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Kernel of the variation operator and periodicity of open books

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Abstract

We consider a parallelizable $2n$-manifold F which has the homotopy type of the wedge product of n-spheres and show that the group of pseudo-isotopy classes of orientation preserving diffeomorphisms that keep the boundary ∂F pointwise fixed and induce the trivial variation operator is a central extension of the group of all homotopy $(2n+1)$-spheres by $H_n(F; S\pi_n(SO(n)))$. Then we apply this result to study the periodicity properties of branched cyclic covers of manifolds with simple open book decompositions and extend the previous results of Durfee, Kauffman and Stevens to dimensions 7 and 15.

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1. Introduction and the results

An open book decomposition of a manifold M^{m+1} is a presentation of this manifold as the union of the mapping torus F_φ and the product $\partial F \times D^2$ along the boundary $\partial F \times S^1$,

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where $\varphi : F^m \to F^m$ is an orientation preserving diffeomorphism which fixes the boundary ∂F pointwise. Open book structures have been used in the study of various topological problems (for short historical overviews see §2 of [22] or Appendix by Winkelnkemper in [23]), and in particular in the study of the isolated complex hypersurface singularities. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial mapping with the only singular point at the origin and with zero locus $V = \{z \in \mathbb{C}^{n+1} \mid f(z) = 0\}$. Consider the intersection of V with a small sphere centered at the origin $K := V \cap S^{2n+1}_\varepsilon$. Milnor has shown in [19] that the mapping

$$\Phi(z) := f(z)/\left|f(z)\right|, \qquad S^{2n+1}_\varepsilon \setminus K \to S^1$$

is the projection map of a smooth fibration such that the fiber $F := \Phi^{-1}(1)$ is a smooth $(n-1)$-connected parallelizable $2n$-manifold homotopically equivalent to the wedge product of n-spheres and $\partial F = K$ is $(n-2)$-connected. This gives the open book structure to the sphere

$$S^{2n+1} = F_\varphi \cup \left(K \times D^2\right).$$

Such an open book decomposition of S^{2n+1} is called a simple fibered knot and the periodicity, in k, of the k-fold cyclic covers of S^{2n+1} branched along K has been studied by Durfee and Kauffman in [8]. Later, Stevens (see [28], Theorem 7 and Proposition 8) generalized Theorems 4.5 and 5.3 of [8] to a wider class of manifolds with simple open book decompositions $M^{2n+1} = F_\varphi \cup (\partial F \times D^2)$ (an open book M^{2n+1} is called *simple* if both M and F are $(n-1)$-connected and M bounds a parallelizable manifold).

Theorem I (Stevens). *Let M_k denote the k-fold cyclic cover of M^{2n+1} branched along ∂F and $n \neq 1$, 3 or 7 odd. If $\operatorname{Var}(\varphi^d) = 0$, then M_k and M_{k+d} are (orientation preserving) homeomorphic, while M_k and M_{d-k}, $k < d$ are orientation reversing homeomorphic. Furthermore, M_{k+d} is diffeomorphic to $(\sigma_d/8)\Sigma \# M_k$.*

Here σ_k is the signature of a parallelizable manifold N_k with the boundary $\partial N_k = M_k$, and Σ is the generator of the finite cyclic group bP_{2n+2} of homotopy $(2n+1)$-spheres that bound parallelizable manifolds. $\operatorname{Var}(h)$ denotes the variation homomorphism of a diffeomorphism $h : F \to F$, which keeps the boundary ∂F pointwise fixed, and defined as follows. Let $[z] \in H_n(F, \partial F)$ be the homology class of a relative cycle z, then one defines $\operatorname{Var}(h) : H_n(F, \partial F) \to H_n(F)$ by the formula $\operatorname{Var}(h)[z] := [h(z) - z]$ (cf. §1 of [28] or §1.1 of [1]).

Stevens also proved topological as well as smooth periodicity for n even (see [28], Theorem 9):

Theorem II. *If for branched cyclic covers M_k of a $(2n+1)$-manifold M with simple open book decomposition $\operatorname{Var}(\varphi^d) = 0$, then M_k and M_{k+2d} are homeomorphic and M_k and M_{k+4d} are diffeomorphic. Moreover, if $n = 2$ or 6, then M_k and M_{k+d} are diffeomorphic.*

Both of the papers viewed the open book M^{2n+1} as the boundary of a $(2n+2)$-manifold and used results of Wall [32], on classification of $(n-1)$-connected $(2n+1)$-manifolds.

Here, in the third section, we are dealing with the same periodicity problems from a different point of view which is based on results of Kreck [15] on the group of isotopy classes of diffeomorphisms of $(n-1)$-connected almost-parallelizable $2n$-manifolds. We give here different proofs of these two theorems of Stevens including the cases $n=3$ and $n=7$ (see Corollaries 2, 3, and 4 below).

As we have just mentioned, our approach is based on the results of Kreck who has computed the group of isotopy classes of diffeomorphisms of closed $(n-1)$-connected almost-parallelizable $2n$-manifolds in terms of exact sequences. In the first part of this paper we use these results to obtain a similar exact sequence for the diffeomorphisms f of a parallelizable handlebody $F \in \mathcal{H}(2n, \mu, n)$, $n \geqslant 2$, that preserve the boundary ∂F pointwise and induce the trivial variation operator $\mathrm{Var}(f): H_*(F, \partial F) \to H_*(F)$. We will denote the group of pseudo-isotopy classes of such diffeomorphisms by $\tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial)$ and prove the following

Theorem 3. *If $n \geqslant 3$ then the following sequence is exact*

$$0 \to \Theta_{2n+1} \to \tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial) \to \mathrm{Hom}\big(H_n(F, \partial F), S\pi_n\big(SO(n)\big)\big) \to 0.$$

If $n=2$ then $\tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial) = 0$.

Here, by $S\pi_n(SO(n))$ we mean the image of $\pi_n(SO(n))$ in $\pi_n(SO(n+1))$ under the natural inclusion $SO(n) \hookrightarrow SO(n+1)$ and by Θ_{2n+1} the group of all homotopy $(2n+1)$-spheres (see Section 2.2 for the details).

Remark. Recently Crowley [7] extended results of Wilkins on the classification of closed $(n-1)$-connected $(2n+1)$-manifolds, $n=3,7$. One could use these results together with the technique of Durfee, Kauffman and Stevens to complete the periodicity theorems for $n=3,7$. However our intention was to show how one can apply the higher dimensional analogs of the mapping class group in studying this kind of problem.

At the end we briefly mention the cyclic coverings of S^3 branched along the trefoil knot as an example which shows that there is no topological periodicity in the case $n=1$.

Let F be a manifold with boundary ∂F and consider two diffeomorphisms φ, ψ of F that are identities on the boundary (in this paper we consider only orientation preserving diffeomorphisms). As usual, two such diffeomorphisms are called *pseudo-isotopic relative to the boundary* if there is a diffeomorphism $\mathcal{H}: F \times I \to F \times I$ which satisfies the following properties:

$$(1)\ \mathcal{H}|_{F\times\{0\}} = \varphi, \qquad (2)\ \mathcal{H}|_{F\times\{1\}} = \psi, \qquad (3)\ \mathcal{H}|_{\partial F \times I} = id.$$

We will denote the group of pseudo-isotopy classes of such diffeomorphisms by $\tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial)$. The group of pseudo-isotopy classes of orientation preserving diffeomorphisms on a closed manifold M will be denoted by $\tilde{\pi}_0\,\mathrm{Diff}(M)$. There is a deep result of Cerf [6] which allows one to replace pseudo-isotopy by isotopy provided that the manifold is simply connected and of dimension at least six. All our manifolds are simply connected here, so $n=2$ is the only case when we actually use pseudo-isotopy. For all other $n \geqslant 3$ we will use the same notations (where tilde ˜ stands for “pseudo”) but mean the usual isotopy. We will call these groups *the mapping class groups*.

