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ON DEFORMATIONS OF A CERTAIN TYPE OF IRREGULAR ALGEBRAIC SURFACE.*

By Arnold Kas.

In this paper, we consider a compact complex analytic surface M which admits a holomorphic map $\Psi: M \to R$ onto a non-singular algebraic curve R where Ψ is everywhere regular (i.e., $d\Psi \neq 0$). We call the triple (M, Ψ, R) non-trivial, if $\Psi: M \to R$ is *not* a holomorphic fibre bundle. In § 1 of this paper, we prove that if (M, Ψ, R) is non-trivial, then $\text{genus}(R) > 1$ and $\text{genus}(C_\tau) > 2$, where $C_\tau = \Psi^{-1}(\tau)$, $\tau \in R$.

In § 2, we give a new construction of a class of surfaces originally constructed by Kodaira [7]. A Kodaira surface is a surface M which may be represented as a cyclic branched covering $M \to S \times R$ of a product of non-singular algebraic curves S and R. We assume that S may be represented as an unramified covering $\pi: S \to R$, and that a group G of automorphisms acting freely on R is given such that each irreducible component in $S \times R$ of the image of the branch locus of $M \to S \times R$, is equal to the graph of a map $g \circ \pi: S \to R$, where $g \in G$.

Kodaira has raised the question of determining all "small" deformations of the surface M. The main result of this paper is the following theorem:

Theorem. *The Kodaira surface M has no obstruction to deformations. Every small deformation of M is induced, in an obvious way, by a deformation of the curve R/G.*

1. Regularly fibred surfaces.

By a "surface," we will mean a compact complex analytic manifold of complex dimension two; hence of topological dimension four.

Definition 1.1. By a "regularly fibred surface," we mean a surface M together with a holomorphic map $\Psi: M \to R$ of M onto a non-singular algebraic curve R, such that Ψ is regular at every point of M.

It follows from this definition that for each point $\tau \in R$, $C_\tau = \Psi^{-1}(\tau)$ is a non-singular algebraic curve in M. We will say that the regularly fibred surface $\Psi: M \to R$ is trivial if every curve C_τ is biholomorphically equivalent

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to a fixed curve C. It follows from a theorem of Grauert and Fischer [3] that if $\Psi: M \rightarrow R$ is a trivial regularly fibred surface, then $\Psi: M \rightarrow R$ is a locally trivial holomorphic fibre bundle.

The following theorem is due in part to Kodaira.

THEOREM 1.1. *Let* $\Psi: M \rightarrow R$ *be a non-trivial regularly fibred surface. Then* $\text{genus}(R) > 1$, *and* $\text{genus}(C_\tau) > 2$.

Proof. Let U be the universal covering space of R, and let T be the Teichmüller space of Teichmüller structures on a compact topological surface of genus $g = \text{genus}(C_\tau)$. There exists a holomorphic map $U \rightarrow T$ which maps each point $u \in U$ to a point of T representing the complex analytic structure of $\tau(u)$ ($u \rightarrow \tau(u)$ is the covering map $U \rightarrow R$). By a theorem of Bers [2], T is a bounded open set in a complex number space $\boldsymbol{C}^n$. If $\text{genus}(R) \leqq 1$, then U is either the Riemann sphere or the complex plane $\boldsymbol{C}$, and the holomorphic map $U \rightarrow T$ must be constant by Liouville' theorem. This contradicts the assumption that $\Psi: M \rightarrow R$ is non-trivial; hence $\text{genus}(R) > 1$.

It is obvious that we must have $\text{genus}(C_\tau) > 0$. If $\text{genus}(C_\tau) = 1$, the function $j(\tau) = J(C_\tau)$ is holomorphic (Cf. Kodaira [6], page 575); hence constant, where $J(C_\tau)$ is the elliptic modular invariant of the elliptic curve C_τ. This again contradicts the assumption that $\Psi: M \rightarrow R$ is non-trivial; hence $\text{genus}(C_\tau) > 1$.

Assume that $\text{genus}(C_\tau) = 2$. Then each curve C_τ admits a unique involution with six fixed points $\kappa_\tau: C_\tau \rightarrow C_\tau$ (Cf. Ahlfors [1], page 51). It is not difficult to see that the involutions κ_τ depend holomorphically on $\tau \in R$, and thus determine an involution, $\kappa: M \rightarrow M$. Let N be the quotient of M by the group $\{1, \kappa\}$. N is a projective line bundle over R, and M is a "fibre preserving" 2-sheeted covering of N. Let $\Lambda \subset N$ be the image of the branch curve of $M \rightarrow N$. Clearly Λ intersects each fibre of N in exactly six points. It follows that the irreducible components of Λ are mutually disjoint unramified coverings of R. Now we claim that there exists a finite unramified covering S of R, such that if N^* is the projective line bundle over S induced from N, and if $\Lambda^* \subset N^*$ is the "pull-back" of Λ, then Λ^* consists of six mutually disjoint sections of N^* over S. To see this, we may suppose that some component Λ^1 of Λ is a k-sheeted covering of R, $k > 1$. Let S_1 be a copy of Λ^1, and let N_1 be the projective line bundle over S_1 induced from N. The "pull-back" of any section of N over R is a section of N_1 over S_1, and the "pull-back" of Λ^1 consists of k disjoint sections of N_1 over S_1. Thus if $\Lambda_1 \subset N_1$ is the pull-back of Λ, then the number of components of Λ_1 which are sections of N_1 over S_1 is greater than the number of components of Λ which are sections of N over R. It follows that if we continue this procedure a finite number of times, we obtain a covering S satisfying our claim. It is clear that $N^* = S \times \boldsymbol{P}^1$, and each component of Λ^* is a "constant" section. Let $\Psi^*: M^* \to S$ be the regularly fibred surface induced from $\Psi: M \to R$ by the covering map $S \to R$. M^* is a 2-sheeted covering of $N^* = S \times \boldsymbol{P}^1$. Thus each fibre C^*_σ, $\sigma \in S$ of M^* is represented as a 2-sheeted covering of $\boldsymbol{P}^1$ branched over six points of $\boldsymbol{P}^1$ which are independent of $\sigma \in S$. It follows that each fibre C^*_σ is biholomorphically equivalent to a fixed curve C; hence each fibre C_τ of M is biholomorphically equivalent to C. This contradicts the assumption that $\Psi: M \to R$ is non-trivial; hence genus$(C_\tau) > 2$. Q. E. D.

