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On a function on the mapping class group of a surface of genus 2

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Abstract

We study representations of subgroups of the mapping class group $\mathcal{M}_g$ of a surface of genus $g \geqslant 2$ arising from the actions of them on the first cohomology groups of the surface with local coefficient systems which are defined by nontrivial homomorphisms $\pi_1(\Sigma_g, *) \to \mathbb{Z}_2 = Aut(\mathbb{Z})$. As an application, in the case of $g = 2$, we construct a function on $\mathcal{M}_2$ which agrees with the Meyer function $\phi : \mathcal{M}_2 \to \mathbb{Q}$ on the Torelli group $\mathcal{J}_2$. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A well-known representation of the mapping class group of a surface of genus g to $Sp(2g, \mathbb{Z})$ is obtained from the action of it on the first cohomology group of the surface. In this case the coefficient group of the cohomology group is $\mathbb{Z}$, more precisely, the local coefficient system obtained from the trivial $\pi_1(\Sigma_g, *)$-module $\mathbb{Z}$. In this paper we take local coefficient systems obtained from nontrivial $\pi_1(\Sigma_g, *)$-module $\mathbb{Z}$'s and consider representations of subgroups of the mapping class group of the surface arising from the action of them on the first cohomology groups of the surface with the above local coefficient systems. Moreover as an application, we construct a function of the mapping class group of the surface of genus 2 and show some properties of it.

We state them more precisely as follows. Let Σ_g be a closed oriented surface of genus $g \geqslant 2$ with a base point $*$. For any nonzero class $w \in H^1(\Sigma_g; \mathbb{Z}_2) = Hom(\pi_1(\Sigma_g, *), \mathbb{Z}_2 = \{0, 1\})$, $\mathbb{Z}$ is regarded as the $\pi_1(\Sigma_g, *)$-module with the action $(\alpha, m) \mapsto (-1)^{w(\alpha)} m$ for $(\alpha, m) \in \pi_1(\Sigma_g, *) \times \mathbb{Z}$, which is denoted by $\mathbb{Z}_w$ and the local coefficient system obtained

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from it is also denoted by the same letter. So we have the cohomology group $H^1(\Sigma_g; \mathbb{Z}_w)$. In this paper, it is identified with the first cohomology group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ of the surface group $\pi_1(\Sigma_g, *)$ with coefficients in $\mathbb{Z}_w$ since Σ_g is an Eilenberg–MacLane space. Let $\mathcal{M}_{g*}$ be the mapping class group of Σ_g relative to the base point and $\mathcal{M}_{g*}^w$ the subgroup of $\mathcal{M}_{g*}$ whose elements preserve the class w. Then the subgroup $\mathcal{M}_{g*}^w$ acts on the cohomology group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\mathit{torsion}$ modulo torsion by the pull back of the inverse.

In order to state our results, we introduce subgroups of $\mathcal{M}_{g*}$ and $\mathcal{M}_g$. Let $\mathcal{J}_g$ be the subgroup of $\mathcal{M}_g$ acting on the first cohomology group $H^1(\Sigma_g, \mathbb{Z})$ trivially. Let $\mathcal{K}_g$ be the subgroup of $\mathcal{M}_g$ generated by all the Dehn twists along separating simple closed curves. Then we have $\mathcal{K}_g \subset \mathcal{J}_g \subset \mathcal{M}_g^w \subset \mathcal{M}_g$. Similarly the subgroups $\mathcal{K}_{g*}$ and $\mathcal{J}_{g*}$ of $\mathcal{M}_{g*}$ are defined and we have $\mathcal{K}_{g*} \subset \mathcal{J}_{g*} \subset \mathcal{M}_{g*}^w \subset \mathcal{M}_{g*}$. The subgroups $\mathcal{J}_g$ and $\mathcal{J}_{g*}$ are called Torelli groups (see [4]).

The following proposition is a collection of some results of Section 2.

Proposition 1. *For each nonzero class* $w \in H^1(\Sigma_g; \mathbb{Z}_2)$*, the above action of* $\mathcal{M}_{g*}^w$ *on* $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\mathit{torsion}$ *gives a surjective homomorphism* $\zeta_{g*}^w : \mathcal{M}_{g*}^w \to Sp(2(g-1), \mathbb{Z})$*, whose restriction to* $\mathcal{K}_{g*}$ *is nontrivial. Moreover it descends to a surjective homomorphism* $\zeta_g^w : \mathcal{M}_g^w \to PSp(2(g-1), \mathbb{Z})$ *whose restriction to* $\mathcal{K}_g$ *is also nontrivial.*

Next we define a representation of the whole group $\mathcal{M}_{g*}$.

Let $\mathit{Aut}(\mathbb{H}, \Omega)$ be the group of automorphisms of $\mathbb{H}$ preserving Ω where $\mathbb{H}$ is the direct sum

$$\bigoplus_{w \in H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\}} H^1\big(\pi_1(\Sigma_g, *), \mathbb{Z}_w\big)/\mathit{torsion}$$

and Ω is the symplectic form defined by the cup product. For any element f of $\mathit{Aut}(\mathbb{H}, \Omega)$, we can put $f = (f_{uv})$, where f_{uv} is an isomorphism or the zero map from $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_v)/\mathit{torsion}$ to $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_u)/\mathit{torsion}$ and u and v run over the nonzero classes of $H^1(\Sigma_g; \mathbb{Z}_2)$. Let $\mathcal{S}$ be the subgroup of $\mathit{Aut}(\mathbb{H}, \Omega)$ consisting of those elements $f = (f_{uv})$ for which there exists $\sigma \in \mathit{Aut}(H^1(\Sigma_g; \mathbb{Z}_2), \cup)$ such that f_{uv} is not 0 if and only if $v = \sigma(u)$. The action of $\mathcal{M}_{g*}$ on $\mathbb{H}$ induces a homomorphism $\zeta_{g*} : \mathcal{M}_{g*} \to \mathcal{S}$. The image of the kernel of the homomorphism $\mathcal{M}_{g*} \to \mathcal{M}_g$ agrees with a normal subgroup $N = \{((-1)^{u(\tau)}\delta_{uv}) \in \mathcal{S} \mid \tau \in \pi_1(\Sigma_g, *)\}$ of $\mathcal{S}$, where δ_{uv} is the identity map on $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_u)$ if $u = v$, and is zero if not. Hence the homomorphism ζ_{g*} descends to a homomorphism $\zeta_g : \mathcal{M}_g \to \mathcal{S}/N$. With these notations, we have the following result which is stated as Corollary 9 in Section 2.

Corollary 2. *The restrictions of* ζ_{g*} *and* ζ_g *to* $\mathcal{K}_{g*}$ *and* $\mathcal{K}_g$*, respectively are nontrivial.*

Here we state some relations between our representations and others. There is a representation of $\mathcal{M}_{g,1}$ constructed by Morita (see [12]) and Trapp (see [15]) which is nontrivial on $\mathcal{J}_{g,1}$, where $\mathcal{M}_{g,1}$ is the mapping class group of Σ_g relative to an embedded disk $(* \in) D^2 \subset \Sigma_g$ and $\mathcal{J}_{g,1}$ is the corresponding Torelli group. It was shown that it descends to a representation of $\mathcal{M}_{g*}$. Morita showed that this representation can be also obtained from his representation of $\mathcal{M}_{g,1}$ to the semi-direct product

$$\tfrac{1}{2} \bigwedge^3 H^1(\Sigma_g; \mathbb{Z}) \rtimes Sp(2g; \mathbb{Z}).$$

Morita's representation is an extension of Johnson's homomorphism from $\mathcal{J}_{g,1}$ to $\bigwedge^3 H^1(\Sigma_g; \mathbb{Z})$. It descends to a representation of $\mathcal{M}_{g*}$ to $\frac{1}{2} \bigwedge^3 H^1(\Sigma_g; \mathbb{Z}) \rtimes Sp(2g; \mathbb{Z})$. Moreover it descends to a representation of $\mathcal{M}_g$ to $\frac{1}{2} \bigwedge^3 H^1(\Sigma_g; \mathbb{Z}) / H^1(\Sigma_g; \mathbb{Z}) \rtimes Sp(2g; \mathbb{Z})$. The kernels of the Morita's representations of $\mathcal{M}_{g,1}$ and $\mathcal{M}_g$ are $\mathcal{K}_{g,1}$ and $\mathcal{K}_g$, respectively (see [5,13]). This fact together with Proposition 1 and Corollary 2 imply that our representations ζ_{g*} and ζ_g are different from the above representations.

