*\chi_k$ would be a symmetric polynomial in the $\delta_2 y_i$ and therefore of the form $B(\mu)^*\chi_k = P(\sigma_2, \cdots, \sigma_{k/2})$. But

$$B(\mu)^*\chi_k^2 = P^2(\sigma_2, \cdots, \sigma_{k/2}) = B(\mu)^*p_k = \sigma_k(\delta_2 y_1, \cdots, \delta_2 y_k).$$

Since k > 0, this is impossible, the σ_i being algebraically independent.

In the case of a complex representation $\tau: \Phi \to U(n)$ we have a factorization

$$\Phi \to T(n) = U(1)^n \stackrel{\iota}{\subseteq} U(n)$$

and the classes $B(\iota)^*c_i$ are given by $\sigma_i(x_1, \cdots, x_n)$. In order to determine the characteristic classes of **R**- and **C**-representations of Φ , it is therefore sufficient to look at irreducible representations $\Phi \to SO(2) = U(1)$ and $\Phi \to O(1) = \mathbb{Z}_2$. Since Φ is finite abelian it is of the form

$$\Phi = \bigoplus_{\nu} \mathbf{Z}_{q_{\nu}}, \text{ where } q_{\nu} \text{ is a prime power } p_{\nu}^{\omega_{\nu}}.$$

It will be convenient to study first the cohomology maps induced by representations $\mathbf{Z}_{p\omega} \rightarrow U(1), O(1)$.

Recall the structure of the cohomology ring of these groups [7] [21].

(5.4)
$$H^*(\mathbf{Z}_{p^{\omega}}, \mathbf{Z}) = \mathbf{Z}[u]/p^{\omega}(u), \ \deg u = 2;$$

(5.5)
$$H^*(\mathbf{Z}_{p\omega}, \mathbf{Z}_2) \cong \mathbf{Z}_2 \text{ for } p > 2;$$

(5.6)
$$H^*(\mathbf{Z}_2,\mathbf{Z}_2) \cong \mathbf{Z}_2[v], \ \deg v = 1, u = \delta_2 v;$$

(5.7)
$$H^*(\mathbf{Z}_{2\omega},\mathbf{Z}_2) \cong \Lambda_{\mathbf{Z}_2}[s] \otimes \mathbf{Z}_2[t],$$

deg s = 1, deg t = 2 for $\omega > 1$; in this case with the canonical class $\iota \in H^1(\mathbf{Z}_{2\omega}, \mathbf{Z}_{2\omega})$ we have $u = \delta_{2\omega}\iota$, $t = \beta_{2\omega}\iota$, $s = \rho_2\iota$, $t = \rho_2 u$, where $\delta_{2\omega}$, $\beta_{2\omega}$ are the Bockstein-homomorphisms associated to

$$0 \to \mathbf{Z} \xrightarrow{2^{\omega}} \mathbf{Z} \to \mathbf{Z}_{2^{\omega}} \to 0 \text{ and } 0 \to \mathbf{Z}_2 \xrightarrow{2^{\omega}} \mathbf{Z}_{2^{\omega+1}} \to \mathbf{Z}_{2^{\omega}} \to 0.$$

Proposition 5.8.

(i) Let
$$\rho^{\lambda}: \mathbb{Z}_{p^{\omega}} \to U(1)$$
 be defined by $\rho^{\lambda}(1) = \exp(\frac{\lambda}{p^{\omega}}), \lambda = 0 \cdots p^{\omega} - 1.$

Then the induced map

$$B(\rho^{\lambda})^* \colon H^*(BU(1), \mathbb{Z}) = \mathbb{Z}[x] \to H^*(\mathbb{Z}_{p^{\omega}}, \mathbb{Z}) = \mathbb{Z}_{p^{\omega}}[u]/p^{\omega}(u)$$

is given by $B(\rho^{\lambda})^*x = \lambda \cdot u$. For \mathbb{Z}_2 -cohomology we have

$$B(\rho^{\lambda})^*x = \begin{cases} 0, & p > 2\\ \lambda \cdot v^2, & p = 2, \omega = 1\\ \lambda \cdot t, & p = 2, \omega > 1 \end{cases}$$

(ii) Let $\bar{\rho}: \mathbb{Z}_{2\omega} \to O(1) = \mathbb{Z}_2$ be the nontrivial representation. Then the map $B(\bar{\rho})^*: H^*(BO(1), \mathbb{Z}_2) = \mathbb{Z}_2[y] \to H^*(\mathbb{Z}_{2\omega}, \mathbb{Z}_2)$ is given by

$$B(\tilde{\rho})^* y = \begin{cases} v, & \omega = 1 \\ s, & \omega > 1 \end{cases}.$$

(i) is proved by (4.25); (ii) is trivial.