If M is embedded into W as a submanifold, then the normal bundle of M in W will be denoted by $\nu(M; W)$. Integer coefficients are understood for all homology and cohomology groups, unless otherwise stated, and symbols $\simeq$ and $\cong$ are used to denote diffeomorphism and isomorphism, respectively.

2. Kernel of the variation operator

2.1. Double of a pair (X, A)

Let (X, A) be a pair of CW complexes, and consider the pair $(X \times I, A \times I)$ (here and later $I = [0, 1]$, and we denote the boundary of I by ∂I).

Definition 1. The subspace $(X \times \partial I) \cup (A \times I)$ of $X \times I$ will be called the double of the pair (X, A), and denoted by $\mathcal{D}_{\mathcal{A}} X$.

We will denote the pair $(X \times \{0\}, A \times \{0\})$ by (X_0, A_0), the product $A \times I$ by A_+ and the union $(X \times \{1\}) \cup A_+$ by X_+. Thus we can write $\mathcal{D}_{\mathcal{A}} X = X_0 \cup X_+$ and $X_0 \cap X_+ = A \times \{0\}$.

Remark. If we take the pair (X, A) to be a manifold with the boundary, then the double $\mathcal{D}_{\mathcal{A}} X$ will be the boundary of the product $X \times I$, which is a closed manifold with the canonically defined smooth structure (see [21]). In this case we will denote the double simply by $\mathcal{D} X$.

Now we construct a natural homomorphism $d_* : H_*(X, A) \to H_*(\mathcal{D}_{\mathcal{A}} X)$. Consider the reduced suspensions of X and A (the common base point is chosen outside of X) and the induced isomorphism between $H_*(X, A)$ and $H_{*+1}(\Sigma X^+, \Sigma A^+)$. The excision property induces a natural isomorphism between $H_{*+1}(\Sigma X^+, \Sigma A^+)$ and $H_{*+1}(X \times I, \mathcal{D}_{\mathcal{A}} X)$, and we define the homomorphism d_* as the composition of these two isomorphisms with the boundary map δ_{*+1} from the exact sequence of the pair $(X \times I, \mathcal{D}_{\mathcal{A}} X)$:

Definition 2.

$$d_q := \delta_{q+1} \circ iso : H_q(X, A) \xrightarrow{\cong} H_{q+1}(X \times I, \mathcal{D}_{\mathcal{A}} X) \xrightarrow{\delta_{q+1}} H_q(\mathcal{D}_{\mathcal{A}} X).$$

The groups $H_*(X, A)$ and $H_*(\mathcal{D}_{\mathcal{A}} X, X)$ are naturally isomorphic and we can rewrite the exact sequence of the pair $(\mathcal{D}_{\mathcal{A}} X, X)$ in the following form:

$$\cdots H_{q+1}(\mathcal{D}_{\mathcal{A}} X) \to H_{q+1}(X, A) \to H_q(X) \xrightarrow{i_q} H_q(\mathcal{D}_{\mathcal{A}} X) \xrightarrow{j_q} H_q(X, A) \cdots.$$

Lemma 1. *For each $q \geqslant 1$ the homomorphism d_q is a splitting homomorphism of the above exact sequence and we have the following short exact sequence that splits*:

$$0 \to H_q(X) \xrightarrow{i_q} H_q(\mathcal{D}_{\mathcal{A}} X) \xrightarrow{j_q} H_q(X, A) \to 0.$$

Proof. It follows rather easily from our definition of d_q that for each $q \geqslant 1$ the composition $j_q \circ d_q$ is the identity map of the group $H_q(X, A)$. This property entails our lemma (cf. [24], Chapter 5, §1.5). □

Let us consider now a homeomorphism $f : X \to X$ which is the identity on A, i.e. $f(x) = x$ for all $x \in A$. For such a map the variation homomorphism $\mathrm{Var}(f) : H_q(X, A) \to H_q(X)$ is defined for all $q \geqslant 1$ by the formula $\mathrm{Var}(f)[z] := [f(z) - z]$ for any relative cycle $z \in H_q(X, A)$ (cf. §1 of [28] or §1.1 of [1]). The map f also induces the map $f^{(r)} : (X, A) \to (X, A)$ and a map $\tilde{f} : \mathcal{D}_\mathcal{A} X \to \mathcal{D}_\mathcal{A} X$ defined as follows:

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in X_0, \\ x & \text{if } x \in X_+. \end{cases}$$

If we denote the corresponding induced maps in homology by f_*, $f_*^{(r)}$, $\tilde{f}_*$ then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_q(X) & \xrightarrow{i_q} & H_q(\mathcal{D}_\mathcal{A} X) & \xrightarrow{j_q} & H_q(X, A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \tilde{f}_*} & & \downarrow{\scriptstyle f_*^{(r)}} & & \\ 0 & \longrightarrow & H_q(X) & \xrightarrow{i_q} & H_q(\mathcal{D}_\mathcal{A} X) & \xrightarrow{j_q} & H_q(X, A) & \longrightarrow & 0 \end{array}$$

Theorem 1. *If* $\mathrm{Var}(f) = 0$, *then* $\tilde{f}_*$ *is the identity map of* $H_q(\mathcal{D}_\mathcal{A} X)$ *for all* q.

Proof. It follows right from the definition of $\mathrm{Var}(f)$ that $f_* - Id = \mathrm{Var}(f) \circ l_*$ and $f_*^{(r)} - Id = l_* \circ \mathrm{Var}(f)$, where $l_* : H_*(X) \to H_*(X, A)$ is induced by the inclusion $(X, \emptyset) \hookrightarrow (X, A)$ (cf. §1.1 of [1]). It is also easy to check that the homomorphisms $\tilde{f}_*$ and d_q are connected with the variation homomorphism via the formula

$$\tilde{f}_* \circ d_q = d_q \circ Id + i_q \circ \mathrm{Var}(f).$$

Hence if $\mathrm{Var}(f) = 0$, then $f_* = Id$, $f_*^{(r)} = Id$ and $\tilde{f}_* \circ d_q = d_q \circ Id$. These three identities together with $j_q \circ d_q = Id$ imply the statement. □

Now we restrict our attention to the case when X is a smooth, simply connected manifold of dimension at least four and $A = \partial X$ is the boundary. Let $\varphi \in \mathrm{Diff}(X, \mathrm{rel}\, \partial)$ and $\tilde{\varphi} \in \mathrm{Diff}(\mathcal{D}X)$ be the extension by the identity to the second half of the double. Define the map $\omega : \mathrm{Diff}(X, \mathrm{rel}\, \partial) \to \mathrm{Diff}(\mathcal{D}X)$ by the formula $\omega(\varphi) := \tilde{\varphi}$.

Theorem 2. *The map* ω *induces a monomorphism* $\tilde{\pi}_0 \,\mathrm{Diff}(X, \mathrm{rel}\, \partial) \to \tilde{\pi}_0 \,\mathrm{Diff}(\mathcal{D}X)$.

Proof. It is easy to see that ω induces a well-defined map of groups of pseudo-isotopy classes of diffeomorphisms, i.e., if φ' is pseudo-isotopic relative to the boundary to φ then $\omega(\varphi')$ is pseudo-isotopic to $\omega(\varphi)$. It is obvious that for any two diffeomorphisms $\varphi, \psi \in \mathrm{Diff}(X, \mathrm{rel}\, \partial)$, $\omega(\varphi \cdot \psi) = \omega(\varphi) \cdot \omega(\psi)$, that is ω induces a homomorphism which we also denote by ω.