COROLLARY 1.1. *Let $\Psi: M \to R$ be a non-trivial regularly fibred surface. Then M is a projective algebraic surface.*

Proof. Clearly, there exists a non-constant meromorphic function on M induced from R. Therefore, by a well-known theorem of Kodaira [5], either M is a projectice algebraic surface or M contains an elliptic curve. By Theorem 1.1, M may not contain an elliptic curve; hence M is a projective algebraic surface. Q. E. D.

2. Kodaira surfaces. The only examples of non-trivial regularly fibred surfaces that we know are of a very special type constructed by Kodaira [7]. Before giving a definition, we will explicitly construct a number of examples.

Let R be a non-singular algebraic curve of genus > 1. Let $G \neq \{1\}$ be a finite group of automorphisms acting freely on R, and let $r > 1$ be an integer such that r divides the order of G. Let S be a finite unramified covering of R with the covering map $\pi: S \to R$. To each element $g \in G$, we let $(g\pi)_*: H_*(S, \boldsymbol{Z}) \to H_*(R, \boldsymbol{Z})$ be the homomorphism on homology groups induced by the covering $g\pi: S \to R$.

Definition 2.1. We say that the quadruple (R, G, r, S) is admissible if the following condition is satisfied: (A) for every $\alpha \in H_1(S, \boldsymbol{Z})$, $\sum_{g \in G} (g\pi)_* \alpha \equiv 0 \mod r$, where for $\beta \in H_1(R, \boldsymbol{Z})$, $\beta \equiv 0 \mod r$ means that there exists $\gamma \in H_1(R, \boldsymbol{Z})$ with $\beta = r\gamma$.

Let (R, G, r, S) be an admissible quadruple. If $g \in G$, we denote by Γ_g, the graph in $S \times R$ of the map $g\pi: S \to R$.

THEOREM 2.1. *Let (R, G, r, S) be an admissible quadruple. There exists a cyclic r-sheeted branched covering $\Phi: M \to S \times R$ such that:*

1) *if* $\Lambda \subset M$ *is the branch locus defined locally by the equation:*

$$\text{jacobian}(\Phi) = 0, \text{ then } \Phi(\Lambda) = \bigcup_{g \in G} \Gamma_g;$$

2) *if* $p \in S \times R$, *then* $\Phi^{-1}(p)$ *consists of either* r *points or* 1 *point, i. e. all of the sheets "come together" at the branch locus.*

Proof. Let $\Gamma = \sum_{g \in G} \Gamma_g$. Let $[\Gamma]$ be the holomorphic line bundle on $S \times R$ determined by the divisor Γ. Assume for the moment that there exists a holomorphic line bundle L on $S \times R$ such that $[\Gamma] = rL$ (the group of line bundles is written additively). Let (f_{jk}) be a system of transition functions defining the line bundle L with respect to a covering $\{U_j\}$ of $S \times R$. Let ζ_j be a local fibre coordinate in the space of $L \mid U_j$ such that if $u \in U_j \cap U_k$, then (u, ζ_j) and (u, ζ_k) represent the same point of L if and only if $\zeta_j = f_{jk}(u)\zeta_k$. Let ϕ be a section of $[\Gamma]$ over $S \times R$ such that divisor $(\phi) = \Gamma$. ϕ may be represented by a system of holomorphic functions (ϕ_j) such that ϕ_j is defined on U_j, and such that if $u \in U_j \cap U_k$, $\phi_j(u) = f_{jk}(u)^r \phi_k(u)$. We define M to be the subvariety of the total space of L such that M is defined in each piece $L \mid U_j$ by the equation: $\zeta_j^r = \phi_j(u)$ (Cf. Wavrik [10], page 11). It is easily verified that M is a non-singular surface, and that together with the natural projection $\Phi: M \to S \times R$, the conditions in the theorem are satisfied.

It remains to be proved that there exists a holomorphic line bundle L on $S \times R$ such that $[\Gamma] = rL$. Let $\mathfrak{F}$ be the group of holomorphic line bndles on $S \times R$, and let P be the Picard (group) variety of $S \times R$. Let $H^{1,1}(S \times R, \mathbf{Z})$ be the subgroup of $H^2(S \times R, \mathbf{Z})$ consisting of those elements whose image in $H^2(S \times R, \mathbf{R})$ is represented under the De Rham isomorphism. by a form of type (1.1). It is well known that there exists an exact sequence:

$$0 \to P \to \mathfrak{F} \xrightarrow{\ c\ } H^{1,1}(S \times R, \mathbf{Z}) \to 0$$

where $c: \mathfrak{F} \to H^{1,1}(S \times R, \mathbf{Z})$ sends each line bundle $F \in \mathfrak{F}$ to its Chern class $c(F) \in H^{1,1}(S \times R, \mathbf{Z})$. Since P is a complex torus; hence an infinitely divisible group, it is clear that a line bundle F is divisible by r in $\mathfrak{F}$ if and only if $c(F)$ is divisible by r in $H^{1,1}(S \times R, \mathbf{Z})$. Moreover, $c(F)$ is divisible by r in $H^{1,1}(S \times R, \mathbf{Z})$ if and only if $c(F)$ is divisible by r in $H^2(S \times R, \mathbf{Z})$. Since $H^2(S \times R, \mathbf{Z})$ contains no torsion, we need only show that the value of $c([\Gamma])$ on any 2-dimensional homology class by the Kronecker product $\langle\ ,\ \rangle$ is divisible by r. Using the Kunneth formula:

$$H_2(S \times R, \mathbf{Z}) = H_2(S) \otimes H_0(R) + H_0(S) \otimes H_2(R) + H_1(S) \otimes H_1(R),$$

we will check the value of $c([\Gamma])$ on each of the three summands separately. Fix points $z \in S$, and $w \in R$. Clearly $H_2(S) \otimes H_0(R)$ is generated by the homology class of $S \times w$, and $H_0(S) \otimes H_2(R)$ is generated by the homology class of $z \times R$. It is well known that by Poincaré duality in $S \times R$, $c([\Gamma])$ is "dual" to the homology class of the divisor $\Gamma = \sum_{g \in G} \Gamma_g$. It follows that the value of $c([\Gamma])$ on the homology class of $S \times w$ is equal to the intersection number of $S \times w$ and $\Gamma = \sum_{g \in G} \Gamma_g$, which is divisible by r since r divides the order of G. By a similar argument, the value of $c([\Gamma])$ on the homology class of $z \times R$ is divisible by r. Finally, let $\alpha \in H_1(S, \boldsymbol{Z})$ and $\beta \in H_1(R, \boldsymbol{Z})$. We let $D_X \colon H_i(X) \to H^{n-i}(V)$ denote Poincaré duality in any manifold X^n. Then it is easy to verify the formula:

$$\langle c([\Gamma]), \alpha \times \beta \rangle = \langle D_{S \times R}(\sum_{g \in G} \Gamma_g), \alpha \times \beta \rangle = -\langle D_R(\beta), \sum (g\pi)_* \alpha \rangle.$$

It follows from assumption (A) that $\langle c([\Gamma]), \alpha \times \beta \rangle \equiv 0 \bmod r$. Q. E. D.

