Signature of Branched Fibrations

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I. Introduction

A branched fibration is a topological analog of a degenerating family of algebraic varieties that is parametrized over a manifold M, with the degenerate fibers lying over a codimension two submanifold $V \subset M$. This is a common situation in algebraic geometry, but there is a wide avenue of choice for the corresponding topological notion. We have chosen a definition that abstracts the main features associated with isolated (complex) hypersurface singularities. This means that the degeneration will be closely associated to a fibered knot; the knot plays the role of the link of the singularity.

In section 2 we review the definitions of knot, fibered knot and Seifert pairing. Theorem 2.9 shows that, over the complex numbers, the Seifert pairing of a fibered knot is non-trivial only on subspaces associated with unit-length eigenvalues of the monodromy.

In section 3 branched fibrations are defined and related to fibered knots and singularities. This is based on joint work of the author and Walter Neumann ([KN]). Theorem 3.2 states the main properties of the knot product construction of [KN]. This construction associates to a knot $\mathcal{K} = (S^k,K)$ and a fibered knot $\mathcal{L} = (S^\ell,L)$ a new product knot $\mathcal{K} \otimes \mathcal{L} = (S^{k+\ell+1},K \otimes L)$. The product knot has a spanning manifold that is defined in terms of branched fibrations. In fact $K \otimes L = \partial M$ where $M = \tau(D^{k+1},F)$, a branched fibration of D^{k+1} along a submanifold $F \subset D^{k+1}$ with $\partial F = K$. The branched fibration is obtained by a pull-back construction from a simpler branched fibration $\tau : D^{\ell+1} \to D^2$, branched over $0 \in D^2$. This branched fibration τ is directly associated with the fibered knot $\mathcal{L}$.

This situation leads to a signature problem: Let $\sigma_\tau(\mathcal{K})$ denote the signature of the branched fibration $\tau(D^{k+1},F)$ when $4 \mid (k + \ell)$. Theorem 3.6 gives a formula for this signature in terms of the eigenvalues of the monodromy for $\mathcal{L}$ and the Seifert forms of $\mathcal{K}$ and $\mathcal{L}$. This result generalizes some computations of signatures of branched coverings (see [DK] and [N]).

In section 4, Theorem 3.6 is applied to some special cases involving Brieskorn singularities, cyclic branched coverings and concordance invariants of links in S^3.

In section 5 we show how to construct a more general class of branched fibrations by mimicking a method due to F. Hirzebruch ([H]) for ramified covers. This leads towards the question of a more general formula for signatures of branched fibrations. This seems to be a difficult problem and may, in fact, require a more general concept of branched fibration or a change in viewpoint.

In any case, I hope to have given an initial framework for these questions and to have shown some of the interesting connections among singularities, knot theory, and signatures.

Throughout the paper all manifolds are smooth; $\cong$ denotes isormorphism or diffeomorphism, while $\approx$ denotes homeomorphism.

II. Knots and Fibered Knots

This section will recall some standard notions in knot theory. The main result (Theorem 2.9) shows that, over the complex numbers, the Seifert pairing of a fibered knot is nontrivial only on subspaces associated with unit-length eigenvalues of the monodromy. This fact will be of use for the signature computations of section III.

Definition 2.1. A knot $\mathcal{K} = (S^n, K)$ is a pair consisting of an oriented n-sphere S^n and a codimension two, compact, closed, oriented submanifold $K \subset S^n$. If K is a homotopy sphere, then the knot is said to be spherical.

When $n = 3$, this definition is intended to include links. That is, a link $K \subset S^3$ is a collection of disjoint embedded oriented circles.

Definition 2.2. A spanning surface for a knot $\mathcal{K} = (S^n, K)$ is a compact oriented $(n-1)$-manifold F, embedded in S^n so that $\partial F = K$. Here the symbol, ∂, denotes oriented boundary.

It is worth remarking that knots always have spanning surfaces. Since the proof of this result is short and it motivates the definition of fibered knot, we include the argument in the next lemma.

Lemma 2.3. If $\mathcal{K} = (S^n, K)$ is any knot, then there exists a spanning surface $F \subset S^n$ for $\mathcal{K}$.

Proof. Let $E = S^n - N°$ where N is a closed tubular neighborhood of K. Note that $H^1(E;\mathbb{Z}) = [E, S^1]$ where $[\ ,\ ]$ denotes homotopy classes of maps. Let $\alpha : E \to S^1$ represent a sum of the generators of $H^1(E;\mathbb{Z})$ with orientations specified by the orientation of $K \subset S^n$. We may assume that α is transverse regular to $* \in S^1$. It is not hard to see that N is diffeomorphic to $K \times D^2$ with $\partial\alpha^{-1}(*)$ corresponding to $K \times *$. Thus, by adding a collar to $\alpha^{-1}(*)$, one obtains $F \subset S^n$ with $\partial F = K$.

Remark. It may happen that the mapping $\alpha : E \to S^1$ described above can be chosen to be a smooth fibration. In this case one says that $\mathcal{K}$ is a fibered knot. The formal definition is as follows:

Definition 2.4. A knot $\mathcal{K} = (S^n, K)$ is fibered with fibered structure $b : S^n \to D^2$, if there is a smooth mapping $b : S^n \to D^2$, transverse to $0 \in D^2$ such that

i) $b^{-1}(0) = K \subset S^n$.

ii) $b/||b|| : S^n - K \to S^1$ is a smooth fibration.

Here $||b||(x)$ denotes the distance from $b(x)$ to the origin in $\mathbb{R}^2$. A fibered knot with fibered structure b will sometimes be indicated by the notation $(S^n,K;b)$.

The first example of a fibered knot is the <u>empty knot of degree</u> a, $[a] = (S^1,\phi;a)$. Here $a:S^1 \to S^1$ is defined by the formula $a(x) = x^a$ (complex multiplication; a is an integer). The map is vacuously transverse to $0 \in D^2$ and a typical fiber is $F = a^{-1}(1) = \{1,\omega,\omega^2,\ldots,\omega^{a-1}\}$ where $\omega = \exp(2\pi i/a)$. Just as all of mathematics unfolds from the empty set, so do many interesting knots come from these empty knots. This comes about by the product construction discussed in the next section.

