CHAPTER I

PROJECTIVE MODULES AND VECTOR BUNDLES

The basic objects studied in algebraic K-theory are projective modules over a ring, and vector bundles over schemes. In this first chapter we introduce the cast of characters. Much of this information is standard, but collected here for ease of reference in later chapters.

Here are a few running conventions we will use. The word *ring* will always mean an associative ring with 1. If R is a ring, the word *R-module* will mean right R-module unless explicitly stated otherwise. We will often use 'f.g.' to denote 'finitely generated.'

§1. Free modules, GL_n and stably free modules

If R is a field, or a division ring, then R-modules are called *vector spaces.* Classical results in linear algebra state that every vector space has a basis, and that the rank (or dimension) of a vector space is independent of the choice of basis. However, much of this fails for arbitrary rings.

As with vector spaces, a *basis* of an R-module M is a subset $\{e_i\}_{i\in I}$ such that every element of M can be expressed in a unique way as a finite sum $\sum e_i r_i$ with $r_i \in R$. If M has a fixed basis we call M a *based free module*, and define the *rank* of the based free module M to be the cardinality of its given basis. Finally, we say that a module M is *free* if there is a basis making it into a based free module.

The canonical example of a based free module is R^n, which consists of n-tuples of elements of R, or "column vectors" of length n.

Unfortunately, there are rings for which $R^n \cong R^{n+t}$, $t \neq 0$. We make the following definition to avoid this pathology, referring the curious reader to the exercises for more details. (If κ is an infinite cardinal number, then every basis of R^κ has cardinality κ. In particular R^κ cannot be isomorphic to R^n for finite n. See ch.2, 5.5 of [Cohn65].)

Definition 1.1 (IBP). We say that a ring R satisfies the (right) *invariant basis property (or IBP)* if R^m and R^n are not isomorphic for $m \neq n$. In this case, the rank of a free R-module M is an invariant, independent of the choice of basis of M.

Most of the rings we will consider satisfy the invariant basis property. For example, commutative rings satisfy the invariant basis property, and so do group rings $\mathbb{Z}[G]$. This is because a ring R must satisfy the IBP if there exists a ring map $f\colon R \to F$ from R to a field or division ring F. (If R is commutative we may take $F = R/\mathfrak{m}$, where $\mathfrak{m}$ is any maximal ideal of R.) To see this, note that any basis of M maps to a basis of the vector space $V = M \otimes_R F$; since $\dim V$ is independent of the choice of basis, any two bases of M must have the same cardinality.

Our choice of right modules dictates that we write R-module homomorphisms on the left. In particular, homomorphisms $R^n \to R^m$ may be thought of as $m \times n$ matrices with entries in R, acting on the column vectors in R^n by matrix multiplication. We write $M_n(R)$ for the ring of $n \times n$ matrices, and write $GL_n(R)$ for the group of invertible $n \times n$ matrices, *i.e.*, the automorphisms of R^n. We will usually write $R^\times$ for the group $GL_1(R)$ of *units* in R.

EXAMPLE 1.1.1. Any finite-dimensional algebra R over a field (or division ring) F must satisfy the IBP, because the rank of a free R-module M is an invariant:

$$\operatorname{rank}(M) = \dim_F(M)/\dim_F(R).$$

For a simple artinian ring R we can say even more. Classical Artin-Wedderburn theory states that $R = M_n(F)$ for some n and F, and that every right R-module M is a direct sum of copies of $V = (F^n)^t$, the R-module consisting of row vectors over F of length n. Moreover, the number of copies of V is an invariant of M, called its *length*; the length is also $\dim_F(M)/n$ since $\dim_F(V) = n$. In this case we also have $\operatorname{rank}(M) = \operatorname{length}(M)/n = \dim_F(M)/n^2$.

There are noncommutative rings which do not satisfy the IBP, *i.e.*, which have $R^m \cong R^n$ for some $m \neq n$. Rank is not an invariant of a free module over these rings. One example is the infinite matrix ring $\operatorname{End}_F(F^\infty)$ of endomorphisms of an infinite-dimensional vector space over a field F. Another is the cone ring $C(R)$ associated to a ring R. (See the exercises.)

Unimodular Rows and Stably Free Modules

DEFINITION 1.2. An R-module P is called *stably free* (of rank $n-m$) if $P \oplus R^m \cong R^n$ for some m and n. (If R satisfies the IBP then the rank of a stably free module is easily seen to be independent of the choice of m and n.) Every stably free module is the kernel of a surjective $m \times n$ matrix $\sigma\colon R^n \to R^m$, because a lift of a basis for R^m yields a decomposition $P \oplus R^m \cong R^n$.

This raises a question: when are stably free modules free? Over some rings every stably free module is free (fields, $\mathbb{Z}$ and the matrix rings $M_n(F)$ of Example 1.1.1 are classical cases), but in general this is not so even if R is commutative; see example 1.2.2 below.