Example 1. The simplest example of an admissible quadruple (i. e., involving the smallest values for the genera of R and S) is the following:

Let R be a non-singular algebraic curve of genus 3 such that R is a 2-sheeted unramified covering of a curve of genus 2. Let $G = \{1, \rho\}$ where $\rho \colon R \to R$ is the involution of sheet interchange, and let $r = 2$. We may choose a basis $\{\beta_1, \beta_2, \cdots, \beta_6\}$ of $H_1(R, \boldsymbol{Z})$ such that that induced map $\rho_* \colon H_1(R, \boldsymbol{Z}) \to H_1(\mathrm{R}, \boldsymbol{Z})$ operates on this basis in the following way:

$$\begin{aligned} \rho_*(\beta_i) &= \beta_i, \quad i = 1, 2 \\ \rho_*(\beta_3) &= \beta_4 \\ \rho_*(\beta_4) &= \beta_3 \\ \rho_*(\beta_5) &= \beta_6 \\ \rho_*(\beta_6) &= \beta_5. \end{aligned}$$

Let $\lambda \colon H_1(R, \boldsymbol{Z}) \to \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ be the homomorphism which sends $\beta = \sum m_\nu \beta_\nu \in H_1(R, \boldsymbol{Z})$ to $([m_3 + m_4], [m_5 + m_6]) \in \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ where $[m] =$ the residue class of m modulo 2. Let $\pi \colon S \to R$ be the 4-sheeted unramified covering of R determined by λ such that the sequence:

$$0 \to \pi_*(H_1(S, \boldsymbol{Z})) \to H_1(R, \boldsymbol{Z}) \to \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \to 0$$

is exact. It is easy to see that the quadruple (R, G, r, S) is admissible.

Example 2. Let R be any non-singular algebraic curve on which a finite group $G \neq \{1\}$ acts freely. Let $r > 1$ be any integer which divides the order of

G, and let $\pi: S \to R$ be the finite unramified covering such that the sequence: $0 \to \pi_*(H_1(S, \mathbf{Z})) \to H_1(R, \mathbf{Z}) \to H_1(R, \mathbf{Z}_r) \to 0$ is exact, where the last homomorphism is determined by the coefficient homomorphism $\mathbf{Z} \to \mathbf{Z}_r$. Clearly $\pi_*(\alpha) \equiv 0 \bmod r$ for every $\alpha \in H_1(S, \mathbf{Z})$. Thus (R, G, r, S) is an admissible quadruple.

More generally, let R be a non-singular algebraic curve of genus > 1, G a finite group of automorphisms of R which acts freely on R, and $\pi: S \to R$ a finite unramified covering of R. As above, we let $\Gamma_g \subset S \times R$ be the graph of the map $g\pi: S \to R$ if $g \in G$.

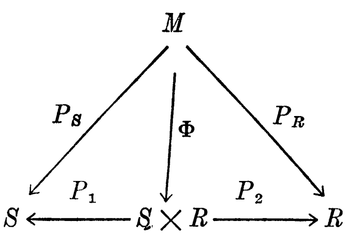
Definition 2.2. By a "Kodaira surface" we mean a compact complex analytic surface M such that there exists a holomorphic map $\Phi: M \to S \times R$ which makes M a r-sheeted cyclic branched covering of $S \times R$ and which satisfies the conditions:

(1) if $p \in S$, then $\Phi^{-1}(p)$ contains either r points or 1 point (i.e., all of the sheets "come together" at the branch locus).

(2) If Δ is an irreducible component of the branch locus of $\Phi: M \to S \times R$, then $\Phi(\Delta) = \Gamma_g$ for some $g \in G$.

Remark. Let $K = \{g \in G \mid \Gamma_g = \Phi(\Delta)$ for some component $\Delta\}$. By replacing the covering map π by $g \circ \pi: S \to R$ where $g \in K$, we may assume that K contains $1_R: R \to R$. Moreover, we may assume that G is generated by K.

Let $P_S: M \to S$ and $P_R: M \to R$ be holomorphic maps such that the diagram:



is commutative, where P_1 and P_2 are the projections of $S \times R$ onto the first and second factors respectively. If M is a Kodaira surface, it is obvious that $P_S: M \to S$ and $P_R: M \to R$ are both non-trivial regularly fibred surfaces. We know of no other examples of non-trivial regularly fibred surfaces.

For a compact complex analytic manifold X, let $h^{1,0}(X) =$ the dimension of the space of holomorphic 1-forms on X. It is well known that if X

is an algebraic manifold, $h^{1,0}(X) = \frac{1}{2}b_1(X) = \dim H^1(X, \mathcal{O})$ where $b_1(X)$ is the first betti number of X, and $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X.

Theorem 2.2. *Let M be a Kodaira surface. Using the notation of Definition 2.2, let $g_S = \text{genus}(S)$ and $g_R = \text{genus}(R)$. Then*

$$h^{1,0}(M) = h^{1,0}(S \times R) = g_S + g_R.$$

Proof. We will prove that $b_1(M) = b_1(S \times R)$. Consider the continuous fibre bundle $P_S: M \to S$, and fix a base point $o \in S$. There exists a natural action of $\pi_1(S)$ on $P_S^{-1}(o)$ defined as follows: given a loop, $\alpha: [0,1] \to S$, $\alpha(0) = \alpha(1) = o$, and a point $p \in P_S^{-1}(o)$, choose a path

$$\alpha': [0,1] \to P_R^{-1}(P_R(p))$$

which "covers" α. α' is uniquely determined for an everywhere dense set of points $p \in P_S^{-1}(o)$ (at least if α is "reasonable"). Define $h_\alpha(p)$ to be the end point of the path α'. Clearly h_α depends only on the homotopy class of α; hence we obtain a homomorphism $\alpha \to h_\alpha$ from $\pi_1(S)$ to the covering transformation group of $P_S^{-1}(o) \to R$. Moreover, since $P_R^{-1}(q)$ is connected for every point $q \in R$, it follows that the homomorphism $\alpha \to h_\alpha$ is surjective. On the other hand, since $P_S: M \to S$ is a continuous fibre bundle, there is a natural action of $\pi_1(S)$ on the homology groups $H_*(P_S^{-1}(o))$ obtained by "deforming cycles." It is easy to see that for $\alpha \in \pi_1(S)$, the action of α on $H_*(P_S^{-1}(o))$ obtained by "deforming cycles over α," is the same as that induced by the automorphism $h_\alpha: P_S^{-1}(o) \to P_S^{-1}(o)$.