To show that ω is actually a monomorphism we use Proposition 1 of Kreck (see [15, p. 650] for the details): *Let A^m be a simply-connected manifold with $m \geqslant 5$ and $h \in \mathrm{Diff}(\partial A)$. h can be extended to a diffeomorphism on A if and only if the twisted double $A \cup_h -A$ bounds a 1-connected manifold B such that all relative homotopy groups $\pi_k(B, A)$ and $\pi_k(B, -A)$ are zero, where A and $-A$ mean the two embeddings of A into the twisted double.* Suppose now that $\omega(\varphi) = \tilde{\varphi}$ is pseudo-isotopic to the identity. Then the mapping torus $\mathcal{D}X_{\tilde{\varphi}}$ is diffeomorphic to the product $\mathcal{D}X \times S^1 = \partial(X \times I \times S^1)$. On the other hand, we can present $\mathcal{D}X_{\tilde{\varphi}}$ as the union of X_φ and $-X \times S^1$ along the boundary $\partial X \times S^1$. Since $\partial(X \times D^2) = X \times S^1 \cup -\partial X \times D^2$ we can paste together $X \times I \times S^1$ and $X \times D^2$ along the common sub-manifold $X \times S^1$ to obtain a new manifold W, which cobounds $X_\varphi \cup -\partial X \times D^2$. Now note that $X_\varphi \cup -\partial X \times D^2$ is diffeomorphic to the twisted double $X \times I \cup_h -X \times I$ where the diffeomorphism $h : \partial(X \times I) \to \partial(X \times I)$ is defined by the identities: $h|_{X_0} = id$, and $h|_{X_+} = \varphi$ (cf. [15], Property 1 of $\widetilde{W}$ on p. 657). The theorem of Seifert and Van Kampen entails that $\pi_1(W) \cong \{1\}$, and hence $\pi_1(W, X \times I) \cong \{1\}$. To show that the other homotopy groups are trivial it is enough to show that $H_*(W, X) \cong \{0\}$ for all $* \geqslant 2$. This can be seen from the relative Mayer–Vietoris exact sequence of pairs $(X \times I \times S^1, X)$ and $(X \times D^2, X)$ where by X we mean a fiber of the product

$$X \times S^1 : H_*\big(X \times S^1, X\big) \xrightarrow{\cong} H_*\big(X \times I \times S^1, X\big) \oplus H_*\big(X \times D^2, X\big) \to H_*(W, X).$$

Thus by Proposition 1 of [15], there is a diffeomorphism of $X \times I$ to itself that gives the required pseudo-isotopy between φ and id. □

2.2. $\tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial)$ *as an extension*

We now let $F \in \mathcal{H}(2n, \mu, n)$ be a parallelizable handlebody, that is, a parallelizable manifold which is obtained by gluing μ n-handles to the $2n$-disk and rounding the corners:

$$F = D^{2n} \cup \bigsqcup_{i=1}^{\mu} \big(D_i^n \times D^n\big).$$

We assume here that $n \geqslant 2$. For the classification of handlebodies in general, see [30]. Obviously F has the homotopy type of the wedge product of n-spheres and nonempty boundary ∂F which is $(n-2)$-connected. The Milnor fibre of an isolated complex hypersurface singularity is an example of such a manifold.

Let us consider now $\varphi \in \tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial)$ and the induced variation homomorphism $\mathrm{Var}(\varphi) : H_n(F, \partial F) \to H_n(F)$. This correspondence gives a well defined map

$$\mathrm{Var} : \tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial) \to \mathrm{Hom}\big(H_n(F, \partial F), H_n(F)\big)$$

which is a derivation (1-cocycle) with respect to the natural action of the group $\tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial)$ on $\mathrm{Hom}(H_n(F, \partial F), H_n(F))$ (cf. [28], §2)

$$\mathrm{Var}(h \circ g) = \mathrm{Var}(h) + h_* \circ \mathrm{Var}(g).$$

This formula implies that the isotopy classes of diffeomorphisms that give trivial variation homomorphisms form a subgroup of $\tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial)$.

Definition 3. The subgroup

$$\tilde{\pi}_0 V\,\mathrm{Diff}(F,\partial) := \{ f \in \tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial) \mid \mathrm{Var}(f)[z] = 0,\ \forall [z] \in H_n(F, \partial F) \}$$

will be called the kernel of the variation operator.

In order to describe the algebraic structure of this kernel we will use the results of Kreck [15] who has computed the group of isotopy classes of diffeomorphisms of closed oriented $(n-1)$-connected almost-parallelizable $2n$-manifolds in terms of exact sequences. First we note that the double of our handlebody F is such a manifold.

Lemma 2. *Let $F \in \mathcal{H}(2n, \mu, n)$ be a parallelizable handlebody $(n \geqslant 2)$, then the double $\mathcal{D}F$ is a closed $(n-1)$-connected stably-parallelizable $2n$-manifold.*

Proof. Since F is simply connected and $\mathcal{D}F = F_0 \cup F_+$, we have $\pi_1(\mathcal{D}F) = 0$. Then using the exact homology sequence of the pair $(F \times I, \partial(F \times I))$ it can be easily seen that $\mathcal{D}F$ is a $(n-1)$-connected manifold. Since F is parallelizable the double will be stably-parallelizable. □

Next we recall the result of Kreck [15]. Let M be a smooth, closed, oriented $(n-1)$-connected almost-parallelizable $2n$-manifold, $n \geqslant 2$. Denote by $\mathrm{Aut}\,H_n(M)$ the group of automorphisms of $H_n(M, \mathbb{Z})$ preserving the intersection form on M and (for $n \geqslant 3$) commuting with the function $\alpha : H_n(M) \to \pi_{n-1}(SO(n))$, which is defined as follows. Represent $x \in H_n(M)$ by an embedded sphere $S^n \hookrightarrow M$. Then function α assigns to x the classifying map of the corresponding normal bundle. Any diffeomorphism $f \in \mathrm{Diff}(M)$ induces a map f_* which lies in $\mathrm{Aut}\,H_n(M)$. This gives a homomorphism

$$\kappa : \tilde{\pi}_0\,\mathrm{Diff}(M) \to \mathrm{Aut}\,H_n(M), \quad [f] \mapsto f_*.$$

The kernel of κ is denoted by $\tilde{\pi}_0 S\,\mathrm{Diff}(M)$ and to each element f from this kernel Kreck assigns a homomorphism $H_n(M) \to S\pi_n(SO(n))$, where $S : \pi_n(SO(n)) \to \pi_n(SO(n+1))$ is induced by the inclusion, in the following way. Represent $x \in H_n(M)$ by an imbedded sphere $S^n \subset M$ and use an isotopy to make $f|_{S^n} = Id$. The stable normal bundle $\nu(S^n) \oplus \varepsilon^1$ of this sphere in M is trivial and therefore the differential of f gives an element of $\pi_n(SO(n+1))$. It is easy to see that this element lies in the image of S. This construction leads to a well defined homomorphism (cf. Lemma 1 of [15])

$$\chi : \tilde{\pi}_0 S\,\mathrm{Diff}(M) \to \mathrm{Hom}\big(H_n(M), S\pi_n\big(SO(n)\big)\big).$$

If $n = 6$ we have $S\pi_n(SO(n)) = 0$, and for all other $n \geqslant 3$ the groups $S\pi_n(SO(n))$ are given in the following table [15, p. 644]:

n (mod 8)	0	1	2	3	4	5	6	7
$S\pi_n(SO(n))$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$

When $n \equiv 3$ (mod 4) the homomorphism $\chi(f)$ can be defined using the Pontryagin class $p_{(n+1)/4}(M_f)$ of the mapping torus M_f. For the details the reader is referred to Lemma 2 of [15].

If M^{2n} bounds a parallelizable manifold and $n \geqslant 3$, then Theorem 2 of [15] gives two short exact sequences:

$$0 \to \tilde{\pi}_0 S\,\mathrm{Diff}(M) \to \tilde{\pi}_0\,\mathrm{Diff}(M) \xrightarrow{\kappa} \mathrm{Aut}\,H_n(M) \to 0, \tag{1}$$

$$0 \to \Theta_{2n+1} \xrightarrow{\iota} \tilde{\pi}_0 S\,\mathrm{Diff}(M) \xrightarrow{\chi} \mathrm{Hom}\big(H_n(M), S\pi_n\big(SO(n)\big)\big) \to 0, \tag{2}$$

where the map ι is induced by the identification of each homotopy $(2n+1)$-sphere with the element of the mapping class group $\tilde{\pi}_0\,\mathrm{Diff}(D^{2n}, \mathrm{rel}\,\partial)$.

