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Algebraic Classification of Linking Pairings on 3-Manifolds

Akio Kawauchi[1]* and Sadayoshi Kojima[2]

[1] Institute for Advanced Study, Princeton, NJ 08540, USA
[2] Columbia University, New York, NY 10027, USA

A *linking* is understood as a pair (G, ϕ) such that G is a finite abelian group and ϕ is a nonsingular, symmetric bilinear pairing $G \times G \to Q/Z$. It is convenient to identify a linking (G, ϕ) with a matrix which represents ϕ relative to the generators of a cyclic splitting of G unless confusion might occur. The linking occurs often in the study of topology. For example, given a closed oriented 3-manifold M, we have a unique linking $(\tau H_1(M), \phi_M)$ defined by the Poincaré duality, where $\tau H_1(M)$ is the torsion part of the integral homology group $H_1(M)$. The purpose of this paper is to determine completely the structure of the abelian semigroup $\mathfrak{N}$ of all linkings (up to isomorphism) under block sum and to observe that any linking is isomorphic to the linking ϕ_M of a closed connected oriented 3-manifold M. To do this, we shall present a complete system of invariants of isomorphic linkings, which arises naturally from our purpose. Such a complete system had already been known by Seifert [11] in the case of odd-primary groups, and in general by Burger [2, Satz 5] in terms of Minkowski's beautiful theory. Though they are related directly or indirectly to each other (cf. Fox [5]), we do not discuss here any relation between them.

$\mathfrak{N}$ is isomorphic to a direct sum $\bigoplus_p \mathfrak{N}_p$ of the abelian semigroups $\mathfrak{N}_p$ of linkings on p-groups (up to isomorphism). Wall [15, Theorem (4)] showed that for each odd p, $\mathfrak{N}_p$ has a presentation with generators (p^{-k}), $(p^{-k}n(p))$ $(k \geqq 1)$ (which are linkings on Z_{p^k}) and relations $2(p^{-k}) = 2(p^{-k}n(p))$, where $n(p)$ is a fixed quadratic non-residue (mod p). Wall [15] gave also generators of $\mathfrak{N}_2$ (cf. van Kampen [13]). They are (in our notations) $A^k(n) = (2^{-k}n)$ $(k \geqq 1)$ (which are linkings on Z_{2^k}), where $n = 1 (k=1)$, $\pm 1 (k=2)$, ± 1, $\pm 5 (k \geqq 3)$, and $E_0^k = \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix} (k \geqq 1)$, $E_1^k = \begin{pmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{pmatrix} (k \geqq 2)$ (which are linkings on $Z_{2^k} \oplus Z_{2^k}$). So, the remaining problem is to give a presentation of $\mathfrak{N}_2$ relative to these generators. For convenience, we will use the symbol $A^k(n)$ for any odd n under the identification

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that $A^1(n)=A^1(1)$, $A^2(n)=A^2((-1)^{(n-1)/2})$ and for $k\geqq 3$ $A^k(n)=A^k(n')$ if and only if $n\equiv n'(\mathrm{mod}\,8)$ (cf. Wall [15], Vinogradov [14, p. 69]).

We shall show the following:

Theorem 0.1. *$\mathfrak{N}_2$ has a presentation with generators $A^k(n)$ ($k\geqq 1$), where $n=1(k=1)$, $\pm 1(k=2)$, ± 1, $\pm 5(k\geqq 3)$, $E_0^k(k\geqq 1)$ and $E_1^k(k\geqq 2)$, and relations*

$$A^k(n_1)\oplus A^k(n_2)=A^k(n_1+4)\oplus A^k(n_2+4)\,(k\geqq 3) \tag{0.1}$$

$$A^k(n)\oplus 2A^k(-n)=A^k(-n)\oplus E_0^k(k\geqq 1) \tag{0.2}$$

$$3A^k(n)=A^k(-n+4)\oplus E_1^k(k\geqq 2) \tag{0.3}$$

$$2E_0^k=2E_1^k(k\geqq 2) \tag{0.4}$$

$$A^k(n_1)\oplus A^{k+1}(n_2)=A^k(n_1+2n_2)\oplus A^{k+1}(n_2+2n_1)\,(k\geqq 1) \tag{1.1}$$

$$A^k(n)\oplus E_1^{k+1}=A^k(n+4)\oplus E_0^{k+1}(k\geqq 1) \tag{1.2}$$

$$E_1^k\oplus A^{k+1}(n)=E_0^k\oplus A^{k+1}(n+4)\,(k\geqq 2) \tag{1.3}$$

$$A^k(n_1)\oplus A^{k+2}(n_2)=A^k(n_1+4)\oplus A^{k+2}(n_2+4)\,(k\geqq 1). \tag{2.1}$$

These relations (0.1), (0.2), ..., (2.1) are easily verified. The true meaning of Theorem 0.1 is that any relation is generated by them.

Section 1 is a preliminary section. In Sect. 2 we define and calculate an invariant of a linking on a 2-group, which comes from the Gaussian sum of a certain associated quadratic function. In Sect. 3 we exhibit all linkings on homogeneous 2-groups (up to isomorphism). In Sect. 4 we classify a linking in terms of our invariants and prove Theorem 0.1. In Sect. 5 we study split linkings and hyperbolic linkings. In particular, we shall calculate the Witt groups of homogeneous linkings and more general linkings. In Sect. 6 we study the linkings of 3-manifolds. Several topological applications are given, besides realizing any linking by ϕ_M of a closed connected oriented 3-manifold M.

1. Preliminaries

Two linkings (G,ϕ), (G',ϕ') are *isomorphic* if there is an isomorphism $f: G\to G'$ such that $\phi(x,y)=\phi'(f(x),f(y))$ for all $x,y\in G$. We denote it by $(G,\phi)\cong(G',\phi')$ but the equality $(G,\phi)=(G',\phi')$ is also used and called a *relation* when both (G,ϕ) and (G',ϕ') are written as block sums of some copies of the given matrices (p^{-k}), $(p^{-k}n(p))$, $A^k(n)$, E_0^k and E_1^k. The isomorphism class of (G,ϕ) is denoted by $[G,\phi]$. Let p be a prime number and k be an integer $\geqq 1$. For any linking (G,ϕ) let $\bar{G}_p^k$ be the subgroup of G generated by elements of order p^s with $s\leqq k$, $G_p=\bigcup_{k\geqq 1}\bar{G}_p^k$ and $\phi_p=\phi|G_p\times G_p$. G_p is the p-component of G and ϕ_p is a linking on G_p. Any linking (G,ϕ) has a primary (orthogonal) splitting $\bigoplus_p(G_p,\phi_p)$, and the isomorphism class $[G,\phi]$ determines the isomorphism class $[G_p,\phi_p]$ uniquely for each p. Let $\tilde{G}_p^k=\bar{G}_p^k/\bar{G}_p^{k-1}+p\bar{G}_p^{k+1}$. Define a pairing

$$\tilde{\phi}_p^k:\tilde{G}_p^k\times\tilde{G}_p^k\to Q/Z$$

by the identity

$$\tilde{\phi}_p^k([x],[y])=p^{k-1}\phi(x,y)$$

for $x, y \in \tilde{G}_p^k$, which is clearly well defined. $(\tilde{G}_p^k, \tilde{\phi}_p^k)$ is also a linking by a fact that (G_p, ϕ_p) has a *homogeneous splitting*, that is, an orthogonal splitting $\bigoplus_{k \geq 1} (G_p^k, \phi_p^k)$ such that G_p^k is isomorphic to a direct sum of copies of Z_{p^k} (see Wall [15]). Note that $\tilde{G}_p^k$ is a vector space over Z_p. Let $r_p^k = r_p^k(G) = \dim_{Z_p} \tilde{G}_p^k$. r_p^k is a group invariant of G. In case (G, ϕ) is a linking of a homogeneous p-group of exponent k, i.e., $(G, \phi) = (G_p^k, \phi_p^k)$, then we can also obtain an induced linking $(I^s G, I^s \phi)$, defined for each $(1 \leqq) s < k$ by $I^s G = G/p^s G$ and $I^s \phi([x], [y]) = p^{k-s} \phi(x, y)$ for $x, y \in G$ and for each $s \geqq k$ by $(I^s G, I^s \phi) = (G, \phi)$. Given a homogeneous splitting $\bigoplus_k (G_p^k, \phi_p^k)$ of (G_p, ϕ_p), then we have always $(I^1 G_p^k, I^1 \phi_p^k) \cong (\tilde{G}_p^k, \tilde{\phi}_p^k)$ for all $k \geqq 1$.

