Appendix: Symplectic geometry

Summary

We collect here the basic tools of symplectic geometry which are used throughout the book. In §1 and §2 we discuss some basic notions concerning symplectic vector spaces and homogeneous symplectic manifolds. All results are well-known and more or less elementary. Hence we shall omit some proofs, and we refer to Arnold [2], Duistermaat [1], Abraham-Mardsen [1] and especially Hörmander [4 Chapter XXI].

In §3 we introduce the inertia index of a triplet of Lagrangian planes. Our presentation is close to that of Lion-Vergne [1] and for the reader's convenience, we give all proofs. We also collect some of its properties, that we need in Chapter 7, as exercises.

A.1. Symplectic vector spaces

In this section and the next one we shall treat symplectic spaces in the real case but the theory is the same in the complex case.

Let E be a real finite-dimensional vector space. A **symplectic form** σ on E is a non-degenerate alternate bilinear form on E. A vector space E endowed with a symplectic form σ is called a **symplectic vector space**. If E is symplectic, then $\dim E$ is even.

If (E_1, σ_1) and (E_2, σ_2) are two symplectic vector spaces, a linear map $u : E_1 \to E_2$ is called symplectic if $u^*\sigma_2 = \sigma_1$. If u is symplectic, then u is necessarily injective.

One denotes by $Sp(E)$ the group of symplectic automorphisms of a symplectic vector space E. It is a closed subgroup of the group $\mathrm{GL}(E)$ of linear automorphisms of E.

Example A.1.1. Let V be a real finite-dimensional vector space, and V^* its dual space. The space $E = V \oplus V^*$ is naturally endowed with a symplectic structure by setting for $(x;\xi)$ and $(x';\xi')$ in $V \oplus V^*$:

$$\sigma((x;\xi),(x';\xi')) = \langle x', \xi\rangle - \langle x, \xi'\rangle \ . \tag{A.1.1}$$

We call this form the natural symplectic form on E.

If $E = V \oplus V^*$ and u is a linear isomorphism of V, then the map $\begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix}$ is a symplectic automorphism of E.

Let (E, σ) be a symplectic vector space. Since σ is non-degenerate, it defines a linear isomorphism H from E^* to E by the formula:

$$\langle \theta, v \rangle = \sigma(v, H(\theta)) \ , \qquad v \in E, \quad \theta \in E^* \ . \tag{A.1.2}$$

This isomorphism H is called the **Hamiltonian isomorphism**. If $\theta \in E^*$ one sometimes writes H_θ instead of $H(\theta)$ and calls H_θ the Hamiltonian vector of θ.

Since H is an isomorphism, the skew bilinear form on $E^* : (u, v) \mapsto \sigma(H_u, H_v)$ is a symplectic form on E^*. It is called the **Poisson bracket**, and denoted $\{u, v\}$. Thus:

$$\{u, v\} = \sigma(H_u, H_v) = \langle v, H_u \rangle \ . \tag{A.1.3}$$

Let ρ be a linear subspace of E. One sets:

$$\rho^\perp = \{x \in E; \sigma(x, \rho) = 0\} \ . \tag{A.1.4}$$

Then:

$$\rho^{\perp\perp} = \rho \ , \qquad (\rho_1 + \rho_2)^\perp = \rho_1^\perp \cap \rho_2^\perp \ , \qquad (\rho_1 \cap \rho_2)^\perp = \rho_1^\perp + \rho_2^\perp \ .$$

The space $\rho^\perp$ is called the orthogonal space to ρ.

Definition A.1.2. *A linear subspace ρ of E is called isotropic (resp. Lagrangian, resp. involutive) if $\rho \subset \rho^\perp$ (resp. $\rho = \rho^\perp$, resp. $\rho \supset \rho^\perp$).*

Some authors use "co-isotropic" instead of "involutive".

Note that if ρ is isotropic (resp. Lagrangian, resp. involutive) then $\dim \rho \leqslant n$ (resp. $= n$, resp. $\geqslant n$) where $n = \frac{1}{2} \dim E$.

A line (resp. a hyperplane) is always isotropic (resp. involutive). If $\dim \rho = n$ and if ρ is isotropic or else involutive, then ρ is Lagrangian.

Assume ρ is isotropic. Then the space $\rho^\perp/\rho$ is naturally endowed with a symplectic structure by setting $\sigma(\dot{x}, \dot{y}) = \sigma(x, y)$, where $\dot{x}$ (resp. $\dot{y}$) is the image of x (resp. y) in $\rho^\perp/\rho$. (We still denote by σ the symplectic form on $\rho^\perp/\rho$.)

If λ is a linear subspace of E one sets:

$$\lambda^\rho = ((\lambda \cap \rho^\perp) + \rho)/\rho \ . \tag{A.1.5}$$

In particular, $E^\rho = \rho^\perp/\rho$. Then it is easily checked that:

$$(\lambda^\perp)^\rho = (\lambda^\rho)^\perp \ . \tag{A.1.6}$$

In particular, if λ is Lagrangian in E, then λ^ρ is Lagrangian in E^ρ.

Conversely, denote by i the embedding $\rho^\perp \hookrightarrow E$ and by j the projection $\rho^\perp \to \rho^\perp/\rho$. Then if μ is a Lagrangian subspace of $\rho^\perp/\rho$, $ij^{-1}(\mu)$ is a Lagrangian subspace of E, containing ρ.

The symplectic space described in Example A.1.1 is not so special. In fact, we have:

Proposition A.1.3. *Let λ_0 be a Lagrangian subspace of E. Then there exists a Lagrangian subspace λ_1 of E such that $E = \lambda_0 \oplus \lambda_1$. Moreover for such a Lagrangian space λ_1, the map $u : \lambda_0 \oplus \lambda_0^* \to E$ defined by $u(x, y) = x - H(y)$ is a symplectic isomorphism. Here $H(y)$ is given by $\lambda_0^* \simeq (E/\lambda_1)^* \to E^* \underset{H}{\to} E$.*

Proof. (i) Let ρ be an isotropic space with $\rho \cap \lambda_0 = \{0\}$. If $\rho \neq \rho^\perp$, then $\rho^\perp \not\subset \lambda_0 + \rho$, otherwise $\rho \supset (\rho^\perp \cap \lambda_0)$, hence $\rho^\perp \cap \lambda_0 = \{0\}$, which contradicts $\dim \rho^\perp > n$. Choose $e \in \rho^\perp \backslash (\lambda_0 + \rho)$. Then $\rho + \mathbb{R}e$ is isotropic and $(\rho + \mathbb{R}e) \cap \lambda_0 = \{0\}$. Then we argue by induction on $\dim \rho$ to get λ_1.

(ii) Let x and x' belong to λ_0 and y and y' to λ_0^*. Then:

$$\sigma(x - H(y), x' - H(y')) = -\sigma(x, H(y')) + \sigma(x', H(y)) = -\langle y', x\rangle + \langle y, x'\rangle \; .$$

Hence the map u is symplectic. Since $\dim(\lambda_0 \oplus \lambda_0^*) = \dim E$, u is an isomorphism. □

Let (E, σ) be a symplectic vector space. We denote by E^a the space E endowed with the symplectic form $-\sigma$, i.e. $E^a = (E, -\sigma)$.

For two symplectic vector spaces (E_1, σ_1) and (E_2, σ_2), $E_1 \oplus E_2$ has also a structure of a symplectic space by $\sigma_1 \oplus \sigma_2$.

Let E_i, $(i = 1, 2, 3)$ be three symplectic vector spaces, and denote by p_{ij} the (i, j)-th projection defined on $E_1 \times E_2 \times E_3$ (e.g. p_{13} is the projection onto $E_1 \times E_3$).

Proposition A.1.4. *Let λ and μ be two Lagrangian subspaces of $E_1 \oplus E_2^a$ and $E_2 \oplus E_3^a$, respectively. Set $\lambda \circ \mu = p_{13}(p_{12}^{-1}\lambda \cap p_{23}^{-1}\mu)$. Then $\lambda \circ \mu$ is a Lagrangian subspace of $E_1 \oplus E_3^a$.*

Proof. The diagonal Δ of $E_2^a \oplus E_2$ is a Lagrangian subspace. Hence $\rho = \{0\} \times \Delta \times \{0\}$ is an isotropic subspace of $E_1 \oplus E_2^a \oplus E_2 \oplus E_3$. Since $E_1 \oplus E_3 = (E_1 \oplus E_2^a \oplus E_2 \oplus E_3)^\rho$ and $\lambda \circ \mu = (\lambda \oplus \mu)^\rho$, we get the result. □

Now we shall study symplectic bases. Let (E, σ) be a symplectic vector space, say of dimension $2n$.

A basis $(e_1, \ldots, e_n; f_1, \ldots, f_n)$ is called symplectic if denoting $(e_1^*, \ldots, e_n^*; f_1^*, \ldots, f_n^*)$ the dual basis on E^*, we have:

$$\sigma = \sum_{j=1}^{n} f_j^* \wedge e_j^* \; . \tag{A.1.7}$$

Of course, (A.1.7) is equivalent to:

(A.1.8) $$\begin{cases} \sigma(e_j, e_k) = \sigma(f_j, f_k) = 0 \ , \\ \sigma(e_j, f_k) = -\sigma(f_k, e_j) = -\delta_{j,k} \qquad 1 \leqslant j, k \leqslant n \ , \end{cases}$$

where δ_{jk} denotes the Kronecker symbol ($\delta_{jk} = 1$ if $j = k$, and is 0 otherwise).

For such a symplectic basis, the symplectic isomorphism H defined in (A.1.2) satisfies:

(A.1.9) $$H(e_j^*) = -f_j \ , \qquad H(f_j^*) = e_j \ , \qquad 1 \leqslant j \leqslant n \ .$$

Let J and K be two subsets of $\{1, \dots, n\}$, and let $((e_j)_{j \in J}, (f_k)_{k \in K})$ be a linearly independent family satisfying the relations (A.1.8). Then it is easily proved that this family can be completed into a symplectic basis. In particular, if ρ is an isotropic (resp. Lagrangian, resp. involutive) subspace, then there exists a symplectic basis such that ρ is generated by $(e_1, \dots, e_j)$ (resp. $(e_1, \dots, e_n)$, resp. $(e_1, \dots, e_n, f_1, \dots, f_k)$) for some j and k.

Denote by $G(E, n)$ the Grassmannian manifold of n-dimensional linear subspaces of E (recall that $\dim E = 2n$). This is a compact manifold (cf. e.g. Griffith-Harris [1]). One denotes by $\Lambda(E)$ the subset of $G(E, n)$ consisting of Lagrangian subspaces. This is a closed (smooth) submanifold, called the **Lagrangian Grassmannian manifold**.

Let $\mu \in \Lambda(E)$. We set:

(A.1.10) $$\Lambda_\mu(E) = \{\lambda \in \Lambda(E); \lambda \cap \mu = \{0\}\}.$$

Assume E is endowed with a symplectic basis, and let $(x; \xi)$ denote the associated linear coordinates, (i.e. each $p \in E$ is written $p = \sum_{j=1}^n (x_j e_j + \xi_j f_j)$). For any $\lambda \in G(E, n)$, there exist $n \times n$ matrices A and B such that:

(A.1.11) the matrix (A, B) has rank n , $\quad \lambda = \{(x; \xi); B\xi = Ax\}$.