Now let $\tau = (\rho^{\lambda_{\nu}})_{\nu}$: $\Phi = \bigoplus_{\nu} \mathbf{Z}_{q_{\nu}} \to U(1)$ be a representation, $q_{\nu} = p_{\nu}^{\omega_{\nu}}$. Then we have by (4.17, i)

T (

(5.9)
$$B(\tau)^* x = \sum_{\nu} B(\rho^{\lambda_{\nu}})^* x = \sum_{\nu} \lambda_{\nu} u_{\nu} \equiv X^{\tau}(\alpha), \alpha = (u_{\nu})_{\nu}$$

and for $\overline{\tau} = (\rho^{\epsilon_{\nu}})_{\nu} \colon \Phi = \bigoplus_{\nu} \mathbb{Z}_{q_{\nu}} \to O(1), \ \epsilon_{\nu} = 0, 1, \ \epsilon_{\nu} = 0$ if $p_{\nu} > 2$ we have

$$B(\bar{\tau})^* y = \sum_{\nu} B(\bar{\rho}^{\epsilon\nu}) y$$

(5.10)

$$= \sum_{\substack{\omega_{\nu}=1\\p_{\nu}=2}} \epsilon_{\nu} v_{\nu} + \sum_{\substack{\omega_{\nu}>1\\p_{\nu}=2}} \epsilon_{\nu} s_{\nu} \equiv Y^{\bar{\tau}}(\beta), \ \beta = (v_{\nu} \cdots; s_{\nu} \cdots).$$

Notation. For the polynomials in (5.2, i, ii) we put by definition:

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$$W_i(\rho_2 x_1, \cdots, \rho_2 x_m; y_1, \cdots, y_k)$$
 and $P_i(x_1, \cdots, x_m; \delta_2 y_1, \cdots, \delta_2 y_k)$.

Combining 5.9 and 5.10 with 5.2, we obtain finally for the characteristic classes of the representation τ the following formulas.

Proposition 5.11.

(i) Let $\tau = (\tau_1, \cdots, \tau_m; \overline{\tau}_1, \cdots, \overline{\tau}_k) : \Phi \to O(n)$ be a representation factored through F_m (5.1). Then we have

$$\begin{aligned} & \text{Def.} \\ w_i(\tau) = B(\tau)^* w_i = W_i(\rho_2 X_1^{\tau_1}, \cdots; Y_1^{\bar{\tau}_1}, \cdots) & \cong \Phi_i^{\tau}(\rho_2 \alpha, \beta) \in H^*(\Phi, \mathbb{Z}_2), \\ & \text{Def.} \\ p_i(\tau) = B(\tau)^* p_i = P_i(X_1^{\tau_1}, \cdots; \delta_2 Y_1^{\bar{\tau}_1}, \cdots) & \equiv \Psi_i^{\tau}(\alpha, \delta_2 \beta) \in H^*(\Phi, \mathbb{Z}). \end{aligned}$$

In the oriented case $\chi(\tau)$ is given via (5.2, iii).

(ii) Let $\tau = (\tau_1, \cdots, \tau_n) : \Phi \to U(n)$ be a representation of Φ factored through T(n). Then we have

$$c_i(\tau) = B(\tau)^* c_i = \sigma_i(X_1^{\tau_1}, \cdots, X_n^{\tau_n}) \stackrel{\text{Def.}}{=} \Theta_i^{\tau}(\alpha) \in H^*(\Phi, \mathbb{Z})$$

The explicit formulas for the case of a cyclic group $\Phi = \mathbf{Z}_p$, p > 2, are as follows.

COROLLARY 5.12.