On the other hand there is a representation of subgroups of the mapping class group $\mathcal{M}_g$ defined by Looijenga using finite abelian coverings. They are defined as follows (see [7]). Let $\pi : \Sigma_h \to \Sigma_g$ be a connected finite abelian covering with covering group G, then we obtain the homomorphism $c : H_1(\Sigma_g; \mathbb{Z}) \to G$ from it. Let $\mathcal{M}_g^c$ be the subgroup of $\mathcal{M}_g$ consisting of those elements which preserve the homomorphism c by pullback. Let Sp_G be the group of symplectic transformations of $H_1(\Sigma_h; \mathbb{Z})$ which commute with the action of the covering group G. The image of G is contained in the center of Sp_G. Any element of $\mathcal{M}_g^c$ lifts to a mapping class of Σ_h and the difference from any other lift is a covering transformation. It follows that we have a well-defined homomorphism $\mathcal{M}_g^c \to Sp_G / G$. We consider the case of $G = \mathbb{Z}_2$, hence $c = w$. If we consider the local coefficient system $\mathbb{Z}_w$ over Σ_g as one over Σ_h via the projection π, then it is trivial. This implies that we have the induced homomorphism $\pi^* : H^1(\Sigma_g; \mathbb{Z}_w) \to H^1(\Sigma_h; \mathbb{Z})$. It turns out that its image is identified with $H^1(\Sigma_g; \mathbb{Z}_w)/torsion$ and is preserved by the image of Looijenga's representation. If we replace the range $Sp_{\mathbb{Z}_2} / \mathbb{Z}_2$ of it by its image, our representation ζ_g^w factors through Looijenga's representation.

As an application, in the case of $g = 2$, we construct a function on the mapping class group $\mathcal{M}_2$ from our representations in the same way as [1]. This function is related to the signature of 4-manifolds which are Σ_2-bundles over surfaces. Its restriction to the Torelli group $\mathcal{J}_2$ agrees with the restriction of the Meyer function ϕ to $\mathcal{J}_2$. We explain them more precisely as follows.

There is a unique conjugacy invariant $\mathbb{Q}$-valued function ψ on $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$ satisfying $\sigma(A, B) = \psi(B) - \psi(AB) + \psi(A)$ for any $A, B \in SL(2, \mathbb{Z})$ where σ is the signature 2-cocycle of $SL(2, \mathbb{Z})$. This function ψ is described explicitly by using the Dedekind sums or the Rademacher ϕ function (see [1,6,9,14]). Composing ψ with ζ_{2*}^w, we obtain the function $\Psi_*^w : \mathcal{M}_{2*}^w \to \mathbb{Q}$.

For any $f \in \mathcal{M}_{2*}^w$, let $M_f \to S^1$ be the Σ_2-bundle over S^1 with the section s where M_f is given by $\Sigma_2 \times [0, 1]/(x, 1) \sim (f^{-1}(x), 0)$ and the section s is determined from the base point of Σ_2. Let w_{M_f} be a unique class of $H^1(M_f; \mathbb{Z}_2)$ such that its restriction to the fiber Σ_2 at the base point $0 \in [0, 1]/\sim$ is w and its restriction to the image of the section s is 0. By considering w_{M_f} as a homomorphism $\pi_1(M_f, *) \to \mathbb{Z}_2 = S^0 \subset U(1)$, we obtain a flat complex line bundle over M_f. Then we have the Atiyah–Patodi–Singer ρ-invariant $\rho_{w_{M_f}}(M_f)$ for (M_f, w_{M_f}) (see [2]). In this case the value of $\rho_{w_{M_f}}(M_f)$ is in $\mathbb{Q}$. We define the function $\mu_*^w : \mathcal{M}_{2*}^w \to \mathbb{Q}$ by $\mu_*^w(f) = \rho_{w_{M_f}}(M_f) + \Psi_*^w(f)$. It turns out that this function μ_*^w descends to a function on $\mathcal{M}_2^w$, which is denoted by $\mu^w(f)$. Hence we can define the function $\mu : \mathcal{M}_2 \to \mathbb{Q}$ by $\mu(f) = \frac{1}{15}\sum_w \mu^w$, where w runs over the nonzero classes of $H^1(\Sigma_g; \mathbb{Z}_2)$ fixed by f^*.

It is known that there is a unique function, which is called the Meyer function, $\phi : M_2 \to \mathbb{Q}$ such that the equality $\operatorname{sign}(a,b) = \phi(b) - \phi(ab) + \phi(a)$ holds for any $a, b \in \mathcal{M}_2$ (see [8,9]). Here the signature cocycle sign of $\mathcal{M}_2$ is defined as follows.

Let P denote the 2 sphere with 3 holes $S^2 \setminus \coprod^3 \operatorname{int} D^2$, then $\pi_1(P,*)$ is a rank 2 free group whose generators are given by α and β. For any $a, b \in \mathcal{M}_2$, we define the homomorphism $h : \pi_1(P,*) \to \mathcal{M}_2$ by $\alpha \mapsto a$, $\beta \mapsto b$. Let Z_h be the Σ_2-bundle over P (the inverse of) whose monodromy is given by h. Since Z_h is also a compact oriented 4-manifold, we have the signature $\operatorname{sign}(Z_h)$ of it which depends only on a and b, so we can put $\operatorname{sign}(a,b) = \operatorname{sign}(Z_h)$.

Let $\mathcal{M}_2^H$ be the subgroup of $\mathcal{M}_2$ acting trivially on $H^1(\Sigma_2, \mathbb{Z}_2)$.

Theorem 3. *The function μ is conjugacy invariant on $\mathcal{M}_2$ and satisfies the equality* $\operatorname{sign}(a,b) = \mu(b) - \mu(ab) + \mu(a)$ *for any $a, b \in \mathcal{M}_2^H$. Its restriction to the Torelli group $\mathcal{J}_2$ is a nontrivial homomorphism, and agrees with the restriction of Meyer function ϕ to $\mathcal{J}_2$.*

This theorem can be obtained by combining Corollary 18 and Proposition 19.

This paper is organized as follows. In Section 2, we construct our representations and show some properties of them. In Section 3, we review a certain function on $SL(2,\mathbb{Z})$ related to the signature 2-cocycle of it. In Section 4, we define the function μ in Theorem 3 and prove this theorem. In Section 5, we give an example, which is needed to show Theorem 3.

2. Cohomologies of surfaces with twisted coefficients

In this section, for any nonzero class of the first cohomology group of a surface with $\mathbb{Z}_2$-coefficient, we construct a representation of the subgroup of the mapping class group of the surface whose elements preserve the class.

Let Σ_g be a closed oriented surface of genus $g \geqslant 2$ and $* \in \Sigma_g$ a base point.

Let $w : \pi_1(\Sigma_g, *) \to \mathbb{Z}_2$ be a homomorphism, then it determines a cohomology class of $H^1(\Sigma_g; \mathbb{Z}_2)$ which is denoted by the same letter w. $\pi_1(\Sigma_g, *)$ acts on $\mathbb{Z}$ by

$$\pi_1(\Sigma_g, *) \times \mathbb{Z} \to \mathbb{Z},$$
$$(\gamma, m) \mapsto \gamma \cdot m := (-1)^{w(\gamma)} m,$$

where the homomorphism w takes values in $\{0,1\}$. So $\mathbb{Z}$ is regarded as a $\pi_1(\Sigma_g, *)$-module, which is denoted by $\mathbb{Z}_w$.

Next we shall compute the first cohomology group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ (see [3]).

Let $Z^1 = Z^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ be the space of w-crossed homomorphisms u from $\pi_1(\Sigma_g, *)$ to $\mathbb{Z}$ which satisfy the equalities $u(\alpha\beta) = u(\alpha) + \alpha \cdot u(\beta)$ for $\forall \alpha, \beta \in \pi_1(\Sigma_g, *)$.

Let $B^1 = B^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ be the space of principal w-crossed homomorphisms u, for which there exists $m \in \mathbb{Z}$ satisfying $u(\alpha) = \alpha \cdot m - m$ for $\forall\alpha \in \pi_1(\Sigma_g, *)$. Then $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ is given by Z^1/B^1.

The fundamental group $\pi_1(\Sigma_g, *)$ is presented by

$$\left\langle \alpha_i, \beta_i \ (1 \leqslant i \leqslant g) \,\middle|\, \prod_{i=1}^{g}[\alpha_i, \beta_i] = 1 \right\rangle.$$

For any $u \in Z^1$, put $u(\alpha_i) = x_i$ and $u(\beta_i) = y_i$. Then we can regard Z^1 as a subset of $\mathbb{Z}^{2g}$ by the inclusion $u \mapsto (x_i, y_i)_{i=1}^{g}$.