Then $\lambda \in \Lambda(E)$ if and only if A^tB is symmetric.

Note that in this case $B\xi = Ax$ iff $x = {}^tBz$, $\xi = {}^tAz$ for some $z \in \mathbb{R}^n$.

If μ is the Lagrangian subspace $\{x = 0\}$, then $\lambda \in \Lambda_\mu(E)$ if $\lambda = \{(x; \xi); \xi = Ax\}$ for some symmetric matrix A. Hence $\Lambda_\mu(E)$ is open and dense in $\Lambda(E)$ and isomorphic to $\mathbb{R}^{n(n+1)/2}$.

We shall sometimes say that a property "P" holds for **generic** λ (λ in $\Lambda(E)$) if there exists an open dense subset Ω of $\Lambda(E)$ such that the property "P" holds for $\lambda \in \Omega$.

We shall also encounter the following situation: λ is Lagrangian in E and contains a line ρ. We are looking for $\mu \in \Lambda(E)$, with $\mu \cap \lambda = \rho$. Consider the maps $i: \rho^\perp \to E$ and $j: \rho^\perp \to \rho^\perp/\rho$. Then it is enough to choose $\mu' \in \Lambda(E^\rho)$ with $\mu' \cap \lambda^\rho = 0$, and set $\mu = i(j^{-1}(\mu'))$. By abuse of language, we shall say that for generic μ with $\mu \supset \rho$, we have $\lambda \cap \mu = \rho$.

A.2. Homogeneous symplectic manifolds

All manifolds and morphisms of manifolds considered here will be real, of class C^∞ or real analytic. Unless otherwise specified, a real function on a manifold is supposed to be of class C^∞ or real analytic. However, most of the results we shall state still hold with suitable modifications for complex analytic manifolds.

Let X be a manifold. We denote by $\tau: TX \to X$ its tangent bundle and by $\pi: T^*X \to X$ its cotangent bundle. We denote by $\dot{T}X$ and $\dot{T}^*X$ the bundles TX and T^*X, with the zero-section removed, and we denote by $\dot{\tau}$ and $\dot{\pi}$ the maps τ and π restricted to $\dot{T}X$ and $\dot{T}^*X$, respectively. Let us recall that if M is a submanifold of X, the normal bundle T_MX and the conormal bundle T_M^*X to M in X are defined by the exact sequences of vector bundles on M:

$$\text{(A.2.1)} \qquad \begin{cases} 0 \to TM \to M \underset{X}{\times} TX \to T_MX \to 0 \ , \\ 0 \to T_M^*X \to M \underset{X}{\times} T^*X \to T^*M \to 0 \ . \end{cases}$$

Let $f: Y \to X$ be a morphism of manifolds. To f are associated the morphisms:

$$\text{(A.2.2)} \qquad \begin{cases} TY \xrightarrow[f']{} Y \underset{X}{\times} TX \xrightarrow[f_\tau]{} TX \ , \\ T^*Y \xleftarrow[{}^tf']{} Y \underset{X}{\times} T^*X \xrightarrow[f_\pi]{} T^*X \ . \end{cases}$$

In particular if one considers the projection $\pi: T^*X \to X$ we get the map ${}^t\pi'$: $T^*X \times_X T^*X \to T^*T^*X$.

If we restrict this map to the diagonal of $T^*X \times_X T^*X$, we get a map $T^*X \to T^*T^*X$, which is a section of the bundle $T^*T^*X \to T^*X$, that is, a differential form of degree 1 (one says: a 1-form). This 1-form on T^*X is called the **canonical 1-form** and denoted by α_X, or simply α if there is no risk of confusion.

Let $(x_1, \ldots, x_n)$ be a system of local coordinates on X. Then the x_j's are real functions defined on some open subset U, satisfying $dx_1 \wedge \cdots \wedge dx_n \neq 0$ on U. At each $x \in U$, $(dx_1, \ldots, dx_n)$ defines a basis of the vector space T_x^*X, and a vector $\xi \in T_x^*X$ is uniquely written as $\xi = \sum_{j=1}^n \xi_j dx_j$. The system $(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$ is called the **coordinate system** on T^*X **associated** to the coordinate system $(x_1, \ldots, x_n)$. It is easily checked that the canonical 1-form α_X is nothing but the form $\sum_{j=1}^n \xi_j dx_j$. Let $\sigma = d\alpha$. Hence $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$ is a symplectic form on T^*X (i.e.: at each $p \in T^*X$, σ induces a symplectic structure on the vector space T_pT^*X). In other word the manifold T^*X is naturally endowed with a symplectic structure, by $d\alpha$. We can then extend to T^*X some of the notions introduced in §1.

A submanifold V of T^*X is called isotropic (resp. Lagrangian, resp. involutive) if at each $p \in V$, the tangent space T_pV has the corresponding property in T_pT^*X. If f is a real function defined on some open subset U of T^*X, the

Hamiltonian vector field H_f of f is the vector field on U, the image of $\mathrm{d}f$ by the Hamiltonian isomorphism $H : T^*T^*X \simeq TT^*X$.

The **Poisson bracket** of two functions f and g is defined by:

(A.2.3) $$\{f,g\} = H_f(g) = \mathrm{d}\alpha(H_f, H_g) \ .$$

One checks the relations:

(A.2.4) $$\begin{cases} \{f,g\} = -\{g,f\} \ , \\ \{f,hg\} = h\{f,g\} + g\{f,h\} \ , \\ \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \ . \end{cases}$$

In particular $[H_f, H_g] = H_{\{f,g\}}$ where $[u,v] = uv - vu$ is the commutator of the vector fields u and v.

If $(x_1, \ldots, x_n)$ is a system of local coordinates on X, $(x;\xi)$ the associated coordinates on T^*X and f is a real function on $U \subset T^*X$, we have:

(A.2.5) $$H_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \right) .$$

A submanifold V of T^*X is involutive iff the Poisson bracket $\{f,g\}$ vanishes on V for any functions f and g which vanish on V. In fact the vector bundle $(TV)^\perp$ is generated by the vector fields H_f, with $f|_V = 0$. Thus $(TV)^\perp \subset TV$ is equivalent to $H_f(g) = 0$ for any f, g with $f|_V = 0, g|_V = 0$. Moreover by (A.2.4), we find that if V is involutive, the sub-bundle $(TV)^\perp$ of TV satisfies the Frobenius integrability conditions (i.e.: the sheaf of sections of $(TV)^\perp$ is closed under bracket $[\cdot,\cdot]$). By the Frobenius theorem (cf. Hörmander [4, Appendix C]), an involutive manifold V admits a foliation, and the leaves of this foliation are called the **bicharacteristic leaves** of V. Note that the dimension of the leaves is the codimension of V. In particular if V is Lagrangian, the leaves are open in V.

Example A.2.1. Let Z be a submanifold of X. The manifold $Z \times_X T^*X$ is involutive and the manifold T_Z^*X is Lagrangian. Notice the extreme case where $Z = X$: we get the Lagrangian manifold T_X^*X, the zero-section of T^*X.

The 1-form α on T^*X induces a richer structure than merely that of a symplectic manifold, and we shall describe this **homogeneous symplectic structure** on T^*X.

Let $H(\alpha)$ be the image of α by the Hamiltonian isomorphism. If we have chosen coordinates $(x;\xi)$ as above, then:

(A.2.6) $$\alpha = \sum_j \xi_j \mathrm{d}x_j \ , \qquad H(\alpha) = -\sum_j \xi_j \frac{\partial}{\partial \xi_j} \ .$$

Thus $-H(\alpha)$ is just the radial vector field on the vector bundle T^*X (i.e. the infinitesimal generator of the action of $\mathbb{R}^+$ on T^*X). This vector field is also called the **Euler vector field**.

We say that a subset S of T^*X is **conic** (resp. **locally conic**) if it is invariant (resp. locally invariant) by the action of $\mathbb{R}^+$. Hence S is locally conic if its intersection with any orbit of $\mathbb{R}^+$ is open in this orbit. A function f defined on an open subset U of T^*X is said to be homogeneous if f satisfies the differential equation $H(\alpha)f = kf$ for some $k \in \mathbb{C}$. Note that a submanifold V is locally conic iff $H(\alpha)$ is tangent to V or equivalently iff V is locally defined by homogeneous equations.

A locally conic submanifold V is isotropic iff $\alpha|_V \equiv 0$, since $\langle \alpha, v \rangle = d\alpha(v, H(\alpha))$, for $v \in TT^*X$.

One says that a locally conic involutive submanifold V is **regular** if $\alpha|_V$ is everywhere different from zero. This is equivalent to the local existence of homogeneous functions $f_1, \dots, f_r$ vanishing on V, with $r = \operatorname{codim} V$, such that:

$$\text{(A.2.7)} \qquad \begin{cases} \{f_i, f_j\} = 0 & \text{on } V \text{ for any } i, j \in \{1, \dots, r\} , \\ df_1 \wedge \cdots \wedge df_r \wedge \alpha \neq 0 & \text{on } V . \end{cases}$$

This is again equivalent to saying that the Euler vector field is not tangent to any bicharacteristic leaves at any point.

Example A.2.2. Let Z be a submanifold of X. Then $Z \times_X T^*X$ is regular involutive outside of $T^*_Z X$.

Convention A.2.3. In this Appendix, unless otherwise specified, all submanifolds of T^*X are locally conic.

Let $\rho(p)$ denote the linear subspace of $T_p T^*X$ generated by the Euler vector field at p. If $p \in T^*_X X$, $\rho(p) = \{0\}$, otherwise $\rho(p)$ is a line.

If V is a (locally conic) submanifold, at each $p \in V$, $T_p V$ contains $\rho(p)$. Let $p \in T^*X$. One sets:

$$\text{(A.2.8)} \qquad \lambda_0(p) = T_p \pi^{-1} \pi(p) .$$

This is a Lagrangian linear subspace of $T_p T^*X$.

Let Λ be a Lagrangian submanifold. The corank of the projection $\pi|_\Lambda : \Lambda \to X$ is, by definition, the dimension of the space $T_p\Lambda \cap \lambda_0(p)$. On $\dot{T}^*X$, this corank is at least one since both $T_p\Lambda$ and $\lambda_0(p)$ contain $\rho(p)$. If this corank is constant, say d, then locally on Λ, $\pi(\Lambda)$ is a smooth submanifold M of X of codimension d, and $\Lambda = T^*_M X$. In particular if this corank is one at some p, then Λ is the conormal bundle to a hypersurface in a neighborhood of p.

Now let X and Y be two manifolds of the same dimension, U_X (resp. U_Y) an open subset of T^*X (resp. T^*Y). Let χ be a diffeomorphism from U_X onto U_Y. If $\chi^*(d\alpha_Y) = d\alpha_X$, one says that χ is a symplectic isomorphism. If moreover χ is homogeneous (i.e.: χ commutes with the action of $\mathbb{R}^+$), then $\chi^*(\alpha_Y) = \alpha_X$ and we shall say that χ is a **contact transformation**, although this is not really correct, since a contact structure is the structure obtained on the quotient space $\dot{T}^*X/\mathbb{R}^+$.