(i) For $\tau = (\rho^{\lambda_1}, \cdots, \rho^{\lambda_m}) : \mathbb{Z}_p \to SO(n), \lambda_i = 0, \cdots, p-1, (2m = n)$ or 2m + 1 = n) we have

$$w_{i}(\tau) = 0, \ i > 0$$

$$p_{i}(\tau) = \sigma_{i}(\lambda_{1}^{2}, \cdots, \lambda_{m}^{2}) \cdot u^{2i} \in H^{*}(\mathbf{Z}_{p}, \mathbf{Z}) = \mathbf{Z}[u]/(p \cdot u),$$

$$i = 1, \cdots, [n/2]$$

$$\chi(\tau) = \begin{cases} 0, & n = 2m + 1 \\ \lambda_{1} \cdots \lambda_{m} \cdot u^{m}, & n = 2m \end{cases}$$
(ii) For $\tau = (\rho^{\lambda_{1}}, \cdots, \rho^{\lambda_{m}}) : \mathbf{Z}_{p} \rightarrow U(n), \ \lambda_{i} = 0, \cdots, p-1 \ we \ have$

$$c_{i}(\tau) = \sigma_{i}(\lambda_{1}, \cdots, \lambda_{n}) \cdot u^{i} \in H^{*}(\mathbf{Z}_{p}, \mathbf{Z}), \ i = 1, \cdots, n.$$

We keep the preceding notations for a fixed representation $\tau: \Phi \to G$ of $\Phi = \bigoplus \mathbb{Z}_{q_{\nu}}, q_{\nu} = p_{\nu}^{\omega_{\nu}}$. In the following X is supposed to be a *CW*-complex. Then we can state the main result of this section as follows.

THEOREM 5.13.

(i) Let
$$\xi: X \to BO(n)$$
 and $\tau: \Phi \to O(n)$. There exists a τ -flat bundle

 $\xi': X \to BO(n)$ with the same Stiefel-Whitney and Pontrjagin classes as ξ if and only if the following holds. There exist classes $\bar{u}_{\nu} \in H^2(X, \mathbb{Z})$ for $p_{\nu} > 2$ satisfying $p_{\nu}^{\omega_{\nu}} \cdot \bar{u}_{\nu} = 0$; classes $\bar{v}_{\nu} \in H^1(X, \mathbb{Z}_2)$ for $p_{\nu} = 2$, $\omega_{\nu} = 1$; and classes $\bar{\iota}_{\nu} \in H^1(X, \mathbb{Z}_{q_{\nu}})$ for $q_{\nu} = p_{\nu}^{\omega_{\nu}}$ with $p_{\nu} = 2$, $\omega_{\nu} > 1$; such that

$$w_i(\xi) = \Phi_i^{\tau}(\rho_2 \bar{\alpha}, \bar{\beta})$$
$$p_i(\xi) = \Psi_i^{\tau}(\bar{\alpha}, \delta_2 \bar{\beta})$$

where $\bar{\alpha} = (\bar{u}_{\nu}), \bar{\beta} = (\bar{v}_{\nu}, \bar{s}_{\nu})$. If ξ and τ are oriented, then so is ξ' and

$$\chi(\xi) = \chi(\xi') \iff \begin{cases} \text{without condition if } n \text{ odd};\\ \chi(\xi) = 0 \text{ if } n \text{ even, } k > 0 \text{ (assuming} \\ H^n(X, \mathbb{Z}) \text{ has no } 2\text{-torsion});\\ \chi(\xi) = \prod_{i=1}^m X_i^{\tau_i}(\bar{\alpha}) \text{ if } n \text{ even, } k = 0 \end{cases}$$

(ii) Let $\xi: X \to BU(n)$ and $\tau: \Phi \to U(n)$. There exists a τ -flat bundle $\xi': X \to BU(n)$ with the same Chern classes as ξ if and only if there exist classes $\tilde{u}_{\nu} \in H^2(X, \mathbb{Z})$ such that $p_{\nu}^{\omega_{\nu}} \cdot \tilde{u}_{\nu} = 0$ and

$$c_i(\xi) = \Theta_i^{\tau}(\bar{\alpha}), \ \bar{\alpha} = (\bar{u}_{\nu})_{\nu}.$$

Proof. We restrict ourselves to the proof of (i); (ii) being proved similarly.