If M is a simply connected manifold of dimension 4, Kreck has proved that κ is a monomorphism ([15], Theorem 1).

Let $F \in \mathcal{H}(2n, \mu, n)$ be a parallelizable handlebody as above, and $\mathcal{D}F$ be the corresponding double. First assume that $n = 2$ and $\varphi \in \tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial)$, then it follows from our Theorems 1 and 2 and Theorem 1 of Kreck [15] that $\tilde{\varphi}$ is the trivial element of $\tilde{\pi}_0\,\mathrm{Diff}(\mathcal{D}F)$, and therefore φ is the identity of $\tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial)$.

Remark. In this case, the handlebody F does not have to be parallelizable and the kernel of the variation operator $\tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial)$ will be trivial for any simply connected 4-manifold F.

Next we consider the case when $n \geqslant 3$ and denote the group $S\pi_n(SO(n))$ by G. Recall also that we can assume that $\mathcal{D}F = F \cup F_+$. Since F is $(n-1)$-connected and the boundary ∂F is $(n-2)$-connected, the universal coefficient theorem together with the cohomology exact sequence of the pair $(\mathcal{D}F, F_+)$ and the excision property give us the following short exact sequence:

$$0 \to \mathrm{Hom}\big(H_n(F, \partial F), G\big) \xrightarrow{j^*} \mathrm{Hom}\big(H_n(\mathcal{D}F), G\big) \xrightarrow{i^*} \mathrm{Hom}\big(H_n(F_+), G\big) \to 0, \tag{3}$$

where $i : F_+ \hookrightarrow \mathcal{D}F$, $j : (\mathcal{D}F, \emptyset) \hookrightarrow (\mathcal{D}F, F)$ are inclusions and i^* and j^* are the corresponding induced maps.

Lemma 3. *$i^*(\chi(\tilde{\varphi}))$ is the trivial map for any $\varphi \in \tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial)$.*

Proof. Take any $[z] \in H_n(F_+)$, then we have $i^*(\chi(\tilde{\varphi}))[z] = \chi(\tilde{\varphi})[i_*(z)]$. Since $H_n(F) \cong \pi_n(F)$ we can present our n-cycle $[z]$ by an imbedded $S^n \hookrightarrow F_+$ and we can also assume that the normal bundle of such a sphere is contained in F_+. We have defined $\tilde{\varphi}$ as the identity on F_+ and this implies $\chi(\tilde{\varphi})[i_*(z)] = 0$ as required. □

Now we define a homomorphism $\chi_r : \tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial) \to \mathrm{Hom}(H_n(F, \partial F), G)$. Take any $\varphi \in \tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial)$ then $\tilde{\varphi} \in \tilde{\pi}_0 S\,\mathrm{Diff}(\mathcal{D}F)$ (recall Theorem 1 above) and $\chi(\tilde{\varphi}) \in \mathrm{Hom}(H_n(\mathcal{D}F), G)$. Since $i^*(\chi(\tilde{\varphi})) = 0$ there exists unique $h \in \mathrm{Hom}(H_n(F, \partial F), G)$ such that $j^*(h) = \chi(\tilde{\varphi})$.

Definition 4. We define the map $\chi_r : \tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial) \to \mathrm{Hom}(H_n(F, \partial F), G)$ by the formula $\chi_r(\varphi) := h$.

It is clear that χ_r is a homomorphism. Here we also consider the map $\iota_r : \Theta_{2n+1} \to \tilde{\pi}_0 \operatorname{Diff}(F, \operatorname{rel} \partial)$ defined as in (2) above: present any homotopy $(2n+1)$-sphere Σ' as the union of two disks via a diffeomorphism $\psi \in \tilde{\pi}_0 \operatorname{Diff}(S^{2n}) \cong \tilde{\pi}_0 \operatorname{Diff}(D^{2n}, \operatorname{rel} \partial) \cong \Theta_{2n+1}$ then take a disk D^{2n} embedded into $\operatorname{int}(F)$ and define the diffeomorphism of F by the formula

$$\iota_r(\Sigma')(x) := \begin{cases} \psi(x) & \text{if } x \in D^{2n} \hookrightarrow F, \\ x & \text{otherwise.} \end{cases}$$

It is obvious that $\operatorname{Im}(\iota_r) \subset \tilde{\pi}_0 V \operatorname{Diff}(F, \partial)$. Now we describe $\tilde{\pi}_0 V \operatorname{Diff}(F, \partial)$ as a central extension of the group Θ_{2n+1} by $H^n(F, \partial F; G) \cong H_n(F; G)$.

Theorem 3. *If $n = 2$ then $\tilde{\pi}_0 V \operatorname{Diff}(F, \partial) = 0$, and for all $n \geqslant 3$ the following sequence is exact*

$$0 \to \Theta_{2n+1} \xrightarrow{\iota_r} \tilde{\pi}_0 V \operatorname{Diff}(F, \partial) \xrightarrow{\chi_r} \operatorname{Hom}\big(H_n(F, \partial F), G\big) \to 0. \tag{4}$$

Proof. We have mentioned already that if $n = 2$, the kernel of the variation operator is trivial. Assume now that $n \geqslant 3$. It follows from Theorems 1 and 2 above that the inclusion map $\omega : \operatorname{Diff}(F, \operatorname{rel} \partial) \to \operatorname{Diff}(\mathcal{D}F)$ induces a monomorphism $s_\omega : \tilde{\pi}_0 V \operatorname{Diff}(F, \partial) \to \tilde{\pi}_0 S \operatorname{Diff}(\mathcal{D}F)$. Since the composition $s_\omega \cdot \iota_r$ coincides with the injective map ι from the exact sequence (2), we see that our ι_r is injective too. It is also clear that $\operatorname{Im}(\iota_r) \subset \operatorname{Ker}(\chi_r)$. Consider now any $\varphi \in \operatorname{Ker}(\chi_r)$, then $\chi(s_\omega(\varphi)) = j^*(\chi_r(\varphi)) = 0$, where j^* is as in (3). Thus $s_\omega(\varphi) \in \operatorname{Ker}(\chi) \cong \Theta_{2n+1} \cong \operatorname{Im}(\iota)$ and since s_ω is a monomorphism we have $\varphi \in \operatorname{Im}(\iota_r)$ as required.

To prove that χ_r is an epimorphism it is enough to show that for a set of generators $\{g_1, \ldots, g_m\}$ of $\operatorname{Hom}(H_n(F, \partial F), G)$ the group $\tilde{\pi}_0 V \operatorname{Diff}(F, \partial)$ contains diffeomorphisms $\{\varphi_1, \ldots, \varphi_m\}$ such that $\chi_r(\varphi_j) = g_j$, $j \in \{1, \ldots, m\}$. Recall that $F = D^{2n} \cup \bigsqcup_{i=1}^{\mu} (D_i^n \times D^n)$ and $H_n(F, \partial F) \cong \mathbb{Z}^\mu$. We can choose the following embedded disks $d_i \hookrightarrow F$, $i \in \{1, \ldots, \mu\}$, as a basis of this homology group:

$$d_i := \{0\}_i \times D^n \hookrightarrow D_i^n \times D^n \hookrightarrow F$$

(here $\{0\}_i$ is the center of the ith handle core disk D_i^n). Take a generator x of G and consider the homomorphism $g_{xi} : H_n(F, \partial F) \to G$ defined by the formula

$$g_{xi}[d_k] := \begin{cases} x & \text{if } k = i, \\ 0 & \text{if } k \neq i, \end{cases} \quad k \in \{1, \ldots, \mu\}$$

end extended linearly to the whole group. The set of such homomorphisms obviously generates $\operatorname{Hom}(H_n(F, \partial F), G)$. Now we will use an analog of the Dehn twist in higher dimensions to construct the diffeomorphism φ_{xi} (cf. [30], Lemma 12).