Direct computation shows

$$Z^1 = \left\{ (x_i, y_i) \in \mathbb{Z}^{2g} \,\middle|\, \sum_{i=1}^{g} \left\{ w(\beta_i)x_i - w(\alpha_i)y_i \right\} = 0 \right\},$$
$$B^1 = \left\{ (x_i, y_i) \in \mathbb{Z}^{2g} \,\middle|\, (x_i, y_i) = -2\big(w(\alpha_i)m, w(\beta_i)m\big),\ m \in \mathbb{Z} \right\},$$

where $w(\alpha_i)$ and $w(\beta_i)$ are in $\mathbb{Z}_2 = \{0, 1\}$. Thus we obtain the following lemma.

Lemma 4.

$$H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w) \cong \begin{cases} \mathbb{Z}^{2g} & w = 0, \\ \mathbb{Z}^{2(g-1)} \oplus \mathbb{Z}_2 & w \neq 0. \end{cases}$$

Remark. Since Σ_g is a $K(\pi, 1)$-space, if we consider $\mathbb{Z}_w$ as a local coefficient system, then we have the identification $H^*(\Sigma_g; \mathbb{Z}_w) \cong H^*(\pi_1(\Sigma, *), \mathbb{Z}_w)$. Hereafter we identify these groups by this identification.

Let $\mathcal{M}_{g*}$ be the mapping class group of orientation and base point preserving diffeomorphisms of Σ_g. For a nonzero class $w \in H^1(\Sigma_g; \mathbb{Z}_2)$, the subgroup of $\mathcal{M}_{g*}$ whose elements preserve the class w is denoted by $\mathcal{M}_{g*}^w$. This subgroup acts on $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/torsion$. In this paper, the action of $f \in \mathcal{M}_{g*}^w$ is defined by the pull-back $(f^{-1})^*$ of the inverse of f. Moreover this action preserves the symplectic form on $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/torsion$ which is given by the cup-product and the identification $H^2(\pi_1(\Sigma_g, *), \mathbb{Z}) \cong \mathbb{Z}$.

Lemma 5. *The above symplectic lattice $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/torsion$ is isomorphic to $\mathbb{Z}^{2(g-1)}$ with the standard symplectic form.*

By Lemma 5, taking a symplectic basis for $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/torsion$, we obtain a homomorphism

$$\zeta_{g*}^w : \mathcal{M}_{g*}^w \to Sp\big(2(g-1), \mathbb{Z}\big).$$

Lemma 6. *The above homomorphism ζ_{g*}^w is surjective.*

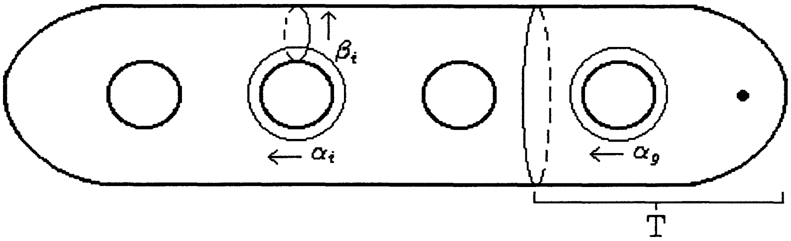

Fig. 1. A basis α_i, β_i and the submanifold T.

Proof of Lemmas 5 and 6. Since the action of the mapping class group of the surface on $H^1(\Sigma_g;\mathbb{Z}_2)\backslash\{0\}$ is transitive, it is sufficient to prove the case of $w=\alpha_g^*$ where α_g^* belongs to the dual basis of the one α_i, β_i $(1\leqslant i\leqslant g)$ of $H_1(\Sigma_g;\mathbb{Z}_2)$ which is given in Fig. 1. Let $T\subset\Sigma_g$ be the submanifold of Σ_g depicted in Fig. 1.

Let $\Sigma_{g-1,1}$ be the closure of the complement of T in Σ_g. By the exact sequence of the pair (Σ_g, T);

$$\begin{aligned}0=H^0(T;\mathbb{Z}_w)&\to H^1(\Sigma_g,T;\mathbb{Z}_w)\to H^1(\Sigma_g;\mathbb{Z}_w)\to H^1(T;\mathbb{Z}_w)\\&\to H^2(\Sigma_g,T;\mathbb{Z}_w)\to H^2(\Sigma_g;\mathbb{Z}_w)\to H^2(T;\mathbb{Z}_w)=0,\end{aligned}$$

and isomorphisms $H^1(T;\mathbb{Z}_w)\cong\mathbb{Z}\oplus\mathbb{Z}_2$, $H^2(\Sigma_g,T;\mathbb{Z}_w)\cong\mathbb{Z}$ and $H^2(\Sigma_g;\mathbb{Z}_w)\cong\mathbb{Z}_2$, we have an isomorphism $H^1(\Sigma_g,T;\mathbb{Z}_w)\cong H^1(\Sigma_g;\mathbb{Z}_w)/\mathit{torsion}$. There is the following commutative diagram:

$$\begin{array}{ccc}H^1(\Sigma_g;\mathbb{Z}_w)\times H^1(\Sigma_g;\mathbb{Z}_w) & \xrightarrow{\ \cup\ } & H^2(\Sigma_g;\mathbb{Z})\\ \uparrow & & \uparrow\cong\\ H^1(\Sigma_g,T;\mathbb{Z}_w)\times H^1(\Sigma_g,T;\mathbb{Z}_w) & \xrightarrow{\ \cup\ } & H^2(\Sigma_g,T;\mathbb{Z})\\ \downarrow\cong & & \downarrow\cong\\ H^1(\Sigma_{g-1,1},S^1;\mathbb{Z})\times H^1(\Sigma_{g-1,1},S^1;\mathbb{Z}) & \xrightarrow{\ \cup\ } & H^2(\Sigma_{g-1,1},S^1;\mathbb{Z})\end{array}$$

where S^1 is the boundary of $\Sigma_{g-1,1}$. From this we can deduce the following commutative diagram:

$$\begin{array}{ccc}H^1(\Sigma_g;\mathbb{Z}_w)/\mathit{torsion}\times H^1(\Sigma_g;\mathbb{Z}_w)/\mathit{torsion} & \xrightarrow{\ \cup\ } & H^2(\Sigma_g;\mathbb{Z})\\ \|\wr & & \|\wr\\ H^1(\Sigma_{g-1,1},S^1;\mathbb{Z})\times H^1(\Sigma_{g-1,1},S^1;\mathbb{Z}) & \xrightarrow{\ \cup\ } & H^2(\Sigma_{g-1,1},S^1;\mathbb{Z})\end{array}$$

Since the symplectic structure on $H^1(\Sigma_{g-1,1},S^1;\mathbb{Z})\cong H^1(\Sigma_{g-1};\mathbb{Z})$ is the standard one, the proof of Lemma 5 is finished.

Let $\mathcal{M}_{g-1,1}$ denote the group of isotopy classes of diffeomorphisms of $\Sigma_{g-1,1}$ whose restriction to the boundary is the identity of it. For any element of $\mathcal{M}_{g-1,1}$, extending it to an isotopy class of diffeomorphism of Σ_g by the identity on T, we have a homomorphism

$\mathcal{M}_{g-1,1} \to \mathcal{M}_{g*}$. By the choice of w, the image of the homomorphism is contained in $\mathcal{M}_{g*}^w$. Clearly the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_{g-1,1} & \longrightarrow & Aut(H^1(\Sigma_{g-1,1}, S^1; \mathbb{Z}), \cup) \\ \downarrow & & \| \wr \\ \mathcal{M}_{g*}^w & \longrightarrow & Aut(H^1(\Sigma_g; \mathbb{Z}_w)/torsion, \cup) \end{array}$$

The fact that $\mathcal{M}_{g-1} \to Aut(H^1(\Sigma_{g-1}; \mathbb{Z}), \cup)$ is surjective implies that the upper homomorphism in the above diagram is surjective, hence so is the lower one. The proof of Lemma 6 is completed. □

Next we see that ζ_{g*}^w descends to a homomorphism from $\mathcal{M}_g^w$ to $PSp(2(g-1), \mathbb{Z})$. There is the following exact sequence

$$1 \to \pi_1(\Sigma_g, *) \xrightarrow{\iota} \mathcal{M}_{g*} \to \mathcal{M}_g \to 1.$$

Since the image of ι is in $\mathcal{M}_{g*}^w$, we have the following exact sequence

$$1 \to \pi_1(\Sigma_g, *) \xrightarrow{\iota} \mathcal{M}_{g*}^w \to \mathcal{M}_g^w \to 1.$$

Lemma 7. *For any $\tau \in \pi_1(\Sigma_g, *)$, the equality $\zeta_{g*}^w \circ \iota(\tau) = (-1)^{w(\tau)} I$ holds. Hence the image of $\zeta_{g*}^w \circ \iota$ is $\{\pm I\} \subset Sp(2(g-1), \mathbb{Z})$.*

Proof. For each $\tau \in \pi_1(\Sigma_g, *)$, the action of $\iota(\tau) \in \mathcal{M}_{g*}$ on $\pi_1(\Sigma_g, *)$ is given by $\iota(\tau)_*(\gamma) = \tau\gamma\tau^{-1}$ for all $\gamma \in \pi_1(\Sigma_g, *)$ (see [11]).