(⇒): Put $\bar{u}_{\nu} = \eta^* u_{\nu}$ for $p_{\nu} > 2$; $\bar{v}_{\nu} = \eta^* v_{\nu}$ for $p_{\nu} = 2$, $\omega_{\nu} = 1$; $\bar{\iota}_{\nu} = \eta^* \iota_{\nu}$ for $p_{\nu} = 2$, $\omega_{\nu} > 1$. These classes satisfy all the requirements by naturality and (5.11).

since $q_{\nu} \cdot \bar{u}_{\nu} = 0 \Rightarrow \exists \eta_{\nu}, \ \delta_{q_{\nu}} \circ \eta_{\nu} \simeq \bar{u}_{\nu}.$ $\underline{p_{\nu} = 2, \ \omega_{\nu} = 1:}$ Take $\eta_{\nu} = \bar{v}_{\nu}: X \to K(\mathbf{Z}_{2}, 1).$ $\underline{p_{\nu} = 2, \ \omega_{\nu} > 1:}$ Take $\eta_{\nu} = \bar{\iota}_{\nu}: X \to K(\mathbf{Z}_{2}\omega, 1).$

This defines a unique class $\eta: X \to \prod_{\nu} K(\mathbf{Z}_{q_{\nu}}, 1) = B\Phi$ such that



is commutative, where π_{ν} is the canonical projection. Then we have clearly $\eta^* u_{\nu} = \bar{u}_{\nu}$ for $p_{\nu} > 2$; $\eta^* v_{\nu} = \bar{v}_{\nu}$ for $p_{\nu} = 2$, $\omega_{\nu} = 1$; $\eta^* \iota_{\nu} = \bar{\iota}_{\nu}$ for $p_{\nu} = 2$, $\omega_{\nu} > 1$; and the result again follows by naturality and (5.11). The statement concerning the Euler-class follows from (5.2, iii). This finishes the proof.

Remark 5.14. The method of this proof is due essentially to Massey-Szczarba [14].

Remark 5.15. The proof implies that for given classes \bar{u}_{ν} , \bar{v}_{ν} , $\bar{\iota}_{\nu}$ as in (5.13) there always exists a τ -flat bundle whose characteristic classes are given by $\Phi_i^{\tau}(\rho_2 \bar{\alpha}, \bar{\beta})$, $\Psi_i^{\tau}(\bar{\alpha}, \delta_2 \bar{\beta})$, resp. by $\Theta_i^{\tau}(\bar{\alpha})$.

Remark 5.16. If the bundles P(X, G) = [X, BG], G = O(n), SO(n), U(n) are classified by their characteristic classes, Theorem (5.13) gives necessary and sufficient conditions for the τ -flatness of a bundle $\xi: X \to BG$ in terms of primary characteristic classes. In the sequel we list some examples illustrating this fact and the content of Theorem (5.13).

Example 5.17. Let X be a CW-complex. The bundles P(X, SO(2)) = [X, BSO(2)] are classified by their Euler-class χ . Given $\tau: \Phi \to SO(2)$, assume that k = 0, i. e. τ irreducible. Then $\xi: X \to BSO(2)$ is τ -flat if and only if there exist classes $\bar{u}_{\nu} \in H^2(X, \mathbb{Z})$ such that $q_{\nu} \cdot \bar{u}_{\nu} = 0$ and

$$\chi(\xi) = \sum_{\nu} \lambda_{\nu} \cdot \tilde{u}_{\nu}, \ \tau = (\rho^{\lambda_{\nu}})_{\nu}.$$

Example 5.18. For any CW-complex X of dim $X \leq 4$ the bundles $P(X, SO(n)) = [X, BSO(n)], n \geq 3$ are classified by w_2, w_4 (resp. χ_4 if n = 4) and p_1 , if $H^4(X, \mathbb{Z})$ has no 2-torsion [11] [20].