First we assume p is odd. We regard $(\tilde{G}_p^k, \tilde{\phi}_p^k)$ as an inner product space over Z_p (cf. [10]) by using the canonical imbedding $Z_p \to Q/Z$ sending 1 to p^{-1}. Denote it by $(\tilde{G}_p^k, \tilde{\phi}_{p*}^k)$.

Definition 1.1. The Legendre's symbol $(\det \tilde{\phi}_{p*}^k / p)$ is denoted by $\chi_p^k(\phi)$.

$\chi_p^k(\phi)$ is an invariant of the isomorphism class $[G, \phi]$ by a property of Legendre's symbol (cf. Seifert [11, p. 199]).

Let $\bigoplus_k (G_p^k, \phi_p^k)$ be any homogeneous splitting of (G_p, ϕ_p). As Wall [15] showed, we have $(G_p^k, \phi_p^k) \cong r_p^k(p^{-k})$ or $(r_p^k - 1)(p^{-k}) \oplus (p^{-k} n(p))$ according to whether $\chi_p^k(\phi) = 1$ or -1.

Hence we have the following:

Lemma 1.1. *The isomorphism class* $[G_p^k, \phi_p^k]$ *is determined uniquely by the invariants* r_p^k *and* $\chi_p^k(\phi)$ *of* $[G, \phi]$.

Now we consider the case $p = 2$. In this case, the arguments will be more complicated because, in general, a homogeneous splitting of (G_2, ϕ_2) is not unique (see the relations in Theorem 0.1).

Definition 1.2. $c^k(\phi)$ is the characteristic element of $\tilde{\phi}_2^k$, which is specified uniquely by the identity

$$\tilde{\phi}_2^k(c^k(\phi), x) = \tilde{\phi}_2^k(x, x)$$

for all $x \in \tilde{G}_2^k$.

$c^k(\phi)$ is an invariant of $[G, \phi]$ in the sense that any isomorphism $(G, \phi) \cong (G', \phi')$ induces an isomorphism $(\tilde{G}_2^k, \tilde{\phi}_2^k) \cong (\tilde{G}_2'^k, \tilde{\phi}_2'^k)$ sending $c^k(\phi)$ to $c^k(\phi')$.

2. The Gaussian Sum and its Associated Invariant

We consider a linking (G, ϕ) with $c^k(\phi) = 0$. Then there exists a function

$$q_k : G_2 / \tilde{G}_2^k \to Q/Z$$

defined by

$$q_k([x]) = 2^{k-1} \phi(x, x)$$

for $x \in G_2$. [Note that if $x' = x + x_0$, $x_0 \in \tilde{G}_2^k$, then

$$2^{k-1} \phi(x', x') = 2^{k-1} \phi(x, x) + \tilde{\phi}_2^k(c^k(\phi), [x_0]).]$$

We denote by $GS_k(\phi)$ the Gaussian sum

$$\sum_{x\in G_2/\tilde{G}_2^k} \exp(2\pi i q_k(x)),$$

where $i=\sqrt{-1}$.

Lemma 2.1. *We have the following properties.*

(1) *If* $(G,\phi)\cong(G',\phi')\oplus(G'',\phi'')$ *and* $c^k(\phi)=0$, *then* $c^k(\phi')=c^k(\phi'')=0$ *and* $GS_k(\phi) = GS_k(\phi')\,GS_k(\phi'')$.

(2) $GS_k(\phi_2^m)=GS_s(I^{m-k+s}\phi_2^m)$ *for* $s<k<m$.

(3) $GS_1(A^m(n))=\begin{cases}2^{(m-1)/2}\exp((-1)^{(n-1)/2}2\pi i/8) & (m\ \textit{even})\\ 2^{(m-1)/2}\exp(2\pi i n/8) & (m\ \textit{odd})\end{cases}(m\geqq 2)$

(4) $GS_1(E_0^m)=2^{m-1}$, $GS_1(E_1^m)=(-1)^{m-1}2^{m-1}(m\geqq 2)$.

Proof. (1) is clear. [Note that $\phi(x,x)=\phi'(x',x')+\phi''(x'',x'')$ for $x=x'+x''$, $x'\in G'$, $x''\in G''$.] For (2) let G be a homogeneous 2-group of exponent m on which ϕ_2^m is defined. Let q_k, q'_s be the functions associated with ϕ_2^m, $I^{m-k+s}\phi_2^m$. It follows that q_k, q'_s define the same function on $G/2^{m-k}G$, showing (2). For (3) we show that

$$GS_1(A^m(n))=2GS_1(A^{m-2}(n))$$

for $m\geqq 4$. In fact,

$$\begin{aligned}GS_1(A^m(n)) &= \sum_{j=1}^{2^{m-1}} \exp(2\pi i n j^2/2^m)\\ &= \sum_{j=1}^{2^{m-2}} \exp(2\pi i n(2j)^2/2^m)+\sum_{j=1}^{2^{m-2}} \exp(2\pi i n(2j-1)^2/2^m)\\ &= \sum_{j=1}^{2^{m-3}} \exp(2\pi i n j^2/2^{m-2})+\sum_{j=1}^{2^{m-3}} \exp(2\pi i n(2^{m-3}+j)^2/2^{m-2})\\ &\quad+\sum_{j=1}^{2^{m-3}} \exp(2\pi i n(2j-1)^2/2^m)+\sum_{j=1}^{2^{m-3}} \exp(2\pi i n[2(2^{m-3}+j)-1]^2/2^m).\end{aligned}$$

Note that $(2^{m-3}+j)^2/2^{m-2}\equiv j^2/2^{m-2}(\mathrm{mod}\,1)$ and $[2(2^{m-3}+j)-1]^2/2^m\equiv 1/2 +(2j-1)^2/2^m(\mathrm{mod}\,1)$. Hence we have

$$GS_1(A^m(n))=2\sum_{j=1}^{2^{m-3}} \exp(2\pi i n j^2/2^{m-2})=2GS_1(A^{m-2}(n)).$$

For $m=2, 3$, $GS_1(A^m(n))$ is easily calculated and (3) is obtained. (4) follows from (1), (3) by applying GS_1 to the relations (0.2) and (0.3). This completes the proof.

Since any linking is isomorphic to a block sum of some copies of $A^m(n)$ $(m\geqq 1)$, $E_0^m(m\geqq 1)$ and $E_1^m(m\geqq 2)$, we see the following:

Corollary 2.1. *For any linking* (G,ϕ) *with* $c^k(\phi)=0$, $GS_k(\phi)$ *is not* zero *and there exists a unique integer* σ (mod 8) *such that*

$$GS_k(\phi)=|GS_k(\phi)|\exp(2\pi i\sigma/8).$$

Let $\bar{Z}_8=Z_8\cup\{\infty\}$ be a semigroup (consisting of 9 elements) whose extra summations are defined by

$$\infty+\infty=\infty+n=n+\infty=\infty,\qquad n\in Z_8.$$

Definition 2.1. $\sigma_k(\phi)$ is the integer $\sigma(\bmod 8)$ or ∞ according to whether $c^k(\phi)$ is 0 or not.

Thus, for each $k \geqq 1$ there exists an invariant $\sigma_k(\phi)$ of $[G, \phi]$ in $\bar{Z}_8$. By definition, a homogeneous linking $\phi_2^m (m \geqq 2)$ has $\sigma_k(\phi_2^m) \neq \infty$ for $k < m$. The invariant $\sigma_1(\phi_2^2)$ was introduced by Brown [1] concerning generalizations of the Kervaire-Brown-Peterson-Browder invariant.

The following is easily obtained from Lemma 2.1

Corollary 2.2. (1) *$\sigma_k : \mathfrak{N} \to \bar{Z}_8$ is a semigroup homomorphism for all $k \geqq 1$,*

(2) $\sigma_k(\phi_2^m) = \sigma_s(I^{m-k+s}\phi_2^m)$ *for $s < k < m$,*

(3) $\sigma_k(A^m(n)) = \begin{cases} (-1)^{(n-1)/2} & (m-k \text{ odd}) \\ n & (m-k \text{ even}), \end{cases} \quad \sigma_k(E_0^m) = 0$

and

$$\sigma_k(E_1^m) = 2[1 - (-1)^{m-k}], \quad \text{where} \quad k < m.$$

For our purpose the following Lemma is important.