By an argument using the spectral sequence of the fibering $P_S: M \to S$, it can be shown that:

$$b_1(M) \leqq b_1(S) + \dim H_0(S, H_1(P_S^{-1}(z), Q))$$

where the last group is a homology group with local coefficients in $H_1(P_S^{-1}(z), Q)$, $z \in S$, and $Q =$ the rational numbers. By the definition of homology with local coefficients, and by the fact that the action of $\pi_1(S)$ on $H_1(P_S^{-1}(o), Q)$ can be factored through a finite group, it follows that

$$H_0(S, H_1(P_S^{-1}(z), Q)) \cong H_1(P_S^{-1}(o), Q)^i,$$

the subspace of $H_1(P_S^{-1}(o), Q)$ invariant under the action of $\pi_1(S)$. Since $\pi_1(S)$ acts on $P_S^{-1}(o)$ as the full covering transformation group of $P_S^{-1}(o) \to R$, we have $H_1(P_S^{-1}(o), Q)^i \cong H_1(R, Q)$; hence we obtain the inequality $b_1(M) \leqq b_1(S) + b_1(R) = b_1(S \times R)$. On the other hand, as

M is a covering of $S \times R$, we obviously have $b_1(M) \geqq b_1(S \times R)$; hence $b_1(M) = b_1(S \times R)$. Q.E.D.

3. The Sheaves $\Phi_*^{(\lambda)}\mathcal{O}$ and $\Phi_*^{(\lambda)}\Theta$. Let $\mathcal{O}_X$ be the sheaf of germs of holomorphic functions on a complex analytic manifold X, and let Θ_X be the sheaf of germs of holomorphic vector fields on X. We may omit the subscript X if there is no possibility of confusion. Using the notation of § 2, let M be a Kodaira surface, and let $\rho: M \to M$ be a generator of the cyclic covering transformation group $\mathfrak{G}$ of the covering $\Phi: M \to S \times R$. ρ induces maps $\rho^*: \mathcal{O}_M \to \mathcal{O}_M$ and $\rho_*: \Theta_M \to \Theta_M$ which "cover" the automorphism $\rho: M \to M$. Explicitly, if $f \in \mathcal{O}_{M,p}$ is the germ of a holomorphic function at $p \in M$, then $\rho^* f = f\rho^{-1} \in \mathcal{O}_{M,\rho(p)}$. If $\theta \in \Theta_{M,p}$, then $\rho_*\theta \in \Theta_{M,\rho(p)}$.

Definition 3.1. By a $\mathfrak{G}$-invariant covering of M, we mean a covering $\mathfrak{U} = \{U\}$ of M by open sets such that: (i) if $U \in \mathfrak{U}$, then $g(U) \in \mathfrak{U}$ for every $g \in \mathfrak{G}$; (ii) if $U \in \mathfrak{U}$ and $g \in \mathfrak{G}$, then either $U = g(U)$ or U and $g(U)$ are disjoint.

It is obvious that every open covering of M may be refined to a $\mathfrak{G}$-invariant covering. Hence the cohomology groups of M with coefficients in any sheaf are equal to the direct limit of "Čech" cohomology groups taken with respect to $\mathfrak{G}$-invariant coverings.

For a $\mathfrak{G}$-invariant covering $\mathfrak{U}$ of M, let $C^q(M, \mathfrak{U}; \mathcal{O})$ and $C^q(M, \mathfrak{U}; \Theta)$ by the groups of q-cochains on M with respect to the covering $\mathfrak{U}$ and with coefficients in the sheaves $\mathcal{O}$ and Θ respectively. The maps $\rho^*: \mathcal{O} \to \mathcal{O}$ and $\rho_*: \Theta \to \Theta$ induce in an obvious way, linear maps $\rho^{\#}: C^q(M, \mathfrak{U}; \mathcal{O}) \to C^q(M, \mathfrak{U}; \mathcal{O})$ and $\rho_{\#}: C^q(M, \mathfrak{U}; \Theta) \to C^q(M, \mathfrak{U}; \Theta)$ respectively. It is easy to verify that these maps commute with the coboundary operations and with the "refinement maps," and therefore they induce linear maps $\rho^{\dagger}: H^q(M, \mathcal{O}) \to H^q(M, \mathcal{O})$ and $\rho_{\dagger}: H^q(M, \Theta) \to H^q(M, \Theta)$. Since the endomorphisms $\rho^{\dagger}$ and $\rho_{\dagger}$ have finite order, the vector spaces $H^q(M, \mathcal{O})$ and $H^q(M, \Theta)$ split into direct sums of eigenspaces. Thus:

$$H^q(M, \mathcal{O}) = \bigoplus_{\lambda^r=1} H^q(M, \mathcal{O})^{\lambda}$$

$$H^q(M, \Theta) = \bigoplus_{\lambda^r=1} H^q(M, \Theta)^{\lambda}$$

where $H^q(M, \mathcal{O})^{\lambda}$ $[H^q(M, \Theta)^{\lambda}]$ is the eigenspace of $H^q(M, \mathcal{O})$ $[H^q(M, \Theta)]$ with respect to $\rho^{\dagger}$ $[\rho_{\dagger}]$ corresponding to the eigenvalue λ.

For each $\mathfrak{G}$-invariant covering $\mathfrak{U}$ of M, let $C^q(M, \mathfrak{U}; \mathcal{O})^{\lambda}$ $[C^q(M, \mathfrak{U}; \Theta)^{\lambda}]$ be the $\rho^{\#}$ $[\rho_{\#}]$ eigenspace of $C^q(M, \mathfrak{U}; \mathcal{O})$ $[C^q(M, \mathfrak{U}; \Theta)]$ corresponding to the eigenvalue λ. Let $H^q(M,\mathfrak{U}:\mathcal{O})^\lambda$ $[H^q(M,\mathfrak{U};\Theta)^\lambda]$ be the cohomology groups of the complex $\{C^q(M,\mathfrak{U};\mathcal{O})^\lambda\}$ $[\{C^q(M,\mathfrak{U};\Theta)^\lambda\}]$. It is easy to verify that:

$$H^q(M,\mathcal{O})^\lambda = \operatorname{dir\,lim} H^q(M,\mathfrak{U};\mathcal{O})^\lambda$$
$$H^q(M,\Theta)^\lambda = \operatorname{dir\,lim} H^q(M,\mathfrak{U};\Theta)^\lambda$$

where the direct limit is taken over all $\mathfrak{G}$-invariant coverings of M.