Let $\tau: \Phi \to SO(n)$. Then $\xi: X \to BSO(n)$ is τ -flat iff there exist classes $\tilde{u}_{\nu}, \, \tilde{v}_{\nu}, \, \tilde{\iota}_{\nu}$ as in (5.13, i) with

$$p_{i}(\xi) = \sigma_{2}(\delta_{2}Y_{1}^{\tau_{1}}, \cdots, \delta_{2}Y_{k}^{\tau_{k}}) - \sigma_{1}((X_{1}^{\tau_{1}})^{2}, \cdots, (X_{m}^{\tau_{m}})^{2})$$

$$w_{2}(\xi) = \sigma_{1}(\rho_{2}X_{1}^{\tau_{1}}, \cdots, \rho_{2}X_{m}^{\tau_{m}}) + \sigma_{2}(Y_{1}^{\tau_{1}}, \cdots, Y_{k}^{\tau_{k}})$$

$$w_{4}(\xi) = \Phi_{4}^{\tau}(\rho_{2}\tilde{\alpha}, \tilde{\beta}).$$

For n = 4 the last condition has to be replaced by

$$\chi(\xi) = \begin{cases} 0 & k > 0 \\ X_1^{\tau_1} \cdot X_2^{\tau_2} & k = 0 \end{cases}$$

Example 5.19. For any CW-complex X of dim $X \leq 2n$, the bundles P(X, U(n)) = [X, BU(n)] are classified by c_i if $H^{2j}(X, \mathbb{Z})$ has no (j-1)!-torsion [19], [23, Thm. 4.7]. The condition in (5.13, ii) is in this case necessary and sufficient for the τ -flatness of $\xi: X \to BU(n)$.

Example 5.20. For any CW-complex X of dim $X \leq 8$, dim $X \leq n-1$, the bundles P(X, O(n)) are classified by w_1, w_2 and p_1, p_2 if $H^4(X, \mathbb{Z})$ has no 2-torsion and $H^8(X, \mathbb{Z})$ no 6-torsion [23, Thms. 4.2, 4.3]. Thus the conditions in (5.13, i) are again equivalent to the τ -flatness of $\xi: X \to BO(n)$.

Take all bundles within the range of the examples (5.17-5.20). By (5.12) and (5.13) we obtain in particular the following two corollaries.

COROLLARY 5.21. Let $\Phi = Z_p$, p > 2.

(i) If $\hat{\tau} \in RO(\mathbb{Z}_p)$ is of dim $n, n \geq 2, \tau = (\rho^{\lambda_1}, \cdots, \rho^{\lambda_m})$, then $\xi \in [X, BO(n)]$ is τ -flat iff $\exists \ \bar{u} \in H^2(X, \mathbb{Z})$ such that $p \cdot \bar{u} = 0$ and $w_i(\xi) = 0$, $p_i(\xi) = \sigma_i(\lambda_1^2, \cdots, \lambda_m^2) \ \bar{u}^{2i}$, i > 0.

(ii) If $\tau \in RSO(\mathbb{Z}_p)$ is given as above, then $\xi \in [X, BSO(n)]$ is τ -flat iff $\exists \ \vec{u} \in H^2(X, \mathbb{Z})$ such that $p \cdot \vec{u} = 0$, $w_i(\xi)$ and $p_i(\xi)$ are given by (i) and

$$\chi(\xi) = egin{cases} 0, & n = 2m+1 \ \lambda_1 \cdot \cdots \cdot \lambda_m ilde{u}^m, & n = 2m \end{cases}$$

(iii) if $\tau \in RU(\mathbb{Z}_p)$ is of dim n, $n \ge 1$, $\tau = (\rho^{\lambda_1}, \cdots, \rho^{\lambda_n})$, then $\xi \in [X, BU(n)]$ is τ -flat iff $\exists \ \bar{u} \in H^2(X, \mathbb{Z})$ such that $p \cdot \bar{u} = 0$ and $c_i(\xi)$ $= \sigma_i(\lambda_1, \cdots, \lambda_n) \cdot \bar{u}^i, \ i = 1, \cdots, n.$

Remark 5.22. Let M be a \mathbb{Z}_p -manifold (p > 2) in the sense of [8], i.e. a compact flat Riemannian manifold with τ -flat tangent bundle for τ : $\mathbb{Z}_p \to O(n), n = \dim M$. By 5.21 we have $w_i(M) = 0$ for i > 0 and $p_i(M)$ is a multiple of u^{2i} . From this follows $p_i(M) = 0$ for i > 0, because $u \in H^2(M, \mathbb{Z})$ is of order p and then $u^{2i} = 0$ for i > 0 by L. Charlap and A. Vasquez, Amer. J. of Math., 87 (1965), p. 557, Remark ii.