Lemma 2.2. *Let (G, ϕ) be a linking with $c^k(\phi) = c^s(\phi) = 0$ for some $k, s (k \leqq s)$. For any homogeneous splitting $\bigoplus_{m \geqq 1} (G_2^m, \phi_2^m)$ of (G_2, ϕ_2), there exists an element $0(\phi, k, s) \in \{0, 4\} \subset \bar{Z}_8$ such that*

$$\sigma_k(\phi) - \sigma_s(\phi) = \sigma_k(\phi_2^{k+1}) + \sigma_k(\phi_2^{k+2}) + \ldots + \sigma_k(\phi_2^s) + 0(\phi, k, s).$$

$0(\phi, k, s)$ does not depend on a choice of a homogeneous splitting of (G_2, ϕ_2) if and only if one of the following cases occurs.

(1) *$s - k$ is even. In this case, $\sigma_k(\phi_2^s) = 0(\phi, k, s) = 0$.*

(2) *$s - k$ is odd and $c^{s+1}(\phi) = 0$. In this case, $0(\phi, k, s) = \sigma_{s+1}(\phi) - \sigma_s(\phi)$.*

(3) *$s - k$ is odd, $r_2^s(G) = 0$ and $c^{s-1}(\phi) = 0$. In this case, $0(\phi, k, s) = \sigma_{s-1}(\phi) - \sigma_s(\phi)$.*

Proof. By Corollary 2.2(1) and $c^k(\phi) = c^s(\phi) = 0$, $\sigma_k(\phi) = \sum_{m \geqq k+1} \sigma_k(\phi_2^m)$ and $\sigma_s(\phi) = \sum_{m \geqq s+1} \sigma_s(\phi_2^m)$, so that

$$\sigma_k(\phi) - \sigma_s(\phi) = \sigma_k(\phi_2^{k+1}) + \ldots + \sigma_k(\phi_2^s) + 0(\phi, k, s),$$

where we let $0(\phi, k, s) = \sum_{m \geqq s+1} (\sigma_k(\phi_2^m) - \sigma_s(\phi_2^m))$. Since ϕ_2^m is a block sum of suitable copies of $A^m(n)$, E_0^m, and E_1^m, it follows from Corollary 2.2(3) that $\sigma_k(\phi_2^m) - \sigma_s(\phi_2^m) \equiv 0 (\bmod 4)$. Hence $0(\phi, k, s) = 0$ or 4, showing the first half. In Case (1) we can also see that $\sigma_k(\phi_2^m) - \sigma_s(\phi_2^m) = 0$ and hence $0(\phi, k, s) = 0$. Further, in this case $\sigma_k(\phi_2^s) = 0$, since ϕ_2^s is a block sum of some copies of E_0^s and E_1^s. In Case (2), by using Case (1) we obtain that

$$\sigma_k(\phi) - \sigma_s(\phi) = \sigma_k(\phi) - \sigma_{s+1}(\phi) + 0(\phi, k, s).$$

Hence $0(\phi,k,s)=\sigma_{s+1}(\phi)-\sigma_s(\phi)$. In Case (3), we see from Case (1) that

$$\sigma_k(\phi)-\sigma_s(\phi)=\sigma_k(\phi)-\sigma_{s-1}(\phi)+0+0(\phi,k,s),$$

so that $0(\phi,k,s)=\sigma_{s-1}(\phi)-\sigma_s(\phi)$.

Let $s-k$ be odd and $c^{s+1}(\phi)\neq 0$. Now we show that if $r_2^s(G)\neq 0$ or $c^{s-1}(\phi)\neq 0$ (in this case $k\leqq s-3$), then $0(\phi,k,s)$ is not uniquely determined. First consider the case $r_2^s(G)\neq 0$. In this case, for any homogeneous splitting $\bigoplus_m \phi_2^m$ of ϕ_2 we can write $\phi_2^s\cong E_0^s$ (or E_1^s)$\oplus\bar{\phi}_2^s$ and $\phi_2^{s+1}=A^{s+1}(n)\oplus\bar{\phi}_2^{s+1}$ for some $\bar{\phi}_2^s$, $A^{s+1}(n)$ and $\bar{\phi}_2^{s+1}$. Let $\phi_2'^s=E_1^s$ (or E_0^s)$\oplus\bar{\phi}_2^s$ and $\phi_2'^{s+1}=A^{s+1}(n+4)\oplus\bar{\phi}_2^{s+1}$. By the relation (1.3) $\phi_2^s\oplus\phi_2^{s+1}\cong\phi_2'^s\oplus\phi_2'^{s+1}$. Consider the new splitting

$$\bigoplus_{m\leqq s-1}\phi_2^m\oplus\phi_2'^s\oplus\phi_2'^{s+1}\bigoplus_{m\geqq s+2}\phi_2^m \quad\text{of}\quad \phi_2.$$

We write

$$\sigma_k(\phi)-\sigma_s(\phi)=\sigma_k(\phi_2^{k+1})+\ldots+\sigma_k(\phi_2'^s)+0'(\phi,k,s).$$

Since $s-k$ is odd, by Corollary 2.2 we have $\sigma_k(\phi_2^s)+4=\sigma_k(\phi_2'^s)$. Hence $0(\phi,k,s)=0'(\phi,k,s)+4$. Next, consider the case $c^{s-1}(\phi)\neq 0$ (hence $k\leqq s-3$). In this case we can write $\phi_2^{s-1}\cong A^{s-1}(n')\oplus\bar{\phi}_2^{s-1}$ and $\phi_2^{s+1}\cong A^{s+1}(n)\oplus\bar{\phi}_2^{s+1}$ for some $A^{s-1}(n')$, ϕ_2^{s-1}, $A^{s+1}(n)$, $\bar{\phi}_2^{s+1}$. Let $\phi_2'^{s+1}=A^{s-1}(n'+4)\oplus\bar{\phi}_2^{s-1}$ and $\phi_2'^{s+1}=A^{s+1}(n+A)\oplus\bar{\phi}_2^{s+1}$. By the relation (2.1), $\phi_2'^{s-1}\oplus\phi_2'^{s+1}\cong\phi_2^{s-1}\oplus\phi_2^{s+1}$. Consider the new splitting

$$\bigoplus_{m\leqq s-1}\phi_2^m\oplus\phi_2'^{s-1}\oplus\phi_2^s\oplus\phi_2'^{s+1}\bigoplus_{m\geqq s+2}\phi_2^m \quad\text{of}\quad \phi_2$$

and write

$$\sigma_k(\phi)-\sigma_s(\phi)=\sigma_k(\phi_2^{k+1})+\ldots+\sigma_k(\phi_2'^{s-1})+\sigma_k(\phi_2^s)+0'(\phi,k,s).$$

Since $(s-1)-k(\geqq 2)$ is even, we see from Corollary 2.2 that $\sigma_k(\phi_2'^{s-1})=\sigma_k(\phi_2^{s-1})+4$, so that $0(\phi,k,s)=4+0'(\phi,k,s)$. This proves Lemma 2.2.

3. Exhibition of Linking on Homogeneous 2-Groups

Let (G,ϕ) be a linking on a homogeneous 2-group of exponent 1. In case $c^1(\phi)=0$, $r=r_2^1(G)$ is even and $(G,\phi)\cong(r/2)E_0^1$. In case $c^1(\phi)\neq 0$ and $\phi(c^1(\phi),c^1(\phi))=0$, r is even and $(G,\phi)\cong 2A^1(1)\oplus[(r-2)/2]E_0^1$. In case $\phi(c^1(\phi),c^1(\phi))\neq 0$, r is odd and $(G,\phi)\cong A^1(1)\oplus[(r-1)/2]E_0^1$. (Cf. Wall [15, p. 290].) Hence we have the following:

Lemma 3.1. *Any linking (G,ϕ) on a homogeneous 2-group of exponent 1 is isomorphic to one of the following linkings, which belong to mutually distinct isomorphism classes:*

$$L_1^1=s_1E_0^1,\quad L_2^1=2A^1(1)\oplus s_2E_0^1,\quad L_3^1=A^1(1)\oplus s_3E_0^1,$$

where $s_1(\neq 0)$, s_2, s_3 are any integers $\geqq 0$. The invariants $r_2^1(G)$ and $\sigma_1(\phi)$ form a complete system of invariants.