Let χ be a homogeneous diffeomorphism $U_X \xrightarrow{\sim} U_Y$, Λ_χ its graph in $U_X \times U_Y$. The inverse image $\chi^*(\beta)$ of a 1-form β on U_Y is characterized by the condition

$(\chi^*(\beta) - \beta)|_{\Lambda_\chi} \equiv 0$. Let Λ_χ^a denote the image of Λ_χ by the antipodal map on T^*Y. Then $\chi^*(\alpha_Y) = \alpha_X$ iff $(\alpha_X + \alpha_Y)|_{\Lambda_\chi^a} \equiv 0$, that is, iff Λ_χ^a is isotropic, hence iff it is Lagrangian. In other words, χ is a contact transformation iff Λ_χ^a is a (locally conic) Lagrangian submanifold of $T^*(X \times Y)$.

We call Λ_χ^a the Lagrangian manifold associated to the graph of the contact transformation χ.

Let (y) be a system of local coordinates on Y, and let $(y;\eta)$ denote the associated coordinates on T^*Y. Then a homogeneous map $\chi : U_X \to U_Y$ is defined by two sets of functions, f_j homogeneous of degree 0, g_k homogeneous of degree 1, $(1 \leqslant j, k \leqslant n)$, with $y_j = f_j$, $\eta_k = g_k$. The map χ is a contact transformation iff Λ_χ^a is involutive, that is, iff:

$$\text{(A.2.9)} \qquad \{f_j, f_k\} = 0 \ , \qquad \{g_j, g_k\} = 0 \ , \qquad \{f_j, g_k\} = -\delta_{j,k} \ .$$

Example A.2.4. Let $(x;\xi)$ denote the coordinates on $T^*\mathbb{R}^n$, and let $\varphi(\xi)$ be a function homogeneous of degree one defined on some open subset U of $(\mathbb{R}^n)^*$ (e.g.: $\varphi(\xi) = (\sum_j \xi_j^2)^{1/2}$ on $\mathbb{R}^n \setminus \{0\}$). Then the map $\chi : (x;\xi) \mapsto (x + \varphi'(\xi); \xi)$ is a contact transformation.

The next result is a useful tool in order to construct contact transformations.

Proposition A.2.5. *Let V be a regular involutive submanifold of T^*X, $p \in V \cap \dot{T}^*X$. Let λ be a Lagrangian linear subspace of T_pT^*X such that $\rho(p) \subset \lambda \subset T_pV$. (Recall that $\rho(p)$ is the line generated by the Euler vector field.) Then there exists a Lagrangian manifold $\Lambda \subset T^*X$ such that $\Lambda \subset V$ and $T_p\Lambda = \lambda$.*

Proof. Let $n = \dim X$, $r = \operatorname{codim} V$. Let $(f_1, \ldots, f_r)$ be a system of homogeneous functions vanishing on V and satisfying (A.2.7). If $r = n - 1$ we set $e = \alpha(p)$. Otherwise we choose $v \in \lambda^\perp \setminus (T_pV + \mathbb{R}H(\alpha))$ and set $e = H^{-1}(v)|_V$. By the classical theory of differential equations, we may find a function g on V such that:

$$\begin{cases} H(\alpha)|_V(g) = 0 \ , & H_{f_j}|_V(g) = 0 \ , \qquad (j \leqslant r) \\ dg(p) = e \ , & g(p) = 0 \ , \end{cases}$$

because e is not tangent to the bicharacteristic leaf passing through p.

Set $V_1 = \{q \in V; g(q) = 0\}$. Then V_1 is a conic manifold which satisfies the required properties if $r = n - 1$ and otherwise V_1 is regular involutive. In this case we argue by induction on r. □

Let X, Y, Z be three manifolds. One denotes by p_1 and p_2 the first and second projection defined on $T^*X \times T^*Y$ or else on $T^*Y \times T^*Z$, and by p_{ij} the (i,j)-th projection defined on $T^*X \times T^*Y \times T^*Z$. We set $p_2^a = a \circ p_2$, where "a" is the antipodal map. Let $\Lambda_1 \subset T^*(X \times Y)$, $\Lambda_2 \subset T^*(Y \times Z)$ be two Lagrangian manifolds. Let $(p_X, p_Y^a) \in \Lambda_1$, $(p_Y, p_Z^a) \in \Lambda_2$, and assume:

$$\text{(A.2.10)} \qquad \begin{cases} \text{the maps } p_2^a|_{\Lambda_1} : \Lambda_1 \to T^*Y \text{ and } p_1|_{\Lambda_2} : \Lambda_2 \to T^*Y \\ \text{are transversal at } p_Y \ . \end{cases}$$

Then replacing Λ_1 and Λ_2 by $\Lambda_1 \cap U$ and $\Lambda_2 \cap V$, where U and V are sufficiently small open neighborhoods of (p_X, p_Y^a) and (p_Y, p_Z^a) respectively, the map p_{13} induces an isomorphism of $\Lambda_1 \times_{T^*Y} \Lambda_2$ with a Lagrangian manifold Λ of $T^*(X \times Z)$, and one sets:

(A.2.11) $$\Lambda = \Lambda_1 \circ \Lambda_2 \ .$$

(Cf. Lemma 7.4.4 and Definition 7.4.5.)

Proposition A.2.6. *Let Λ_1 be a Lagrangian submanifold of $T^*(X \times Y)$, $(p_X, p_Y^a) \in \Lambda_1$ with $p_Y \notin T_Y^*Y$. Assume that the map $p_1|_{\Lambda_1} : \Lambda_1 \to T^*X$ is smooth. Then there exists a manifold Z of the same dimension as Y, and a Lagrangian manifold $\Lambda_2 \subset T^*(Y \times Z)$ defined in a neighborhood of (p_Y, p_Z^a), such that:*

(i) *$\Lambda_2 = T_S^*(Y \times Z)$, where S is a hypersurface of $Y \times Z$,*
(ii) *Λ_2 is associated to the graph of a contact transformation (i.e.: $p_1|_{\Lambda_2}$ and $p_2^a|_{\Lambda_2}$ are local isomorphisms),*
(iii) *$\Lambda_1 \circ \Lambda_2 = T_{S'}^*(X \times Z)$, where S' is a hypersurface of $X \times Z$.*

Proof. Set $E_X = T_{p_X} T^*X$, $E_Y = T_{p_Y} T^*Y$ and let E_Y^a denote the space E_Y endowed with the opposite symplectic structure. Then (p_1, p_2^a) defines a symplectic isomorphism $T_{(p_X, p_Y^a)} T^*(X \times Y) \simeq E_X \times E_Y^a$, and we shall identify a Lagrangian space in $T_{(p_X, p_Y^a)} T^*(X \times Y)$ and its image in $E_X \times E_Y^a$.

Let ρ_X denote the linear space generated by the Euler vector field at p_X in E_X and define similarly ρ_Y in E_Y, ρ_{XY} in $E_X \times E_Y^a$, ρ_{YY} in $E_Y \times E_Y^a$. Set:

$$\lambda_1 = T_{(p_X, p_Y^a)} \Lambda_1 \ , \qquad \lambda_{0X} = T_{p_X} \pi^{-1}\pi(p_X) \ , \qquad \lambda_{0Y} = T_{p_Y} \pi^{-1}\pi(p_Y) \ ,$$

and identify λ_1 and λ_{0X} to Lagrangian subspaces of $E_X \times E_Y^a$ and E_X, respectively.

By the hypothesis that $p_1|_{\lambda_1} : \lambda_1 \to E_X$ is surjective, we get that $p_2|_{\lambda_1} : \lambda_1 \to E_Y$ is injective (cf. Exercise A.4). Moreover $p_2^{-1}(\rho_Y) \cap \lambda_1 = \rho_{XY}$. Since $p_2(p_1^{-1}(\lambda_{0X}) \cap \lambda_1)$ is Lagrangian in E_Y (Proposition A.1.4), we have for a generic Lagrangian space $\lambda \subset E_Y$, with $\rho_Y \subset \lambda$:

$$\lambda \cap p_2(p_1^{-1}(\lambda_{0X}) \cap \lambda_1) = \rho_Y \ .$$

This implies $\rho_{XY} = p_2^{-1}(\lambda) \cap p_1^{-1}(\lambda_{0X}) \cap \lambda_1$, thus:

(A.2.12) $\rho_{XY} = (\lambda_{0X} \times \lambda) \cap \lambda_1$, for a generic Lagrangian space $\lambda \subset E_Y^a$ such that $\rho_Y \subset \lambda$.

Then for a generic Lagrangian space $\mu \subset E_Y \times E_Y^a$ with $\rho_{YY} \subset \mu$, we have:

(A.2.13) $p_1|_\mu : \mu \to E_Y$ and $p_2|_\mu : \mu \to E_Y^a$ are isomorphisms ,

(A.2.14) $$(\lambda_{0Y} \times \lambda_{0Y}) \cap \mu = \rho_{YY} \ .$$

Since $\lambda = p_1(\mu \cap p_2^{-1}(\lambda_{0Y}))$ is generic we may assume further λ satisfies the condition (A.2.12). Then we have:

(A.2.15) $$(\lambda_1 \circ \mu) \cap (\lambda_{0X} \times \lambda_{0Y}) = \rho_{XY} .$$

Now take Y, p_Y and λ_{0Y} as Z, p_Z and λ_{0Z} respectively. By Proposition A.2.5 we may find a Lagrangian manifold $\Lambda_2 \subset T^*(Y \times Z)$ such that $T_{(p_Y, p_Z^a)}\Lambda_2 = \mu$. Then Λ_2 will satisfy all the required conditions, by (A.2.13), (A.2.14) and (A.2.15). □

Corollary A.2.7. *Let $\Lambda \subset \dot{T}^*X$ be a Lagrangian manifold, $p \in \Lambda$. Then there exists a contact transformation χ defined in a neighborhood of p such that $\chi(\Lambda)$ is the conormal bundle to a hypersurface, and moreover the Lagrangian manifold associated to the graph of χ is the conormal bundle to a hypersurface.*

Proof. This is nothing but Proposition A.2.5 when $X = \{\mathrm{pt}\}$. □

Corollary A.2.8. *Let $\Lambda \subset \dot{T}^*(X \times Y)$ be a Lagrangian manifold associated with a contact transformation (i.e.: $p_1|_\Lambda$ and $p_2^a|_\Lambda$ are local isomorphisms). Then, locally on Λ, there exists a manifold Z of the same dimension as X and two Lagrangian manifolds $\Lambda_1 \subset T^*(X \times Z)$, $\Lambda_2 \subset T^*(Z \times Y)$ such that Λ_1 and Λ_2 are associated with contact transformations, Λ_1 and Λ_2 are the conormal bundles to hypersurfaces of $X \times Z$ and $Z \times Y$ respectively, and $\Lambda = \Lambda_1 \circ \Lambda_2$.*

Proof. By Proposition A.2.6 there exists a contact transformation χ such that if Λ_2 is the Lagrangian manifold associated to the graph of χ, then Λ_2 and $\Lambda \circ \Lambda_2$ are the conormal bundles to hypersurfaces. Then $\Lambda = (\Lambda \circ \Lambda_2) \circ \Lambda_3$, where Λ_3 is the Lagrangian manifold associated to χ^{-1}. □

To end this section, let us recall the following well-known result.