COROLLARY 5.23. Let $\Phi = Q(n) = \mathbb{Z}_2^n$ and $\mu: Q(n) \to O(n)$ the natural inclusion. Then a O(n)-bundle $\xi: X \to BO(n)$ is μ -flat (i.e. a sum of line bundles) iff \exists classes $\bar{y}_1, \dots, \bar{y}_n \in H^1(X, \mathbb{Z}_2)$ such that $p_i(\xi) = 0$, $w_i(\xi) = \sigma_i(\bar{y}_1, \dots, \bar{y}_n)$. 6. Rational characteristic classes of flat bundles. It is known that the rational Pontrjagin classes of flat $GL(n, \mathbf{R})$ -bundles and the rational Euler class of flat SO(n)-bundles are trivial. See [15] where this is proved via the Chern-Weil theorem. Here we first study flat C^* - resp. SO(2) = U(1)bundles. Then we prove by topological methods the triviality of the rational Chern classes for flat U(n)-bundles under some assumptions on the fundamental group $\pi_1 X$ of the base space.⁴

THEOREM 6.1. Consider a bundle $\xi: X \to BC^*$. Then ξ is flat if and only if $\rho_0 c_1(\xi) = c_1(\xi) q = 0$

 $(c_1(\xi)_{\mathbf{Q}} = rational \ Chern \ class; \rho_0 \ induced \ by \ \mathbf{Z} \rightarrow \mathbf{Q}).$

Proof. Consider the commutative diagram of sheaves with exact rows. (cf. 4.17):

Since these sheaves are abelian, we have exact cohomology sequences

$$\begin{array}{ccc} & \cdots \to H^{1}(X, \mathbf{C}_{d}) \to H^{1}(X, \mathbf{C}_{d}^{*}) \xrightarrow{\delta''} H^{2}(X, \mathbf{Z}) \xrightarrow{\alpha} H^{2}(X, \mathbf{C}_{d}) \to \cdots \\ (6.2) & & \downarrow & \downarrow \\ & \cdots \to H^{1}(X, \mathbf{C}) \to H^{1}(X, \mathbf{C}^{*}) \xrightarrow{\delta} H^{2}(X, \mathbf{Z}) \longrightarrow H^{2}(X, \mathbf{C}) \to \cdots . \end{array}$$

By (4.22) we have $\delta = c_1$; since \underline{C} is a soft sheaf, $H^p(X, \underline{C}) = 0$ for p > 0and so c_1 is an isomorphism. By exactness we have now for $\xi \in H^1(X, \underline{C}^*)$: $\rho_0 c_1(\xi) = 0 \iff \alpha c_1(\xi) = 0 \iff \exists \eta \in H^1(X, \underline{C}_d^*), \xi = \iota_* \eta \iff \xi$ is flat by (2.5).

PROPOSITION 6.3. Let ξ be a C^* -bundle on X and assume $\pi_1 X$ is finite. Then the following assertions are equivalent:

- (i) ξ is flat,
- (ii) $\rho_0 c_1(\xi) = 0$,

⁴ Added 7.27.66: In a forthocming paper "On the characteristic homomorphism of flat bundles" the following will be shown: Let X be a CW-complex, ξ a flat G-bundle, where G is a compact or complex and reductive Lie group. Then the real (or rational) characteristic homomorphism $\xi^*: H^*(BG, \mathbb{R}) \to H^*(X, \mathbb{R})$ is trivial. Added in proof: See Topology, vol. 6 (1967), pp. 153-159.

(iii) $\pi^*c_1(\xi) = 0$, where $\zeta = (\pi \colon X \to X)$ is the universal covering bundle.

Moreover we have:

- (iv) ι_* in (6.2) is injective, i.e. ξ has at most one flat structure. The flat C^* -bundles are classified by $H^2(\pi_1 X, \mathbb{Z}) \cong \operatorname{Hom}(\pi_1 X, \mathbb{C}^*)$.
- (v) If $H^2(\tilde{X}, \mathbb{Z}) = 0$, then $H^2(\pi_1 X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$, i. e. all C*-bundles on X are flat.

Proof. (i) \Leftrightarrow (ii) by (6.1).