Lemma 3.2. *Any linking (G,ϕ) on a homogeneous 2-group of exponent 2 is isomorphic to one of the following linkings, which belong to mutually distinct isomorphism classes:*

$$L^2_{1,1}=s_1E^2_0,\quad L^2_{1,2}=E^2_1\oplus s_2E^2_0,$$
$$L^2_{2,1}=2A^2(1)\oplus s_3E^2_0,\quad L^2_{2,2}=2A^2(-1)\oplus s_4E^2_0,$$
$$L^2_{2,3}=A^2(1)\oplus A^2(-1)\oplus s_5E^2_0,\quad L^2_{2,4}=A^2(1)\oplus A^2(-1)\oplus E^2_1\oplus s_6E^2_0,$$
$$L^2_{3,1}=A^2(1)\oplus s_7E^2_0,\quad L^2_{3,2}=A^2(1)\oplus E^2_1\oplus s_8E^2_0,$$
$$L^2_{3,3}=A^2(-1)\oplus s_9E^2_0,\quad L^2_{3,4}=A^2(-1)\oplus E^2_1\oplus s_{10}E^2_0,$$

where $s_1(\neq 0), s_2, \ldots, s_{10}$ are any integers $\geqq 0$. The invariants $r^2_2(G)$, $\sigma_1(\phi)$, $\sigma_2(\phi)$ form a complete system of invariants in this case.

Proof. By Lemma 3.1, $I^1\phi$ is isomorphic to one of L^1_1, L^1_2, and L^1_3. If $I^1\phi\cong L^1_1$ (or L^1_3, respectively), then ϕ is isomorphic to $L^2_{1,1}$ or $L^2_{1,2}$ (or $L^2_{3,1}$, $L^2_{3,2}$, $L^2_{3,3}$ or $L^2_{3,4}$, respectively) by using the relation $2E^2_0=2E^2_1$ in (0.4). In case $I^1\phi\cong L^1_2$, ϕ is isomorphic to $L^2_{2,1}$, $L^2_{2,2}$, $L^2_{2,3}$ or $L^2_{3,4}$ by using $2E^2_0=2E^2_1$ and the relation $2A^2(n)\oplus E^2_1=2A^2(-n)\oplus E^2_0(n=\pm 1)$ obtained from the relations (0.2) and (0.3). By Lemma 3.1, the invariants $r^2_2(G)$ and $\sigma_2(\phi)$ distinguish between $L^2_{1,j}$, $L^2_{2,j'}$, and $L^2_{3,j''}$. Further, we have that $\sigma_1(L^2_{1,j})=0,4$ according as $j=1,2$, $\sigma_1(L^2_{2,j'})=2,-2,0,4$ according as $j'=1, 2, 3, 4$ and $\sigma_1(L^2_{3,j''})=1, 5, -1, 3$ according as $j''=1, 2, 3, 4$. This completes the proof.

Lemma 3.3. *Any linking (G,ϕ) on a homogeneous 2-group of exponent $k\geqq 3$ is isomorphic to one of the following linkings, which belong to mutually distinct isomorphism classes:*

$$L^k_{1,1}=s_1E^k_0,\quad L^k_{1,2}=E^k_1\oplus s_2E^k_0,\quad L^k_{2,1,1}=2A^k(1)\oplus s_3E^k_0,$$
$$L^k_{2,1,2}=A^k(1)\oplus A^k(5)\oplus s'_3E^k_0,\quad L^k_{2,2,1}=2A^k(-1)\oplus s_4E^k_0,$$
$$L^k_{2,2,2}=A^k(-1)\oplus A^k(-5)\oplus s'_4E^k_0,\quad L^k_{2,3,1}=A^k(1)\oplus A^k(-1)\oplus s_5E^k_0,$$
$$L^k_{2,3,2}=A^k(1)\oplus A^k(-5)\oplus s'_5E^k_0,\quad L^k_{2,4,1}=A^k(1)\oplus A^k(-1)\oplus E^k_1\oplus s_6E^k_0,$$
$$L^k_{2,4,2}=A^k(1)\oplus A^k(-5)\oplus E^k_1\oplus s'_6E^k_0,\quad L^k_{3,1,1}=A^k(1)\oplus s_7E^k_0,$$
$$L^k_{3,1,2}=A^k(5)\oplus s'_7E^k_0,\quad L^k_{3,2,1}=A^k(1)\oplus E^k_1\oplus s_8E^k_0,$$
$$L^k_{3,2,2}=A^k(5)\oplus E^k_1\oplus s'_8E^k_0,\quad L^k_{3,3,1}=A^k(-1)\oplus s_9E^k_0,$$
$$L^k_{3,3,2}=A^k(-5)\oplus s'_9E^k_0,$$
$$L^k_{3,4,1}=A^k(-1)\oplus E^k_1\oplus s_{10}E^k_0,\quad L^k_{3,4,2}=A^k(-5)\oplus E^k_1\oplus s'_{10}E^k_0,$$

where $s_1(\neq 0), s_2, s_3, s'_3, \ldots, s_{10}, s'_{10}$ are any integers $\geqq 0$. The invariants $r^k_2(G)$, $\sigma_k(\phi)$, $\sigma_{k-1}(\phi)$ and $\sigma_{k-2}(\phi)$ form a complete system of invariants in this case.

Proof. By Lemma 3.2, the system of invariants $r^k_2(G)$, $\sigma_k(\phi)$, and $\sigma_{k-1}(\phi)$ determines the isomorphism class of $(I^2G, I^2\phi)$ uniquely. Assume $(I^2G, I^2\phi)\cong L^2_{a,b}$ for some a, b. If $a=1$, then clearly $(G,\phi)\cong L^k_{1,b}$. In case $a\geqq 2$, we have $(G,\phi)\cong L^k_{a,b,1}$ or $L^k_{a,b,2}$ by using the relation (0.1). $L^k_{a,b,1}$ and $L^k_{a,b,2}$ belong to distinct isomorphism classes, since $\sigma_{k-2}(A^k(n))=n\neq 5n=\sigma_{k-2}(A^k(5n))$, $n=\pm 1$. This completes the proof.

4. Classification of Linkings in the General Case and Proof of Theorem 0.1

Theorem 4.1. *Two linkings (G,ϕ), (G',ϕ') are isomorphic if and only if $r_p^k(G)=r_p^k(G')$, $\chi_{p'}^k(\phi)=\chi_{p'}^k(\phi')$ and $\sigma_k(\phi)=\sigma_k(\phi')$ for all prime p, odd prime p' and $k\geqq 1$.*

Proof. By Lemma 1.1 it suffices to prove that if $r_2^k(G)=r_2^k(G')$ and $\sigma_k(\phi)=\sigma_k(\phi')$ for all $k\geqq 1$, then $(G_2,\phi_2)\cong(G'_2,\phi'_2)$. The proof consists of the following two steps:

Step 1. There exist homogeneous splittings $\bigoplus_m \phi_2^m$, $\bigoplus_m \phi_2'^m$ of ϕ_2, ϕ'_2 such that $I^2\phi_2^m\cong I^2\phi_2'^m$ for all $m\geqq 1$.

Step 2. Assume there are homogeneous splittings $\bigoplus_m \phi_2^m$, $\bigoplus_m \phi_2'^m$ of ϕ_2, ϕ'_2 such that $I^2\phi_2^m\cong I^2\phi_2'^m$ for all $m\geqq 1$. Then there exist homogeneous splittings $\bigoplus_m \hat\phi_2^m$, $\bigoplus_m \hat\phi_2'^m$ of ϕ_2, ϕ'_2 such that $\hat\phi_2^m\cong\hat\phi_2'^m$ for all $m\geqq 1$, so that $\phi_2\cong\phi'_2$.