Proposition A.2.9. *Let Λ be a conic submanifold of $\dot{T}^*X$, $p \in \Lambda$. Assume Λ is isotropic (resp. Lagrangian, resp. regular involutive). Then there exists a contact transformation χ defined in a neighborhood of p such that $\chi(p) = (0; dx_n) \in T^*\mathbb{R}^n$ and $\Lambda = \{(x;\xi); x = 0, \xi_1 = \cdots = \xi_r = 0\ (r < n)\}$ (resp. $\Lambda = \{(x;\xi); x = 0\}$, resp. $\Lambda = \{(x;\xi); \xi_1 = \cdots = \xi_p = 0, (p < n)\}$).*

The proof is left as an exercise.

A.3. Inertia index

Let (E, σ) be a real symplectic vector space of dimension $2n$ and let λ_1, λ_2, λ_3 be three Lagrangian subspaces. (In this section, unless otherwise specified, a subspace means a linear subspace.)

Definition A.3.1. *The inertia index of the triplet $(\lambda_1, \lambda_2, \lambda_3)$, denoted $\tau_E(\lambda_1, \lambda_2, \lambda_3)$* (or simply $\tau(\lambda_1, \lambda_2, \lambda_3)$) *is the signature of the quadratic form q defined on the $3n$*

dimensional vector space $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ *by:*

$$q(x_1, x_2, x_3) = \sigma(x_1, x_2) + \sigma(x_2, x_3) + \sigma(x_3, x_1) \ .$$

This index is also sometimes called the "Maslov index".

In a suitable basis of $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ one can represent q by a diagonal matrix whose diagonal entries consist of p_+-uples of $+1$, p_--uples of -1 and $3n - p_+ - p_-$-uples of 0. Then the **signature** of q, denoted $\mathrm{sgn}(q)$ is equal to $p_+ - p_-$.

The index τ has the following properties.

Theorem A.3.2. (i) $\tau(\lambda_1, \lambda_2, \lambda_3)$ *is alternating with respect to the permutations of the* λ_j's, *that is:*

$$\tau(\lambda_1, \lambda_2, \lambda_3) = -\tau(\lambda_2, \lambda_1, \lambda_3) = -\tau(\lambda_1, \lambda_3, \lambda_2) \ .$$

(ii) τ *satisfies the "cocycle condition": for any quadruplet* λ_1, λ_2, λ_3, λ_4 *of Lagrangian spaces, we have:*

$$\tau(\lambda_1, \lambda_2, \lambda_3) - \tau(\lambda_1, \lambda_2, \lambda_4) + \tau(\lambda_1, \lambda_3, \lambda_4) - \tau(\lambda_2, \lambda_3, \lambda_4) = 0 \ .$$

(iii) *When the Lagrangian subspaces* λ_1, λ_2, λ_3 *move continuously in the Lagrangian Grassmannian* $\Lambda(E)$ *in such a manner that* $\dim(\lambda_1 \cap \lambda_2)$, $\dim(\lambda_2 \cap \lambda_3)$ *and* $\dim(\lambda_3 \cap \lambda_1)$ *remain constant, then* $\tau(\lambda_1, \lambda_2, \lambda_3)$ *remains constant.*

(iv) $\tau(\lambda_1, \lambda_2, \lambda_3) \equiv n + \dim(\lambda_1 \cap \lambda_2) + \dim(\lambda_2 \cap \lambda_3) + \dim(\lambda_3 \cap \lambda_1) \bmod 2\mathbb{Z}$.

(v) *If* ρ *is an isotropic space contained in* $(\lambda_1 \cap \lambda_2) + (\lambda_2 \cap \lambda_3) + (\lambda_3 \cap \lambda_1)$ *then:*

$$\tau_E(\lambda_1, \lambda_2, \lambda_3) = \tau_{E^\rho}(\lambda_1^\rho, \lambda_2^\rho, \lambda_3^\rho) \ .$$

(vi) *Let* $E^a = (E, -\sigma)$ *be the vector space E endowed with the symplectic form* $-\sigma$. *Then:*

$$\tau_{E^a}(\lambda_1, \lambda_2, \lambda_3) = -\tau_E(\lambda_1, \lambda_2, \lambda_3) \ .$$

(vii) *Let* (E_1, σ_1) *and* (E_2, σ_2) *be two symplectic vector spaces, and let* λ_1, λ_2, λ_3 *(resp.* μ_1, μ_2, μ_3*) be a triplet of Lagrangian subspaces of* E_1 *(resp.* E_2*). Then:*

$$\tau_{E_1 \oplus E_2}(\lambda_1 \oplus \mu_2, \lambda_2 \oplus \mu_2, \lambda_3 \oplus \mu_3) = \tau_{E_1}(\lambda_1, \lambda_2, \lambda_3) + \tau_{E_2}(\mu_1, \mu_2, \mu_3) \ .$$

(Here $E_1 \oplus E_2$ *is endowed with the symplectic form* $\sigma_1 \oplus \sigma_2$.*)*

Proof. (i) is clear since the quadratic form $q(x_1, x_2, x_3)$ is alternating with respect to the permutations of the λ_j's.

(iii) Set, for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$:

$$B(x, y) = \sigma(x_1, y_2) + \sigma(x_2, y_3) + \sigma(x_3, y_1) + \sigma(y_1, x_2) + \sigma(y_2, x_3) + \sigma(y_3, x_1) \ .$$

The totally isotropic space I of q is by the definition the space:

(A.3.1) $$I = \{x \in \lambda_1 \oplus \lambda_2 \oplus \lambda_3; B(x,y) = 0 \text{ for any } y\} .$$

since $B(x,y) = \sigma(y_1, x_2 - x_3) + \sigma(y_2, x_3 - x_1) + \sigma(y_3, x_1 - x_2)$, we have:

$$I = \{(x_1, x_2, x_3) \in \lambda_1 \oplus \lambda_2 \oplus \lambda_3; x_2 - x_3 \in \lambda_1, x_3 - x_1 \in \lambda_2, x_1 - x_2 \in \lambda_3\} .$$

Set $y_1 = x_2 + x_3 - x_1$, $y_2 = x_3 + x_1 - x_2$, $y_3 = x_1 + x_2 - x_3$. Then:

$$y_1 \in \lambda_2 \cap \lambda_3 , \qquad y_2 \in \lambda_3 \cap \lambda_1 , \qquad y_3 \in \lambda_1 \cap \lambda_2 .$$

Moreover

$$2x_1 = y_2 + y_3 , \qquad 2x_2 = y_3 + y_1 , \qquad 2x_3 = y_1 + y_2 .$$

Therefore, by the linear transformation $(x_1, x_2, x_3) \mapsto (y_1, y_2, y_3)$, I is isomorphic to $(\lambda_1 \cap \lambda_2) \oplus (\lambda_2 \cap \lambda_3) \oplus (\lambda_3 \cap \lambda_1)$. Since the rank of q (denoted $\mathrm{rk}(q)$) is $3n - \dim I$, we get:

(A.3.2) $$\mathrm{rk}(q) = 3n - \dim(\lambda_1 \cap \lambda_2) - \dim(\lambda_2 \cap \lambda_3) - \dim(\lambda_3 \cap \lambda_1) .$$

With the hypotheses of (iii), we get that $\mathrm{rk}(q)$ is constant. Since q moves continuously when the λ_j's move continuously, this implies (iii).

(iv) We have:

(A.3.3) $$\mathrm{sgn}(q) \equiv \mathrm{rk}(q) \bmod 2\mathbb{Z} .$$

Hence (iv) follows from (A.3.2).

(ii) Assume for a while that λ_1 and λ_2 intersect transversally and denote by p_1 and p_2 the projections:

$$p_i : E = \lambda_1 \oplus \lambda_2 \to \lambda_i \qquad (i = 1, 2) .$$

For x and y in E we have:

$$\begin{aligned} \sigma(p_1(x), y) &= \sigma(p_1(x), p_1(y) + p_2(y)) \\ &= \sigma(p_1(x), p_2(y)) \\ &= \sigma(x, p_2(y)) . \end{aligned}$$

Lemma A.3.3. *Suppose λ_1 and λ_2 are transversal. Then $\tau(\lambda_1, \lambda_2, \lambda_3)$ is equal to the signature of the quadratic form q_3 on λ_3 defined by:*

$$q_3(x_3) = -\sigma(p_1(x_3), p_2(x_3)) = -\sigma(p_1(x_3), x_3) .$$

Proof of Lemma A.3.3. Let $x = (x_1, x_2, x_3) \in \lambda_1 \oplus \lambda_2 \oplus \lambda_3$. Then:

$$\begin{aligned} q(x) &= \sigma(x_1, x_2) + \sigma(x_2, x_3) + \sigma(x_3, x_1) \\ &= \sigma(x_1, x_2) - \sigma(p_1(x_3), x_2) - \sigma(x_1, p_2(x_3)) \\ &= \sigma(x_1 - p_1(x_3), x_2 - p_2(x_3)) - \sigma(p_1(x_3), p_2(x_3)) . \end{aligned}$$

By the linear transformation $(x_1, x_2, x_3) \mapsto (x_1 - p_1(x_3), x_2 - p_2(x_3), x_3)$, the quadratic form $(x_1, x_2, x_3) \mapsto \sigma(x_1 - p_1(x_3), x_2 - p_2(x_3))$ is equivalent to the quadratic form $(x_1, x_2, x_3) \mapsto \sigma(x_1, x_2)$. Hence its signature is zero, which proves the Lemma. □

Lemma A.3.4. *Let* λ_j $(j = 1,2,3)$ *and* μ *be four Lagrangian subspaces such that* $\lambda_j \cap \mu = \{0\}$, $(j = 1,2,3)$. *Then*:

(A.3.4) $$\tau(\lambda_1, \lambda_2, \lambda_3) = \tau(\lambda_1, \lambda_2, \mu) + \tau(\lambda_2, \lambda_3, \mu) + \tau(\lambda_3, \lambda_1, \mu) \ .$$

Proof of Lemma A.3.4. By Lemma (A.3.3), the right hand side of (A.3.4) is the signature of the quadratic form on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$:

$$q'(y_1, y_2, y_3) = \sigma(p_1(y_2), y_2) + \sigma(p_2(y_3), y_3) + \sigma(p_3(y_1), y_1) \ ,$$

where now p_j denote the projection:

$$p_j : \lambda_j \oplus \mu \to \lambda_j \qquad (j = 1,2,3) \ .$$

Consider the linear automorphism of $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ defined by:

$$x_1 = y_1 + p_1(y_2) \ , \qquad x_2 = y_2 + p_2(y_3) \ , \qquad x_3 = y_3 + p_3(y_1)$$

$$y_1 = (x_1 - p_1(x_2) + p_1(x_3))/2 \ , \qquad y_2 = (x_2 - p_2(x_3) + p_2(x_1))/2 \ ,$$

$$y_3 = (x_3 - p_3(x_1) + p_3(x_2))/2 \ .$$

An easy calculation shows:

(A.3.5) $$q(x_1, x_2, x_3) = q'(y_1, y_2, y_3) \ ,$$

which proves the lemma. □

End of the proof of Theorem A.3.2. Choose a Lagrangian subspace μ transversal to all λ_j's $(j = 1,2,3,4)$ and apply (A.3.4). Then (ii) follows in view of (i).

(v) We shall decompose the proof in several steps.

(a) First assume $\rho \subset \lambda_1 \cap \lambda_2 \cap \lambda_3$. Then the quadratic form q on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ is the pull-back of the corresponding quadratic form on $\lambda_1^\rho \oplus \lambda_2^\rho \oplus \lambda_3^\rho$, by the surjective map $\lambda_1 \oplus \lambda_2 \oplus \lambda_3 \to \lambda_1^\rho \oplus \lambda_2^\rho \oplus \lambda_3^\rho$. The assertion follows in that case.