For any $u \in Z^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$, we have

$$\begin{aligned} \iota(\tau^{-1})^* u(\gamma) = u(\tau^{-1}\gamma\tau) &= -\tau^{-1} \cdot u(\tau) + \tau^{-1} \cdot u(\gamma) + \tau^{-1}\gamma \cdot u(\tau) \\ &= \tau^{-1} \cdot u(\gamma) + (\gamma - 1)\tau^{-1} \cdot u(\tau). \end{aligned}$$

Since the map $\gamma \mapsto (\gamma - 1)\tau^{-1} \cdot u(\tau)$ belongs to $B^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$, we obtain $\iota(\tau^{-1})^*[u] = [\tau^{-1} \cdot u] = (-1)^{w(\tau^{-1})}[u] = (-1)^{w(\tau)}[u]$, where $[u] \in H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ is the class of u. The proof is completed. □

By this lemma, the homomorphism ζ_{g*}^w descends to a homomorphism

$$\zeta_g^w : \mathcal{M}_g^w \to PSp\big(2(g-1), \mathbb{Z}\big).$$

Let $\mathcal{K}_{g*}$ be the subgroup of $\mathcal{M}_{g*}$ generated by all the Dehn twists along separating simple closed curves on Σ_g. Similarly the subgroup $\mathcal{K}_g \subset \mathcal{M}_g$ is defined. Clearly $\mathcal{K}_{g*}$ and $\mathcal{K}_g$ are subgroups of $\mathcal{M}_{g*}^w$ and $\mathcal{M}_g^w$, respectively for any $w \in H^1(\Sigma_g; \mathbb{Z}_2)$.

The example in Section 5 shows the following proposition in the case of $g = 2$. For $g > 2$, similar examples can be given.

Proposition 8. *The restrictions of the homomorphisms ζ_{g*}^w and ζ_g^w to the subgroups $\mathcal{K}_{g*}$ and $\mathcal{K}_g$, respectively are nontrivial.*

We define a representation of the whole group $\mathcal{M}_{g*}$ as follows. Let $\mathbb{H}$ be the direct sum

$$\bigoplus_{w\in H^1(\Sigma_g;\mathbb{Z}_2)\setminus\{0\}} H^1\big(\pi_1(\Sigma_g,*),\mathbb{Z}_w\big)/\mathit{torsion}$$

and Ω be the symplectic form on it given by the direct sum of the cup products on $H^1(\pi_1(\Sigma_g,*),\mathbb{Z}_w)/\mathit{torsion}$'s. By Lemma 5, $(\mathbb{H},\Omega)$ is isomorphic to $\mathbb{Z}^{2(2^{2g}-1)(g-1)}$ with the standard symplectic form. Clearly $\mathcal{M}_{g*}$ acts on $\mathbb{H}$ and preserves the symplectic form Ω. Let $\mathit{Aut}(\mathbb{H},\Omega)$ be the group of automorphisms of $\mathbb{H}$ preserving Ω. For any $f\in \mathit{Aut}(\mathbb{H},\Omega)$, we put $f=(f_{uv})$ where f_{uv} is an isomorphism or the zero map from $H^1(\pi_1(\Sigma_g,*),\mathbb{Z}_v)/\mathit{torsion}$ to $H^1(\pi_1(\Sigma_g,*),\mathbb{Z}_u)/\mathit{torsion}$ and u and v run over the nonzero classes of $H^1(\Sigma;\mathbb{Z}_2)$. Let $\mathcal{S}$ be the subgroup of $\mathit{Aut}(\mathbb{H},\Omega)$ consisting of those elements $f=(f_{uv})$ for which there exists $\sigma\in \mathit{Aut}(H^1(\Sigma_g;\mathbb{Z}_2),\cup)$ such that f_{uv} is not 0 if and only if $v=\sigma(u)$. Clearly the action of $\mathcal{M}_{g*}$ on $\mathbb{H}$ induces a homomorphism

$$\zeta_{g*}:\mathcal{M}_{g*}\to\mathcal{S}.$$

By Lemma 7, the image of $\zeta_{g*}\circ\iota$ agrees with $N=\{((-1)^{u(\tau)}\delta_{uv})\in\mathcal{S}\mid \tau\in\pi_1(\Sigma_g,*)\}\subset\mathcal{S}$, where δ_{uv} is the identity map on $H^1(\pi_1(\Sigma_g,*),\mathbb{Z}_u)$ if $u=v$, and is zero if not. It is easy to see that N is a normal subgroup of $\mathcal{S}$. Hence the homomorphism ζ_{g*} descends to a homomorphism

$$\zeta_g:\mathcal{M}_g\to\mathcal{S}/N.$$

Clearly the restriction of ζ_{g*} to $\mathcal{M}^H_{g*}$ is identified with

$$\prod_{w\in H^1(\Sigma_g;\mathbb{Z}_2)\setminus\{0\}} \zeta^w_{g*}:\mathcal{M}^H_{g*}\to Sp\big(2(g-1);\mathbb{Z}\big)^{2^{2g}-1}.$$

For ζ_g, a similar result holds. Proposition 8 implies the following corollary.

Corollary 9. *The restrictions of ζ_{g*} and ζ_g to $\mathcal{K}_{g*}$ and $\mathcal{K}_g$, respectively are nontrivial.*

Remark. The above homomorphisms ζ^w_{g*} are also obtained from the action of diffeomorphisms of Σ_g on the moduli spaces of flat $O(2)$-connections on Σ_g. More precisely, for nonzero class $w\in H^1(\Sigma_g,\mathbb{Z}_2)$, we regard it as a homomorphism from $\pi_1(\Sigma_g,*)$ to $\mathbb{Z}_2\subset O(2)$ whose image is not contained in the identity component of $O(2)$. Then it defines a nontrivial flat $O(2)$ bundle over Σ_g. We consider the moduli space of flat $O(2)$ connections on this bundle modulo automorphisms of the bundle which preserve an orientation of the fiber at the base point. For any diffeomorphism of Σ_g which preserves the base point, orientation and the class w, by taking a lift of it to an automorphism of the above $O(2)$-bundle which preserves an orientation of the fiber at the base point, we obtain the action of the diffeomorphism on the moduli space. This action is independent of the choice of a lift and a representative of an isotopy class of a diffeomorphism. It is easy to see that the moduli space can be identified with a $2(g-1)$-dim torus $T^{2(g-1)}$. The above action induces the action of $\mathcal{M}^w_{g*}$ on the first homology group of the moduli space, so we obtain a homomorphism from $\mathcal{M}^w_{g*}$ to $\mathit{Aut}(H_1(T^{2(g-1)},\mathbb{Z}))$. It is easy to check that this homomorphism agrees with ζ^w_{g*}.

3. Some functions on $\mathcal{M}_{2*}^{w}$ and $\mathcal{M}_{2*}$

In this section we consider the case of $g=2$ and construct some functions on mapping class groups with values in $\mathbb{Q}$.

Let w be a nonzero class of $H^1(\Sigma_g;\mathbb{Z}_2)$. Take an oriented basis for $H^1(\pi_1(\Sigma_2,*),\mathbb{Z}_w)/$ *torsion* and fix it. Then by Lemma 4, $H^1(\pi_1(\Sigma_2,*),\mathbb{Z}_w)/$*torsion* is identified with $\mathbb{Z}^2$. Hence the action of $\mathcal{M}_{2*}^{w}$ on the cohomology group induces the homomorphism

$$\zeta_{2*}^{w}:\mathcal{M}_{2*}^{w}\to SL(2,\mathbb{Z}).$$

Next we introduce some functions on $SL(2,\mathbb{Z})$ and $PSL(2,\mathbb{Z})$ (see [1,6,9,14]).

Let

$$\phi:PSL(2,\mathbb{Z})\to\mathbb{Z}$$

be the Rademacher ϕ function which is defined by

$$A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto\phi(A)=\begin{cases}b/d & \text{if } c=0,\\ (a+d)/c-12\,\mathrm{sign}(c)s(a,c) & \text{if } c\neq 0,\end{cases}$$

where $A\in SL(2,\mathbb{Z})$ is a lift of $[A]\in PSL(2,\mathbb{Z})$ and

$$s(a,c):=\sum_{k=1}^{|c|-1}\left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{ka}{c}\right)\right)$$

is the Dedekind sums for coprime integers a and c. Here $((x))=0$ if x is an integer, and $=x-[x]-1/2$ if not.