Proof of Step 1. By Lemma 3.1 note that $I^1\phi_2^m\cong I^1\phi_2^m$, and in particular, $\phi_2^1\cong\phi_2'^1$ for any homogeneous splittings $\bigoplus_m \phi_2^m$, $\bigoplus_m \phi_2'^m$ of ϕ_2, ϕ'_2, since $\sigma_m(\phi)=\infty$ if and only if $\sigma_m(\phi_2^m)=\infty$. Suppose there exist homogeneous splittings $\bigoplus_m \phi_2^m$, $\bigoplus_m \phi_2'^m$ of ϕ_2, ϕ'_2 such that $I^2\phi_2^m\cong I^2\phi_2'^m$ for all $m\leqq k-1 (k\geqq 2)$. Let $I^1\phi_2^k\cong I^1\phi_2'^k\cong L_a^1$. Then by Lemma 3.2 we have $I^2\phi_2^k\cong L_{a,b}^2$, $I^2\phi_2'^k\cong L_{a,b'}^2$ for some b, b'. If $\sigma_{k+1}(\phi)=\infty$, then by the relation (1,1) and (1,3) we find new splittings $\bigoplus_m \hat\phi_2^m$, $\bigoplus_m \hat\phi_2'^m$ of ϕ_2, ϕ'_2 such that $I^2\hat\phi_2^m\cong I^2\hat\phi_2'^m$ for all $m\leqq k$. If $\sigma_k(\phi)\neq\infty$ and $\sigma_{k-1}(\phi)=\infty$ or if $\sigma_m(\phi)=\infty$ for all $m\leqq k-1$, then the relation (1.2) or a successive use of the relation (1.1) also enables us to find such splittings of ϕ_2, ϕ'_2. [Notice a fact (obtained from (1.1)) that for any odd $n_2,\dots,n_k$ there exist odd $n'_2,\dots,n'_k$ with $n'_i\equiv n_i \pmod 4$, $i\leqq k-1$, and $n'_k\equiv n_k+2 \pmod 4$ such that

$$A^1(1)\oplus A^2(n_2)\oplus\dots\oplus A^k(n_k)=A^1(1)\oplus A^2(n'_2)\oplus\dots\oplus A^k(n'_k).]$$

If $\sigma_{k+1}(\phi)\neq\infty$ and $\sigma_{k-1}(\phi)\neq\infty$, then we see from Lemma 2.2, Case (1), that $\sigma_{k-1}(\phi_2^k)=\sigma_{k-1}(\phi_2'^k)$, so that $\sigma_1(I^2\phi_2^k)=\sigma_1(I^2\phi_2'^k)$ by Corollary 2.2(2). Since $I^1\phi_2^k\cong I^1\phi_2'^k$, it follows from Lemma 3.2 that $I^2\phi_2^k\cong I^2\phi_2'^k$. Now assume that $\sigma_{k+1}(\phi)\neq\infty$, $\sigma_k(\phi)=\sigma_{k-1}(\phi)=\infty$, and $\sigma_j(\phi)\neq\infty$ for some $j<k-1$. Since $\sigma_k(\phi)=\infty$, the label a of $L_{a,b}^2$ and $L_{a,b'}^2$ is 2 or 3. We use the following

Sublemma. $\sigma_1(I^2\phi_2^k)\equiv\sigma_1(I^2\phi_2'^k) \pmod 4$.

From this sublemma we see that $b,b'\in\{1,2\}$ or $b,b'\in\{3,4\}$. Since $\sigma_{k-1}(\phi)=\infty$, write $\phi_2^{k-1}\cong A^{k-1}(n)\oplus\bar\phi_2^{k-1}$ for some $A^{k-1}(n)$, $\bar\phi_2^{k-1}$. If $a=2$ and $b,b'\in\{1,2\}$, then we use the relation

$$A^{k-1}(n)\oplus A^k(n_1)\oplus A^k(n_2)$$
$$=A^{k-1}(n+2n_1+2n_2)\oplus A^k(n_1+2n)\oplus A^k(n_2+2n+4n_1)$$

obtained from the relation (1.1). If $a=2$ and $b,b'\in\{3,4\}$ or if $a=3$, then we use the relation (1.2). It follows that there exist new splittings $\bigoplus_m \hat\phi_2^m$, $\bigoplus_m \hat\phi_2'^m$ of ϕ_2, ϕ'_2 such that $I^2\hat\phi_2^m\cong I^2\hat\phi_2'^m$ for all $m\leqq k$. By induction on k, we complete the proof of Step 1 except for the proof of the sublemma.

Proof of Sublemma. Using $\sigma_m(\phi)=\sigma_m(\phi')$ for all m, we obtain from Lemma 2.2 the congruence

$$\sigma_j(\phi_3^{j+1})+\ldots+\sigma_j(\phi_2^k)+\sigma_j(\phi_2^{k+1})\equiv\sigma_j(\phi_2'^{j+1})+\ldots+\sigma_j(\phi_2'^k)+\sigma_j(\phi_2'^{k+1})\,(\mathrm{mod}\,4).$$

By Corollary 2.2 the assumption $I^2\phi_2^m\cong I^2\phi_2'^m$, $m\leqq k-1$, implies that $\sigma_j(\phi_2^m)\equiv\sigma_j(\phi_2'^m)\,(\mathrm{mod}\ 4)$ for $j<m\leqq k-1$. Since $\sigma_{k+1}(\phi)\neq\infty$, ϕ_2^{k+1} is a block sum of copies of E_0^{k+1} and E_1^{k+1}. So, by Corollary 2.2, $\sigma_j(\phi_2^{k+1})\equiv 0\ (\mathrm{mod}\,4)$. Similarly, $\sigma_j(\phi_2'^{k+1})\equiv 0\ (\mathrm{mod}\,4)$. By the above congruence we have $\sigma_j(\phi_2^k)\equiv\sigma_j(\phi_2'^k)\ (\mathrm{mod}\,4)$, and hence by Corollary 2.2, $\sigma_1(I^2\phi_2^k)\equiv\sigma_1(I^2\phi_2'^k)\ (\mathrm{mod}\,4)$. This proves the sublemma.

Proof of Step 2. Assume the splittings $\bigoplus_m\phi_2^m$, $\bigoplus_m\phi_2'^m$ of ϕ_2, ϕ_2' have $I^2\phi_2^m\cong I^2\phi_2'^m$ for all m. For $m=1$, 2 this implies $\phi_2^m\cong\phi_2'^m$. Suppose $\phi_2^m\cong\phi_2'^m$ for all $m\leqq k-1(k\geqq 3)$. If $\sigma_k(\phi)\neq\infty$, then the isomorphism $I^2\phi_2^k\cong I^2\phi_2'^k$ lifts to an isomorphism $\phi_2^k\cong\phi_2'^k$. Let $\sigma_k(\phi)=\infty$. Let $I^2\phi_2^k\cong I^2\phi_2'^k\cong L_{a,b}^2$ for some $a(\neq 1)$ and b. By Lemma 3.3, ϕ_2^k, $\phi_2'^k$ are isomorphic to $L_{a,b,1}^k$ or $L_{a,b,2}^k$. If $\sigma_{k+2}(\phi)=\infty$ or if $\sigma_{k-2m}(\phi)=\infty$ for all integers m with $1\leqq k-2m\leqq k-2$, then by the relation (2.1) we can find new splittings $\bigoplus_m\hat{\phi}_2^m$, $\bigoplus_m\hat{\phi}_2'^m$ of ϕ_2, ϕ_2' such that $\hat{\phi}_2^m\cong\hat{\phi}_2'^m$ for all $m\leqq k$ and $I^2\hat{\phi}_2^m\cong I^2\hat{\phi}_2'^m$ for all $m\geqq k+1$. [Notice the following relations derived from (2.1):

$$\begin{aligned}&A^1(1)\oplus A^3(n_1)\oplus\ldots\oplus A^{2s-1}(n_{s-1})\oplus A^{2s+1}(n_s)\\&\quad=A^1(1)\oplus A^3(n_1)\oplus\ldots\oplus A^{2s-1}(n_{s-1})\oplus A^{2s+1}(n_s+4),\\&A^2(n_1)\oplus A^4(n_2)\oplus\ldots\oplus A^{2s}(n_s)\oplus A^{2s+2}(n_{s+1})\\&\quad=A^2(n_1)\oplus A^4(n_2)\oplus\ldots\oplus A^{2s}(n_s)\oplus A^{2s+2}(n_{s+1}+4).]\end{aligned}$$

Assume $\sigma_{k+2}(\phi)\neq\infty$ and $\sigma_j(\phi)\neq\infty$ for some $j=k-2m$ (m, an integer $\geqq 1$). Since $k+2-j$ is even, we obtain from Lemma 2.2(1) and $\sigma_m(\phi)=\sigma_m(\phi')$ for any m that

$$\sigma_j(\phi_2^{j+1})+\ldots+\sigma_j(\phi_2^k)+\sigma_j(\phi_2^{k+1})=\sigma_j(\phi_2'^{j+1})+\ldots+\sigma_j(\phi_2'^k)+\sigma_j(\phi_2'^{k+1}).$$