(b) Now assume $\rho \subset \lambda_2 \cap \lambda_3$ and $\lambda_3^\rho = \lambda_1^\rho$. Consider the quadratic form q'' on $\lambda_1 \oplus \lambda_2 \oplus (\lambda_1 \cap \rho^\perp) \oplus \rho$ defined by:

$$q''(x_1, x_2, u, v) = \sigma(x_1, x_2) + \sigma(x_2, u + v) + \sigma(u + v, x_1) \ .$$

Then $q''(x_1, x_2, u, v) = \sigma(x_1, x_2) + \sigma(x_2, u) + \sigma(v, x_1) = \sigma(x_1 - u, x_2 - v)$. Hence the signature of q'' is zero. By the hypothesis, $\lambda_3 = (\lambda_1 \cap \rho^\perp) + \rho$. This implies that the signature of q on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ is that of q''. Thus $\tau = 0$ in that case.

(c) Set $\tilde{\lambda}_i = (\lambda_i \cap \rho^\perp) + \rho = (\lambda_i + \rho) \cap \rho^\perp$, $i = 1, 2, 3$. We have:

(A.3.6) $$\lambda_1^{\tilde{\lambda}_1 \cap \lambda_2} = \tilde{\lambda}_1^{\tilde{\lambda}_1 \cap \lambda_2} \ .$$

In fact $\rho \subset \lambda_1 + \lambda_2 \cap \lambda_3 \subset \lambda_1 + \lambda_2 \cap \rho^\perp$ implies

$$\rho \subset \lambda_1 + (\lambda_1 + \rho) \cap \lambda_2 \cap \rho^\perp \subset \lambda_1 + \tilde{\lambda}_1 \cap \lambda_2 \ .$$

We obtain $\tilde{\lambda}_1 \subset \lambda_1 + (\tilde{\lambda}_1 \cap \lambda_2)$ thus $\tilde{\lambda}_1 \subset [\lambda_1 + (\tilde{\lambda}_1 \cap \lambda_2)] \cap (\tilde{\lambda}_1 + \lambda_2)$ which gives:

(A.3.7) $$\tilde{\lambda}_1^{\tilde{\lambda}_1 \cap \lambda_2} = \lambda_1^{\tilde{\lambda}_1 \cap \lambda_2} \ .$$

Since both spaces in (A.3.7) are Lagrangian, (A.3.6) follows. Similarly we have:

(A.3.8) $$\lambda_1^{\tilde{\lambda}_1 \cap \lambda_3} = \tilde{\lambda}_1^{\tilde{\lambda}_1 \cap \lambda_3} \ .$$

Therefore we get by (b):

(A.3.9) $$\tau(\lambda_1, \tilde{\lambda}_1, \lambda_j) = 0 \qquad \text{for} \qquad j = 2, 3 \ .$$

Now we have:

$$\begin{aligned} \tau(\lambda_1, \lambda_2, \lambda_3) &= \tau(\lambda_1, \lambda_2, \tilde{\lambda}_1) + \tau(\lambda_2, \lambda_3, \tilde{\lambda}_1) + \tau(\lambda_3, \lambda_1, \tilde{\lambda}_1) \\ &= \tau(\tilde{\lambda}_1, \lambda_2, \lambda_3) \ . \end{aligned}$$

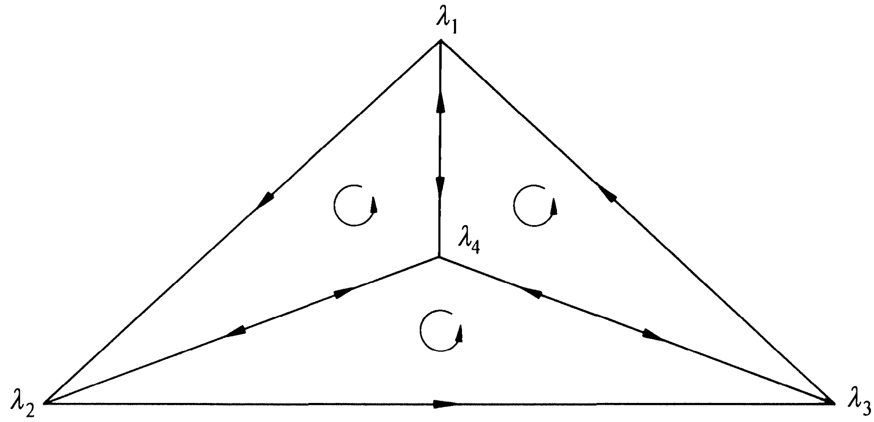
Repeating this argument, we obtain

$$\tau(\lambda_1, \lambda_2, \lambda_3) = \tau(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$$

and the term on the right-hand side equals $\tau(\lambda_1^\rho, \lambda_2^\rho, \lambda_3^\rho)$ by (a).

(vi) and (vii) are obvious. □

Remark A.3.5. The "cocycle condition", written in Theorem A.3.2 (ii), may be visualized by figure A.3.1.



$$\tau(\lambda_1, \lambda_2, \lambda_3) = \tau(\lambda_1, \lambda_2, \lambda_4) + \tau(\lambda_2, \lambda_3, \lambda_4) + \tau(\lambda_3, \lambda_1, \lambda_4)$$

Fig. A.3.1

We shall calculate explicitly the Maslov index in a special case. Assume E endowed with a symplectic basis, and denote by $(x;\xi)$ the associated linear coordinates. Let A and B be two $(n \times n)$-matrices satisfying the hypothesis (A.1.11) with A^tB symmetric, and define:

$$\lambda_1 = \{x = 0\} \ , \qquad \lambda_2 = \{\xi = 0\} \ , \qquad \lambda_3 = \{(x;\xi);\, Ax = B\xi\} \ .$$

Proposition A.3.6. *One has:*

$$\tau(\lambda_1, \lambda_2, \lambda_3) = -\operatorname{sgn}(A^tB) \ .$$

Proof. Denote by p_1 and p_2 the projections from $E = \lambda_1 \oplus \lambda_2$ to λ_1 and λ_2 respectively. By Lemma A.3.3, $-\tau(\lambda_1, \lambda_2, \lambda_3)$ is the signature of the quadratic form q_3 on λ_3, defined by:

$$\begin{aligned} q_3((x;\xi)) &= \sigma(p_1(x;\xi), p_2(x;\xi)) \\ &= \sigma(\xi, x) \\ &= \langle \xi, x \rangle \ . \end{aligned}$$

The map from $\mathbb{R}^n$ to λ_3 given by $z \mapsto ({}^tBz, {}^tAz)$ is a linear isomorphism. Hence $\langle \xi, x \rangle = \langle z, A^tBz \rangle$, and the result follows. □

Let $\lambda_1, \ldots, \lambda_N$ be Lagrangian subspaces with $N \geqslant 3$ and let μ be another Lagrangian subspace. By Theorem A.3.2 (ii) we have the identity:

$$\text{(A.3.10)} \quad \begin{cases} \tau(\lambda_1, \lambda_2, \lambda_3) + \tau(\lambda_1, \lambda_3, \lambda_4) + \cdots + \tau(\lambda_1, \lambda_{N-1}, \lambda_N) \\ \quad = \tau(\lambda_1, \lambda_2, \mu) + \tau(\lambda_2, \lambda_3, \mu) + \cdots + \tau(\lambda_{N-1}, \lambda_N, \mu) + \tau(\lambda_N, \lambda_1, \mu) \ . \end{cases}$$

This can be visualized by Figure A.3.2 (in which $N = 5$).

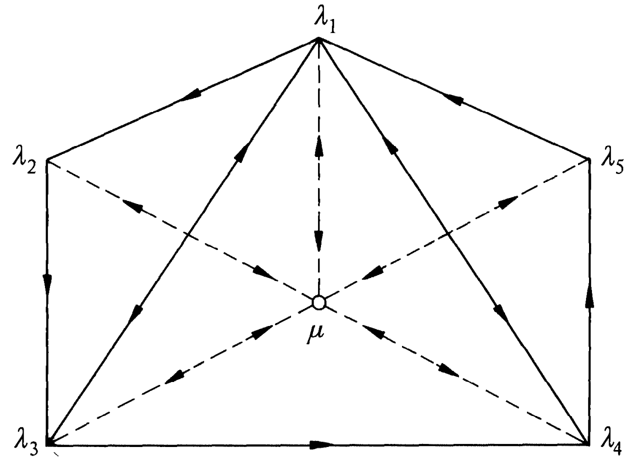


Fig. A.3.2

Definition A.3.7. *Let $\lambda_1, \dots, \lambda_N$ be Lagrangian subspaces, $N \geqslant 3$. One defines the index $\tau(\lambda_1, \dots, \lambda_N)$ as the left hand side of* (A.3.10).

Proposition A.3.8.

(i) $\tau(\lambda_1, \lambda_2, \dots, \lambda_N) = \tau(\lambda_2, \lambda_3, \dots, \lambda_N, \lambda_1) = -\tau(\lambda_N, \lambda_{N-1}, \dots, \lambda_1)$.

(ii) *Assume* $N \geqslant 4$ *and let* $j \in \{3, \dots, N-1\}$. *Then* $\tau(\lambda_1, \dots, \lambda_N) = \tau(\lambda_1, \dots, \lambda_j) + \tau(\lambda_1, \lambda_j, \lambda_{j+1}, \dots, \lambda_N)$.

(iii) *When the λ_j's move continuously in such a manner that* $\dim(\lambda_1 \cap \lambda_2)$, $\dim(\lambda_2 \cap \lambda_3)$, ..., $\dim(\lambda_{N-1} \cap \lambda_N)$ *and* $\dim(\lambda_N \cap \lambda_1)$ *remain constant, then* $\tau(\lambda_1, \dots, \lambda_N)$ *remains constant.*

(iv) $\tau(\lambda_1, \dots, \lambda_N) \equiv nN + \dim(\lambda_1 \cap \lambda_2) + \cdots + \dim(\lambda_N \cap \lambda_1) \bmod 2\mathbb{Z}$.

Proof. (i) and (ii) are obvious by the definition and Theorem A.3.2. (iii) and (iv) follow immediately from Theorem A.3.2 and (A.3.10) when choosing the Lagrangian space μ transversal to all λ_j's. □

As before, denote by E^a the space E endowed with the symplectic form $-\sigma$, and denote by "a" the identity $E \to E^a$. If λ is a subspace of E, we denote by λ^a its image by a. Then, we have $\tau_{E^a}(\lambda_1^a, \lambda_2^a, \lambda_3^a) = -\tau_E(\lambda_1, \lambda_2, \lambda_3)$.