The bundle $\zeta: X \to B(\pi_1 X)$ maps the sequence (cf. 4.17)

$$\cdots \to H^1(\pi_1 X, \mathbb{C}) \to H^1(\pi_1 X, \mathbb{C}^*) \longrightarrow H^2(\pi_1 X, \mathbb{Z}) \to H^2(\pi_1 X, \mathbb{C}) \to \cdots$$

into the upper line of (6.2). Since $\pi_1 X$ is finite, we have $H^p(\pi_1 X, \mathbf{C}) = 0$, p > 0 and hence δ' is an isomorphism. Thus we obtain the following diagram with exact rows:

$$(6.4) \qquad \begin{array}{c} 0 & 0 \\ \downarrow & \downarrow \\ 0 \rightarrow H^{1}(\pi_{1}X, \mathbb{C}^{*}) \longrightarrow H^{2}(\pi_{1}X, \mathbb{Z}) \rightarrow 0 \\ \cong \downarrow s = \zeta^{*} & \downarrow \zeta^{*} \\ 0 \rightarrow H^{1}(X, \mathbb{C}_{d}^{*}) \longrightarrow H^{2}(X, \mathbb{Z}) & \xrightarrow{\alpha} H^{2}(X, \mathbb{C}_{d}) \\ \downarrow & \downarrow \\ 0 & H^{2}(\tilde{X}, \mathbb{Z}) \end{array}$$

where s is the Steenrod-map of (2.4) and hence an isomorphism. That the vertical row is exact follows from the spectral sequence of the covering π since $H^1(\tilde{X}, \mathbb{Z}) = 0$ [7, p. 356]. The equivalence of (ii) and (iii), as well as (iv), (v) can now be read off from (6.4), using of course (6.1).

COROLLARY 6.5. (i) For SO(2) = U(1)-bundles and the Euler class χ Theorem 6.1 and Prop. 6.3 remain valid. In this case one has to use the exact sequence of sheaves

$$0 \to \mathbf{Z} \to \mathbf{R} \xrightarrow{\text{exp}} \underline{SO}(2) \to 0.$$

Moreover Prop. 6.3 may even be formulated for $GL^+(2, \mathbf{R})$ -bundles, since each

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conjugacy-class of homomorphisms $\pi_1 X \to GL^+(2, \mathbf{R})$ contains an element factorizing through $SO(2), \pi_1 X$ being finite.

(ii) By (4.16) we have $c_1(\xi)_{\mathbf{Q}} = 0$ for any flat $GL(n, \mathbf{C})$ - resp. U(n)-bundle.

Example 6.6. Let M be a compact orientable 2-manifold. We observe that a flat C^* - resp. SO(2)-bundle ξ on M is trivial, because (6.1) and $H^2(M, \mathbb{Z}) = \mathbb{Z}$ imply $c_1(\xi) = 0$ resp. $\chi(\xi) = 0$. Hence the nontrivial flat $GL^+(2, \mathbb{R})$ -bundles on M discussed in [15] are not flat C^* - or SO(2)-bundles.

Example 6.7. For $X = P_n \mathbf{R}$ $(n \ge 2)$ we have $H^2(\mathbf{Z}_2, \mathbf{Z}) = H^2(P_n \mathbf{R}, \mathbf{Z}) \cong \mathbf{Z}_2$. The nontrivial SO(2)-bundle ξ $(\chi(\xi) \ne 0)$ on $P_n \mathbf{R}$ is flat. Theorem 6.1 is also illustrated by Example 5.17.

Remark 6.8. (6.1) and (6.7) should be compared with the proof of Thm. 1 in [24] which contains implicitly the statement that a flat SO(2)-bundle is trivial. The author has informed us that his proof is indeed incorrect.

We now prove the triviality of the rational Chern classes for flat U(n)bundles satisfying a certain splitting principle to be defined below. Recall the splitting principle of [3] [13] described by the following diagram



The lifted bundle $\pi^* \xi = \xi \circ \pi$ factors through BT(n), as $\overline{\xi}$ is U(n)-equivariant, and hence splits into a sum of U(1)-bundles. Moreover π^* is injective in integral cohomology.

Definition 6.10. Let $\xi: X \to BU(n)$ be a flat U(n)-bundle: $\xi = \gamma_* \zeta_X = B(\gamma) \circ \zeta_X$. A map $f: Y \to X$ is called a γ -splitting (with respect to Q), if

- (i) $f^*: H^*(X, \mathbf{Q}) \to H^*(Y, \mathbf{Q})$ is injective,
- (ii) \exists a homomorphism $\beta = (\beta_1, \dots, \beta_n) : \pi_1 Y \to T(n) = U(1)^n$ such that in the following diagram the big rectangle is homotopy commutative

PROPOSITION 6.12. Let ξ be flat, $\xi = B(\gamma) \circ \zeta_X$ and $f: Y \to X$ a γ -splitting. Then $\rho_0 c(\xi) = c(\xi) q = 1$.