Consider the direct image sheaves $\Phi_*\mathcal{O}_M$ and $\Phi_*\Theta_M$ on $S\times R$. The maps $\rho^*\colon \mathcal{O}_M\to\mathcal{O}_M$ and $\rho_*\colon \Theta_M\to\Theta_M$ induce automorphisms (which we denote by the same symbols) $\rho^*\colon \Phi_*\mathcal{O}_M\to\Phi_*\mathcal{O}_M$ and $\rho_*\colon \Phi_*\Theta_M\to\Phi_*\Theta_M$ respectively, covering the identity on $S\times R$.

Definition 3.2.

$$\Phi_*^{(\lambda)}\mathcal{O} = \{f\in\Phi_*\mathcal{O}_M \mid \rho^*f=\lambda f\}$$
$$\Phi_*^{(\lambda)}\Theta = \{\theta\in\Phi_*\Theta_M \mid \rho_*\theta=\lambda\theta\}$$

(Compare Grothendieck [4]).

We describe these sheaves explicitly. If $p\in S\times R$, we distinguish two cases:

(i) p is not contained in the image of the branch locus of $\Phi\colon M\to S\times R$. Let $\Phi^{-1}(p)=\{q_1,q_2,\cdots,q_r\}$ where the points q_i are indexed by integers modulo r and arranged in such a way that $\rho(q_i)=q_{i+1}$. Then,

$$(\Phi_*^{(\lambda)}\mathcal{O})_p = \{(f_1,f_2,\cdots,f_r) \mid f_i\in\mathcal{O}_{M,q_i}, f_i=\lambda f_{i+1}\rho\}$$
$$(\Phi_*^{(\lambda)}\Theta)_p = \{(\theta_1,\theta_2,\cdots,\theta_r) \mid \theta_i\in\Theta_{M,q_i}, \rho_*\theta_i=\lambda\theta_{i+1}\}.$$

(ii) p is contained in the image of the branch locus of $\Phi\colon M\to S\times R$. Let $q=\Phi^{-1}(p)$. Then,

$$(\Phi_*^{(\lambda)}\mathcal{O})_p = \{f\in\mathcal{O}_{M,q} \mid f=\lambda f\rho\}$$
$$(\Phi_*^{(\lambda)}\Theta)_p = \{\theta\in\Theta_{M,q} \mid \rho_*\theta=\lambda\theta\}.$$

Let $\mathfrak{U}=\{U\}$ be a $\mathfrak{G}$-invariant covering of M, and let $\Phi(\mathfrak{U})=\{\Phi(U)\}$ be the covering of $S\times R$ consisting of the sets $\{\Phi(U)\mid U\in\mathfrak{U}\}$. It is immediately obvious that:

$$C^q(M,\mathfrak{U};\mathcal{O})^\lambda \cong C^q(S\times R,\Phi(\mathfrak{U});\Phi_*^{(\lambda)}\mathcal{O})$$
$$C^q(M,\mathfrak{U};\Theta)^\lambda \cong C^q(S\times R,\Phi(\mathfrak{U});\Phi_*^{(\lambda)}\Theta).$$

The following theorem is clear from the above remarks.

THEOREM 3.1.

$$H^q(M, \mathcal{O}) \cong \bigoplus_{\lambda^r=1} H^q(S \times R, \Phi_*^{(\lambda)}\mathcal{O})$$

$$H^q(M, \Theta) \cong \bigoplus_{\lambda^r=1} H^q(S \times R, \Phi_*^{(\lambda)}\Theta).$$

4. Deformations of Kodaira surfaces. We will prove a series of lemmas describing the structure of the sheaves $\Phi_*^{(\lambda)}\mathcal{O}$ and $\Phi_*^{(\lambda)}\Theta$.

LEMMA 4.1. *The sheaves $\Phi_*^{(\lambda)}\mathcal{O}$ and $\Phi_*^{(\lambda)}\Theta$ are locally free.*

Proof. We need only consider the restriction of these sheaves to a neighborhood of a point $p = \Phi(q) \in S \times R$ where $q \in M$ lies on the branch locus. We may choose local coordinates $(\tilde{z}, \tilde{w})$ on M with the center at q, and local coordinates (z, w) on $S \times R$ with the center at p such that in terms of these coordinates:

$$\Phi : (\tilde{z}, \tilde{w}) \to (z, w) = (\tilde{z}^r, \tilde{w})$$

$$\rho : (\tilde{z}, \tilde{w}) \to (\epsilon^k \tilde{z}, \tilde{w})$$

where $\epsilon = \exp(2\pi i/r)$, $0 < k < r$, and k and r are relatively prime. Let $\lambda = \epsilon^j$, $0 \leqq j < r$. Then $(\Phi_*^{(\lambda)}\mathcal{O})_p = \{f \in \mathcal{O}_{M,q} \mid f = \lambda f \circ \rho\}$. Let $f \in (\Phi_*^{(\lambda)}\mathcal{O})_p$. In local coordinates, let $f(\tilde{z}, \tilde{w}) = \sum f_n(\tilde{w})\tilde{z}^n$. The condition that $f = \lambda f \circ \rho$ becomes:

$$\sum f_n(\tilde{w})\tilde{z}^n = \sum f_n(\tilde{w})\epsilon^{nk+j}\tilde{z}^n.$$

It follows that $f(\tilde{z}, \tilde{w})$ must have the form:

$$f(\tilde{z}, \tilde{w}) = z^h \sum f_n(\tilde{w})\tilde{z}^{rn} = \tilde{z}^h g(z, w)$$

where $0 \leqq h < r$, $hk \equiv -j$ modulo r. The correspondence $f(\tilde{z}, \tilde{w}) \to g(z, w)$ clearly determines an isomorphism $\Phi_*^{(\lambda)}\mathcal{O} \mid U \cong \mathcal{O}_{S\times R} \mid U$ for a suitably small neighborhood U of p. Hence the sheaf $\Phi_*^{(\lambda)}\mathcal{O}$ is locally free. By a similar argument, the sheaf $\Phi_*^{(\lambda)}\Theta$ is also locally free.

LEMMA 4.2. $\Phi_*^{(1)}\mathcal{O} = \mathcal{O}_{S\times R}$.

Proof. Obvious.