For each disk d_k consider the "half-handle" $= (\frac{1}{2} D_k^n) \times D^n$ and notice that the closure of the complement to all these "half-handles" in F

$$\bar{\mathcal{F}} := cl\left(F \setminus \bigsqcup_{k=1}^{\mu} \left(\frac{1}{2} D_k^n\right) \times D^n\right)$$

is diffeomorphic to the closed $2n$-disk D^{2n}, and the intersection of each "half-handle" with the boundary $\partial \bar{\mathcal{F}} \simeq S^{2n-1}$ is $\partial(\frac{1}{2} D_k^n) \times D^n \simeq S_k^{n-1} \times D^n$. We take a smooth map $\varphi_x : (D^n, S^{n-1}) \to (SO(n), id)$ that sends a neighborhood of S^{n-1} to id and represents an element $[\varphi_x] \in \pi_n(SO(n))$ such that $S([\varphi_x]) = x$ and define the diffeomorphism $\varphi_{xi}|_{\bigsqcup_{k=1}^{\mu}(\frac{1}{2}D_k^n)\times D^n}$ by the formula

$$\varphi_{xi}(t,s) := \begin{cases} (\varphi_x(s)\circ t, s) & \text{if } (t,s) \in (\frac{1}{2}D_i^n)\times D^n, \\ (t,s) & \text{if } (t,s) \in (\frac{1}{2}D_k^n)\times D^n \text{ and } k \neq i. \end{cases} \tag{5}$$

In particular, this gives a diffeomorphism $\phi \in \mathrm{Diff}(\partial\bar{\mathcal{F}})$ which is defined on $S_i^{n-1}\times D^n \hookrightarrow \partial\bar{\mathcal{F}}$ by restricting t to the boundary of $\frac{1}{2}D_i^n$ (see (5) above) and by the identity everywhere else. We will show now that ϕ is isotopic to the identity. Consider the handlebody

$$F_i := D^{2n} \cup \left(D_i^n \times D^n\right) = cl\left(F \setminus \bigsqcup_{k=1,k\neq i}^{\mu}\left(\frac{1}{2}D_k^n\right)\times D^n\right)$$

and denote by $\hat{F}_i$ the manifold obtained from F_i by removing the open disk $\frac{1}{2}D^{2n}$ from D^{2n}. Hence $\partial\hat{F}_i \simeq \partial F_i \sqcup S^{2n-1}$. The first equation of (5) together with the identity map define a diffeomorphism Φ of $\hat{F}_i$ such that $\Phi|_{S^{2n-1}} = \phi$ and $\Phi|_{\partial F_i} = Id$. We use the identity again to extend this Φ to a diffeomorphism $\tilde{\Phi}$ of $\mathcal{D}\hat{F}_i$ where

$$\mathcal{D}\hat{F}_i := \mathcal{D}F_i \setminus \tfrac{1}{2}D^{2n} \simeq \hat{F}_i \cup_{\partial F_i} F_i \quad \text{and} \quad \tilde{\Phi}|_{F_i} = Id, \quad \tilde{\Phi}|_{\hat{F}_i} = \Phi.$$

Thus ϕ is the restriction of $\tilde{\Phi}$ to the boundary $\partial\mathcal{D}\hat{F}_i = S^{2n-1}$ and hence can be considered as an element of the inertia group of $\mathcal{D}F_i$ (cf. [15], Proposition 3). Now it follows from Lemma 2 above and results of Kosinski ([14], see §3) and Wall [29] that ϕ is isotopic to the identity. In particular, we can use this isotopy on $S^{2n-1}\times[\frac{1}{2},\frac{1}{4}] \subset \frac{1}{2}D^{2n}$ to extend the diffeomorphism $\varphi_{xi}|_{\bigsqcup_{k=1}^{\mu}(\frac{1}{2}D_k^n)\times D^n}$ to a diffeomorphism of the whole handlebody F. Denote the result of this extension by φ_{xi}. Clearly $\varphi_{xi} \in \tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial)$, and we leave it to the reader to check that $\chi_r(\varphi_{xi}) = g_{xi}$. □

Corollary 1. *We have the following commutative diagram*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Theta_{2n+1} & \xrightarrow{\iota_r} & \tilde{\pi}_0 V\,\mathrm{Diff}(F,\partial) & \xrightarrow{\chi_r} & \mathrm{Hom}(H_n(F,\partial F),G) & \longrightarrow & 0 \\
 & & \updownarrow\equiv & & \downarrow s_\omega & & \downarrow j^* & & \\
0 & \longrightarrow & \Theta_{2n+1} & \xrightarrow{\iota} & \tilde{\pi}_0 S\,\mathrm{Diff}(\mathcal{D}F) & \xrightarrow{\chi} & \mathrm{Hom}(H_n(\mathcal{D}F),G) & \longrightarrow & 0 \\
 & & & & \downarrow i^*\cdot\chi & & \downarrow i^* & & \\
 & & & & \mathrm{Hom}(H_n(F_+),G) & \xleftrightarrow{\equiv} & \mathrm{Hom}(H_n(F_+),G) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where all horizontal and vertical sequences are exact.

Proof. The standard diagram chasing procedure is left to the reader. □

Example 1. Consider the case when $F = S^3 \times D^3$. Then $\mathcal{D}F = S^3 \times S^3$, $\mathrm{Hom}(H_n(F, \partial F), G) \cong G \cong \mathbb{Z}$, $\Theta_7 \cong \mathbb{Z}_{28}$ and $\tilde{\pi}_0 S\,\mathrm{Diff}(\mathcal{D}F) \cong \mathcal{H}_{28}$, that is the factor group of the group $\mathcal{H}$ (upper unitriangular 3×3 matrices with integer coefficients) modulo the cyclic subgroup $28\mathbb{Z}$, where $\mathbb{Z}$ is the center of $\mathcal{H}$ (cf. [9] or §1.3 of [16]). Thus $\tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial) \cong S\pi_3(SO(3)) \oplus \Theta_7 \cong \mathbb{Z} \oplus \mathbb{Z}_{28}$ and the first vertical short exact sequence from the previous corollary can be written as follows

$$0 \to S\pi_3\big(SO(3)\big) \oplus \Theta_7 \to \tilde{\pi}_0 S\,\mathrm{Diff}(S^3 \times S^3) \to S\pi_3\big(SO(3)\big) \to 0.$$

Such exact sequence was obtained by Levine ([17], Theorems 2.4 and 3.3) and Sato ([27], Theorem II) for the group $\tilde{\pi}_0 S\,\mathrm{Diff}(S^p \times S^p)$. See another example at the end of Section 3.1 where it is shown that the extension of Theorem 3 can be nontrivial.

3. Manifolds with open book decompositions

3.1. Periodicity in higher dimensions

In this section we will apply our exact sequence (4) to study the periodicity of branched cyclic covers of manifolds with open book decompositions.

Definition 5. We will say that a smooth closed $(m+1)$-dimensional manifold M has an open book decomposition if it is diffeomorphic to the union

$$M \simeq F_\varphi \cup_r \big(\partial F \times D^2\big),$$

where F is m-dimensional manifold with boundary ∂F, $\varphi \in \mathrm{Diff}(F, \mathrm{rel}\,\partial)$ is an orientation preserving diffeomorphism of F that keeps the boundary pointwise fixed, F_φ is the mapping torus of φ

$$F_\varphi := F \times [0,1]/(x,0) \sim \big(\varphi(x), 1\big)$$

and $r : \partial F_\varphi \to \partial F \times S^1$ is a diffeomorphism that makes the following diagram commute

$$\begin{CD}
\partial F_\varphi @>{in}>> F_\varphi \\
@V{r}VV @VV{\pi}V \\
\partial F \times S^1 @>{p_2}>> S^1
\end{CD}$$

(here p_2 is the projection onto the second factor and π is the bundle projection of the mapping torus onto the base circle).