Proposition A.3.9. *Let $\lambda_1, \lambda_2, \mu_1, \mu_2$ be four Lagrangian subspaces of E, and denote by Δ the diagonal of $E^a \oplus E$ (which is Lagrangian). Then:*

(A.3.11) $$\tau_{E^a \oplus E}(\lambda_1^a \oplus \lambda_2, \mu_1^a \oplus \mu_2, \Delta) = \tau_E(\lambda_1, \lambda_2, \mu_2, \mu_1) \ .$$

Proof. Denote by τ the left hand side of (A.3.11). Then $\tau = \tau_1 + \tau_2 + \tau_3$, with:

$$\tau_1 = \tau(\lambda_1^a \oplus \lambda_2, \mu_1^a \oplus \mu_2, \lambda_1^a \oplus \mu_2) \ ,$$

$$\tau_2 = \tau(\mu_1^a \oplus \mu_2, \Delta, \lambda_1^a \oplus \mu_2) \ ,$$

$$\tau_3 = \tau(\Delta, \lambda_1^a \oplus \lambda_2, \lambda_1^a \oplus \mu_2) \ .$$

Then:

$$\tau_1 = \tau(\lambda_1^a, \mu_1^a, \lambda_1^a) + \tau(\lambda_2, \mu_2, \mu_2) = 0 \ ,$$

$$\tau_2 = \tau(\mu_1^a, \mu_2^a, \lambda_1^a) \ ,$$

$$\tau_3 = \tau(\lambda_1, \lambda_2, \mu_2) \ .$$

(To calculate τ_2 and τ_3 we apply Theorem A.3.2 (v) with $\rho = \{0\} \oplus \mu_2$ and $\rho = \lambda_1^a \oplus \{0\}$, respectively.) Therefore $\tau = \tau(\lambda_1, \lambda_2, \mu_2) - \tau(\mu_1, \mu_2, \lambda_1) = \tau(\lambda_1, \lambda_2, \mu_2, \mu_1)$. □

To end this section, let us describe the action of the symplectic group $Sp(E)$ on the space $\Lambda^3(E)$ of triplets of Lagrangian subspaces of E.

For $r = (r_0, r_1, r_2, r_3, d) \in \mathbb{N}^4 \times \mathbb{Z}$ we set:

$$\begin{aligned} W_r = \{(\lambda_1, \lambda_2, \lambda_3) \in \Lambda^3(E)\ ; \quad & \dim(\lambda_1 \cap \lambda_2 \cap \lambda_3) = r_0\ , \\ \dim(\lambda_1 \cap \lambda_2) = r_3\ , \quad & \dim(\lambda_2 \cap \lambda_3) = r_1\ , \\ \dim(\lambda_3 \cap \lambda_1) = r_2\ , \quad & \tau(\lambda_1, \lambda_2, \lambda_3) = d\}\ . \end{aligned}$$

Then $Sp(E)$ acts naturally on $\Lambda^3(E)$, and the W_r's are invariant by $Sp(E)$. Consider the conditions:

$$\text{(A.3.12)} \quad \begin{cases} 0 \leqslant r_0 \leqslant r_1, r_2, r_3 \leqslant n\ , & r_1 + r_2 + r_3 \leqslant n + 2r_0\ , \\ |d| \leqslant n + 2r_0 - (r_1 + r_2 + r_3)\ , & d \equiv n + r_1 + r_2 + r_3 \mod 2\mathbb{Z}\ . \end{cases}$$

It can be easily shown that (A.3.12) is a necessary and sufficient condition in order that W_r is non-empty. The action of $Sp(E)$ on $\Lambda^3(E)$ has only finitely many orbits, and these orbits are the W_r, where r satisfies (A.3.12). Since we do not use this result, we leave the proof as an exercise.

Exercises to the Appendix

Exercise A.1. Let $(x) = (x_1, \ldots, x_n)$ be a system of local coordinates on X, and $(x;\xi)$ the associated coordinates on T^*X. Let $\varphi = (\varphi_1, \ldots, \varphi_p): X \to \mathbb{R}^p$ be a smooth map, and let $S = \{x; \varphi(x) = 0\}$, (hence $d\varphi_1 \wedge \cdots \wedge d\varphi_p \neq 0$ and S is smooth). Let $x_\circ \in S$, $p = (x_\circ; \sum_i a_i\, d\varphi_i(x_\circ)) \in T_S^*X$. Prove that:

$$T_S^*X = \left\{(x;\xi); \varphi(x) = 0, \xi_j = \sum_{i=1}^{p} \lambda_i \frac{\partial \varphi_i}{\partial x_j}, \lambda_i \in \mathbb{R}\right\}\ ,$$

$$\begin{aligned} T_p T_S^*X = \Bigg\{(x;\xi); & \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial x_j}(x_\circ) x_j = 0, i = 1, \ldots, p, \\ & \xi_j = \sum_{i=1}^{p} \frac{\partial \varphi_i}{\partial x_j}(x_\circ)\lambda_i + \sum_{k=1}^{n}\sum_{i=1}^{p} a_i \frac{\partial^2 \varphi_i(x_\circ)}{\partial x_k \partial x_j} x_k, \lambda_i \in \mathbb{R}\Bigg\}\ . \end{aligned}$$

Exercise A.2. Let Λ be a closed conic Lagrangian submanifold of T^*X. Prove that there exists a submanifold M of X such that $\Lambda = T_M^*X$.

Exercise A.3. Let X and Y be two manifolds of the same dimension, f a real function on $X \times Y$ with $df \neq 0$ on $S = \{f = 0\}$. Set $\Lambda = \dot{T}_S^*(X \times Y)$.

Prove that $p_1|_\Lambda$ and $p_2^a|_\Lambda$ are local isomorphisms (from Λ to T^*X and Λ to T^*Y, respectively) if and only if the determinant $\begin{pmatrix} 0 & d_y f \\ d_x f & d_{xy}^2 f \end{pmatrix}$ does not vanish on S.

Exercise A.4. Let (E_1, σ_1) and (E_2, σ_2) be two symplectic vector spaces and let λ be a Lagrangian linear subspace of $E_1 \oplus E_2$. Prove that $p_1|_\lambda$ is injective if and only if $p_2|_\lambda$ is surjective.

Exercise A.5. Let $f: Y \to X$ be a morphism of manifolds and let $p \in Y \times_X T^*X$.
(a) Let Λ_Y be a Lagrangian submanifold of T^*Y. Assume that ${}^t f'$ is clean with respect to Λ_Y at p_Y. Prove that for U a sufficiently small neighborhood of p, $f_\pi(U \cap {}^t f'^{-1}(\Lambda_Y))$ is a smooth Lagrangian manifold.
(b) Let Λ_X be a Lagrangian submanifold of T^*X. Assume that f_π is clean with respect to Λ_X at p_X. Prove that for U a sufficiently small neighborhood of p, ${}^t f'(U \cap f_\pi^{-1}(\Lambda_X))$ is a smooth Lagrangian manifold.

Exercise A.6. Let E_1 and E_2 be two symplectic vector spaces, ν a Lagrangian subspace of $E_1 \oplus E_2^a$ and let λ_i and μ_i be two Lagrangian subspaces of E_i $(i = 1, 2)$. Prove:

$$\begin{aligned}\tau_{E_1 \oplus E_2^a}(\lambda_1 \oplus \lambda_2^a, \mu_1 \oplus \mu_2^a, \nu) &= \tau_{E_1}(\lambda_1, \mu_1, \nu \circ \mu_2) - \tau_{E_2}(\lambda_2, \mu_2, \lambda_1 \circ \nu^a) \\ &= \tau_{E_1}(\lambda_1, \mu_1, \nu \circ \lambda_2) - \tau_{E_2}(\lambda_2, \mu_2, \mu_1 \circ \nu^a) \ .\end{aligned}$$

Exercise A.7. Let (E, σ) be a complex symplectic vector space, and endow the real underlying vector space $E^{\mathbb{R}}$ with the real symplectic form $2 \operatorname{Re} \sigma$. Let λ_1, λ_2, λ_3 be three complex Lagrangian subspaces of E. Prove that $\tau_{E^{\mathbb{R}}}(\lambda_1, \lambda_2, \lambda_3) = 0$.

Exercise A.8. Let E_i $(i = 1, 2, 3, 4)$ be a symplectic vector space and let $\lambda_i \subset E_i \oplus E_{i+1}^a$ $(i = 1, 2, 3)$ be a Lagrangian plane.

Prove that $(\lambda_1 \circ \lambda_2) \circ \lambda_3 = \lambda_1 \circ (\lambda_2 \circ \lambda_3)$.

Exercise A.9. Let (E_i, μ_i) be a pair of a symplectic vector space E_i and a Lagrangian plane μ_i of E_i $(i = 1, 2, 3, 4)$.

Then for Lagrangian planes $\lambda_1 \subset E_1 \oplus E_2^a$ and $\lambda_2 \subset E_2 \oplus E_3^a$, we define (cf. (7.5.10)):

$$\tau(\lambda_1 : \lambda_2) = \tau_{E_2}(\mu_2, \lambda_2 \circ \mu_3, \mu_1 \circ \lambda_1^a) \ .$$

Now, let λ_i be a Lagrangian plane of $E_i \oplus E_{i+1}^a$ $(i = 1, 2, 3)$. Prove the equality:

$$\tau(\lambda_1 : \lambda_2 \circ \lambda_3) + \tau(\lambda_2 : \lambda_3) = \tau(\lambda_1 \circ \lambda_2 : \lambda_3) + \tau(\lambda_1 : \lambda_2) \ .$$

(Hint: Using Exercise A.6, prove that both sides are equal to:

$$\tau_{E_2 \oplus E_3^a}(\mu_2 \oplus \mu_3^a, \mu_1 \circ \lambda_1^a \oplus \lambda_3^a \circ \mu_4^a, \lambda_2).)$$

Exercise A.10. Let E_i be a symplectic vector space and λ_i and μ_i two Lagrangian subspaces of E_i $(i = 1, 2, 3)$. Let ν and ν' be Lagrangian subspaces of $E_1 \oplus E_2^a$ and $E_2 \oplus E_3^a$, respectively. Prove:

$$\tau_{E_1 \oplus E_2^a}(\lambda_1 \oplus \lambda_2^a, \nu, \mu_1 \oplus \mu_2^a) + \tau_{E_2 \oplus E_3^a}(\lambda_2 \oplus \lambda_3^a, \nu', \mu_2 \oplus \mu_3^a)$$

$$- \tau_{E_1 \oplus E_3^a}(\lambda_1 \oplus \lambda_3^a, \nu \circ \nu', \mu_1 \oplus \mu_3^a)$$

$$= \tau_{E_2}(\lambda_2, \nu' \circ \lambda_3, \lambda_1 \circ \nu^a) - \tau_{E_2}(\mu_2, \nu' \circ \mu_3, \mu_1 \circ \nu^a).$$

(Hint: use Exercise A.6.)

Notes

Symplectic and contact geometry are classical subjects which go back to Hamilton and Jacobi, and that we shall not review here. As pointed out at the beginning of this appendix, the results of §1 and §2 are well-known and may be found for example in Hörmander [4].

In 1965, in order to calculate asymptotic expansions "in a neighborhood of a caustic" (i.e. when the projection of a smooth Lagrangian manifold has not a constant rank), Maslov [1] (cf. also Keller [1]) introduced the index of a closed curve in a Lagrangian submanifold of a symplectic space. His theory was clarified and reformulated by Arnold [1], then by Hörmander [2] and Leray [4] who defined the index of three Lagrangian planes intersecting transversally, until Kashiwara (cf. Lion-Vergne [1]) defined the index τ in the general case by the simple method we have given here. Note that in Lion-Vergne (loc. cit.) the index τ is generalized to the local field case.