Let $L^{(\lambda)}$ be a holomorphic line bundle on $S \times R$ such that $\Phi_*^{(\lambda)}\mathcal{O} \cong \mathcal{O}(L^{(\lambda)})$, the sheaf of germs of holomorphic sections of $L^{(\lambda)}$.

LEMMA 4.3. *Let $\lambda = \epsilon^j \neq 1$, where $\epsilon = \exp(2\pi i/r)$, $0 < j < r$. Then*

$$\left(\frac{r}{(j,r)}\right) L^{(\lambda)} = -\sum_{g \in K} \left(\frac{h_g}{(j,r)}\right) [\Gamma_g]$$

(Cf. *Definition* 2.2), where h_g is an integer, $0 < h_g < r$, and where (j, r) is the greatest common divisor of j and r.

Proof. Let $\mathcal{H} = \mathcal{O}(-\sum_{g \in K}(\frac{h_g}{(j,r)})[\Gamma_g])$. We identify $\mathcal{H}$ with the sheaf of germs of holomorphic functions on $S \times R$ whose divisors are multiples of the divisor $\sum_{g \in K}(\frac{h_g}{(j,r)})\Gamma_g$. For each point $p \in S \times R$ we define an isomorphism σ_p: $\Phi_*{}^{(\lambda)}\mathcal{O}_p{}^{\otimes \frac{r}{(j,r)}} \cong \mathcal{H}_p$ as follows:

(i) if $p \notin \Gamma_g$, $g \in K$, let $\Phi^{-1}(p) = \{q_1, \cdots, q_r\}$, $\rho(q_i) = q_{i+1}$ (Cf. § 3). Let $f^{(1)} \otimes f^{(2)} \otimes \cdots \otimes f^{(m)} \in (\Phi_*{}^{(\lambda)}\mathcal{O})_p{}^{\otimes m}$, where $m = \frac{r}{(j,r)}$, and let

$$f^{(i)} = (f_1{}^{(i)}, f_2{}^{(i)}, \cdots, f_r{}^{(i)}), f_l{}^{(i)} = \lambda f_{l+1}{}^{(i)} \circ \rho.$$

Then it is easy to verify that $\prod_{i=1}^{m} f_l{}^{(i)} = (\prod_{i=1}^{m} f_{l+1}{}^{(i)}) \circ \rho$; hence $\prod f_l{}^{(i)} = g \circ \Phi$, $g \in \mathcal{O}_{S \times R, p}$. We let $\sigma_p(f^{(1)} \otimes \cdots \otimes f^{(m)}) = g$. σ_p may be extended linearly to an isomorphism σ_p: $(\Phi_*{}^{(\lambda)}\mathcal{O})_p{}^{\otimes m} \cong \mathcal{H}_p$. (ii) If $p \in \Gamma_g$, $g \in K$, $p = \Phi(q)$, we choose local coordinates (z, w) at p and $(\tilde{z}, \tilde{w})$ at q, and we use the notation in the proof of Lemma 4.1. Thus,

$$\Phi: (\tilde{z}, \tilde{w}) \to (z, w) = (\tilde{z}^r, \tilde{w}), \rho: (\tilde{z}, \tilde{w}) \to (\epsilon^k \tilde{z}, \tilde{w})$$

where $(k, r) = 1$. Let h_g be the unique integer such that $0 < h_g < r$, $kh \equiv -j$ modulo r. Then $(\Phi_*{}^{(\lambda)}\mathcal{O})_p$ is isomorphic to the germs of functions $f(\tilde{z}, \tilde{w})$ of the form $f(\tilde{z}, \tilde{w}) = \tilde{z}^{h_g} F(z, w)$. If $f^{(1)} \otimes \cdots \otimes f^{(m)} \in (\Phi_*{}^{(\lambda)}\mathcal{O})_p{}^{\otimes m}$, where in terms of the local coordinates $f^{(i)} = \tilde{z}^{h_g} F^{(i)}(z, w)$, then

$$\prod_{i=1}^{m} f^{(i)} = \tilde{z}^{h_g m} \prod_{i=1}^{m} F^{(i)} = z^{h_g/(j,r)} F(z, w).$$

We let $\sigma_p(f^{(1)} \otimes \cdots \otimes f^{(m)}) = z^{h_g/(j,r)} F(z, w) \in \mathcal{H}_p$. It is easy to see that σ_p may be extended linearly to an isomorphism σ_p: $(\Phi_*{}^{(\lambda)}\mathcal{O})_p{}^{\otimes m} \cong \mathcal{H}_p$ which does not depend on a choice of coordinates. One can easily verify that the isomorphisms σ_p "fit together" coherently to determine an isomorphism σ: $(\Phi_*{}^{(\lambda)}\mathcal{O})^{\otimes m} \cong \mathcal{H}$. The lemma follows immediately from the definition of these sheaves.

Let $\mathcal{N}_g$ be the sheaf, concentrated on Γ_g, of germs of holomorphic sections of the normal bundle of Γ_g in $S \times R$. There is a natural projection $\Theta_{S \times R} \to \mathcal{N}_g$. Let Ξ be the kernel of the map $\Theta_{S \times R} \to \sum_{g \in K} \mathcal{N}_g$, so that the sequence of sheaves:

$$0 \to \Xi \to \Theta \to \sum_{g \in K} \mathcal{N}_g \to 0$$

is exact. Clearly Ξ is the sheaf of germs of holomorphic vector fields on $S \times R$ whose restrictions to each curve Γ_g, $g \in K$, is tangent to Γ_g.

LEMMA 4.4. $\Phi_*^{(1)}\Theta \cong \Xi$.