Another construction that gives rise to fibered knots involves the notion of the <u>link of a singularity</u>. Let $f:\mathbb{C}^n \to \mathbb{C}$ be a complex polynomial mapping such that $f(0) = 0$. One says that f has an <u>isolated singularity at 0</u> if the gradient $\nabla f = (\partial f/\partial z_1,\partial f/\partial z_2,\ldots,\partial f/\partial z_n)$ vanishes at $0 \in \mathbb{C}^n$ and is non-zero in some deleted neighborhood of $0 \in \mathbb{C}^n$. Under these conditions one can define a knot, the <u>link of f</u> by $L(f) = (S_\varepsilon^{2n-1},L(f))$ where $L(f) = f^{-1}(0) \cap S_\varepsilon^{2n-1}$ for $0 < \varepsilon << 1$. For ε sufficiently small, $L(f)$ is independent of the choice of ε. In ([M]) Milnor shows that $L(f)$ is a fibered knot. The fibration of the complement is given by the mapping $f/||f||:S_\varepsilon^{2n-1} - L(f) \to S^1$. The Brieskorn polynomials (see [B]) $f(z) = z_0^{a_0} + z_1^{a_1} + \ldots + z_n^{a_n}$ provide one such class of isolated singularities.

One method for studying a knot is to consider invariants for the embedding of a spanning surface. The simplest of these is the Seifert pairing:

<u>Definition 2.5</u>. Let $K = (S^n,K)$ be a knot with spanning surface $F \subset S^n$. The <u>Seifert pairing</u> $\theta_K:H_*(F) \times H_{n-*-1}(F) \to \mathbb{Z}$ is defined as follows: Let $i:F \to S^n - F$ be the mapping obtained by pushing F into its complement along the positive normal direction. Then $\theta_K(x,y) = \ell(i_*x,y)$ where ℓ denotes linking numbers in S^n,

Here H_* denotes the free part of the reduced homology. That is, $H_*(X) = \overline{H}_*(X)/T_*(X)$ where $T_*(X) \subset \overline{H}_*(X)$ is the torsion subgroup of the reduced integral homology group. While the Seifert pairing actually depends upon the choice of spanning surface F, we have chosen to omit this dependence in the notation.

If the submanifold F has a middle dimension, then there is a well-known relationship between the Seifert pairing and the intersection form on F. This is given by the following theorem of J.Levine ([L_1]).

<u>Theorem 2.6</u>. Let $F^{2n} \subset S^{2n+1}$ be a compact oriented 2n-dimensional manifold with boundary embedded in S^{2n+1}. Let $\theta:H_n(F) \times H_n(F) \to \mathbb{Z}$ be the middle-dimensional Seifert pairing, and let $\langle,\rangle:H_n(F) \times H_n(F) \to \mathbb{Z}$ be the intersection form on F. Then, for $x,y \in H_n(F)$, $\langle x,y\rangle = \theta(x,y) + (-1)^n\theta(y,x)$.

This is a key relationship for signature computations. If F^{2n} is given as above, and n is even, then $\langle,\rangle$ is a symmetric form and the <u>signature of F</u>

$\sigma(F)$, is defined to be the signature of the form $(H_n(F),\langle,\rangle)$. Thus the Seifert pairing determines the signature of F.

In the case of fibered knots, extra structure is provided by the monodromy $h_*:H_*(F)\to H_*(F)$ where F is a typical fiber. Here $h:F\to F$ is the diffeomorphism of the fiber that defines the structure for the corresponding fiber bundle over S^1. The following proposition is well known.

Proposition 2.7. Let $L=(S^n,L)$ be a fibered knot with fiber G and monodromy h. Let Θ_L denote the Seifert pairing with respect to G. Then $\Theta_L(x,h_*y) = (-1)^{|x||y|}\Theta_L(y,x)$. Here if $x \in H_*(F)$, then $|x| = * + 1$.

Remark: In matrix terms this proposition becomes $H = \Theta^{-1}\Theta^T$ where H is a matrix of h_* with respect to some basis for $H_*(F)$ and Θ is a matrix of the Seifert pairing. The graded transpose Θ^T is defined by the equations $\Theta_p^T = (-1)^{(p-1)(n-p)}\Theta_p^t$ where Θ_p is that part of the Seifert matrix coming from the Seifert pairing on $H_p(F)\times H_{n-p-1}(F)$ and t denotes ordinary transpose.

In what follows, we write h instead of h_*.

Corollary 2.8. Let $K=(S^{2n+1},K)$ be a fibered knot with fiber F and monodromy h. Let $\langle,\rangle:H_n(F)\times H_n(F)\to\mathbb{Z}$ denote the intersection pairing and $\Theta:H_n(F)\times H_n(F)\to\mathbb{Z}$ denote the Seifert pairing in the middle dimension. Then for $x,y\in H_n(F)$, $\langle x,y\rangle = \Theta(x,(I-h)y)$.

Proof. By 2.7, $(-1)^{(n+1)(n+1)}\Theta(y,x) = \Theta(x,hy)$. By 2.6, $\langle x,y\rangle = \Theta(x,y) + (-1)^n\Theta(y,x) = \Theta(x,y) - \Theta(x,hy) = \Theta(x,(I-h)y)$. This proves the corollary.

In the rest of this section we will analyze the relationship between the eigenvalues of the monodromy and the structure of the Seifert pairing in the middle dimension. Thus we fix a fibered knot $K=(S^{2n+1},K)$ with fiber F, monodromy h, and Seifert and intersection forms as in Corollary 2.8. Let $A = H_n(F)\otimes\mathbb{C}$. Then $\langle,\rangle$ and Θ extend to forms on A via the formulas $\langle x\otimes\alpha, y\otimes\beta\rangle = \bar\alpha\beta\langle x,y\rangle$ and $\Theta(x\otimes\alpha,y\otimes\beta) = \bar\alpha\beta\Theta(x,y)$ for $\alpha,\beta\in\mathbb{C}$ and $x,y\in H_n(F)$. The bar denotes complex conjugation. Thus $\langle,\rangle$ be comes a Hermitian or skew-Hermitian form on A, and Θ a sesquilinear form. Let $h:A\to A$ denote the usual monodromy tensored with $1\mathbb{C}$.

Consider the Jordan normal form for $h:A\to A$. Then $A \cong \underset{\lambda}{\oplus} A_\lambda$ where λ is an eigenvalue for h and A_λ is the corresponding subspace. (A basis can be chosen for A_λ such that $h_\lambda = h|A_\lambda$ is a λ-Jordan block, and $h \cong \oplus h_\lambda$).

Theorem 2.9. Let K be a fibered knot as above with monodromy h and complex Seifert pairing $\Theta:A\times A\to\mathbb{C}$. Then Θ is an orthogonal direct sum of its restrictions to the Jordan subspaces A_λ. That is, if $x\in A_\lambda$ and $y\in A_\mu$ with $\lambda\neq\mu$ then $\Theta(x,y)=0$. Furthermore, if $\Theta|A_\lambda \not\equiv 0$ then $||\lambda||=1$. Thus, only unit length eigenvalues of the monodromy are relevant to the Seifert pairing.