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List of notations and conventions

General notations

$\mathbb{N}$: set of non-negative integers
$\mathbb{Z}$: ring of integers
$\mathbb{Q}$: field of rational numbers
$\mathbb{R}$: field of real numbers
$\mathbb{C}$: field of complex numbers
$\mathbb{R}^+$: multiplicative group of positive real numbers
$\mathbb{R}^-$: the set of negative real numbers
$\mathbb{R}_{\geqslant 0}$ (resp. $\mathbb{R}_{\leqslant 0}$): $\{c \in \mathbb{R}; c \geqslant 0$, (resp. $c \leqslant 0)\}$
$\mathbb{C}^\times$: multiplicative group of non-zero complex numbers
$\# A$: number of elements of a finite set A
$A \backslash B$: the complementary set to B in A
$\delta_{i,j}$: Kronecker symbol, $\delta_{i,j} = 0$ for $i \neq j$ and $\delta_{i,j} = 1$ for $i = j$
$\bar{S}$: the closure of a subset S
∂S: $\bar{S} \backslash S$ cf. (9.2.4)
$X \times_S Y$: fiber product over S cf. Notations 2.3.13
$\mathbb{R}^n$: Euclidian n-space
$\varinjlim$: inductive limit
$\varprojlim$: projective limit
"$\varinjlim$": ind-object cf. I §11
"$\varprojlim$": pro-object cf. I §11
$\{x_n\}_{n \in I}$: a sequence indexed by I; ($I = \mathbb{N}$ or $\mathbb{Z}$)
$x_n \underset{n}{\to} x$: the sequence $\{x_n\}_n$ converges to x
$\{\mathrm{pt}\}$: the set consisting of a single element
$\square$: means the square is Cartesian

Manifolds

$X, Y, \ldots$: real or complex manifolds
δ or $\delta_X : X \hookrightarrow X \times X$: the diagonal embedding
$\tau : TX \to X$: the tangent vector bundle to X
$\pi : T^*X \to X$: the cotangent vector bundle to X
$T_M X$: the normal vector bundle to a submanifold M of X cf. A.2.1

T_M^*X: the conormal vector bundle to a submanifold M of X cf. A.2.1
$T_X^*X \simeq X$: the zero section, identified to X
E_γ: the vector space E endowed with the γ-topology cf. III §5
X_γ: the space X endowed with the γ-topology cf. III §5
φ_γ: the continuous map $X \to X_\gamma$ cf. III §5
$f: Y \to X$: a morphism of manifolds
$f|_N: N \to M$: the morphism induced by f, ($N \subset Y, M \subset X$)
Tf: $TY \xrightarrow[f']{} T \times_X TX \xrightarrow[f_\tau]{} TX$ cf. (4.1.8)
$T_N f$: $T_N Y \xrightarrow[f'_N]{} N \times_M T_M X \xrightarrow[f_{N\tau}]{} T_M X$ cf. (4.1.9)
$T^*Y \xleftarrow[{}^t f']{} Y \times_X T^*X \xrightarrow[f_\pi]{} T^*X$ cf. (4.3.2)
$T_N^*Y \xleftarrow[{}^t f'_N]{} N \times_M T_M^*X \xrightarrow[f_{N\pi}]{} T_M^*X$ cf. (4.3.3)
T_Y^*X: $\ker({}^t f': Y \times_X T^*X \to T^*Y)$ cf. (4.3.4)
a_X: the map $X \to \{\mathrm{pt}\}$
$\bar{X}$: the complex conjugate manifold associated to a complex manifold X cf. X §1
$X^{\mathbb{R}}$: the real underlying manifold to a complex manifold X cf. X §1
$\dim Y/X$, $\operatorname{codim}_X Y$: the relative dimension and codimension cf. Notation 3.3.8
$\dim X$, $\dim_{\mathbb{R}} X$: dimension of X cf. II §9 and Notation 3.3.8 (at the exception of XI §3)
$\dim_{\mathbb{C}} X$: complex dimension of X cf. II §9

Vector bundles

$\tau: E \to Z$: a vector bundle over Z
$\pi: E^* \to Z$: the dual vector bundle
Z is identified to the zero-section of a vector bundle over Z
$\dot{E} = E \backslash Z$, $\dot{E}^* = E^* \backslash Z$
$\dot{\tau} = \tau|_{\dot{E}}$, $\dot{\pi} = \pi|_{\dot{E}^*}$
a: the antipodal map on E
S^a: the image of S by a, $S \subset E$
S°: the polar set to S cf. (3.7.6)
e: the Euler vector field cf. V §5
$A + B$: sum in a vector bundle cf. (5.4.5)
$S = \dot{E}/\mathbb{R}^+$: sphere bundle associated to the vector bundle E cf. (3.6.3)
$T^*(E/Z)$: the relative cotangent bundle cf. (5.5.3)

Normal cones

$C_M(S)$: normal cone of S along M cf. Definition 4.1.1
$C(S_1, S_2)$: normal cone of $S_1 \times S_2$ along the diagonal cf. Definition 4.1.1

$C_\mu(A,B)$, $f^{\#}(A,B)$, $f_\infty^{\#}(A,B)$, $f^{\#}(A)$, $f_\infty^{\#}(A)$, $A \hat{+} B$, $A \underset{\infty}{\hat{+}} B$: cf. Definition 6.2.3

$N(S) = TX\backslash C(X\backslash S, S)$: cf. Definition 5.3.6

$N^*(S) = N(S)^\circ$: cf. Definition 5.3.6

$\tilde{X}_M$: normal deformation of M in X cf. IV §1

Symplectic geometry

σ: symplectic form cf. A §1

α_X or α: canonical 1-form on T^*X cf. A §2

H: Hamiltonian isomorphism cf. A §1

H_f: Hamiltonian vector field cf. A §2

$\{\cdot,\cdot\}$: Poisson bracket cf. A §2

$\tau(\cdot,\cdot,\cdot)$: inertia index cf. A §3

E^ρ: $\rho^\perp/\rho$, ρ isotropic in E cf. A §1

$\lambda^\rho = (\lambda \cap \rho^\perp + \rho)/\rho$, λ Lagrangian, ρ isotropic cf. A §1

$\lambda_1 \circ \lambda_2$: composition of Lagrangian planes cf. A §1

$\Lambda_1 \circ \Lambda_2$: composition of Lagrangian manifolds cf. Definition 7.4.5

Λ_φ: Lagrangian manifold associated to φ cf. (7.5.1)

$\lambda_\circ(p) = T_p\pi^{-1}\pi(p)$ cf. (7.5.2)

$\lambda_\Lambda(p) = T_p\Lambda$

τ_φ: cf. (7.5.3)

$\tau(\lambda_1 : \lambda_2)$: cf. (7.5.10)

Algebra

A: a ring (all rings are unitary)

A-module: left A-module (all modules are unitary)

A^{op}: the opposite ring to A (an A^{op}-module is a right A-module)

$\mathscr{C}$: a category, $\mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}_{\mathscr{C}}(\cdot,\cdot)$ cf. I §1

$\mathscr{C}^\circ$: the opposite category cf. I §1

id: the identity morphism

$\mathscr{C}^\vee$: the category of functors from $\mathscr{C}^\circ$ to $\mathfrak{Set}$

$\mathscr{C}^\wedge$: $\mathscr{C}^{\circ\vee\circ}$

Ker, Coker, Im, Coim: cf. I §2

$\mathbf{C}^*(\mathscr{C})$: $* = \emptyset, +, -, b$: categories of complexes of $\mathscr{C}$ cf. I §3

$\mathbf{K}^*(\mathscr{C})$: $* = \emptyset, +, -, b$: cf. Definition 1.3.4

$\mathrm{Ht}(X,Y)$: the group of morphisms from X to Y homotopic to zero cf. I §3

$\mathrm{K}(\mathscr{C})$: Grothendieck group of a category $\mathscr{C}$ cf. Exercise I.27

$X[k]$: translated complex cf. Definition 1.3.2

$\tau^{\leqslant}, \tau^{\geqslant}$: truncation functors cf. (1.3.10), (1.3.11), X §1

$H^k(X)$: k-th cohomology object of a complex X cf. Definition 1.3.5

$M(f)$: mapping cone of f cf. I §4
$X \longrightarrow Y \longrightarrow Z \xrightarrow[+1]{}$: triangle cf. Notation 1.5.8
$\mathscr{C}_S$: localization of $\mathscr{C}$ by S cf. Definition 1.6.2
$\mathscr{C}/N$: localization of $\mathscr{C}$ by N cf. Notation 1.6.8
$\mathbf{D}^*(\mathscr{C})$, $* = \emptyset, +, -, b$: derived categories cf. Definition 1.7.1
RF: right derived functor of F cf. Definition 1.8.1
LF: left derived functor of F cf. I §8
$\mathbf{D}^*_{\mathscr{C}'}(\mathscr{C})$: the full subcategory of $\mathbf{D}^*(\mathscr{C})$ consisting of complexes whose cohomology objects belong to $\mathscr{C}'$ cf. I §7
$H_I(X)$, $H_{II}(X)$, $s(X)$: complexes associated to a double complex cf. I §9
$N \otimes_A M$ or $N \otimes M$: tensor product (over a ring A)
$\mathrm{Hom}_A(N, M)$ or $\mathrm{Hom}(N, M)$: group of homomorphisms (over a ring A)
$\mathrm{Tor}_n^A(N, M) = H^{-n}(N \otimes_A^L M)$ cf. Example 1.10.12
$\mathrm{Ext}_A^n(N, M) = H^n(R\mathrm{Hom}_A(N, M))$
M-L : Mittag-Leffler cf. I §12
$\oplus$: direct sum cf. I §2
$X \times_Z Y$: product over Z cf. Exercise I.6
$X \oplus_Z Y$: direct sum over Z cf. Exercise I.6
$\mathrm{Ext}^j(X, Y) = \mathrm{Hom}_{\mathbf{D}(\mathscr{C})}(X, Y[j])$ cf. Exercise I.17
$\mathrm{hd}(\mathscr{C})$: homological dimension of $\mathscr{C}$ cf. Exercise I.17
$\mathrm{gld}(A)$: global homological dimension of A cf. Exercise I.28
$\mathrm{wgld}(A)$: weak global homological dimension of A cf. Exercise I.29
tr: trace cf. Exercise I.32
χ: Euler-Poincaré index cf. Exercise I.32
$\mathrm{b}_j(V) = \dim H^j(V)$ cf. (5.4.17)
$\mathrm{b}_j^*(V) = (-1)^j \sum_{k \leqslant j} (-1)^k \mathrm{b}_k(V)$ cf. Exercise I.34 and (5.4.18)

Sheaves

$F, G, H, \ldots$: sheaves, $\mathscr{R}$ a sheaf of rings on a space X
$\mathscr{H}om_{\mathscr{R}}(F, G)$ or $\mathscr{H}om(F, G)$: sheaf of $\mathscr{R}$-homomorphisms of F in G cf. Definition 2.2.7
$\mathrm{Hom}(F, G) = \Gamma(X; \mathscr{H}om(F, G))$
$\mathscr{R}^{\mathrm{op}}$: the opposite ring
$F \otimes_{\mathscr{R}} G$ or $F \otimes G$: tensor product sheaf of F and G (over $\mathscr{R}$) cf. Definition 2.2.8
$F|_Z$: inverse image of F on Z
$F_x = F|_{\{x\}}$: the stalk of F at x
$s|_Z$, s_x: the restriction of a section s to Z and the germ of s at x
$\mathrm{supp}(s)$: the support of a section s
$\Gamma(X; F)$: global section of F on X
$\Gamma(Z; F) = \Gamma(Z; F|_Z)$
$f^{-1}F$: inverse image of a sheaf F cf. Definition 2.3.1
f_*F: direct image of a sheaf F cf. Definition 2.3.1