Proof. It is enough to consider the stalks of these sheaves at a point $p \in \Gamma_g$, $p \in K$. Then, using the notation of the pereceding proofs, $(\Phi_*^{(1)}\Theta)_p = \{\theta \in \Theta_{M,q} \mid \rho_*\theta = \theta\}$. In local coordinates, if $\theta = a(\tilde{z}, \tilde{w})\partial/\partial\tilde{z} + b(\tilde{z}, \tilde{w})\partial/\partial\tilde{w}$, then $\rho_*\theta = \epsilon^k a(\epsilon^{-k}\tilde{z}, \tilde{w})\partial/\partial\tilde{z} + b(\epsilon^{-k}\tilde{z}, \tilde{w})\partial/\partial\tilde{w}$. Then the condition that $\rho_*\theta = \theta$ becomes:

$$\begin{cases} \epsilon^k a(\epsilon^{-k}\tilde{z}, \tilde{w}) = a(\tilde{z}, \tilde{w}) \\ b(\epsilon^{-k}\tilde{z}, \tilde{w}) = b(\tilde{z}, \tilde{w}). \end{cases}$$

It is easily verified that these conditions imply:

$$\begin{cases} a(\tilde{z}, \tilde{w}) = \tilde{z}c(z, w) \\ b(\tilde{z}, \tilde{w}) = d(z, w) \end{cases}$$

where $c(z, w)$ and $d(z, w)$ are holomorphic functions of (z, w). It follows that the holomorphic vector field $\theta = a(\tilde{z}, \tilde{w})\partial/\partial\tilde{z} + b(\tilde{z}, \tilde{w})\partial/\partial\tilde{w}$ is "projectable" and $\Phi_*\theta = rzc(z, w)\partial/\partial w + d(z, w)\partial/\partial w$. Clearly $\Phi_*\theta \in \Xi_p$. It is easily verified that $\theta \to \Phi_*\theta$ determines an isomorphism $\Phi_*^{(1)}\Theta \cong \Xi$.

LEMMA 4.5. $\Phi_*^{(\lambda)}\Theta \cong (\Phi_*^{(1)}\Theta) \otimes (\Phi_*^{(\lambda)}\mathcal{O})$.

Proof. Obvious.

Let W be the holomorphic vector bundle on $S \times R$ such that $\Xi = \mathcal{O}(W)$. It follows from Lemma 4.5 that $\Phi_*^{(\lambda)}\Theta = \mathcal{O}(W \otimes L^{(\lambda)})$.

THEOREM 4.1. $H^1(M, \Theta) \cong H^1(S \times R, \Xi)$.

Proof. By Theorem 3.1, $H^1(M, \Theta) \cong \bigoplus_{\lambda^r=1} H^1(S \times R, \Phi_*^{(\lambda)}\Theta)$. By Lemmas 4.4 and 4.5, it is sufficient to prove that $H^1(S \times R, W \otimes L^{(\lambda)}) = 0$ if $\lambda \neq 1$. Let T be the holomorphic tangent bundle on $S \times R$. We have the exact sequence:

$$0 \to \Phi_*^{(\lambda)}\Theta \to \Theta_{S\times R} \otimes \Phi_*^{(\lambda)}\mathcal{O} \to \sum_{g \in K} \mathcal{O}(([\Gamma_g] + L^{(\lambda)})_{\Gamma_g}) \to 0$$

and the exact cohomology sequence:

$$\cdots \to \sum_{g \in K} H^0(\Gamma_g, ([\Gamma_g] + L^{(\lambda)})_{\Gamma_g}) \to H^1(S \times R, W \otimes L^{(\lambda)})$$
$$\to H^1(S \times R, T \otimes L^{(\lambda)}) \to \cdots.$$

Thus it is sufficient to prove:

(i) $H^0(\Gamma_g, ([\Gamma_g] + L^{(\lambda)})_{\Gamma_g}) = 0$, if $\lambda \neq 1$

(ii) $H^1(S \times R, T \otimes L^{(\lambda)}) = 0$, $\lambda \neq 1$.

Proof of (i). Obviously $(\Gamma_g^2) = 2 - 2g_S < 0$. By Lemma 4.3, the intersection number $(\Gamma_g + L^{(\lambda)}) \cdot \Gamma_g = (1 - h_g/r)(\Gamma_g^2) < 0$ since $h_g < r$; hence $H^0(\Gamma_g, ([\Gamma_g] + L^{(\lambda)})_{\Gamma_g}) = 0$.

Proof of (ii). Let $\sum z_j$ be a canonical divisor on S, where $z_j \in S$, and let $\sum w_k$ be a canonical divisor on R. Let $C_j' = z_j \times R$ and $C_k'' = S \times w_k$. Then $T = [-\sum C_j'] \oplus [-\sum C_k'']$, and

$$H^1(S \times R, T \otimes L^{(\lambda)}) = H^1(S \times R, [-\sum C_j'] + L^{(\lambda)}) \oplus H^1(S \times R, [-\sum C_k''] + L^{(\lambda)}).$$

We consider each of these summands separately. We have the exact sequence:

$$0 \to \mathcal{O}([-\sum C_j'] + L^{(\lambda)}) \to \mathcal{O}(L^{(\lambda)}) \to \sum_j \mathcal{O}(L^{(\lambda)} \mid C_j') \to 0,$$

and the exact cohomology sequence:

$$\cdots \to \sum H^0(C_j', L^{(\lambda)} \mid C_j') \to H^1(S \times R, [-\sum C_j'] + L^{(\lambda)}) \to H^1(S \times R, L^{(\lambda)}) \to \cdots.$$

By Theorem 2.2, $\dim H^1(M, \mathcal{O}_M) = \dim H^1(S \times R, \mathcal{O}_{S \times R})$. Hence by Theorem 3.1 and Lemma 4.2, we have $H^1(S \times R, L^{(\lambda)}) = 0$. It follows from Lemma 4.3 that the intersection number $(L^{(\lambda)} \cdot C_j') < 0$; hence $H^0(C_j', L^{(\lambda)} \mid C_j') = 0$. It follows from the exact sequence that $H^1(S \times R, [-\sum C_j'] + L^{(\lambda)}) = 0$ if $\lambda \neq 1$. By a similar argument, $H^1(S \times R, [-\sum C_k''] + L^{(\lambda)}) = 0$ if $\lambda \neq 1$; hence $H^1(S \times R, T \otimes L^{(\lambda)}) = 0$, $\lambda \neq 1$, and the theorem is proved. Q. E. D.

Let P_1 and P_2 denote the projections of $S \times R$ onto S and R respectively. Let $T(S)$, $T(R)$, $T(\Gamma_g)$ denote the tangent bundles of S, R and Γ_g, and let $N(\Gamma_g)$ denote the normal bundle of Γ_g in $S \times R$. We identify the tangent bundle T of $S \times R$ with $P_1^*(T(S)) \oplus P_2^*(T(R))$ in an obvious way, and if $(w, z) \in S \times R$, we identify the fibre $T_{(w,z)}$ with $T_w(S) \oplus T_z(R)$. Let $T \mid \Gamma_g$ be the restriction of T to Γ_g, and consider the inclusion map $\iota\colon T(\Gamma_g) \to T \mid \Gamma_g$. One easily verifies that under our identifications, if $(w, z) \in \Gamma_g$;

$$\iota(T_{(w,z)}(\Gamma_g)) = \{(u, v) \in T_w(S) \oplus T_z(R) \mid v = (g\pi)_*(u)\}.$$

We get a splitting, $T \mid \Gamma_g = T(\Gamma_g) \oplus N(\Gamma_g)$ if we identify $T(\Gamma_g)$ with $\iota(T(\Gamma_g))$ and let: $N_{(z,w)}(\Gamma_g) = \{(u, v) \in T_w(S) \oplus T_z(R) \mid v = -(g\pi)_*(u)\}$. In terms of this splitting, the projection; $T \mid \Gamma_g \to N(\Gamma_g)$ is defined at each point (w, z) by

$$(u,v) \to \tfrac{1}{2}(u - (g\pi)_*^{-1}(v), v - (g\pi)_*(u)),$$

where $(u,v) \in T_w(S) \oplus T_z(R)$.