Proof. To prove that $\Theta|A_\lambda \not\equiv 0 \Rightarrow ||\lambda|| = 1$ it will suffice to assume that $\lambda \neq 1$ and that $h|A_\lambda$ corresponds to a single Jordan block. That is, we assume that A_λ has a basis $\{e_0, e_1, \ldots, e_s\}$ such that $he_0 = \lambda e_0$ and $he_k = \lambda e_k + e_{k-1}$ for $k = 1, 2, \ldots, s$. We shall let Θ denote $\Theta|A_\lambda$ with this basis. The proof will proceed by induction on s.

Note that $\overline{\langle x,y\rangle} = (-1)^n\langle y,x\rangle$. Thus we have $\langle e_0,e_0\rangle = \Theta(e,(I-h)e_0) = \Theta(e_0,(1-\lambda)e_0)$. Hence $\langle e_0,e_0\rangle = (1-\lambda)\Theta(e_0,e_0)$. Therefore $(1-\bar{\lambda})\overline{\Theta(e_0,e_0)} = (-1)^n(1-\lambda)\Theta(e_0,e_0)$. Since we also know that $\langle x,y\rangle = \Theta(x,y) + (-1)^n\overline{\Theta(y,x)}$ and are assuming that $\lambda \neq 1$, this implies that $(1-\lambda)\Theta(e_0,e_0) = \Theta(e_0,e_0) + [(1-\lambda)/(1-\bar{\lambda})]\Theta(e_0,e_0)$. Hence, if $\Theta(e_0,e_0) \neq 0$, then $(1-\bar{\lambda})(1-\lambda) = (1-\bar{\lambda}) + (1-\lambda)$. Therefore $\lambda\bar{\lambda} = 1$ and $||\lambda|| = 1$. This completes the proof for $s = 0$.

Continuing by induction, we assume that the result has been shown for all Jordan blocks of size less than s. Let $B_{s-1} = [e_0, e_1, \ldots, e_{s-1}] \subsetneq A_\lambda$ denote the subspace spanned by these basis vectors. By induction, if $\Theta|B_{s-1} \not\equiv 0$ then $||\lambda|| = 1$. Thus we may assume that $\Theta|B_{s-1} \equiv 0$. Under this assumption we now make a second induction on s to show that $\Theta(e_s,e_k) \neq 0$ or $\Theta(e_k,e_s) \neq 0$ for any k satisfying $0 \leq k < s$ implies that $||\lambda|| = 1$.

To start this second induction, note that $\langle e_s,e_0\rangle = \Theta(e_s,(I-h)e_0) = (1-\lambda)\Theta(e_s,e_0)$ while $\langle e_0,e_s\rangle = \Theta(e_0,(I-h)e_s) = \Theta(e_0,(1-\lambda)e_s - e_{s-1}) = (1-\lambda)\Theta(e_0,e_s)$ (since $\Theta|B_{s-1} \equiv 0$, $\Theta(e_0,e_{s-1}) = 0$). The same argument as in the first induction at $s = 0$ now shows that $\Theta(e_0,e_s) \neq 0 \Rightarrow ||\lambda|| = 1$. To complete this second induction, suppose that $\Theta(e_s,e_k) = \Theta(e_k,e_s) = 0$ for $0 \leq k < \ell \leq s - 1$. Then a similar computation shows that $\Theta(e_s,e_\ell) \neq 0$ or $\Theta(e_\ell,e_s) \neq 0 \Rightarrow ||\lambda|| = 1$. This completes the second induction.

We now may therefore assume that $\Theta(e_i,e_j) = 0$ for $i \neq s$ and $j \neq s$. Thus, since $\Theta \not\equiv 0$, $\Theta(e_s,e_s) \neq 0$. But $\langle e_s,e_s\rangle = \Theta(e_s,(1-\lambda)e_s - e_{s-1}) = (1-\lambda)\Theta(e_s,e_s)$. Hence the same calculuation as before shows that $||\lambda|| = 1$. This completes the first induction and hence the proof that $\Theta|A_\lambda \not\equiv 0 \Rightarrow ||\lambda|| = 1$.

The rest of the proof is obtained by very similar induction arguments. The details will be omitted. Different Jordan blocks corresponding to the same eigenvalue λ are, in fact, orthogonal for Θ.

Remark. For eigenvectors there is a direct relationship between the eigenvalue and the value of the Seifert pairing. Suppose that $\Theta(x,x) = \alpha \neq 0$ and that $hx = \lambda x$. Then (as in the proof above) $(1-\bar{\lambda})\bar{\alpha} = (-1)^n(1-\lambda)\alpha$. If $\lambda \neq 1$, then $(1-\lambda)/(1-\bar{\lambda}) = -\lambda$. Hence $\lambda\alpha = (-1)^{n+1}\bar{\alpha}$. For example, if $\alpha = 1 - \omega$ with $\omega \neq 1$, $||\omega|| = 1$ and $n = 0$ then $\lambda = -(1-\bar{\omega})/(1-\omega) = \bar{\omega}$. If $\alpha = (1-\omega_0)\ldots(1-\omega_n)$ with $\omega_i \neq 1$, $||\omega_i|| = 1$ then $\lambda = \bar{\omega}_0\bar{\omega}_1\ldots\bar{\omega}_n$.

An example. Consider the empty knot of degree a, $[a] = (S^1,\phi;a)$, where a is a positive integer. This knot has spanning surface $F = \{1,\omega,\omega^2,\ldots,\omega^{a-1}\}$ where $\omega = \exp(2\pi i/a)$. Thus $\tilde{H}_0(F) \cong \mathbb{Z}^{a-1}$ with basis $\{e_0,e_1,\ldots,e_{a-2}\}$ where

$e_k = [\omega^k] - [\omega^{k+1}]$ and $[p]$ denotes the integral homology class of the point $p \in F$. The monodromy acts via rotation by $2\pi/a$, hence $h[\omega^k] = [\omega^{k+1}]$ and therefore $he_k = e_{k+1}$ (but note that $1 + \omega + \omega^2 + \ldots + \omega^{a-1} = 0$ and $e_{a-1} = -(e_0 + e_1 + \ldots + e_{a-2}))$. Let $A = \bar{H}_0(F;\mathbb{C})$. The eigenvalues of the mondromy are $\bar{\omega}, \bar{\omega}^2, \ldots, \bar{\omega}^{a-1}$. A corresponding basis of eigenvectors is given by $E_k = e_0 + \omega^k e_1 + \omega^{2k} e_2 + \ldots + \omega^{(a-1)k} e_{a-1}$. The integral Seifert pairing has matrix Λ_a with respect to the basis $\{e_0, \ldots, e_{a-2}\}$, where

$$\Lambda_a = \begin{bmatrix} 1 & -1 & & & 0 \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ 0 & & & & 1 \end{bmatrix}$$ is an $(a-1) \times (a-1)$ matrix.