Moreover we define the $\mathbb{Z}$-valued function

$$\nu:SL(2,\mathbb{Z})\to\mathbb{Z}$$

by

$$A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto\nu(A)=\begin{cases}\mathrm{sign}(b) & \text{if } A=\left(\begin{smallmatrix}1&k\\0&1\end{smallmatrix}\right),\ k\in\mathbb{Z},\\ \mathrm{sign}(c(a+d-2)) & \text{if not},\end{cases}$$

and the $\mathbb{Q}$-valued function

$$\psi:SL(2,\mathbb{Z})\to\mathbb{Q}$$

by

$$A\mapsto\psi(A)=\tfrac{1}{3}\phi(A)-\nu(A).$$

It is known that ψ is a unique function on $SL(2,\mathbb{Z})$ with values in $\mathbb{Q}$ satisfying $\sigma=\delta\psi$, where

$$\sigma:SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})\to\mathbb{Z}$$

is the signature 2-cocycle and δ is the coboundary operator.

We define the function

$$\Psi_{*}^{w}:\mathcal{M}_{2*}^{w}\to\mathbb{Q}$$

by

$$\Psi_*^w(f) = \psi\big(\zeta_{2*}^w(f)\big).$$

Since ψ is conjugacy invariant on $SL(2,\mathbb{Z})$, Ψ_*^w is independent of the choice of a basis for $H^1(\pi_1(\Sigma_2,*),\mathbb{Z}_w)/torsion$.

We also define the function

$$\Psi_* : \mathcal{M}_{2*} \to \mathbb{Q}$$

by

$$\Psi_*(f) = \frac{1}{15} \sum_{\substack{w\in H^1(\Sigma_2;\mathbb{Z}_2)\setminus\{0\} \\ f^*w=w}} \Psi_*^w(f).$$

The following lemma is clear from the conjugacy invariance of ψ.

Lemma 10. *Ψ_*^w is $\mathcal{M}_{2*}^w$-conjugacy invariant and Ψ_* is $\mathcal{M}_{2*}$-conjugacy invariant.*

4. The signature of surface bundles

In this section we define the function μ in Theorem 3 and show some properties of it.

Let X be a compact oriented surface with a base point x_0 and possibly with boundaries.

Let $\pi : Z \to X$ be a Σ_g-bundle with a section $s : X \to Z$ where $g \geqslant 2$. Considering (the inverse of) the monodromies of it, we have a homomorphism

$$h : \pi_1(X, x_0) \to \mathcal{M}_{g*},$$

where we identify Σ_g with the fiber Z_{x_0} at the base point x_0 of X. Conversely if a homomorphism from $\pi_1(X,x_0)$ to $\mathcal{M}_{g*}$ is given, then we can construct a surface bundle with a section since $BDiff^+(\Sigma_g,*) \simeq K(\mathcal{M}_{g*},1)$ holds for $g \geqslant 2$ (see [10]).

Lemma 11. *Let w be a nonzero class of $H^1(\Sigma_g;\mathbb{Z}_2)$. Under the above setting, suppose $\operatorname{Im} h \subset \mathcal{M}_{g*}^w$, then there exists a unique class $w_Z \in H^1(Z;\mathbb{Z}_2)$ satisfying $w_Z|_{\Sigma_g} = w$ in $H^1(\Sigma_g,\mathbb{Z}_2)$ and $s^*w_Z = 0$ in $H^1(X;\mathbb{Z}_2)$.*

Proof. We consider a spectral sequence $(E_r^{p,q}, d_r)$ for the cohomology $H^*(Z;\mathbb{Z}_2)$ with $E_2^{p,q} = H^p(X; H^q(\Sigma_g;\mathbb{Z}_2))$ of the fibration $\pi : Z \to X$. In this case, we have isomorphisms $E_2^{0,1} \cong H^1(\Sigma_g;\mathbb{Z}_2)^{\pi_1(X,*)}$ and $E_2^{1,0} = H^1(X;\mathbb{Z}_2)$. Clearly we have $E_2^{1,0} = E_3^{1,0} = \cdots = E_\infty^{1,0}$. It is easy to see that $d_2 : E_2^{0,1} \to E_2^{2,0}$ is 0 because of the existence of a section s of the fibration by the assumption. So we get $E_2^{0,1} = E_3^{0,1} = \cdots = E_\infty^{0,1}$. Hence we obtain isomorphisms

$$H^1(Z;\mathbb{Z}_2) \cong E_\infty^{1,0} \oplus E_\infty^{0,1} = E_2^{1,0} \oplus E_2^{0,1} \cong H^1(X;\mathbb{Z}_2) \oplus H^1(\Sigma_g;\mathbb{Z}_2)^{\pi_1(X,x_0)}.$$

For the above splitting of $H^1(Z;\mathbb{Z}_2)$, the projection to the second factor agrees with the restriction and one to the first factor may be given by pullback of s. By the hypothesis, the class w is in $H^1(\Sigma_g;\mathbb{Z}_2)^{\pi_1(X,x_0)}$. This completes the proof. □

Hereafter we assume the hypothesis of Lemma 11, that is $w\neq 0$ and $\operatorname{Im} h\subset \mathcal{M}^w_{g*}$.

Let $\mathbb{C}_{w_Z}$ be the $\pi_1(Z,*)$-module $\mathbb{Z}_{w_Z}\otimes\mathbb{C}$ over $\mathbb{C}$. It defines the flat complex line bundle over Z, which is also denoted by $\mathbb{C}_{w_Z}$, so the cohomology groups $H^*(Z;\mathbb{C}_{w_Z})$ and $H^*(Z,\partial Z;\mathbb{C}_{w_Z})$ are defined. Put

$$\widehat{H}^*(Z;\mathbb{C}_{w_Z}):=\operatorname{Im}\bigl[H^*(Z,\partial Z;\mathbb{C}_{w_Z})\to H^*(Z;\mathbb{C}_{w_Z})\bigr].$$

There is the nondegenerate hermitian form on $\widehat{H}^2(Z;\mathbb{C}_{w_Z})$ which is defined by the cup-product, the inner product on $\mathbb{C}$ and evaluation on the fundamental cycle $[Z,\partial Z]$. Its signature is denoted by $\operatorname{sign}_{w_Z}(Z)$.

On the other hand, from the surface bundle $Z\to X$ with a section s, we construct the flat vector bundle $\mathcal{H}$ over X with the hermitian form as follows.

The fiber $\mathcal{H}_x$ at $x\in X$ is the cohomology group

$$H^1\bigl(\pi_1\bigl(\pi^{-1}(x),s(x)\bigr);\mathbb{C}_{w_Z}|_{\pi^{-1}(x)}\bigr)\cong H^1(\Sigma_g;\mathbb{C}_w)\cong\mathbb{C}^{2(g-1)}.$$

The hermitian form is given by i times the cup-product, the inner product on $\mathbb{C}$ and evaluation on the fundamental class of the fiber $\pi^{-1}(x)$.

By the definition of $\mathcal{H}$, (the inverse of) the holonomy homomorphism of the flat bundle $\mathcal{H}$ is given by

$$\tau:\pi_1(X,x_0)\to Sp\bigl(2(g-1),\mathbb{Z}\bigr)\hookrightarrow U(g-1,g-1),$$

which agrees with the homomorphism $\pi_1(X,*)\ni\alpha\mapsto(h(\alpha)^{-1})^*\in Aut(H^1(\Sigma_g;\mathbb{C}_w), i\times(\cdot\cup\cdot))$. By considering $\mathcal{H}$ as a local coefficient system, we obtain the first cohomology group $H^1(X;\mathcal{H})$ with the skew-hermitian form which is also defined by the cup-product and the hermitian form of the bundle. The skew-hermitian form multiplied by i is a hermitian form on $H^1(X;\mathcal{H})$, so we get its signature, which is denoted by $\operatorname{sign}(X;\mathcal{H})$.