By assumption, $\phi_2^m\cong\phi_2'^m$ for $m\leqq k-1$, so that $\sigma_j(\phi_2^m)=\sigma_j(\phi_2'^m)$ for $j<m\leqq k-1$. Since $I^2\phi_2^{k+1}\cong I^2\phi_2'^{k+1}$, $\phi_2^{k+1}\cong\phi_2'^{k+1}$ or ϕ_2^{k+1} and $\phi_2'^{k+1}$ are isomorphic to $L_{a',b',1}^{k+1}$ or $L_{a',b',2}^{k+1}$ for the same $a'(\neq 1)$, b'. By Corollary 2.2, $\sigma_j(A^{k+1}(n))=\sigma_j(A^{k+1}(5n))$, since $k+1-j$ is odd. Hence $\sigma_j(\phi_2^{k+1})=\sigma_j(\phi_2'^{k+1})$. It follows that $\sigma_j(\phi_2^k)=\sigma_j(\phi_2'^k)$. By Corollary 2.2, $\sigma_j(A^k(n))\neq\sigma_j(A^k(5n))$. Using that ϕ_2^k and $\phi_2'^k$ are isomorphic to $L_{a,b,1}^k$ or $L_{a,b,2}^k$, we see that $\phi_2^k\cong\phi_2'^k$. By induction on k, we complete the proof of Step 2.

This completes the proof of Theorem 4.1.

Proof of Theorem 0.1. First we note that any block sum

$$\phi_2^m=A^m(n_1)\oplus\ldots\oplus A^m(n_s)\oplus r_0E_0^m\oplus r_1E_1^m$$

is deformed only by the relations (0.1), (0.2), and (0.3) so that $I^1\phi_2^m=L_1^1$, L_2^1 or L_3^1. To see this, it suffices to show that for any odd n_1, n_2, and n_3, there is an odd n_4 such that

$$A^m(n_1)\oplus A^m(n_2)\oplus A^m(n_3)=A^m(n_4)\oplus E_0^m(\text{or } E_1^m).$$

But we can take $n_i = \pm 1, \pm 5$. So this is easily done. Consider any relation $(R \bigoplus_m \phi_2^m = \bigoplus_m \phi_2'^m$ such that $\phi_2^m, \phi_2'^m$ are block sums of copies of $A^m(n)$ $(n = \pm 1, \pm 3)$, E_0^m and E_1^m. We show that the relations in Theorem 0.1 are sufficient to deform both sides of the relation (R) so that $\phi_2^m = \phi_2'^m = L_a^1(m=1)$, $L_{a,b}^2(m=2)$, or $L_{a,b,b'}^m(m \geqq 3)$ for some a, b, b'. By Lemma 3.1 and the above remark, we can assume $I^1\phi_2^m = I^1\phi_2'^m = L_1^1$, L_2^1 or L_3^1 for all $m \geqq 1$. Consider the proof of Theorem 4.1. The proofs of Step 1 and Lemma 3.2[1] show that the relations in Theorem 0.1 are sufficient to deform both sides of (R) so that $I^2\phi_2^m = I^2\phi_2'^m = L_{a,b}^2$ for each $m \geqq 2$ and some a, b. Then the proofs of Step 2 and Lemma 3.3 show that the relations in Theorem 0.1 are sufficient to deform both sides of (R) so that $\phi_2^m = \phi_2'^m = L_{a,b,b'}^m$ for each $m \geqq 3$ and some a, b, b'. This completes the proof of Theorem 0.1.

5. Split Linkings and Hyperbolic Linkings

A linking (G, ϕ) is *split*, if there exists a direct summand H of G that is a self-orthogonal complement (i.e., $H^\perp = H$) with respect to ϕ.

A split linking can be stated in terms of invariants as follows:

Proposition 5.1. *A linking (G, ϕ) is split if and only if $r_p^k(G) \equiv 0 (\bmod 2)$, $\chi_{p'}^k(\phi) = (-1)^{r_{p'}^k(p'-1)/4}$ and $\sigma_k(\phi) = 0$ or ∞ for all prime p, odd prime p' and $k \geqq 1$.*

Proof. Suppose (G, ϕ) is split by a direct summand H with $H^\perp = H$. The p-primary component H_p of H is a direct summand of G_p such that $H_p^\perp = H_p$ with respect to ϕ_p, so that $G_p \cong H_p \oplus H_p$. Given an element $x \in H_p$ of the highest order, p^k, then there exists an element $x' \in G_p$ of order p^k with $\phi_p(x, x') = p^{-k}$. Since $\phi_p(x, x) = 0$, we see from [15, Lemma (1)] that $(G_p, \phi_p) \cong \begin{pmatrix} 0 & p^{-k} \\ p^{-k} & u \end{pmatrix} \oplus (G_p', \phi_p')$ for some linking (G_p', ϕ_p') and $u \in Q/Z$. Let $H_p' = H_p \cap G_p'$. Clearly, $H_p'^\perp = H_p'$ with respect to ϕ_p'. Further, H_p' is a direct summand of G_p'. [Note that $|H_p| = |H_p'|p^k$ since $|G_p'| = |H_p'|^2$ and $|G_p| = |G_p'|p^{2k} = |H_p|^2$. Then the inclusion $H_p' \oplus \langle x \rangle \subset H_p$ is an isomorphism, where $\langle x \rangle$ is the subgroup of H_p generated by x. Let $pr_1 : H_p = H_p' \oplus \langle x \rangle \to H_p'$ be the projection to the first factor. For an epimorphism $r : G_p \to H_p$ with $r|H_p = \mathrm{id}$, we define $r' : G_p' \to H_p'$ to be the composite $G_p' \subset G_p \xrightarrow{r} H_p \xrightarrow{pr_1} H_p'$. We have $r'|H_p' = \mathrm{id}$, that is, H_p' is a direct summand of G_p'.] Hence (G_p', ϕ_p') is split. When p is odd,

$$\begin{pmatrix} 0 & p^{-k} \\ p^{-k} & u \end{pmatrix} \cong \begin{pmatrix} 0 & p^{-k} \\ p^{-k} & 0 \end{pmatrix} \cong (p^{-k}) \oplus (-p^{-k}),$$

so that by induction (G_p, ϕ_p) is isomorphic to a block sum of copies of $(p^{-k}) \oplus (-p^{-k})$ $(k \geqq 1)$. It follows that $r_p^k(G) \equiv 0 (\bmod 2)$ and $\chi_p^k(\phi) = (-1)^{r_p^k(p-1)/4}$ for all odd prime p and $k \geqq 1$. When $p = 2$, $\begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & u \end{pmatrix} \cong B_0^k$ or $A^k(1) \oplus A^k(-1)$, so that by induction (G_2, ϕ_2) is isomorphic to a block sum of copies of B_0^k, $A^k(1) \oplus A^k(-1)$ $(k \geqq 1)$. It follows from Corollary 2.2 that $\sigma_k(\phi) = 0$ or ∞ for all

[1] For $m \geqq 3$ note the relation $2A^m(n) \oplus E_0^m$ (or E_1^m, respectively) $= A^m(-n) \oplus A^m(-n+4) \oplus E_1^m$ (or E_0^m, respectively) obtained from (0.2) and (0.3) for any odd n, which lifts the relation $2A^2(n) \oplus E_0^2 = 2A^2(-n) \oplus E_1^2 (n = \pm 1)$

$k \geqq 1$. To see the converse, we use, when p is odd, the matrix $r(p^{-k}) \oplus r(-p^{-k})$ with $2r = r_p^k(G)$, and when $p=2$, the matrix rB_0^k or $rA^k(1) \oplus rA^k(-1)$ with $2r = r_2^k(G)$ according to whether $\sigma_k(\phi)=0$ or ∞. Take a block sum of these matrices for all prime p and $k \geqq 1$. The resulting split linking (G', ϕ') has $r_p^k(G') = r_p^k(G)$, $\chi_{p'}^k(\phi') = \chi_{p'}^k(\phi)$, and $\sigma_k(\phi') = \sigma_k(\phi)$ for all prime p, odd prime p' and $k \geqq 1$. By Theorem 4.1, $(G, \phi) \cong (G', \phi')$ and (G, ϕ) is split. This completes the proof.