Such a union is also called the relative mapping torus with page F and binding ∂F (cf. [22] or [28]). When M has dimension $(2n+1)$ and F has the homotopy type of a n-dimensional CW-complex, it is said that the page is *almost canonical*. The diffeomorphism φ is called the geometric monodromy and the induced map $\varphi_* : H_n(F) \to H_n(F)$ is the (algebraic) monodromy. If instead of φ we take some positive power of this diffeomorphism, say φ^k, we obtain the k-fold cyclic cover M_k of M, branched along ∂F, i.e.

$$M_k = F_{\varphi^k} \cup_r \left(\partial F \times D^2\right).$$

It was shown in [8] (Theorem 4.5) that if a fibered knot ∂F is a rational homology sphere and $\varphi^d = id$ for some $d > 0$, then the k-fold cyclic covers M_k of S^{2n+1} branched along ∂F have the periodic behavior in d. In case of the links of isolated complex polynomial singularities these restrictions on ∂F and φ are equivalent to the condition $\mathrm{Var}(\varphi^d) = 0$.

Remarks.

(i) Notice that the conditions $\varphi_*^d = id$ and ∂F is a rational homology sphere imply that $\mathrm{Var}(\varphi^d) = 0$, but the converse is not true (see [28, p. 231]).
(ii) Proposition 3.3 of [11] proves that an open book M^{2n+1} with page F and monodromy φ is a homotopy sphere if and only if $\mathrm{Var}(\varphi)$ is an isomorphism.

In addition to the almost canonical page requirement we will need to assume more about M (cf. [28], §3, p. 232), i.e. we assume from now on that M has a *simple open book decomposition*. It implies, in particular, that M bounds a simply connected parallelizable manifold. We will also assume that $n \geqslant 3$, $\pi_1(\partial F) = 1$ and $\mathrm{Var}(\varphi^d) = 0$ for some $d \geqslant 1$ (where φ is the diffeomorphism that gives M the open book structure). A parallelizable simply connected manifold bounded by M will be denoted by N.

Before we give proofs of the periodicity theorems (Corollaries 2, 3 and 4 below) we will first obtain some auxiliary results. It is clear that F is a parallelizable manifold. Take now any $z \in H_n(F, \partial F) \cong \pi_n(F, \partial F)$ and choose an embedded disk $(D^n, \partial D^n) \hookrightarrow (F, \partial F)$ that represents this relative cycle. Inside of $\mathcal{D}F = F \cup F_+$ we consider the double $\mathcal{D}D^n = D^n \cup D^n_+$, and since the boundary ∂D^n has trivial normal bundle in ∂F we can add to F one n-handle along this sphere to obtain the manifold $F(z) := F \cup (D^n_+ \times d^n)$. As we have done above, we extend a diffeomorphism $\varphi \in \mathrm{Diff}(F, \mathrm{rel}\,\partial)$ to a diffeomorphism $\varphi_z \in \mathrm{Diff}(F(z), \mathrm{rel}\,\partial)$ using the identity on $D^n_+ \times d^n$. Then we obviously have $\mathcal{D}D^n \hookrightarrow F(z) \hookrightarrow \mathcal{D}F$ and $\varphi_z = \tilde{\varphi}|_{F(z)}$.

Lemma 4. *The mapping torus F_φ of φ is framed if and only if the mapping torus $F(z)_{\varphi_z}$ of φ_z is framed.*

Proof. We will show that any framing of F_φ can be extended to a framing of $F(z)_{\varphi_z}$. The other direction is trivial. Since φ is the identity on the boundary, we have $S^{n-1} \times S^1 \hookrightarrow \partial F_\varphi = (\partial F) \times S^1$, where S^{n-1} is the boundary of our relative homology class z. We can assume that F has a collar $\partial F \times [0,1]$ and φ is the identity map on this collar. Now we have $D^n \hookrightarrow F \hookrightarrow F \cup (\partial F \times [0,1])$ and we use the disk theorem to change φ by an isotopy to a diffeomorphism φ' such that $\varphi'|_{D^{2n}} = \varphi'|_{\partial F \times 1} = id$ and $D^n \subset \mathrm{int}(D^{2n}) \subset F \cup (\partial F \times [0, \frac{1}{2}])$. Then clearly $F_{\varphi'} \simeq F_\varphi$ and $D^n \times S^1 \hookrightarrow F_{\varphi'}$ with the trivial normal bundle. Furthermore since $S^{n-1} \times [0,1] \hookrightarrow \partial F \times [0,1]$ with trivial normal bundle too, we can connect $\partial D^n \times S^1 \hookrightarrow F_{\varphi'}$ with $S^{n-1} \times S^1 \hookrightarrow (\partial F \times 1) \times S^1 = \partial(F_{\varphi'})$, using the collar $(S^{n-1} \times [0,1]) \times S^1$. This implies that the trivial normal bundle of $S^{n-1} \times S^1$ in $\partial(F_\varphi)$

comes from the trivial normal bundle of $D^n \times S^1$ in F_φ. Now notice that the mapping torus $F(z)_{\varphi_z}$ is the union of F_φ and $D^n_+ \times d^n \times S^1$ along $S^{n-1} \times d^n \times S^1 \hookrightarrow \partial(F_\varphi)$. Therefore the restriction of the framing of F_φ to $S^{n-1} \times d^n \times S^1 = (\partial D^n) \times d^n \times S^1$ (where $D^n \times d^n \times S^1 \hookrightarrow F_\varphi$) can be extended to a framing of $F(z)_{\varphi_z}$. □

Theorem 4 (n is odd, $\neq 1$). *Suppose* $[\psi] \in \tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial)$ *and* $M^{2n+1} \simeq F_\psi \cup_r (\partial F \times D^2)$ *bounds a parallelizable manifold* N. *Then* $\chi_r(\psi) = 0$.

Proof. It is enough to show that $\chi_r(\psi)[z] = 0$ for an arbitrary relative homology class $z \in H_n(F, \partial F)$. As we just did above, we represent such a class by an embedded disk $(D^n, \partial D^n) \hookrightarrow (F, \partial F)$ and take the double $\mathcal{D}D^n = S^n \hookrightarrow \mathcal{D}F$. We will denote this double by dz (*to avoid cumbersome notations we denote by* dz *both the homology class and the embedded sphere* $\mathcal{D}D^n$ *that represents this class*) and its normal bundle in $\mathcal{D}F$ by $\nu(dz; \mathcal{D}F)$ respectively. Note that $\nu(dz; \mathcal{D}F)$ is trivial. The proof now splits into two parts.

(1) $n \geqslant 5$: It is clear that $\psi_{z*} = id$ on $H_n(F(z))$ and we can isotope ψ_z to a diffeomorphism ψ'_z such that $\psi'_z|_{dz} = id$ (see [10]). Extending this new diffeomorphism by the identity to the diffeomorphism $\tilde{\psi}' \in \mathrm{Diff}(\mathcal{D}F)$ we obtain an element of $\tilde{\pi}_0 S\,\mathrm{Diff}(\mathcal{D}F)$ which pointwise fixes dz and maps $F(z)$ to itself. Now it follows from the commutative diagram of Corollary 1 that it is enough to show that $\chi(\tilde{\psi}')[dz] = 0$. Since by Lemma 4 the mapping torus $F(z)_{\psi'_z}$ is framed, the normal bundle $\nu(dz \times S^1; \mathcal{D}F_{\tilde{\psi}'})$ is stably trivial. Since n is odd, the map $G = S\pi_n(SO(n)) \hookrightarrow \pi_n(SO(n+1)) \to \pi_n(SO(n+2))$ is a monomorphism (see [31]) and therefore the map

$$l^* : \mathrm{Hom}\big(H_n(\mathcal{D}F), G\big) \to \mathrm{Hom}\big(H_n(\mathcal{D}F), \pi_n(SO)\big)$$

is a monomorphism too. Hence $l^*(\chi(\tilde{\psi}'))[dz]$ is the obstruction to triviality of the stable normal bundle $\nu(dz \times S^1; \mathcal{D}F_{\tilde{\psi}'})$ and since this bundle is trivial we have $\chi(\tilde{\psi}')[dz] = 0$, as required.