It is an easy calculation to see that the Seifert pairing over $\mathbb{C}$ is diagonal. Its matrix Δ_a, with respect to the basis $\{E_1, E_2, \ldots, E_{a-1}\}$ is given by the diagonal matrix

$$\Delta_a = \begin{bmatrix} 1-\omega & & & & 0 \\ & 1-\omega^2 & & & \\ & & 1-\omega^3 & & \\ & & & \ddots & \\ 0 & & & & 1-\omega^{a-1} \end{bmatrix}$$

III. Branched Fibrations and Knot Products

In this section we define branched fibrations and explain their relationship to singularities, fibered knots and knot products.

Definition 3.1. A branched fibration over D^2 is a smooth mapping $\tau: D^{n+1} \to D^2$ such that $\tau^{-1}(0) \approx CL$ where $L \subset S^n$ is a knot, and CL denotes the cone on L. The mapping τ must satisfy the following two conditions:

i) $\tau|S^n: S^n \to D^2$ is a fibered structure for (S^n, L). (see 2.4).

ii) $\tau|D^{n+1} - CL: D^{n+1} - CL \to D^2 - \{0\}$ is a smooth fibration.

Any fibered knot gives rise to a branched fibration as follows: Let $\mathcal{L} = (S^n, L; b)$ be a fibered knot. We may alter the structure map b so that it has only regular values in the interior of D^2. Then define $cb: D^{n+1} \to D^2$ by the equation $cb(rx) = rb(x)$ where $0 \leq r \leq 1$ and $x \in S^n$. Let $\tau: D^{n+1} \to D^2$ be the result of smoothing cb at the cone point. Then $\tau: D^{n+1} \to D^2$ is a branched fibration so that $\tau|S^n$ is the given fibered structure for $\mathcal{L}$.

This correspondence between branched fibrations over the disk and fibered knots is an abstraction from the case of isolated singularities. If $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an isolated complex polynomial singularity with link $\mathcal{L}(f) = (S_\epsilon^{2n-1}, L(f))$, then we let $N_f = \{z \in D_\varepsilon^{2n} \mid ||f(z)|| \leq \delta\}$ where $0 < \delta << \varepsilon$. Then $f:N_f \to D^2$ is a branched fibration.

Note that the most elementary branched fibration over D^2 is simply a ramified covering of the disk: $\mu_a : D^2 \to D^2$, $\mu_a(z) = z^a$. This corresponds to the empty knot of degree a .

Just as $\mu_a : D^2 \to D^2$ is a local model for a cyclic branched covering, a branched fibration $\tau : D^{n+1} \to D^2$ may be used as a local model for more general branched fibrations.

Thus suppose we are given a manifold M containing a properly embedded codimension two submanifold $V \subset M$. Suppose that there exists a smooth mapping $\alpha : M \to D^2$, transverse to $0 \in D^2$, so that $V = \alpha^{-1}(0)$. Then a new branched fibration $\tau_M : \tau(M,V) \to M$ is formed by constructing the pull-back

$$\begin{array}{ccc} \tau(M,V) & \longrightarrow & D^{n+1} \\ \big\downarrow{\scriptstyle \tau_M} & & \big\downarrow{\scriptstyle \tau} \\ M & \xrightarrow{\alpha} & D^2 \end{array} .$$

That is, $\tau(M,V) = \{(m,x) \in M \times D^{n+1} \mid \alpha(m) = \tau(x)\}$ and $\tau_M(m,x) = m$. The mapping τ_M is a smooth fibration away from $V \subset M$ and $\tau_M^{-1}(v) \cong CL$ for each $v \in V$ ($L \subset S^n$ is the fibered knot corresponding to τ) .

If $\tau = \mu_a : D^2 \to D^2$, then this pull-back yields an a-fold cyclic branched covering with branch set V . In the general case, the restriction of τ_M to a normal disk to $v \in V$ recovers τ . The construction $\tau(M,V)$ will not always be independent of α ; this is the case when M is 2-connected (see [KN]) . See ([KN]) also for a general definition of a branched fibration of M along V .

<u>Product Construction</u>. Let $K \subset S^n$ be any knot and $L \subset S^n$ be a fibered knot with associated branched fibration $\tau : D^{m+1} \to D^2$. Then there is a well-defined <u>knot product</u> $K \otimes L = (S^{n+m+1}, K \otimes L)$. It is defined as follows:

Let $F^n \subset D^{n+1}$ be a properly embedded submanifold of D^{n+1} such that $\partial F = K \subset S^n$. Let $\alpha : D^{n+1} \to D^2$ be a mapping transverse to $0 \in D^2$ so that $\alpha^{-1}(0) = F$. We may then form the pull-back

$$\begin{array}{ccc} \tau(D^{n+1},F) & \longrightarrow & D^{m+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ D^{n+1} & \xrightarrow{\alpha} & D^2 \end{array} .$$

The knot product is defined by taking boundaries:

$$(S^{n+m+1}, K \otimes L) = (\partial(D^{n+1} \times D^{m+1}), \partial\tau(D^{n+1},F)) .$$

Note that when $L = [a]$, the empty knot of degree a , then $K \otimes [a]$ is the a-fold cyclic branched covering of S^n along K and the construction gives an embedding $K \otimes [a] \subset S^{n+2}$. This new knot, $K \otimes [a]$, is called the a-fold cyclic suspension of K .

This product construction enjoys a number of useful properties, as summarized in the next theorem (proved in [KN]).