$f_!F$: direct image with proper supports cf. (2.5.1)
$f^{\#}$: see Definition 2.7.4
F_Z: sheaf on X such that $F_Z|_Z = F|_Z, F_Z|_{X \setminus Z} = 0$, Z locally closed cf. II §3
$\Gamma_Z(F)$: subsheaf of F consisting of sections supported by Z cf. II §3
$M_X = a_X^{-1}M$: constant sheaf on X with stalk M (M: an A-module)
$M_Z = (M_X)_Z$, $Z \subset X$, Z locally closed
$F \boxtimes_S G$: external tensor product (a sheaf on $X \times_S Y$) cf. Notation 2.3.12
$\Gamma_c(X; F) = a_{X!}F$: global section with compact supports cf. (2.5.2)
Rf_*, $R\Gamma_Z$, $R\Gamma(X, \cdot)$, $Rf_!$, $\otimes^L$, $\boxtimes^L$, $R\mathcal{H}om$: derived functors of the preceding ones cf. II §6
$H^j_Z(F)$, $H^j_Z(X; F)$, $H^j_c(X; F)$, $\mathcal{E}xt^j_R(F, G)$, $\mathrm{Ext}^j_R(F, G)$, $\mathcal{T}or^j_R(F, G)$: cf. Notation 2.6.8
wgld($\mathcal{R}$): cf. Definition 2.6.2
supp(F) = closure of $\bigcup_j \operatorname{supp} H^j(F)$ cf. (2.6.34) and II §2
$\mathcal{C}^\cdot(\mathcal{U}; F)$: Čech complex associated to a family of open subsets cf. II §8
$f^!$: right adjoint of $Rf_!$ cf. III §1
$D_X F = R\mathcal{H}om(F, \omega_X)$ cf. Definition 3.1.16
$D'_X F = R\mathcal{H}om(F, A_X)$ cf. Definition 3.1.16
$\int_X$: the morphism $H^n_c(X; or_X) \to A$ cf. (3.3.15)
$F^\wedge$: Fourier-Sato transform cf. Definition 3.7.8
$F^\vee$: inverse Fourier-Sato transform cf. Definition 3.7.8
Φ_K, Ψ_K: functors associated to the kernel K cf. Definitions 3.6.1 and 7.1.3
$K_1 \circ K_2$: composition of kernels, cf. (3.6.2) and Proposition 7.1.2
$K_1 \circ_\mu K_2$: microlocal composition of kernels cf. Definition 7.3.2
$\nu_M(F)$: specialisation of F along M cf. Definition 4.2.2
$\mu_M(F)$: microlocalization of F along M cf. Definition 4.3.1
f_μ^{-1}, $f_\mu^!$, f_*^μ, $f_!^\mu$: microlocal operations cf. VI §1
$\mu hom(G \to F)$, $\mu hom(F \leftarrow G)$, $\mu hom(F, G)$: microlocalisation functors cf. Definition 4.4.1
SS(F): micro-support of F cf. V §1
$\mathbf{D}^*(X) = \mathbf{D}^*(A_X)$, ($* = \emptyset, +, b, -$): derived category of the category of sheaves of A-modules cf. II §6
$N(X, Y; \Omega_X, \Omega_Y)$: category of kernels cf. Definition 7.1.1
$\mathbf{N}(X, Y; p_X, p_Y)$: category of kernels cf. Definition 7.3.7
ϕ_f: vanishing-cycle functor cf. VIII §6
Ψ_f: nearby-cycle functor cf. VIII §6
${}^pH^k$: perverse cohomology cf. X §2

Special sheaves

or_X: orientation sheaf cf. Definition 3.3.3
$or_{Y/X}$: relative orientation sheaf, cf. (3.3.3)
ω_X: dualizing complex cf. Definition 3.1.16
$\omega_{Y/X}$: relative dualizing complex cf. Definition 3.1.16

$\mathscr{D}b_M$: sheaf of distributions cf. II §9.6
$\mathscr{B}_M$: sheaf of hyperfunctions cf. II §9.6 and Definition 10.5.1
$\mathscr{V}_M$: sheaf of densities on a manifold cf. II §9.5
$\mathscr{C}_M$: sheaf of microfunctions cf. Definition 11.5.1

Sheaves on complex manifolds

$\mathscr{O}_X$: sheaf of holomorphic functions on X
$\mathscr{O}_X^{(p)}$: sheaf of holomorphic p-forms
Ω_X: $\mathscr{O}_X^{(n)} \otimes or_X$ ($n = \dim_{\mathbb{C}} X$)
$\mathscr{D}_X$: sheaf of rings of holomorphic differential operators of finite order
$\mathscr{D}_{Y\to X}$: bimodule of differential operators from Y to X cf. Definition 11.2.8
$\mathrm{Icar}(\mathscr{M})$: cf. (11.2.7)
$\mathrm{char}(\mathscr{M})$: characteristic variety of a $\mathscr{D}_X$-module $\mathscr{M}$ cf. (11.2.14)
$\mathscr{K}_X = \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}[\dim_{\mathbb{C}} X]$ cf. (11.2.18)
$\mathscr{N}^* = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{K}_X)$ cf. (11.2.19)
$\underline{\boxtimes}$,: external tensor product in the category of $\mathscr{D}_X$-modules cf. (11.2.21)
$\underline{f}^{-1}\mathscr{M}$: inverse image in the category of $\mathscr{D}_X$-modules cf. Definition 11.2.10
$\underline{f}_*\mathscr{N}$, $\underline{f}_!\mathscr{N}$: direct image in the category of $\mathscr{D}_X$-modules cf. Definition 11.2.10
$\mathscr{E}_X^{\mathbb{R}}$, $\mathscr{E}_{Y\to X}^{\mathbb{R}}$, $\mathscr{E}_{X\leftarrow Y}^{\mathbb{R}}$: ring and bimodules of holomorphic microlocal operators cf. Definition 11.4.2 and Proposition 11.4.3
$\mathscr{C}_{S|X}^{\mathbb{R}}$: sheaf of microfunctions on a complex submanifold S cf. Definition 11.4.2

Categories

$\mathfrak{Set}$: category of sets
$\mathfrak{Ab}$: category of abelian groups
$\mathfrak{Mod}(A)$: abelian category of left A-modules
$\mathfrak{Mod}^f(A)$: category of finitely generated left A-modules
$\mathfrak{Mod}(\mathscr{R})$: category of sheaves of $\mathscr{R}$-modules ($\mathscr{R}$: a sheaf of rings on X)
$\mathfrak{Mod}(A_X)$: category of sheaves of A-modules on X
$\mathfrak{Sh}(X) = \mathfrak{Mod}(\mathbb{Z}_X)$: category of sheaves of abelian groups on X
$\mathbf{D}(A) = \mathbf{D}(\mathfrak{Mod}(A))$ cf. Notations 1.7.14 and 2.6.1
$\mathbf{D}(X) = \mathbf{D}(A_X) = \mathbf{D}(\mathfrak{Mod}(A_X))$ cf. Notations 2.6.11
$\mathbf{D}^b(X;\Omega)$: localization of $\mathbf{D}^b(X)$ on $\Omega \subset T^*X$ cf. VI §1
$\mathbf{D}^b(X;p) = \mathbf{D}^b(X;\{p\})$: localization of $\mathbf{D}^b(X)$ at p cf. VI §1
$\mathbf{D}^+_{\mathbb{R}^+}(E)$: subcategory of $\mathbf{D}^+(E)$ consisting of conic objects cf. Definition 3.7.1
$\mathfrak{Cons}(\mathbf{S})$, w-$\mathfrak{Cons}(\mathbf{S})$: categories of constructible and weakly constructible sheaves on $\mathbf{S}$ cf. VIII §1

$\mathbf{D}^b_{\mathrm{S}-c}(\mathrm{S})$ and $\mathbf{D}^b_{w-\mathrm{S}-c}(\mathrm{S})$: subcategories of $\mathbf{D}^b(|\mathrm{S}|)$ consisting of objects with constructible and weakly constructible cohomologies cf. VIII §1

$\mathbb{R}$-$\mathfrak{Cons}(X)$ and w-$\mathbb{R}$-$\mathfrak{Cons}(X)$: categories of $\mathbb{R}$-constructible and weakly $\mathbb{R}$-constructible sheaves on X cf. Definition 8.4.3

$\mathbf{D}^b_{\mathbb{R}-c}(X)$ and $\mathbf{D}^b_{w-\mathbb{R}-c}(X)$: subcategories of $\mathbf{D}^b(X)$ consisting of objects with $\mathbb{R}$-constructible and weakly $\mathbb{R}$-constructible cohomology cf. VIII §4

$\mathbf{D}^b_{\mathbb{C}-c}(X)$ and $\mathbf{D}^b_{w-\mathbb{C}-c}(X)$: cf. VIII §5

$\mathrm{K}_{\mathbb{R}-c}(X)$: Grothendieck group of $\mathbf{D}^b_{\mathbb{R}-c}(X)$ cf. IX §7

${}^p\mathbf{D}^{\leqslant 0}_{w-\mathbb{R}-c}(X)$, ${}^p\mathbf{D}^{\geqslant 0}_{w-\mathbb{R}-c}(X)$, etc.: t-structure associated to a perversity p cf. X §2 and §3

${}^\mu\mathbf{D}^{\leqslant 0}_{w-\mathbb{R}-c}(X)$, ${}^\mu\mathbf{D}^{\geqslant 0}_{w-\mathbb{R}-c}(X)$, etc.: t-structure defined microlocaly cf. X §3

Cycles, traces and constructible functions

$\chi(F)(x)$, $\chi_c(F)(x)$, $\chi(X;F)$, $\chi_c(X;F)$: Euler-Poincaré indices cf. IX §1

tr_X: trace morphism cf. (9.1.3)

$\int_X$: $H^0_c(X,\omega_X)\to k$ cf. III §3 and IX §1

$\mathscr{CS}_p(F)$: sheaf of subanalytic p-chains with values in F cf. Definition 9.2.1

$\mathscr{CS}^X_p = \mathscr{CS}_p(A_X) = \mathscr{CS}_p$

$\partial_p : \mathscr{CS}_p \to \mathscr{CS}_{p-1}$: boundary operator cf. (9.2.10)

$\mathscr{ZS}^X_p$: sheaf of subanalytic p-cycles cf. Definition 9.2.5

$C_1 \cap C_2$: intersection of two cycles cf. Definition 9.2.12

$\#(C_1 \cap C_2)$: intersection number of two cycles cf. Definition 9.2.12

$\mathscr{L}_X$: sheaf of Lagrangian cycles cf. Definition 9.3.1

f^*, f_*: inverse and direct images of Lagrangian cycles cf. Definition 9.3.3

$\boxtimes$: external product of cycles cf. (9.3.2)

$[T^*_YX]$: the Lagrangian cycle associated to $Y \hookrightarrow X$ cf. Example 9.3.4

$[\sigma_0]$: cycle associated to the zero section cf. Definition 9.3.5

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