(2) $n \equiv 3 \pmod 8$: Since $\partial N = M^{2n+1} \simeq F_\psi \cup_r (\partial F \times D^2)$ and $\partial(F \times D^2) = (\partial F \times D^2) \cup (F \times S^1)$, we can paste together the manifolds N and $F \times D^2$ along the common part of the boundary $\partial F \times D^2$ (respecting orientations of course) to obtain a manifold (after smoothing the corner)

$$W^{2n+2} := N \cup_{\partial F \times D^2} \big(F \times D^2\big) \quad \text{with } \partial W = F_\psi \cup \big(F \times S^1\big) \simeq \mathcal{D}F_{\tilde{\psi}}.$$

We use elementary obstruction theory to show that this W is stably parallelizable. Fix a frame field of the stable tangent bundle of $N \subset W$. Obstructions to the extension of this frame field over the whole manifold lie in the groups $H^{q+1}(W, N; \pi_q(SO)) \cong H^{q+1}(F, \partial F; \pi_q(SO)) \cong H_{2n-q-1}(F; \pi_q(SO))$. If $q = n-1$ or $q = 2n-1$ then $\pi_q(SO) \cong 0$ because $n \equiv 3 \pmod 8$, if $q \neq n-1$ or $q \neq 2n-1$ then $H_{2n-q-1}(F; \mathbb{Z}) \cong 0$ and all obstructions lie in the trivial groups anyway. Hence $\mathcal{D}F_{\tilde{\psi}}$ is stably parallelizable and $\chi(\tilde{\psi}) = 0$ (see [15], Lemma 2) which entails $\chi_r(\psi) = 0$. □

Now we can prove the following theorem of Stevens including the cases when $n = 3, 7$ (cf. [28], Theorem 7).

Corollary 2. *Let M_k be the k-fold branched cyclic cover of a $(2n+1)$-manifold $M = F_\varphi \cup_r (\partial F \times D^2)$ with simple open book decomposition, where n is odd, $\neq 1$. Suppose* $\mathrm{Var}(\varphi^d) = 0$*, then M_k and M_{k+d} are (orientation preserving) homeomorphic, while M_k and M_{d-k}, $d > k$, are orientation reversing homeomorphic.*

Proof. Since $\mathrm{Var}(\varphi^d) = 0$ and M_d bounds a parallelizable manifold (see Lemma 5 of [28]) we have $\chi_r(\varphi^d) = 0$ by the previous theorem. The exact sequence (4) implies that φ^d is isotopic to a diffeomorphism which belongs to the image $\iota(\Theta_{2n+1})$ and therefore $F_{\varphi^{d+k}}$ is diffeomorphic to $F_{\varphi^k}\#\Sigma'$ (cf. Lemma 1 of [2]) for some $\Sigma' \in \Theta_{2n+1}$. In particular, it means that $F_{\varphi^{d+k}}$ is homeomorphic (via some homeomorphism that preserves orientation) to F_{φ^k}, and hence M_{d+k} is homeomorphic to M_k. To see the orientation reversing case, notice that the mapping torus F_g is diffeomorphic to $F_{g^{-1}}$ via an orientation reversing diffeomorphism induced, for instance, by the map $(x,t) \mapsto (g(x), 1-t)$ from $F \times I$ to $F \times I$. This diffeomorphism extends to an orientation reversing homeomorphism of the corresponding open books M and M_{-1}. Hence in our situation $M_k = F_{\varphi^k} \cup_r (\partial F \times D^2)$ is homeomorphic (orient. revers.) to $F_{\varphi^{-k}} \cup_r (\partial F \times D^2)$ which is homeomorphic (orient. pres.) to $F_{\varphi^{-k}}\#\Sigma' \cup_r (\partial F \times D^2) \simeq F_{\varphi^{d-k}} \cup_r (\partial F \times D^2) = M_{d-k}$. □

Remark. If one defines $M_k := F_{\varphi^k} \cup_r (\partial F \times D^2)$ for any $k \in \mathbb{Z}$, then the first statement that M_k is homeomorphic to M_{k+d} remains true, and the restriction $d > k$ in the second part can be omitted.

To show diffeomorphism type periodicity we will basically use the same argument plus the fact that the homotopy sphere Σ' bounds a parallelizable manifold. We start with proving this fact. Thus for $n \in \mathbb{N}$, $n \geqslant 2$, we consider a diffeomorphism h with $[h] \in \tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial)$ such that our simple open book $M^{2n+1} = F_h \cup_r (\partial F \times D^2)$ bounds a simply connected parallelizable manifold N and $\chi_r(h) = 0$. In particular, we can assume that $h \in \mathrm{Im}(\iota)$ is the identity except on a small closed disk $\mathcal{D}^{2n} \hookrightarrow \mathrm{int}(F)$ embedded into the interior of F.

Lemma 5. *The natural inclusions $i_1 : F \hookrightarrow F_h$ and $i_2 : F_h \hookrightarrow M$ induce isomorphisms $i_{1*} : H_n(F) \to H_n(F_h)$ and $i_{2*} : H_n(F_h) \to H_n(M)$, respectively, and every $[z] \in H_n(M)$ can be represented by an embedded sphere $S^n \hookrightarrow M$ with trivial normal bundle $\nu(S^n; M)$. In addition, $H_n(M) \cong H_{n+1}(M)$.*

Proof. That i_{1*} is an isomorphism follows immediately from the Wang exact sequence, and the other two isomorphisms follow from the exact sequence of Stevens:

$$0 \to H_{n+1}(M) \to H_n(F, \partial F) \xrightarrow{\mathrm{Var}(h)} H_n(F) \to H_n(M) \to 0$$

which arises from the exact sequence of the pair (M, F) (see Proposition 1 of [28]). Since the normal bundle of any $S^n \hookrightarrow M$ is stable and M bounds a parallelizable manifold, the bundle $\nu(S^n; M)$ must be trivial. □

Now we would like to kill $H_n(M)$ using surgery, and as a result obtain a homotopy sphere $\Sigma_h \in \Theta_{2n+1}$ (we again assume $n \geqslant 3$). For each generator $[z_i] \in H_n(F)$ we fix an embedding $\phi_i : S_i^n \times d_i^{n+1} \hookrightarrow F \times (0,1) \hookrightarrow M$ disjoint from $\mathcal{D}^{2n} \times (0,1)$. Then we attach handles $D_i^{n+1} \times d_i^{n+1}$ to the product $M \times I$ along these embeddings into $M \times \{1\}$ to obtain a cobordism W between $M = M \times \{0\}$ and the homotopy sphere Σ_h which is the result of these ϕ_i-surgeries on M. Furthermore, we can choose the embeddings ϕ_i compatible with the framing of M that comes from the framing of N (see Lemma 6.2 of [13]), and hence we get W as a framed manifold. Taking the union of N and W along M we obtain a parallelizable manifold with boundary Σ_h, i.e. $\Sigma_h \in bP_{2n+2}$. Following Kreck's construction of the isomorphism $\sigma : \ker(\chi) \to \Theta_{2n+1}$ (see [15], pp. 655–656) one can easily show that Σ_h is well defined (depends only on the isotopy class of h) and that $\iota_r(\Sigma_h) = [h]$.

Let us denote the signature of a parallelizable manifold N_k with boundary $\partial N_k = M_k = F_{\varphi^k} \cup_r (\partial F \times D^2)$ by σ_k, and the generator of bP_{2n+2} by Σ.