Theorem 3.2. Let $K = (S^n, K)$ be a knot and $L = (S^m, L)$ a fibered knot. Then

i) If K and L are both fibered then so is $K \otimes L$, and $L \otimes K = (-1)^{(n-1)(m-1)} K \otimes L$. (Here $-(S,K) = (-S,-K)$.)

ii) The product operation is associative.

iii) Suppose that $F \subset D^{n+1}$ is obtained by pushing the interior of a spanning surface $\tilde{F}$ for K into the interior of D^{n+1} . Let $\tau : D^{m+1} \to D^2$ be the branched fibration corresponding to L; let G be a spanning manifold for L . Then $K \otimes L$ has a spanning manifold M that is diffeomorphic to $\tau(D^{n+1}, F)$. Furthermore, M has the homotopy type of the join $F*G$.

iv) With notation as in iii), note that $\bar{H}_*(M) \cong \bar{H}_*(F) \otimes \bar{H}_*(G)$ $(\bar{H}_* = H_{*-1}$, H as in section 2). If Θ_K and Θ_L are Seifert pairings of K and L with respect to the spannning surfaces $\tilde{F}$ and G , and $\Theta_{K\otimes L}$ the Seifert pairing of $K \otimes L$ with respect to M , then

$$\Theta_{K\otimes L} \simeq \Theta_K \otimes \Theta_L$$

using the above decomposition of the homology of M . That is, for elements of homogeneous degree

$$\Theta_{K\otimes L}(a\otimes a', b\otimes b') = (-1)^{|a'||b|}\Theta_K(a,b)\Theta_L(a',b')$$

(where $x \in \bar{H}_{|x|} = H_{|x|-1}$ defines the $\bar{H}_*$ grading).

v) If $f:(\mathbb{C}^n,0) \to (\mathbb{C},0)$ and $g"(\mathbb{C}^m,0) \to (\mathbb{C},0)$ are isolated complex hypersurface singularities, then $f + g:(\mathbb{C}^n \times \mathbb{C}^m, 0) \to (\mathbb{C},0)$, $(f+g)(x,y) = f(x) + g(y)$ is also an isolated singularity and $L(f+g) \simeq L(f) \otimes L(g)$.

<u>Signature problems</u>. Given a knot $K = (S^{4n+1}, K)$ with spanning surface F of dimension 4n one defines the <u>signature of K</u> , $\sigma(K)$, to be the signature of the spanning surface. Thus $\sigma(K) = \sigma(F)$. Standard arguments using the Novikov addition theorem (see [AS],[KT]) show that this signature is independent of the choice of spanning surface. In order to generalize this notion, let $\tau : D^{\ell+1} \to D^2$ be a branched fibration corresponding to a fibered knot $L = (S^\ell, L)$. Let $L = (S^k, K)$ be any knot and assume that $4 | (k+\ell)$. Define the <u>τ-signature of K</u> , $\sigma_\tau(K)$, by the formula $\sigma_\tau(K) = \sigma(K \otimes L)$.

Since we know from 3.2 that $K \otimes L$ has a spanning surface $M \simeq \tau(D^{k+1}, F)$ where F is a pushed-in spanning surface for K , the τ-signature is the signature of this branched fibration. Thus $\sigma_\tau(K) = \sigma(\tau(D^{k+1}, F))$. The rest of this section will be devoted to showing how to compute this signature. We shall use part iv) of Theorem 3.2 to reduce the problem to signatures of forms related to the Seifert pairings of the two knots.

Lemma 3.3. Let $K = (S^k, K)$ be any knot, $L = (S^\ell, L)$ be a fibered knot with associated branched fibration τ. Let $F \subset D^{k+1}$ be a pushed in spanning surface for K, and G be a fiber for L. Let $M = \tau(D^{k+1}, F)$. Assume that $4 | (k + \ell)$. Then $\sigma(M) = 0$ unless k and ℓ are both odd. Given that k and ℓ are odd, let $A = H_p(F) \otimes H_q(G)$ where $p = (k-1)/2$, $q = (\ell-1)/2$. Let $<,>: A \times A \to \mathbb{Z}$ denote the restriction of the intersection form on $H_n(M)$ to A (via 3.2, iv) with $n = (k + \ell)/2)$. Then $\sigma_\tau(K) = \sigma(A, <,>)$.

Proof: We know that $H_n(M) \cong \bigoplus_{s+t=n-1} H_s(F) \otimes H_t(G)$. The intersection form decomposes as an orthogonal sum on parts $A(s,t) \oplus A(k-s-1, \ell-t-1) = B(s,t)$, $A(s,t) = H_s(F) \otimes H_t(G) \times H_{k-s-1}(F) \otimes H_{\ell-t-1}$. Let $\sigma(B(s,t))$ denote the signature of the form on $H_n(M) \times H_n(M)$ restrictied to $B(s,t)$. Now suppose that $s \neq k-s-1$ or $t \neq \ell-t-1$. Then the form on $B(s,t)$ will have a matrix $N = \begin{bmatrix} 0 & X \\ X^t & 0 \end{bmatrix}$ if a basis of $B(s,t)$ is chosen with respect to the tensor decomposition. But certainly $\sigma(N) = 0$ and this suffices to prove the lemma.

In the light of this lemma we let $K = (S^k, K)$, $L = (S^\ell, L)$ where $k = 2p + 1$ and $\ell = 2q + 1$. Assume that $4 | (k + \ell)$. Let Θ be the Seifert pairing for K on $H_p(F)$ and Ω be the Seifert pairing for L on $H_q(G)$. We will continue to use this notation for the rest of the section.

Definition 3.4. Let $\Theta_\lambda^\varepsilon$ be the complex hermitian or skew-hermitian form defined by the equation $\Theta_\lambda^\varepsilon = (1-\bar{\lambda})\Theta + \varepsilon(1-\lambda)\Theta^*$ where $\varepsilon = \pm 1$ and λ is any complex number of unit length. (* denotes conjugate transpose here.) Define $\sigma(K; \lambda, \varepsilon)$ by the formula $\sigma(K; \lambda, \varepsilon) = \sigma(\Theta_\lambda^\varepsilon)$. Note that if a form $\Theta_\lambda^\varepsilon$ is skew-hermitian then $\sigma(\Theta_\lambda^\varepsilon)$ is, by definition, the signature of the hermitian form $(-i)\Theta_\lambda^\varepsilon$.

The notation that follows will be useful in formulating the next theorem. Let F and G be spanning surfaces for K and L as described above. Let $A = \bar{H}_p(F;\mathbb{C}) \otimes \bar{H}_q(G;\mathbb{C})$ and write $\bar{H}_q(G;\mathbb{C}) = \bigoplus_\lambda B_\lambda$ where B_λ is the Jordan subspace of $\bar{H}_q(G;\mathbb{C})$ corresponding to an eigenvalue λ of the monodromy $h: \bar{H}_q(G;\mathbb{C}) \to \bar{H}_q(G;\mathbb{C})$. Finally, let $\Omega(\lambda)$ denote the restriction of the Seifert pairing Ω to the subspace B_λ.