Let $\mathfrak{N}_p^k$ be the abelian semigroup of linkings on homogeneous p-groups of exponent k. The abelian semigroup $\mathfrak{N}$, $\mathfrak{N}_p$ or $\mathfrak{N}_p^k$ modulo split linkings forms an abelian group called the *Witt group of linkings*, the *Witt group of linkings on p-groups*, or the *Witt group of linkings on homogeneous p-groups of exponent k* and denoted by W, W_p or W_p^k, respectively. We shall show the following

Proposition 5.2. *$W \cong \bigoplus_p W_p$. When p is odd, $W_p \cong \bigoplus_k W_p^k$ and, for each $k \geqq 1$, $W_p^k \cong Z_2 \oplus Z_2$ or Z_4 according to whether $p \equiv 1 (\mathrm{mod}\, 4)$ or $3 (\mathrm{mod}\, 4)$*[2]*. When $p=2$, W_2 is isomorphic to a direct sum of infinite copies of Z_2, and $W_2^1 \cong Z_2$, $W_2^2 \cong Z_8$*[3] *and, for each $k \geqq 3$, $W_2^k \cong Z_8 \oplus Z_2$.*

Proof. Clearly the canonical isomorphism $\mathfrak{N} \cong \bigoplus_p \mathfrak{N}_p$ induces an isomorphism $W \cong \bigoplus_p W_p$. For any odd p, we obtain from Lemma 1.1 and Proposition 5.1 the remaining first half, where W_p^k is generated by (p^{-k}) and $(p^{-k} n(p))$ when $p \equiv 1 (\mathrm{mod}\, 4)$ or (p^{-k}) alone when $p \equiv 3 (\mathrm{mod}\, 4)$ (cf. [10, Lemma IV.1.5]). To calculate W_2^k we use Lemmas 3.1, 3.2 and 3.3. By Lemma 3.1 W_2^1 is generated by $A^1(1)$. By Proposition 5.1, $r_2^1 (\mathrm{mod}\, 2)$ induces an isomorphism $W_2^1 \cong Z_2$. By Lemma 3.2, W_2^2 is generated by at most 8 elements. Since $\sigma_1(A^2(1)) = 1$, we see from Proposition 5.1 that σ_1 induces an isomorphism $W_2^2 \cong Z_8$ sending $A^2(1)$ to 1. By Lemma 3.3, W_2^k is generated by at most 16 elements. By Proposition 5.1 note that $8A^k(1)$ and $2[A^k(1) \oplus A^k(-5)]$ represent 0 in W_2^k. Since $\sigma_{k-1}(A^k(1)) = 1$, $\sigma_{k-1}(A^k(1) \oplus A^k(-5)) = 0$, and $\sigma_{k-2}(A^k(1) \oplus A^k(-5)) = -4$, we obtain an isomorphism $Z_8 \oplus Z_2 \cong W_2^k$ sending $(1,0)$ to $A^k(1)$ and $(0,1)$ to $A^k(1) \oplus A^k(-5)$. To calculate W_2, note that any linking $A^k(n)$ represents the same element as $A^k(1)$ in W_2, since for $k \geqq 2$, Theorem 4.1 shows that $\bigoplus_{m=1}^{k-1} A^m(1) \oplus A^k(n) = \bigoplus_{m=1}^{k} A^m(1)$. Then from the calculation of W_2^k it follows that W_2 is generated by $A^k(1)$, $k \geqq 1$, which represent the elements of order 2. By Proposition 5.1, $r_2^k (\mathrm{mod}\, 2)$ guarantees us that these elements are linearly independent over Z_2. This completes the proof.

A linking (G, ϕ) is *hyperbolic* if (G, ϕ) is isomorphic to a block sum of matrices of the form $\begin{pmatrix} 0 & p^{-k} \\ p^{-k} & 0 \end{pmatrix}$ ($p = a$ prime number, $k \geqq 1$). The following is direct from Proposition 5.1.

Corollary 5.1. *A linking (G, ϕ) is hyperbolic if and only if $r_p^k(G) \equiv 0 (\mathrm{mod}\, 2)$, $\chi_p^k(\phi) = (-1)^{r_p^k(p-1)/4}$ and $\sigma_k(\phi) = 0$ for all odd prime p and $k \geqq 1$.*

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[2] Thanks to A. Ranicki for this statement

[3] This was also proved by L. Guillou and A. Marin [6] (see also E. H. Brown, Jr. [1])

6. The Linking Pairings of 3-Manifolds

We consider the linking $(\tau H_1(M), \phi_M)$ of a closed connected oriented 3-manifold M. The invariants $r_p^k(\tau H_1(M))$, $\chi_p^k(\phi_M)$, and $\sigma_k(\phi_M)$ are denoted by $r_p^k(M)$, $\chi_p^k(M)$, and $\sigma_k(M)$, respectively.

The following Lemma improves a classical result of Hantzch [7].

Lemma 6.1. *If M is imbedded piecewise-linearly in a 4-sphere S^4, then the linking ϕ_M is hyperbolic.*

Proof. M divides S^4 into two compact, connected, orientable 4-manifolds V_1, V_2. By Mayer-Vietoris sequence, the homomorphism

$$\bar{i} = i_{1*} + i_{2*} : \tau H_1(M) \to \tau H_1(V_1) \oplus \tau H_1(V_2)$$

is an isomorphism (τH_* = the torsion part of H_*). By Poincaré duality and the universal-coefficient theorem, the homology exact sequence of the pair (V_j, M) induces the following exact sequence

$$0 \to \tau H_2(V_j, M) \xrightarrow{\partial_j} \tau H_1(M) \xrightarrow{i_{j*}} \tau H_1(V_j) \to 0\,,$$

$j = 1, 2$. [Note that $|\tau H_1(M)| = |\tau H_1(V_j)|^2$, $j = 1, 2$.] Let $\bar{\phi}_j : \tau H_2(V_j, M) \times \tau H_1(V_j) \to Q/Z$ be the pairing defined by the Poincaré duality. Let $G_j^0 = \bar{i}^{-1}[\tau H_1(V_j)]$ $(\subset \tau H_1(M))$, $j = 1, 2$. Let $x, x' \in G_1^0$. Then $i_{2*}(x) = i_{2*}(x') = 0$. By $(\#)_2$ there is $\bar{x} \in \tau H_2(V_2, M)$ with $\partial_2 \bar{x} = x$. Hence $\phi_M(x, x') = \phi_M(\partial_2 \bar{x}, x') = \bar{\phi}_2(\bar{x}, i_{2*}(x')) = 0$. Similarly, $\phi_M(x, x') = 0$ for $x, x' \in G_2^0$. Since ϕ_M is nonsingular, $\tau H_1(M) = G_1^0 \oplus G_2^0$ and $|G_1^0| = |G_1^0|$, ϕ_M induces an isomorphism $G_2^0 \to \mathrm{Hom}[G_1^0, Q/Z]$. Hence by taking the generators of a primary cyclic splitting of G_1^0 and then taking the dual generators of G_2^0, we see that ϕ_M is hyperbolic. This completes the proof.

Let N be an open oriented 3-manifold. The linking pairing $\phi_N : \tau H_1(N) \times \tau H_1(N) \to Q/Z$ is still defined, though it may be singular (cf. [12]).

Corollary 6.1. *If N is imbedded piecewise-linearly in S^4, then we have $2^{k-1}\phi_N(x, x) = 0$ for $x \in \tau H_1(N)$ with $2^k x = 0$, $k = 1, 2, 3 \ldots$.*

Proof. Suppose $2^{k-1}\phi_N(x, x) \neq 0$ for some x with $2^k x = 0$. Then for a compact submanifold N' of N we also have $2^{k-1}\phi_{N'}(x', x') \neq 0$ for some x' with $2^k x' = 0$. If N and hence N' imbed in S^4, then so does the double $D(N')$ of N'. But $c^k(D(N')) \neq 0$, i.e., $\sigma_k(D(N')) = \infty$. This contradicts Lemma 6.1.

For our application we take, as N, a punctured manifold M_0 of M, that is, $M_0 = M - \{x\}$ for some $x \in M$. In this case, ϕ_N is isomorphic to ϕ_M.