Corollary 3 (cf. [28], Proposition 8). *Let $M^{2n+1} = F_\varphi \cup_r (\partial F \times D^2)$ be the manifold with simple open book decomposition where n is odd, $\neq 1$ and M_k be the k-fold branched cyclic cover of M. If* $\operatorname{Var}(\varphi^d) = 0$ *then M_{k+d} is diffeomorphic to $(\frac{\sigma_d}{8} \cdot \Sigma)\#M_k$.*

Proof. We have just seen above that $M_{k+d} \simeq \Sigma' \# M_k$ with $\Sigma' = m \cdot \Sigma \in bP_{2n+2}$ for some $m \in \mathbb{N}$. Since $M_d = F_{\varphi^d} \cup_r (\partial F \times D^2) \simeq \partial(F \times D^2)\# m\Sigma$ and $m\Sigma$ bounds a parallelizable manifold, say W_m, with signature $\sigma(W_m) = 8m$ and $\partial(F \times D^2)$ bounds $F \times D^2$ (which is also parallelizable) with signature zero, the connected sum of W_m and $F \times D^2$ along the boundary (cf. §2 of [13]) will give us a parallelizable manifold $N_d := W_m \# F \times D^2$ with boundary $\partial N_d = M_d$ and signature $\sigma(N_d) = \sigma(W_m) + \sigma(F \times D^2) = 8m + 0$. Thus $m = \frac{\sigma(N_d)}{8} \equiv \frac{\sigma_d}{8}$ mod (order of bP_{2n+2}) and the corollary follows. □

When $n = even$, the periodicity of M_k is more complicated. Consider the link of the singularity $z_0^2 + z_1^2 + \cdots + z_n^2 = 0$ with $n = 2m$ and denote the $(4m+1)$-dimensional Kervaire sphere by Σ and the tangent S^n-sphere bundle to S^{n+1} by T. Then M_{k+8} is diffeomorphic to M_k and the diffeomorphism types are listed in the table (see [8], Proposition 6.1)

$M_1 \simeq M_7$	M_2	$M_3 \simeq M_5$	M_4	M_6	M_8
S^{2n+1}	T	Σ	$(S^n \times S^{n+1})\#\Sigma$	$T\#\Sigma$	$S^n \times S^{n+1}$

The following result is due to Stevens ([28], Theorem 9).

Corollary 4. *If for branched cyclic covers M_k of a $(2n+1)$-manifold M with simple open book decomposition* $\operatorname{Var}(\varphi^d) = 0$, *then M_k and M_{k+2d} are homeomorphic and M_k and M_{k+4d} are diffeomorphic. Moreover, if $n = 2$ or 6, then M_k and M_{k+d} are diffeomorphic.*

Proof. When $n = 2$ the mapping class group is trivial and $[\varphi^d] = Id$. If $n = 6$ then $G \cong 0$ and $bP_{14} \cong 0$ (see [13, Lemma 7.2]) which implies that $[\varphi^d] = Id$. For the other even n we know that the group G is isomorphic either to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and hence $\chi_r(\varphi^d)$ has order two. Therefore $\varphi^{2d} \in bP_{4m+2}$ which means that M_k is homeomorphic to M_{k+2d}, and since the group bP_{4m+2} is either trivial or $\mathbb{Z}_2$ (see [13]), φ^{4d} must be pseudo-isotopic to the identity. □

Example 2. (The authors are indebted to the referee for suggesting this example.) Consider again the singularity $z_0^2 + z_1^2 + \cdots + z_n^2 = 0$ with $n = 2m$. Assume in addition that $m \neq 0 \pmod 4$ and that the Kervaire sphere $\Sigma \in bP_{4m+2}$ is exotic, e.g., when $4m + 2 \neq 2^l - 2$ (see [3]). Here the Milnor fiber F, is the tangent disc bundle to the sphere S^{2m} and hence $\mathcal{D}F \simeq S^{2m} \times S^{2m}$. It is also well known that the geometric monodromy φ of this singularity satisfies the properties: $\varphi_* = -Id$, $\mathrm{Var}(\varphi^2) = 0$ and $\mathrm{Var}(\varphi) \neq 0$ (cf. [18], Chapter 3). Since M_0 is not diffeomorphic to M_2 and M_1 is not diffeomorphic to M_5, $\chi_r([\varphi^2])$ will be a generator of $\mathrm{Hom}(H_n(F, \partial F), G) \cong \mathbb{Z}_2$ and $[\varphi^4]$ will be a generator of $bP_{4m+2} \cong \mathbb{Z}_2$. Since $\Theta_{4m+1} \cong bP_{4m+2} \oplus \mathrm{Coker}(J_{4m+1})$ (cf. [4]) we see that in this case $\tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial) \cong \mathbb{Z}_4 \oplus \mathrm{Coker}(J_{4m+1})$ and the exact sequence (4) does not split. Furthermore, one can deduce from our Theorem 2 and results of Sato and Levine (see [27], Proposition 2.2 or [17], §1.2) that the monodromy $[\varphi]$ generates the quotient $\tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial)/\tilde{\pi}_0 V\,\mathrm{Diff}(F, \partial)$ and therefore $\tilde{\pi}_0\,\mathrm{Diff}(F, \mathrm{rel}\,\partial) \cong \mathbb{Z}_8 \oplus \mathrm{Coker}(J_{4m+1})$.

3.2. Periodicity in dimension 3

It is known that if the dimension of the open book M^{2n+1} is three, then there is homological periodicity (see references in [8]) but there is no topological one. For the sake of completeness we illustrate this with the following classical example (cf. [26], Chapter 10.D). Let $f(z_0, z_1) = z_0^2 + z_1^3$ be the complex polynomial which defines the curve $V = \{f(z_0, z_1) = 0\}$ in $\mathbb{C}^2$ with the cusp at the origin. The corresponding Milnor fibration has monodromy φ of order six, the boundary of the fiber F is the trefoil knot K and $\mathrm{Var}(\varphi^6) = 0$. This fibration gives the open book structure to the standard 3-sphere $S^3 = M_1 = F_\varphi \cup (K \times D^2)$. We show that $M_7 \neq M_1$ and $M_6 \neq M_0 = (F \times S^1) \cup (K \times D^2)$.

Let us first compare $\pi_1(M_0)$ with $\pi_1(M_6)$. The theorem of Seifert and Van Kampen entails that $\pi_1(M_0) \cong \pi_1(F)$ which is the free group on two generators. As for M_6 one can easily find using the Reidemeister–Schreier theorem a presentation for $\pi_1(F_{\varphi^6})$ and then show that $\pi_1(M_6)$ admits the following presentation:

$$\langle Z_1, Z_2, \ldots, Z_6 \mid Z_1 = Z_6 Z_2, \ldots, Z_j = Z_{j-1} Z_{j+1}, \ldots, Z_6 = Z_5 Z_1 \rangle.$$

It takes a bit more effort to show that this group is isomorphic to the group of upper unitriangular 3×3 matrices with integer coefficients (cf. [20], §8)

$$\mathcal{H} \cong \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z} \right\}.$$

Suppose now that M_7 were homeomorphic to the sphere. Then we could take the union of N_7 and D^4 (recall that N_7 is the cyclic covering of D^4 branched along the fiber $(F, K) \hookrightarrow (D^4, S^3)$ where $F \cap S^3 = K$):

$$W^4 := N_7 \cup_{S^3} D^4.$$

Since N_7 is parallelizable (see [5], Theorem 5 or [12], Chapter XII), W^4 would be a closed spin-manifold. Hence its signature $\sigma(W^4) = \sigma(N_7)$ must be a multiple of 16 by the theorem of Rokhlin [25]. But $\sigma(N_7) = -8$ as one can find using the Seifert pairing on $H_1(F)$ (cf. [5,12]), and hence $M_7 \neq M_1$. Actually much more is known. Milnor in [20] proved that $\pi_1(M_r)$ is isomorphic to the commutator subgroup $[\Gamma, \Gamma]$ of the centrally extended triangle group Γ which has a presentation

$$\Gamma \cong \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^2 = \gamma_2^3 = \gamma_3^r = \gamma_1 \cdot \gamma_2 \cdot \gamma_3 \rangle.$$

This group Γ is infinite when $r \geqslant 6$ (see [20], §2,3) and hence $[\Gamma, \Gamma]$, that has index $r - 6$, is infinite too. In particular, none of the cyclic coverings of S^3 branched along the trefoil knot can be simply connected.

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Further reading

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