Definition 3.5. With notation as above, let $\varepsilon = (-1)^q$ and $\mu = (-1)^{(p+1)(q+1)}$. Define $\sigma(\tau;\lambda)$ by the formula $\sigma(\tau;\lambda) = \sigma(\Omega(\lambda) + \varepsilon\Omega(\lambda)^*)$. Define $\sigma_\tau^\lambda(K)$ by the formula $\sigma_\tau^\lambda(K) = \mu\sigma(\Theta \otimes \Omega(\lambda) + \Theta^* \otimes \Omega(\lambda)^*)$.

Theorem 3.6. Let $K = (S^{2p+1}, K)$ be any knot and τ a branched fibration corresponding to the fibered knot $L = (S^{2q+1}, L)$. Let $\varepsilon = (-1)^q$ and $\mu = (-1)^{(p+1)(q+1)}$. Then, using the notation developed above, one has the following formulas for $\sigma_\tau(K)$.

1) $\sigma_\tau(K) = \sum_\lambda \sigma_\tau^\lambda(K)$ where λ runs over all eigenvalues of the monodromy h for τ satisfying $||\lambda|| = 1$.

2) If each Jordan subspace B_λ for $||\lambda|| = 1$, $\lambda \neq 1$ is in fact the λ-eigenspace (i.e., $h|B_\lambda$ has no nilpotent part), then $\sigma_\tau^\lambda(K) = \mu\sigma(\tau;\lambda)\sigma(K;\lambda,\varepsilon)$. Hence

$$\sigma_\tau(K) = \mu\sum_{\lambda\in E}\sigma(\tau;\lambda)\sigma(K;\lambda,\varepsilon) + \sigma_\tau^1(K)$$

where $E = \{\lambda \mid ||\lambda|| = 1,\ \lambda \neq 1,\ \lambda \text{ an eigenvalue of } h\}$.

Proof. First note that the sign $\mu = (-1)^{(p+1)(q+1)}$ comes from the grading convention on the Seifert pairing for $K \otimes L$ as given in 3.2(iv). By 3.3, we need only consider $A = H_p(F) \otimes H_q(G)$ for signature computations. Since by Theorem 2.9 the decomposition $A = \underset{\lambda}{\oplus} B_\lambda$ gives an orthogonal decomposition of Θ_L, we conclude that the signature $\sigma_\tau(K) = \sigma(K \otimes L)$ is the sum of the signatures obtained from tensoring $\overline{H}_p(F;\mathbb{C})$ with B_λ. This gives part 1).

To see the reduction in part 2) it is convenient to use matrix notation. Let V denote a matrix for $\Omega(\lambda)$. Let H be the monodromy on this subspace. Let W be a matrix for Θ_K on $\overline{H}_p(F;\mathbb{C})$. Then by the remark after Proposition 2.7, $H = -\varepsilon V^{-1}V^*$ where $\varepsilon = (-1)^q$ and $*$ denotes conjugate transpose. For part 2) we are given that $H = \lambda I$, I an identity matrix, and $\lambda \neq 1$, $||\lambda|| = 1$. Thus $V^* = -\varepsilon\lambda V$. Let $X = W \otimes V + W^* \otimes V^*$. Then we know that $\sigma_\tau^\lambda(K) = \mu\sigma(X)$. But $X = W \otimes V - \varepsilon\lambda W^* \otimes V$ and it is easy to see from this that $X = [(1-\overline{\lambda})W + \varepsilon(1-\lambda)W^*] \otimes [((1-\lambda)(1-\overline{\lambda}))^{-1}(V + \varepsilon V^*)]$. Since the signature of a tensor product of (skew)hermitian forms is the product of the signatures, this shows that $\sigma(X) = \sigma(K;\lambda,\varepsilon)\sigma(\tau;\lambda)$. This completes the proof of the theorem.

Remark. The signatures $\sigma(K;\lambda,\varepsilon)$ are well-known (see [DK], [L2] and [T]). It is also worth remarking that computing $\sigma_\tau(K)$ actually amounts to finding $\sigma(\tau(D^{k+1},F))$ where $F \subset D^{k+1}$ is any properly embedded codimension-two submanifold with $\partial F = K \subset S^k$. We defined $\sigma_\tau(K)$ when F was a pushed-in spanning surface. A standard (see [KT]) Novikov addition argument shows that $\sigma_\tau(K) = \sigma(\tau(D^{k+1},F))$ for an arbitrary surface as above.

IV. Applications and Examples

Brieskorn manifolds. Here τ is the branched fibration corresponding to the singularity $f(z) = z_0^{a_0} + a_1^{a_1} + \dots + z_n^{a_n}$ and $L = (S^{2n+1}, L)$ where $L = L(f)$. By Theorem 3.2, we know that $\Theta_L \simeq (-1)^{\frac{n(n+1)}{2}} \Lambda_{a_0} \otimes \Lambda_{a_1} \otimes \dots \otimes \Lambda_{a_n}$ where Λ_a is the Seifert pairing for the empty knot of degree a. Since (as in section 2) $\Lambda_a \simeq \mathrm{Diag}(1-\omega, 1-\omega^2, \dots, 1-\omega^{a-1})$ where $\omega = \exp(2\pi i/a)$, we let $\omega_k = \exp(2\pi i/a_k)$. Then Θ is isomorphic to a diagonal matrix with entries $e(\vec{\kappa}) = (-1)^{\frac{n(n+1)}{2}}(1-\omega_0^{\kappa_0})(1-\omega_1^{\kappa_1})\dots(1-\omega_n^{\kappa_n})$ where $\vec{\kappa} = (\kappa_0,\dots,\kappa_n)$ and $0 < \kappa_j < a_j$ for $j = 0,1,\dots,n$. The monodromy matrix H is also diagonal with eigenvalues $\lambda_{\vec{\kappa}} = \overline{\omega}_0^{\kappa_0}\overline{\omega}_1^{\kappa_1}\dots\overline{\omega}_n^{\kappa_n}$. Thus $\sigma(\tau;\lambda_{\vec{\kappa}}) = \begin{cases}\mathrm{Sign}(\mathrm{Re}(e(\vec{\kappa}))), & n \text{ even}\\ \mathrm{Sign}(\mathrm{Im}(e(\vec{\kappa}))), & n \text{ odd}\end{cases}$

where Re and Im stand for real and imaginary parts respectively. It then follows (as in [B]) that if $\varepsilon(\vec{\kappa}) = \sigma(\tau;\lambda^{\vec{\kappa}})$ then

$$\varepsilon(\vec{\kappa}) = \begin{cases} 1 & \text{if } 0 < \sum_s \kappa_s/a_s < 1 \pmod 2 \\ -1 & \text{if } 1 < \sum_s \kappa_s/a_s < 2 \pmod 2 \\ 0 & \text{otherwise} \end{cases} .$$

Thus, given $K = (S^{2p+1},K)$ with $4\,|\,(p+n)$ we conclude that

$$\sigma_\tau(K) = (-1)^{(n+1)(p+1)} \sum_{\vec{0}<\vec{\kappa}<\vec{a}} \varepsilon(\vec{\kappa})\sigma(K;\lambda^{\vec{\kappa}},\varepsilon)$$

where $\varepsilon = (-1)^n$ and $\vec{\kappa} < \vec{a}$ means $\kappa_i < a_i$ for all i.