Proof.
$$f_{\rho_0}c(\xi) = \rho_0 c(f^*\xi) = \prod_{i=1}^n (1 + \rho_0 c_1(\beta_{i*}\zeta_Y))$$
. But by (6.1)
 $\rho_0 c_1(\beta_{i*}\zeta_Y) = 0$ and hence $f^*\rho_0 c(\xi) = 1$. Condition (i) now implies $\rho_0 c(\xi) = 1$.

We have not been able to decide if a flat U(n)-bundle $\xi = B(\gamma) \circ \zeta_x$ always admits a γ -splitting. However in the following cases we can apply 6.12.

THEOREM 6.13. Assume $\pi_1 X$ is of the form $0 \to N \to \pi_1 X \to \Phi \to 1$ with Φ finite and N abelian.

(i) For any flat U(n)-bundle ξ on X we have $\rho_0 c(\xi) = 1$.

(ii) For any flat O(n)-bundle ξ on X we have $\rho_0 p(\xi) = 1$.

Proof. Let $f: Y \to X$ be the regular covering corresponding to the invariant subgroup N of $\pi_1 X$. Φ operates properly on Y and $Y/\Phi \cong X$. The spectral sequence for f in rational cohomology collapses since $H^*(\Phi, Q) = Q$ and hence $f^*: H^*(X, Q) \cong H^*(Y, Q) \Phi \subset H^*(Y, Q)$ [7, p. 355]. Thus f satisfies (i) of 6.10. By assumption, $\xi = B(\gamma) \circ \zeta_X$ with $\gamma: \pi_1 X \to U(n)$ and hence $f^*\xi = \xi \circ f = B(\gamma) \circ B\pi_1 f \circ \zeta_Y = B(\gamma \circ \pi_1 f) \circ \zeta_Y$. Since N is abelian, a conjugate of $\gamma \circ \pi_1 f: \pi_1 Y \to U(n)$ factors through $T(n) \subset U(n)$ and f is a γ -splitting. (i) follows now from 6.12. (ii) follows from (i) by the definition of Pontrjagin classes.

Remark 6.14. Let $\xi = \gamma_* \zeta$ a flat bundle. (6.13) remains valid if $\gamma: \pi_1 X \to U(n)$ factors through a group Γ of the form $0 \to N \to \Gamma \to \Phi \to 1$ with Φ finite and N abelian. Recall that by (2.1) $B\Gamma$ is a space of type $K(\Gamma, 1)$ and hence has a structure of a CW-complex.

THEOREM 6.15. Let $\Gamma \subset \Phi$ be a subgroup of finite index m in a discrete group Φ , $\lambda \in RU(\Gamma)$ a 1-dimensional representation. Then for the induced representation $i_1\lambda$ we have $\rho_0 c(i_1\lambda) = 1$.

Proof. By [12] we have the factorization $i_! \lambda \colon \Phi \to \Gamma^m \times S(m) \xrightarrow{\lambda^m \times 1}$

 $N = T(m) \times S(m) \subset U(m)$ where S(m) is the symmetric group in *m* letters, \times means semidirect product and $N = T(m) \times S(m)$ is the normalizer of T(m) in U(m). The assertion follows now from 6.13 and 6.14.

COROLLARY 6.16. Let $f: Y \to X$ be a finite covering of X and η a U(1)-bundle on Y with $\rho_0 c_1(\eta) = 0$, i.e., η flat by 6.1. Then $f_{!}\eta = \xi$ is flat and $\rho_0 c(\xi) = 1$.

Proof. By 6.1 $\eta = \lambda_* \zeta_Y$ with $\lambda : \pi_1 Y \to U(1)$. Then by 3.1 we have $\xi = f_!(\lambda_* \zeta_Y) = (i!\lambda)_* \zeta_X$, $i = \pi_1 f$ and finally $\rho_0 c(\xi) = 1$ by 6.15.

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