If X and Y are compact complex-analytic manifolds and $\phi: X \to Y$ is a holomorphic mapping which makes X an unramified covering of Y, then there are natural maps $\phi_*^{-1}: H^q(Y, \Theta_Y) \to H^q(X, \Theta_X)$. To describe these maps explicitly, we consider open coverings $\mathfrak{U} = \{U_\alpha\}$ of X and $\mathfrak{V} = \{V_j\}$ of Y such that there exists a surjective map $\alpha \to j(\alpha)$ from the index set of $\mathfrak{U}$ onto the index set of $\mathfrak{V}$ such that for every $U_\alpha \in \mathfrak{U}$, $\phi \mid U_\alpha$ is a biholomorphic map from U_α onto $V_{j(\alpha)}$. If $\xi \in H^q(Y, \Theta_Y)$ is represented by a q-cocycle $(\xi_{j_0 j_1 \cdots j_q})$, then $\phi_*^{-1}(\xi) = \tilde{\xi} \in H^q(X, \Theta_X)$ is represented by the q-cocycle $(\tilde{\xi}_{\alpha_0\alpha_1\cdots\alpha_q})$ such that: $\phi_*\tilde{\xi}_{\alpha_0\alpha_1\cdots\alpha_q} = \xi_{j(\alpha_0)\cdots j(\alpha_q)}$. The following lemma is trivial.

LEMMA 4.6. *The linear maps* $\phi_*^{-1}: H^q(Y, \Theta_Y) \to H^q(X, \Theta_X)$ *are injective.*

We return to our Kodaria surface M. Let $W = R/G$ (Cf. *Definition* 2.2 and the remark following that definition). We have the unramified coverings:

(i) $R \to W = R/G$

(ii) $S \to W$, defined by $S \xrightarrow{\pi} R \to W$

(iii) $g\pi: S \to R$.

These coverings give rise to linear maps:

(i) $\ell_R: H^1(W, \Theta_W) \to H^1(R, \Theta_R)$

(ii) $\ell_S: H^1(W, \Theta_W) \to H^1(S, \Theta_S)$

(iii) $(g\pi)_*^{-1}: H^1(R, \Theta_R) \to H^1(S, \Theta_S)$.

It is well known (Künneth formula) and easily verified that $H^1(S \times R, \Theta) \cong H_1(S, \Theta_S) \oplus H^1(R, \Theta_R)$. Let $\sigma: H^1(W, \Theta_W) \to H^1(S \times R, \Theta)$ be the monomorphism determined by

$$\ell_S \oplus \ell_R: H^1(W, \Theta_W) \to H^1(S, \Theta_S) \oplus H^1(R, \Theta_R).$$

LEMMA 4.7. $\sigma(H^1(W, \Theta_W)) = H^1(S \times R, \Xi)$.

Proof. We need to prove that the sequence:

$$0 \to H^1(W, \Theta_W) \xrightarrow{\sigma} H^1(S \times R, \Theta) \xrightarrow{\kappa} \sum_{g \in K} H^1(\Gamma_g, \mathfrak{N}_g)$$

is exact. Identifying $H^1(S \times R, \Theta)$ with $H^1(S, \Theta_S) \oplus H^1(R, \Theta_R)$, one easily verifies that the kernel of κ corresponds to

$$\{(\xi, \eta) \in H^1(S, \Theta_S) \oplus H^1(R, \Theta_R) \mid (g\pi)_*^{-1}(\eta) = \xi, \text{ for every } g \in K\}$$

(Cf. our remarks above on the bundle $T \mid \Gamma_g$). It follows that if $(\xi, \eta) \in$ kernel κ, then $\eta = g_* \eta$ for every $g \in G$ and $\xi = \pi_*^{-1}(\eta)$. Since η is invariant under G, and $\xi = \pi_*^{-1}(\eta)$, there exists some $\tau \in H^1(W, \Theta_W)$ with $(\xi, \eta) = (\ell_S(\tau), \ell_R(\tau))$. It follows that the sequence is exact.

Let $\{W_t\}_{t \in A}$ be a complex analytic family of curves parametrized by a complex manifold A, such that $W_0 = W$ where $o \in A$ is a base point. This family induces in an obvious way a family $\{M_t\}_{t \in A}$ where $M_0 = M$.

THEOREM 4.2. *If the complex analytic family of curves* $\{W_t\}_{t \in A}$ *is complete and effectively parametrized at a point* $t_0 \in A$, *then the family* $\{M_t\}$ *is complete and effectively parametrized at* $t_0 \in A$. *(For the definitions cf. Kodaira and Spencer* [9]*)*.

Proof. By Theorem 4.1 and Lemma 4.7, we have an isomorphism $H^1(W_{t_0}, \Theta_{t_0}) \cong H^1(M_{t_0}, \Theta_{t_0})$. Let

$$\rho_{t_0}\colon T_{t_0}(A) \to H^1(W_{t_0}, \Theta_{t_0}), \qquad \rho'_{t_0}\colon T_{t_0}(A) \to H^1(M_{t_0}, \Theta_{t_0})$$

be the maps of infinitesimal deformation (cf. [9], page 364). One can easily verify that the diagram

$$\begin{array}{ccc} H^1(W_{t_0}, \Theta_{t_0}) & \cong & H^1(M_{t_0}, \Theta_{t_0}) \\ {\scriptstyle\rho_{t_0}}\nwarrow & & \nearrow{\scriptstyle\rho'_{t_0}} \\ & T_{t_0}(A) & \end{array}$$

is commutative. It follows that $\rho'_{t_0}\colon T_{t_0}(A) \to H^1(M_{t_0}, \Theta_{t_0})$ is an isomorphism and $\{M_t\}_{t \in A}$ is effectively parametrized at t_0. By a theorem of Kodaira and Spencer [8], the family $\{M_t\}_{t \in A}$ is complete at t_0.

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