Lemma 12. *Under the above setting, the equality* $\operatorname{sign}_{w_Z}(Z)=\operatorname{sign}(X;\mathcal{H})$ *holds.*

Proof. There exists a spectral sequence $(E_r^{p,q},d_r)$ for the cohomology group $H^*(Z,\partial Z;\mathbb{C}_{w_Z})$ with $E_2^{p,q}\cong H^p(X,\partial X;H^q(\Sigma_g;\mathbb{C}_w))$ of the fibration $Z\to X$. Note that the cohomology group $H^p(X,\partial X;H^q(\Sigma_g;\mathbb{C}_w))$ depends on the section s, hence the isomorphism from $E_2^{p,q}$ to it does so, and note that $E_2^{1,1}=H^1(X,\partial X;H^1(\Sigma_g;\mathbb{C}_w))=H^1(X,\partial X;\mathcal{H})$. Since we have $H^0(\Sigma_g;\mathbb{C}_w)=H^2(\Sigma_g;\mathbb{C}_w)=0$, we obtain $E_2^{2,0}=E_2^{0,2}=0$. Hence we get the isomorphisms

$$H^2(Z,\partial Z;\mathbb{C}_{w_Z})\cong\sum_{p+q=2}E_\infty^{p,q}\cong E_2^{1,1}\cong H^1(X,\partial X;\mathcal{H}).$$

Similarly we obtain isomorphisms $H^2(Z;\mathbb{C}_{w_Z}) \cong H^1(X;\mathcal{H})$ and $H^2(X,\partial X; H^2(\Sigma_g;\mathbb{C})) \cong H^4(Z,\partial Z;\mathbb{C})$. Moreover we have the following commutative diagrams:

$$\begin{array}{ccc} H^2(Z,\partial Z;\mathbb{C}_{w_Z}) & \longrightarrow & H^2(Z;\mathbb{C}_{w_Z}) \\ \| \wr & & \| \wr \\ H^1(X,\partial X;\mathcal{H}) & \longrightarrow & H^1(X;\mathcal{H}) \end{array}$$

$$\begin{array}{ccc} H^2(Z,\partial Z;\mathbb{C}_{w_Z}) \times H^2(Z;\mathbb{C}_{w_Z}) & \xrightarrow{\ \cup\ } & H^4(Z,\partial Z;\mathbb{C}) \\ \| \wr & & \| \wr \\ H^1(X,\partial X;\mathcal{H}) \times H^1(X;\mathcal{H}) & \xrightarrow{-(\cup)} & H^2(X,\partial X; H^2(\Sigma_g;\mathbb{C})) \cong H^2(X,\partial X;\mathbb{C}) \end{array}$$

where $-(\cup)$ on the lower arrow denotes minus the cup product on cohomology with local coefficient system. In this case $-(\cup)$ agrees with the cup product in the spectral sequence. Taking the two times multiple by i in the definition of $\operatorname{sign}(X;\mathcal{H})$ into consideration, we obtain the equality $\operatorname{sign}_{w_Z}(Z) = \operatorname{sign}(X;\mathcal{H})$. $\square$

Put $P := S^2 \setminus \coprod^3 \operatorname{int} D^2$, then $\pi_1(P,*)$ is a rank 2 free group whose generators are given by α and β.

Any homomorphism ξ from $\pi_1(P,*)$ to $U(p,q)$ defines a flat vector bundle E_ξ over P with a hermitian form. In the same way as above, we have the signature $\operatorname{sign}(P,E_\xi)$ of the hermitian form on $\widehat{H}^1(P;E_\xi) := \operatorname{Im}[H^1(P,\partial P;E_\xi) \to H^1(P;E_\xi)]$. The flat bundle E_ξ depends only on the homomorphism ξ, hence on the two elements $(\xi(\alpha),\xi(\beta)) =: (A,B)$ of $U(p,q)$. So we can put $\operatorname{sign}(A,B) := \operatorname{sign}(P,E_\xi)$ (see [1]).

Now we shall consider the case of $g=2$.

Since the restriction of sign to $SL(2,\mathbb{Z}) \hookrightarrow U(1,1)$ agrees with the signature 2-cocycle σ (see [1]), we have $\operatorname{sign} = \sigma = \delta\psi$. Thus we obtain

$$\operatorname{sign}(A,B) = \psi(B) - \psi(AB) + \psi(A).$$

Let X be a compact oriented surface with boundary $\partial X = \coprod_i S^1_i$. Let

$$\xi' : \pi_1(X,x_0) \to SL(2,\mathbb{Z})$$

be a homomorphism and ξ the composition of ξ' with $SL(2,\mathbb{Z}) \hookrightarrow U(1,1)$, then by the signature additivity, we have

$$\operatorname{sign}(X,E_\xi) = \sum_i \psi\big(\xi'(S^1_i)\big),$$

where S^1_i denotes also a class in $\pi_1(X,x_0)$ corresponding to the boundary S^1_i which is determined up to conjugation. Here we note that, since ψ is conjugacy invariant on $SL(2,\mathbb{Z})$, the right hand side of the above equality is well-defined.

Lemma 13.

$$\operatorname{sign}_{w_Z}(Z) = \sum_i \Psi^w_*\big(h(S^1_i)\big).$$

Proof. The flat bundle $\mathcal{H}$ in Lemma 12 is constructed from the representation $\tau = \xi : \pi_1(X, x_0) \to SL(2,\mathbb{Z}) \hookrightarrow U(1,1)$, so we get $\mathcal{H} = E_\xi$. Hence we have

$$\mathrm{sign}_{w_Z}(Z) = \mathrm{sign}(X, \mathcal{H}) = \mathrm{sign}(X, E_\xi) = \sum_i \psi\bigl(\xi'(S_i^1)\bigr).$$

Since $\xi' = \zeta_{2*}^w h$ and $\Psi_*^w = \psi \circ \zeta_{2*}^w$ holds by definitions, we obtain

$$\psi\bigl(\xi'(S_i^1)\bigr) = \psi\bigl(\zeta_{2*}^w h(S_i^1)\bigr) = \Psi_*^w h(S_i^1),$$

hence $\mathrm{sign}_{w_Z}(Z) = \sum_i \Psi_*^w(h(S_i^1))$. This completes the proof. $\square$

Now we recall the definition of the ρ-invariants (see [2]).

Let M be an oriented Riemannian manifold of dimension $2l-1$ and $\alpha : \pi_1(M) \to U(n)$ a unitary representation. The self-adjoint operator on even forms on M:

$$D : \Omega^{even}(M;\mathbb{C}) \to \Omega^{even}(M;\mathbb{C})$$

is defined by

$$D(\phi) = i^l(-1)^{p+1}(*d - d*)\phi,$$

where ϕ is a form of degree $2p$. Moreover the extension of D to the even forms with coefficients in the flat vector bundle of rank n defined by α is denoted by D_α, which is also a self-adjoint operator.

For the operator D_α, we define the function $\eta_\alpha(s)$ by

$$\eta_\alpha(s) = \sum_{\lambda\neq 0}(\mathrm{sign}\,\lambda)|\lambda|^{-s},$$

where λ runs over the eigenvalues of D_α. The corresponding function to D is denoted by $\eta(s)$. These functions extend at $s = 0$ and have finite values here. Its values $\eta(0)$ and $\eta_\alpha(0)$ are called the η-invariants of a Riemannian manifold.

Put $\tilde{\eta}_\alpha(s) := \eta_\alpha(s) - n\eta(s)$.

Theorem 14 (Atiyah, Patodi and Singer [2]). $\tilde{\eta}_\alpha(0)$ *is independent of the metric. It is a diffeomorphism invariant of M and α which we shall denote by $\rho_\alpha(M)$. If $M = \partial N$ with α extending to a unitary representation of $\pi_1(N)$ then*

$$\rho_\alpha(M) = n\,\mathrm{sign}(N) - \mathrm{sign}_\alpha(N).$$

Now we return to our setting. Let $\partial Z = \coprod_i \partial_i Z$ be the decomposition of the boundary of Z by the connected components. Then $\partial_i Z$ is the Σ_2 bundle over S_i^1 with (the inverse of) a monodromy $h(S_i^1) \in \mathcal{M}_{2*}^w$ where S_i^1 is a connected component of the boundary $\partial X = \coprod_i S_i^1$. In Theorem 14, we put $N = Z$, $M = \partial Z$ and $\alpha = w_Z|_{\partial Z}$. Then using Lemma 13 we obtain

$$\mathrm{sign}(Z) = \sum_i \rho_{w_{\partial_i Z}}(\partial_i Z) + \mathrm{sign}_{w_Z}(Z) = \sum_i \bigl\{\rho_{w_{\partial_i Z}}(\partial_i Z) + \Psi_*^w\bigl(h(S_i^1)\bigr)\bigr\}.$$

For an orientation and base point preserving diffeomorphism f of Σ_2 or its isotopy class, let M_f denote the Σ_2 bundle over S^1 with a monodromy f^{-1}.