Cyclic Covers. A case of special interest is $f(z_0) = z_0^{a_0}$. This corresponds to a standard cyclic cover. Thus $\sigma_{[a]}(K) = \sum_{s=1}^{a-1} \sigma(K;\lambda^s,1)$ where $K = (S^{2q+1},K)$, q is odd, and $\lambda = \exp(2\pi i/a)$.

Note that $\sigma_{[a]}(K) = \sigma([a](D^{2q+2},F)) = \mathrm{Sign}(F_a)$ where $F \subset D^{2q+1}$ is a spanning manifold for K and F_a is the a-fold cyclic branched covering of D^{2q+2} along F. In this case the decomposition corresponds to an eigenspace decomposition of $\overline{H}_{q+1}(F_a;\mathbb{C})$ with respect to the covering translation for the branched covering $F_a \to D^{2q+2}$. The monodromy of the empty knot [a] corresponds directly to this deck transformation.

Links in S^3. Recall that two oriented links L and $L' \subset S^3$ are concordant if there is an embedding $j: C \times [0,1] \hookrightarrow S^3 \times [0,1]$ such that $(S^3,L) = (S^3 \times 0, j(C \times 0))$ and $(S^3,L') = (S^3 \times 1, j(C \times 1))$ where C is a disjoint collection of (oriented) circles. If a is a power of a prime number, then it follows as in [GD] that $\sigma_{[a]}(L)$ is a concordance invariant of $L \subset S^3$. Similarly, $\sigma(L;\omega,1)$ is a concordance invariant for $\omega = \exp(2\pi i\kappa/a)$, $0 < \kappa < a$, $a = p^s$, p a prime. Let $\sigma_\omega(L) = \sigma(L;\omega,1)$. By using our approach for cyclic covers, we can extract information about these signatures. For example, we shall give a well-known (see [T]) relationship among signature, genus and nullity.

First recall some further definitions: The multiplicity, $\mu(L)$, of a link $L \subset S^3$ is the number of components of the link. Given $L \subset S^3$, let $F' \subset D^4$ be a pushed-in connected spanning surface for L, and let F'_a denote the a-fold cyclic cover of D^4 branched along E', and L_a the a-fold cyclic cover of S^3 branched along L. With ω as above, let $\langle,\rangle_\omega = (1-\bar{\omega})\theta + (1-\omega)\theta^*$ where $\theta: \overline{H}_1(F';\mathbb{C}) \times \overline{H}_1(F';\mathbb{C}) \to \mathbb{C}$ is the Seifert pairing. The ω-nullity is defined by the formula, $\eta_\omega(L) = (\text{nullity } \langle,\rangle_\omega) + 1$. It is easy to see that $\eta_\omega(L) - 1$ equals the rank of the ω-eigenspace of $H_1(L_a;\mathbb{C})$ with respect to the covering transformation.

Theorem 4.1. Let $F \subset D^4$ be any connected orientable, properly embedded surface so that $\partial F = L \subset S^3$. Let $g(F)$ denote the genus of F. Let ω be a primitive a^{th} root of unity where $a = p^s$, p a prime. Then
$|\sigma_\omega(L)| \le g(F) + \mu(L) - \eta_\omega(L)$.

Proof. Let N denote the a-fold cyclic branched cover of D^4 along F. It is easy to see that $H_1(N) = 0$, and the relative sequence for $(N, \partial N)$ then shows that $\beta_1(N, \partial N) = 0$ (β_1 = first Betti number). Hence $\beta_3(N) = 0$ by Lefschetz duality. Let X denote Euler characteristic. Then $X(N) = a - (a-1)X(F)$ and $X(F) = 1 - \beta_1(F) = 1 - (2g + \mu - 1) = (2 - 2g) - \mu$ where $g = g(F)$ and $\mu = \mu(L)$. Hence $\beta_2(N) = (a-1)\beta_1(F)$. Let $H_2(N;\mathbb{C}) = \oplus_{\kappa=1}^{a-1} H(\kappa)$ where $H(\kappa)$ denotes the eigenspace corresponding to ω^κ. Then $\dim H(\kappa) = (\frac{1}{a-1})\dim H_2(N;\mathbb{C}) = \beta_1(F)$. Let $<,> = <,>_\omega$ = the intersection pairing restricted to $H(1)$. Thus

$$\begin{aligned} |\sigma_\omega(L)| &\le \dim H(1) - \text{nullity } <,> \\ &= \beta_1(F) - \text{nullity } <,> \\ &= 2g + \mu - 1 - \eta_\omega(L) + 1 \\ &= 2g + \mu - \eta_\omega . \end{aligned}$$

Hence $|\sigma_\omega(L)| \le 2g(F) + \mu(L) - \eta_\omega(L)$. This proves the theorem.

G-signatures. For cyclic covers these methods also let us calculate g-signatures where g is some power of the covering translation. Let $K = (S^{2q+1}, K)$ with q odd, and let $F \subset D^{2q+2}$ be a spanning manifold in the disk so that $\partial F = K$. Let $g: F_a \to F_a$ be a covering translation so that $g_*: H_{n+1}(F_a;\mathbb{C}) \to H_{n+1}(F_a;\mathbb{C})$ is represented on the eigenspaces $H(\kappa)$ by multiplication by ξ^κ for ξ an a^{th} root of unity. The usual definition of g-signature (see [AS]) becomes $\sigma(g, F_a) = \Sigma_{\kappa=1}^{a-1} \xi^\kappa(H(\kappa))$ in this context. Since our method gives $\sigma(H(\kappa))$ in terms of the Seifert pairing, this formula may be computed explicitly.