Example 6.1. The lens space $L(n, m)$ has $H_1(L(n, m)) \cong Z_n$ and $\phi_{L(n,m)} \cong (n^{-1}m)$ (cf. [12]). For odd n, $L(n, m)_0$ is imbeddable in S^4 by Zeeman [16], whereas for even n, $L(n, m)_0$ is still non-imbeddable in S^4 by Corollary 6.1 (cf. Epstein [3]).

Example 6.1 shows that for odd n the connected sum $L(n, m) \# -L(n, m)$ is imbeddable in S^4. On the other hand, the invariant χ_p^k enables us to see the following

Proposition 6.1. *If $L(n, m) \# L(n, m')$ is imbedded piecewise-linearly in S^4, then $L(n, m')$ and $-L(n, m)$ have the same oriented homotopy type.*

Proof. Since $-L(n, m) = L(n, -m)$, it suffices to show that $-mm'$ is a quadratic residue (mod n). Let $n = p_1^{e_1} \ldots p_s^{e_s}$, where p_i are distinct odd prime numbers and

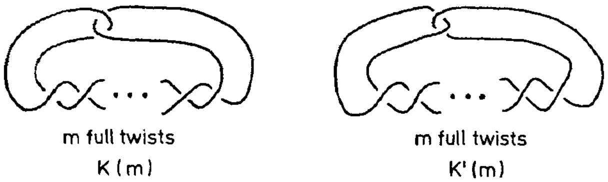

Fig. 1

$e_i \geqq 1$. Since $(n,m)=(n,m')=1$, there are unique splittings $n^{-1}m = p_1^{-e_1}m_1 + \ldots + p_2^{-e_s}m_s$ and $n^{-1}m' = p_1^{-e_1}m'_1 + \ldots + p_s^{-e_s}m'_s$. Since $\chi_{p_i}^{e_i}(L(n,m) \# L(n,m')) = (m_i m'_i/p_i) = (-1)^{(p_i-1)/2} = (-1/p_i)$ by Corollary 5.1 and Lemma 6.1 – it follows that $-m_i m'_i$ is a quadratic residue $(\bmod p_i)$ and hence $(\bmod p_i^{e_i})$, $i=1, 2, \ldots, s$. Then $-m_i m'_i \equiv k^2 (\bmod p_i^{e_i})$, $i=1, 2, \ldots, s$, taking k as the sum $\sum_{i=1}^{s} k_i u_i p_1^{e_1} \ldots \widehat{p_i^{e_i}} \ldots p_s^{e_s}$ where $-m_i m'_i \equiv k_i^2 (\bmod p_i^{e_i})$ and $u_i p_1^{e_1} \ldots \widehat{p_i^{e_i}} \ldots p_s^{e_s} \equiv 1 (\bmod p_i^{e_i})$. This implies that $-mm'$ is a quadratic residue $(\bmod n)$ (cf. [14, p. 49]), completing the proof.

The μ-invariant is also known as an invariant of the imbeddability of a Z_2-*homology* 3-*sphere into* S^4 ([8]). It is independent of our invariant χ_p^k. For example, $\mu(L(7,1) \# L(7,3)) \neq 0$ but $L(7,3)$ and $-L(7,1)$ have the same oriented homotopy type, whereas $\mu(L(9,2) \# L(9,5)) = 0$ but $\chi_3^2(L(9,2) \# L(9,5)) = 1 \neq (-1)^{2(3-1)/4}$. Using all of their information, we can decide the imbeddability of $L(n,m) \# L(n,m')$ for $n<11$. The first unknown example is $L(11,2) \# L(11,3)$.

Example 6.2. For each $k \geqq 1$ there exists a closed connected oriented 3-manifold M with $H_1(M) = Z_{2^k} \oplus Z_{2^k}$ and $\phi_M \cong B_0^k$.

In fact, by [9] we construct M with $H_1(M) = Z_{2^k} \oplus Z_{2^k}$ such that M is imbedded in S^4. By Lemma 6.1, $\phi_M \cong B_0^k$.

Lemma 6.2. *If M_0 is a fiber of a fibered 2-knot $K \subset S^4$ and $H_1(M) = Z_{2^k} \oplus Z_{2^k} (k \geqq 2)$, then $\phi_M \cong B_1^k$.*

Proof. Let $X = S^4 - K$ and $\tilde{X}$ be its infinite cyclic cover, homeomorphic to $M_0 \times R^1$. Let $G = H_1(\tilde{X})$. Note that $G \cong H_1(M) = Z_{2^k} \oplus Z_{2^k}$. There is a linking ϕ on G, isomorphic to ϕ_M such that $\phi(tx, ty) = \phi(x, y)$ for all $x, y \in G$, where $t: G \to G$ is an automorphism induced by a generator of the infinite cyclic transformation group of $\tilde{X}$. By Corollary 6.1, $\sigma_k(\phi) = \sigma_k(\phi_M) \neq \infty$ (i.e., $c^k(\phi) = 0$), so that $\phi_M \cong B_0^k$ or B_1^k (cf. Lemmas 3.2, 3.3). It suffices to show that $\sigma_{k-1}(\phi) \neq 0$. Recall that the function $q_{k-1}: G_2/\bar{G}_2^{k-1} (= \tilde{G}_2^k) \to Q/Z$ is defined by $q_{k-1}([x]) = 2^{k-2}\phi(x,x)$. $2q_{k-1} = 0$, since $2q_{k-1}([x]) = \tilde{\phi}_2^k([x], c^k(\phi))$ and $c^k(\phi) = 0$. By Wang exact sequence, $t-1: \tilde{G}_2^k \to \tilde{G}_2^k$ is an isomorphism. It follows that $\tilde{\phi}_2^k(tx, x) = 2^{-1}$ for any $x \neq 0$ in $\tilde{G}_2^k$, since $tx \neq x$ and hence $\tilde{G}_2^k = \{0, x, tx, x+tx\}$. For any $x \neq 0$ in $\tilde{G}_2^k$ write $x = (t-1)x'$. Then $q_{k-1}(x) = q_{k-1}(tx' - x') = q_{k-1}(tx') + q_{k-1}(-x') + \tilde{\phi}_2^k(zx', -x') = \tilde{\phi}_2^k(tx', x') = 2^{-1}$. Hence $GS_{k-1}(\phi) = 1 - 1 - 1 - 1 = -2$ and $\sigma_{k-1}(\phi) = 4$. This completes the proof.

Example 6.3. For each $k \geqq 2$ there exists a closed connected oriented 3-manifold M with $H_1(M) = Z_{2^k} \oplus Z_{2^k}$ such that $\phi_M \cong B_1^k$.

To see this, we show that for each $k \geqq 1$ there is a closed connected oriented 3-manifold M with $H_1(M) = Z_{2^k} \oplus Z_{2^k}$ such that M_0 is a fiber of a fibered 2-knot. Then the assertion follows from Lemma 6.2. Consider the knots $K(m)$ and $K'(m) (m \geqq 1)$ illustrated in Fig. 1.

$K(1)$ is a trefoil knot and $K'(1)$ is a figure-eight knot. Note that $K(m)$, $K'(m)$ have Seifert matrices $\begin{pmatrix} m & 0 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} m & 1 \\ 0 & -1 \end{pmatrix}$, respectively. Let $M(m)$, $M'(m)$ be the 3-fold branched covers of S^3 along $K(m)$, $K'(m)$, respectively. By Fox [4], $H_1(M(m)) \cong Z_{3m-1} \oplus Z_{3m-1}$ and $H_1(M'(m)) \cong Z_{3m+1} \oplus Z_{3m+1}$. By Zeeman [16], $M(m)_0$, $M'(m)_0$ are fibers of some fibered 2-knots. Since for each $k \geqq 1$ there is an integer m such that $2^k = 3m-1$ or $3m+1$, we have constructed an M with $H_1(M) = Z_{2^k} \oplus Z_{2^k}$ such that M_0 is a fiber of some fibered 2-knot for each $k \geqq 1$.

By taking connected sums of some copies of manifolds in Examples 6.1, 6.2, and 6.3, we see the following:

Theorem 6.1 *For any linking* (G, ϕ) *there exists a closed connected oriented 3-manifold* M *with* $H_1(M) \cong G$ *and* $\phi_M \cong \phi$.

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Note added in proof. We are informed by J. S. Birman that from a different viewpoint she and D. Johnsen have obtained several results of this paper independently at the same time.