For a nonzero class $w \in H^1(\Sigma_2; \mathbb{Z}_2)$, the function

$$\mu_*^w : \mathcal{M}_{2*}^w \to \mathbb{Q}$$

is defined by

$$\mu_*^w(f) = \rho_{w_{M_f}}(M_f) + \Psi_*^w(f)$$

for $f \in \mathcal{M}_{2*}^w$. Then we can rewrite the above equality as follows:

$$\operatorname{sign}(Z) = \sum_i \mu_*^w\bigl(h(S_i^1)\bigr).$$

Since the signature of a surface bundle Z_h over a compact surface X is determined by the action of the corresponding monodromy h^{-1} on the first cohomology group of the fiber Σ_2, the equality $\operatorname{sign}(Z_h) = 0$ holds for any homomorphism $h : \pi_1(X, x_0) \to \mathcal{J}_{2*}$. This and the above equality imply that the restriction of μ_*^w to $\mathcal{J}_{2*}$ is a homomorphism.

Let $\mu_* : \mathcal{M}_{2*} \to \mathbb{Q}$ be function defined by

$$\mu_*(f) = \frac{1}{15} \sum_{\substack{w \in H^1(\Sigma_2;\mathbb{Z}_2)\setminus\{0\} \\ f^*w = w}} \mu_*^w(f)$$

for $f \in \mathcal{M}_{2*}$. The above argument and the example in Section 5 imply the following proposition.

Proposition 15. *The restrictions of μ_* and μ_*^w $(w \neq 0)$ to $\mathcal{J}_{2*}$ are nontrivial homomorphisms.*

Next we shall prove that, for any nonzero class w in $H^1(\Sigma_2; \mathbb{Z}_w)$, the map $\mu_*^w : \mathcal{M}_{2*}^w \to \mathbb{Q}$ descends to the map

$$\mu^w : \mathcal{M}_2^w \to \mathbb{Q}.$$

For any $a, b \in \mathcal{M}_{2*}$, let $\operatorname{sign}(a, b)$ be the signature $\operatorname{sign}(Z_h)$ of the surface bundle Z_h over P constructed from the homomorphism $h : \pi_1(P, *) \to \mathcal{M}_{2*}$ defined by $\alpha \mapsto a$ and $\beta \mapsto b$. Then we have $\operatorname{sign}(a, b) = \operatorname{sign}(A, B)$, where $A, B \in U(2, 2)$ is the image of a, b by the obvious homomorphism

$$\mathcal{M}_{2*} \to Aut\bigl(H^1(\Sigma_2; \mathbb{Z}), \cup\bigr) \to Sp(4, \mathbb{Z}) \to U(2, 2),$$

where we used a fixed basis of $H^1(\Sigma_2; \mathbb{Z})$.

Note that $\operatorname{sign}(A, B) = 0$ if $A = 1$, $B = 1$ or $AB = 1$ (see [1]). Hence we have $\operatorname{sign}(a, b) = 0$ if $a \in \pi_1(\Sigma_2, *)$, $b \in \pi_1(\Sigma_2, *)$ or $ab \in \pi_1(\Sigma_2, *)$.

The following lemma is easy or have already been proven.

Lemma 16. *For any nonzero class $w \in H^1(\Sigma_2; \mathbb{Z}_w)$, the map μ_*^w satisfies the following properties.*

(1) $\mu_*^w(1) = 0$,
(2) $\mu_*^w(a^{-1}) = -\mu_*^w(a)$,

(3) $\mu_*^w(faf^{-1}) = \mu_*^{f^*w}(a)$,
(4) $\mathrm{sign}(a,b) = \mu_*^w(b) - \mu_*^w(ab) + \mu_*^w(a)$,
where $a, b \in \mathcal{M}_{2*}^w$ *and* $f \in \mathcal{M}_{2*}$.

Lemma 17. *For any* $\alpha \in \pi_1(\Sigma_g, *)$ *and* $f \in \mathcal{M}_{2*}^w$*, the followings hold.*
(1) $\mu_*^w(\alpha) = 0$,
(2) $\mu_*^w(\alpha f) = \mu_*^w(f\alpha) = \mu_*^w(f)$.

Proof. (1) Since we have $M_\alpha \cong \Sigma_2 \times S^1 = \partial(\Sigma_2 \times D^2)$, there exists $\widetilde{w} \in H^1(\Sigma_2 \times D^2; \mathbb{Z}_2)$ satisfying $\widetilde{w}|_{\partial(\Sigma_2 \times D^2)} = w_{M_\alpha}$. By Theorem 14 and $H^2(\Sigma_2 \times D^2; \mathbb{C}_{\widetilde{w}}) = 0$, we have

$$\rho_{w_{M_\alpha}}(M_\alpha) = \mathrm{sign}(\Sigma_2 \times D^2) - \mathrm{sign}_{\widetilde{w}}(\Sigma_2 \times D^2) = 0.$$

By Lemma 7, we have

$$\Psi_*^w(\alpha) = \psi\big(\zeta_{2*}^w(\alpha)\big) = \psi(\pm 1) = 0.$$

So we obtain $\mu_*^w(\alpha) = \rho_{w_{M_\alpha}}(M_\alpha) + \Psi_*^w(\alpha) = 0$.

(2) $\mathrm{sign}(f, \alpha) = 0$, (4) of Lemma 16 and (1) of this lemma imply (2). □

Corollary 18. *The functions* μ_*^w *and* μ_* *descend to the functions* μ^w *and* μ *on* $\mathcal{M}_2^w$ *and* $\mathcal{M}_2$*, respectively. For the functions* μ^w *and* μ*, the similar properties to Lemma* 16 *hold except for* (4) *for* μ*, but the restriction of* μ *to* $\mathcal{M}_2^H$ *satisfies the corresponding property to* (4) *of Lemma* 16*. Their restrictions to* $\mathcal{J}_2$ *are nontrivial homomorphisms.*

It is well known that sign defines a 2-cocycle on $\mathcal{M}_2$ over $\mathbb{Z}$, which is called the signature cocycle, and that it is a coboundary over $\mathbb{Q}$. By the fact that $H^1(\mathcal{M}_2, \mathbb{Q}) = 0$, there exists a unique function, which is called Meyer function,

$$\phi : \mathcal{M}_2 \to \mathbb{Q}$$

such that $\mathrm{sign}(a,b) = \phi(b) - \phi(ab) + \phi(a)$. It is known that ϕ satisfies the corresponding properties to Lemma 16 on $\mathcal{M}_2$ and its image is in $\frac{1}{5}\mathbb{Z}$ (see [8,9]).

Proposition 19. *On the Torelli group* $\mathcal{J}_2$*, the function* μ *agrees with the Meyer function* ϕ.

Proof. The Torelli group $\mathcal{J}_2$ is normally generated in $\mathcal{M}_2$ by the Dehn twist along a separating simple closed curve on Σ_2 (see [4]). Such a Dehn twist is given by the diffeomorphism f in the example in Section 5. The functions μ and ϕ are conjugacy invariant on $\mathcal{M}_2$ and are homomorphisms on $\mathcal{J}_2$. Thus in order to prove this proposition, we have only to show the equality $\mu(f) = \phi(f)$, but it is true from Corollary 3.7 in [8] and the example in Section 5. □

5. An example

In this section we shall give an example.

Let α_i, β_i $(i = 1, 2)$ be the generators for the fundamental group $\pi_1(\Sigma_2, *)$ of Σ_2 depicted in Fig. 2.

We use the same letters α_i, β_i for the corresponding basis for $H_1(\Sigma_2; \mathbb{Z}_2)$.

Take a class $w \in H^1(\Sigma_2; \mathbb{Z}_2)$ and fix it. We assume $w(\alpha_1) = 1$.

We can identify the space $Z^1 := Z^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w)$ of w-crossed homomorphisms from $\pi_1(\Sigma_2, *)$ to $\mathbb{Z}$ with the subset $\{(x_i, y_i)_{i=1}^2 \in \mathbb{Z}^4 \mid \sum_{i=1}^2 \{w(\beta_i)x_i - w(\alpha_i)y_i\} = 0\}$ of $\mathbb{Z}^4$ by the map $\iota : u \mapsto (x_1, y_1, x_2, y_2) := (u(\alpha_1), u(\beta_1), u(\alpha_2), u(\beta_2))$. We give elements E, F and W of Z^1 by $\iota E = (0, -w(\alpha_2), 0, 1)$, $\iota F = (0, w(\beta_2), 1, 0)$ and $\iota W = (w(\alpha_1), w(\beta_1), w(\alpha_2), w(\beta_2))$. It is easy to check that these elements form a basis for $Z^1 \cong \mathbb{Z}^3$ and $2W$ is a basis for the space $B^1 := B^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w) \cong 2\mathbb{Z}$ of the principal w-crossed homomorphisms. Thus we obtain

$$H^1\big(\pi_1(\Sigma_2, *), \mathbb{Z}_w\big) \cong \mathbb{Z}E \oplus \mathbb{Z}F \oplus \mathbb{Z}_2 W.$$

Let $f : \Sigma_2 \to \Sigma_2$ be the base point preserving diffeomorphism of Σ_2 which is the positive Dehn twist along the loop in Fig. 3, hence f belongs to $\mathcal{K}_{2*}$.