An interesting special case is given by $V \xrightarrow{\pi} S^2$, a d-fold cyclic cover branched along d points. The branched covering space V is the completion of the covering of $X = S^2 - \{d \text{ points}\}$ which corresponds to the representation $\pi_1(X) \xrightarrow{\alpha} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$ where α takes each standardly oriented generator to $1 \in \mathbb{Z}$. It is easy to see that $V - \{d\text{-disks}\} \cong F_d$ where F_d is the fiber of the branched fibration corresponding to $z_0^d + z_1^d$. From this one computes directly that $\sigma(g,V) = d \coth(i\theta/2)$ where g corresponds to $\xi = e^{i\theta}$. This is, of course, a very special case of the Atiyah-Singer G-signature theorem. This approach has been used by Patrick Gilmer in [G]. It is also instructive to form V directly by a cut-and-paste construction and then do the calculations from a direct geometric base.

V. A Generalized Hirzebruch Construction

So far we have restricted ourselves to branched fibrations closely related to knot theory. In order to form more general branched fibrations, we generalize a construction due to Hirzebruch ([H]) .

Let $V^{m-2} \subset M^m$ be an inclusion of closed manifolds. Assume that the homology class of V , $[V] \in H_{m-2}(M;\mathbb{Z})$, is Poincare dual to dx for $x \in H^2(M;\mathbb{Z})$ where d is a (non-zero) integer. Under these conditions there is a well-defined d-fold cyclic branched cover of M with branch set V . We assert the corresponding statement for a certain class of branched fibrations.

Definition 5.1. Call a branched fibration $\tau : D^{2k} \to D^2$ d-equivariant if it satisfies the following conditions: Let S^1 act on $D^{2k} = \{z \in \mathbb{C} \mid \|z\| = 1\}$ by the formula $\lambda \cdot (z_1, z_2, \ldots, z_k) = (\lambda^{\alpha_1} z_1, \lambda^{\alpha_2} z_2, \ldots, \lambda^{\alpha_k} z_k)$ for a chosen set of positive integers $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then, with respect to this action, $\tau(\lambda \cdot z) = \lambda^d \tau(z)$.

For example, the branched fibration corresponding to a Brieskorn polynomial $f(z_1, \ldots, z_k) = z_1^{a_1} + \ldots + z_k^{a_k}$ is d-equivariant for d = least common multiple $(a_1, a_2, \ldots, a_k)$.

Proposition 5.2. Let $\tau : D^{2k} \to D^2$ be a d-equivariant branched fibration. Let $V^{m-2} \subset M^m$ be an embedding of closed manifolds so that the Poincare dual of $[V] \in H_{m-2}(M;\mathbb{Z})$ is divisible by d . Then there is a manifold $\hat{M}$ and a smooth mapping $\pi : \hat{M} \to M$ forming a τ-branched fibration of M along V .

Proof. Let $\bar{\Lambda} \to M$ be the complex disk bundle with first Chern class $c_1(\bar{\Lambda}) = x$ where dx = Poincare dual of $[V]$. Let D_0^2 denote D^2 with the circle action $\lambda \cdot z = \lambda^d z$. Note that if X and Y are S^1-spaces, then one defines

$$X \times_{S^1} Y = \{[x,y] \mid (\lambda x, y) \sim (x, \lambda y) \text{ for } \lambda \in S^1 ;\ [x,y] = \text{equivalence class of } (x,y)\}$$

Thus $D_0^2 \times_{S^1} \Lambda = \bar{\Lambda}^d$ where Λ denotes the circle bundle associated with $\bar{\Lambda}$ and $\bar{\Lambda}^d$ is the d-fold tensor product of $\bar{\Lambda}$ with itself.

Let $E = D^{2k} \times_{S^1} \Lambda$. Then τ induces a mapping $T : E \to \bar{\Lambda}^d$ by the formula $T[z,x] = [\tau(z), x]$. Note that T is a τ-branched fibration of $\bar{\Lambda}^d$ along its zero section $M \subset \bar{\Lambda}^d$.

Since $c_1(\bar{\Lambda}^d) = dx$ is Poincaré dual to $V \subset M$, there exists a section $s : M \to \bar{\Lambda}^d$ so that s is transverse to M and $s(M) \cap M = V$.

Let $\hat{M} = T^{-1} s(M)$ and $\pi : \hat{M} \to M$ be the restriction of $a : E \to M$ where $a[z,x] = p(x)$, and $p : \Lambda \to M$ is the bundle projection. This constructs the desired branched fibration.

Remark. For branched coverings this construction appears in an article of Hirzebruch ([H]) . The construction is summarized by the following diagram:

$$\begin{tikzcd}
\hat{M} = \mathcal{T}^{-1}s(M) \arrow[r, hook, "i"] \arrow[drr, "\pi"'] & E \arrow[r, "\mathcal{T}"] \arrow[dr, "a"] & \Lambda^d \arrow[d] \\
& & M \arrow[u, bend right, "s"']
\end{tikzcd}$$.

The next theorem generalizes a result obtained by P.Gilmer for cyclic coverings ([G]) .

Theorem 5.3. With notation as above, we may obtain information on the tangent bundle of $\hat{M}$ as follows: Let $T(\ )$ denote the real tangent bundle. Let $\boldsymbol{\Lambda}^d$ denote the underlying real vector bundle of $\overline{\Lambda}^d$ and $\mathbb{E}$ the underlying real vector bundle of $E \to M$. Then

$$T(\hat{M}) \oplus \pi^*(\boldsymbol{\Lambda}^d) \cong \pi^*(T(M) \oplus \mathbb{E}) .$$

Proof. The proof is identical in form to that given by Gilmer. We have

$$\begin{aligned} T(\hat{M}) \oplus \nu^E_{\hat{M}} &= i^*T(E) \\ &= i^*(a^*T(M) \oplus a^*\mathbb{E}) \\ &= \pi^*(T(M) \oplus \mathbb{E}) \end{aligned}$$

while $\nu^E_{\hat{M}} = i^*\mathcal{T}^*\nu^{\boldsymbol{\Lambda}^d}_{S(M)} = \pi^*(\boldsymbol{\Lambda}^d)$. Here ν^Y_X denotes the normal bundle of X in Y .

Remark. For cyclic branched covers, this result is extraordinarily useful, since $\dim \hat{M} = \dim M$ and $\hat{M}$ is closed if M is closed. Thus the formula of 5.3 in conjunction with the Hirzebruch Index Theorem gives formulas for the signature of cyclic branched covers.

The situation is not all so fortunate for branched fibrations. Now $\hat{M}$ need not be closed and it will have higher dimension than M . Thus 5.3 is only a first step towards the signature of branched fibrations. We hope to continue this study in another paper.

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