The action of f on the fundamental group of Σ_2 is given by

$$f_* : \pi_1(\Sigma_2, *) \to \pi_1(\Sigma_2, *),$$
$$\begin{cases} \alpha_1 \mapsto l\alpha_1 l^{-1} \\ \beta_1 \mapsto l\beta_1 l^{-1} \\ \alpha_2 \mapsto \alpha_2 \\ \beta_2 \mapsto \beta_2, \end{cases}$$

where $l = [\beta_1, \alpha_1]$.

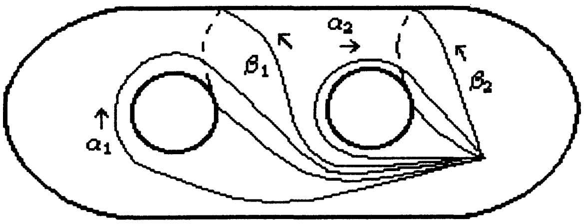

Fig. 2. Generators of $\pi_1(\Sigma_2, *)$.

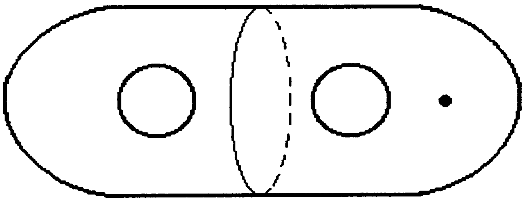

Fig. 3. The loop defining f.

Direct computation shows the following equalities:

$$\begin{aligned}
f^*E &= \big(1+4w(\alpha_2)w(\beta_2)\big)E + 4w(\alpha_2)^2F - 4w(\alpha_2)W,\\
f^*F &= -4w(\beta_2)^2E + \big(1-4w(\beta_2)w(\alpha_2)\big)F + 4w(\beta_2)W,\\
f^*W &= W.
\end{aligned}$$

Then the representation matrix of the homomorphism

$$f^*: H^1\big(\pi_1(\Sigma_2,*),\mathbb{Z}_w\big)/\mathit{torsion} \to H^1\big(\pi_1(\Sigma_2,*),\mathbb{Z}_w\big)/\mathit{torsion}$$

with respect to the basis E, F is given by

$$\begin{pmatrix} 1+4w(\alpha_2)w(\beta_2) & -4w(\beta_2)^2 \\ 4w(\alpha_2)^2 & 1-4w(\beta_2)w(\alpha_2) \end{pmatrix}.$$

For $w \in H^1(\Sigma_2,\mathbb{Z}_2)$ such that $(w(\alpha_2), w(\beta_2)) = (0,0), (0,1), (1,0), (1,1)$, the representation matrices of $(f^{-1})^*$, which are $\zeta_{2*}^w(f)$ by the definition, are obtained as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, \quad \begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix},$$

respectively. Here we note that, the basis E, F is oriented in the case of $(w(\alpha_2), w(\beta_2)) = (0,1), (1,0), (1,1)$ and is not in the case of $(w(\alpha_2), w(\beta_2)) = (0,0)$. But in the latter case, since the representation matrix of $(f^{-1})^*$ is identity, the one with respect to an oriented basis is also identity. Thus we have eight matrices in all in the case of $w(\alpha_1) = 1$.

Similarly in the case of $w(\alpha_1) = 0$, under appropriate choices of oriented bases, we obtain the same representation matrices of $(f^{-1})^*$ as above, but the number of the matrices are 4, 1, 1 and 1, respectively. Hence we obtain seven matrices in all.

Since we have

$$\psi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0, \quad \psi\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \psi\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} = \psi\begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix} = \frac{1}{3},$$

we obtain

$$\Psi_*^w(f) = \begin{cases} 0 & \text{if } \zeta_{2*}^w(f) = \mathrm{id}, \\ \frac{1}{3} & \text{if not.} \end{cases}$$

Next we shall compute $\mu_*^w(f)$.

Let w be a class of $H^1(\Sigma_2;\mathbb{Z}_2)$ satisfying $(w(\alpha_1), w(\alpha_2), w(\beta_2)) = (1,0,1)$. Then we have the representation matrix

$$(f^{-n})^* = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 4n \\ 0 & 1 \end{pmatrix},$$

with respect to the above basis. Hence we obtain

$$\Psi_*^w(f^n) = \tfrac{1}{3}\phi\begin{pmatrix} 1 & 4n \\ 0 & 1 \end{pmatrix} - \nu\begin{pmatrix} 1 & 4n \\ 0 & 1 \end{pmatrix} = \tfrac{4}{3}n - \operatorname{sgn}(n).$$

Lemma 20. *For any $w \in H^1(\Sigma_2;\mathbb{Z}_2)$, the set $\{\rho_{w(M_{f^n})}(M_{f^n}) \mid n \in \mathbb{Z}\}$ is bounded.*

Proof. We only prove the case of the class w given by $(w(\alpha_1), w(\beta_1), w(\alpha_2), w(\beta_2)) = (1, 0, 1, 0)$, since the other cases are shown in the same way.

Let H_2 be a handlebody of genus 2 with boundary Σ_2 such that the loop in Fig. 3 is a boundary of an embedded disk. The diffeomorphisms f^n $(n \in \mathbb{Z})$ of $\Sigma_2 = \partial H_2$ extend to diffeomorphisms of H_2. Let

$$H_{2,f^n} = H_2 \times [0,1]/H_2 \times 1 \overset{f^{-n}}{\sim} H_2 \times 0$$

be the mapping torus, then it is a compact 4-manifold with a boundary $\partial H_{2,f^n} = M_{f^n}$. Clearly there exists a class $w(n) \in H^1(H_{2,f^n}; \mathbb{Z}_2)$ whose restriction to the boundary M_{f^n} is $w_{M_{f^n}}$. In order to prove this lemma, by Theorem 14 and the definition of signature, we have only to show that the set $\{\dim_{\mathbb{Q}} H^2(H_{2,f^n}; \mathbb{Q}), \dim_{\mathbb{C}} H^2(H_{2,f^n}; \mathbb{C}_{w(n)})\}$ is bounded. But it is true since the manifolds H_{2,f^n} are homotopic to each other, so the proof of this lemma is finished. □

Since f^n belongs to $\mathcal{J}_{2*}$ and μ_*^w is a homomorphism on $\mathcal{J}_{2*}$, we have

$$\mu_*^w(f^n) = n\mu_*^w(f) = n\big(\rho_{w_{M_f}}(M_f) + \Psi_*^w(f)\big) = n\big(\rho_{w_{M_f}}(M_f) + \tfrac{1}{3}\big).$$

On the other hand, we have

$$\mu_*^w(f^n) = \rho_{w_{M_{f^n}}}(M_{f^n}) + \Psi_*^w(f^n) = \rho_{w_{M_{f^n}}}(M_{f^n}) + \tfrac{4}{3}n - \operatorname{sgn}(n).$$

We consider $n \to \infty$. By Lemma 20, we obtain $\rho_{w_{M_f}}(M_f) = 1$, hence $\mu_*^w(f^n) = \frac{4}{3}n$ and $\rho_{w_{M_{f^n}}}(M_{f^n}) = \operatorname{sgn}(n)$.

Similarly, for any $w \in H^1(\Sigma_2; \mathbb{Z}_2)$ such that the representation matrix of $(f^{-1})^*$ is not the identity, we obtain $\mu_*^w(f^n) = \frac{4}{3}n$. As a result, we have $\mu_*^w(f^n) = \frac{4}{3}n$ for the nine of the fifteen nonzero classes of $H^1(\Sigma_2; \mathbb{Z}_2)$ and $\mu_*^w(f^n) = 0$ for the other classes. Finally we obtain

$$\mu_*(f^n) = \frac{1}{15} \sum_{\substack{w \in H^1(\Sigma_2;\mathbb{Z}_2)\setminus\{0\} \\ (f^n)^* w = w}} \mu_*^w(f^n) = \frac{1}{15} \cdot 9 \cdot \frac{4}{3}n = \frac{4}{5}n.$$

By Corollary 18, the functions μ^w and μ take the same values at f^n as μ_*^w and μ_* do, respectively.

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