1.2.1. The most important special case, at least for inductive purposes, is when $m = 1$, *i.e.*, $P \oplus R \cong R^n$. In this case σ is a row vector, and we call σ a *unimodular row*. It is not hard to see that the following conditions on a sequence $\sigma = (r_1, ..., r_n)$ of elements in R are equivalent for each n:

- σ is a unimodular row;
- $R^n \cong P \oplus R$, where $P = \ker(\sigma)$;
- $R = r_1R + \cdots + r_nR$;
- $1 = r_1s_1 + \cdots + r_ns_n$ for some $s_i \in R$.

If $R^n \cong P \oplus R$ with P free, then a basis of P would yield a new basis for R^n and hence an invertible matrix g whose first row is the unimodular row $\sigma\colon R^n \to R$ corresponding to P. This gives us a general criterion: P is a free module iff the corresponding unimodular row may be completed to an invertible matrix. (The invertible matrix is in $GL_n(R)$ if R satisfies the IBP).

When R is commutative, every unimodular row of length 2 may be completed. Indeed, if $r_1s_1 + r_2s_2 = 1$, then the desired matrix is:

$$\begin{pmatrix} r_1 & r_2 \\ -s_2 & s_1 \end{pmatrix}$$

Hence $R^2 \cong R \oplus P$ implies that $P \cong R$. In §3 we will obtain a stronger result: every stably free module of rank 1 is free. The fact that R is commutative is crucial; in Ex. 1.6 we give an example of a unimodular row of length 2 which cannot be completed over $D[x, y]$, D a division ring.

Example 1.2.2. Here is an example of a unimodular row σ of length 3 which cannot be completed to an element of $GL_3(R)$. Hence $P = \ker(\sigma)$ is a rank 2 stably free module P which is not free, yet $P \oplus R \cong R^3$. Let σ be the unimodular row $\sigma = (x, y, z)$ over the commutative ring $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 = 1)$. Every element (f, g, h) of R^3 yields a vector field in 3-space ($\mathbb{R}^3$), and σ is the vector field pointing radially outward. Therefore an element in P yields a vector field in 3-space tangent to the 2-sphere S^2. If P were free, a basis of P would yield two tangent vector fields on S^2 which are linearly independent at every point of S^2 (because together with σ they span the tangent space of 3-space at every point). It is well known that this is impossible: you can't comb the hair on a coconut. Hence P cannot be free.

The following theorem describes a "stable range" in which stably free modules are free (see 2.3 for a stronger version). A proof may be found in [Bass, V.3.5], using the "stable range" condition (S_n) of Ex. 1.5 below. Example 1.2.2 shows that this range is sharp.

Bass Cancellation Theorem for Stably Free Modules 1.3. *Let R be a commutative noetherian ring of Krull dimension d. Then every stably free R-module of rank $> d$ is a free module. Equivalently, every unimodular row of length $n \geq d + 2$ may be completed to an invertible matrix.*

The study of stably free modules has a rich history, and we cannot do it justice here. An excellent source for further information is the book [Lam].

EXERCISES

1.1. *Semisimple rings.* An R-module M is called *simple* if it has no submodules other than 0 and M, and *semisimple* if it is the direct sum of simple modules. A ring R is called *semisimple* if R is a semisimple R-module. If R is semisimple, show that R is a direct sum of a finite (say n) number of simple modules. Then use the Jordan-Hölder Theorem, which states that the length of a semisimple module is an invariant, to show that every stably free module is free. In particular, this shows that semisimple rings satisfy the IBP. *Hint:* Show that length$= n \cdot$ rank is an invariant of free R-modules.

1.2. (P.M.Cohn) Consider the following conditions on a ring R:
(I) R satisfies the invariant basis property (IBP);
(II) For all m and n, if $R^m \cong R^n \oplus P$ then $m \geq n$;

(III) For all n, if $R^n \cong R^n \oplus P$ then $P = 0$.

Show that (III) $\Rightarrow$ (II) $\Rightarrow$ (I). For examples of rings satisfying (III) but not (II), resp. (II) but not (I), see [Cohn66].

1.3. Show that (III) and the following matrix conditions are equivalent:

(a) For all n, every surjection $R^n \to R^n$ is an isomorphism;

(b) For all n, and $f, g \in M_n(R)$, if $fg = 1_n$, then $gf = 1_n$ and $g \in GL_n(R)$.

Then show that commutative rings satisfy (b), hence (III).

1.4. Show that right noetherian rings satisfy condition (b) of the previous exercise. Hence they satisfy (III), and have the right invariant basis property.

1.5. *Stable Range Conditions.* We say that a ring R satisfies condition (S_n) if for every unimodular row $(r_0, r_1, ..., r_n)$ in R^{n+1} there is a unimodular row $(r'_1, ..., r'_n)$ in R^n with $r'_i = r_i - r_0 t_i$ for some $t_1, ..., t_n$ in R. The *stable range of* R, $sr(R)$, is defined to be the smallest n such that R satisfies condition (S_n). (Warning: our (S_n) is the stable range condition SR_{n+1} of [Bass].)

(a) (Vaserstein) Show that (S_n) holds for all $n \geq sr(R)$.

(b) If $sr(R) = n$, show that all stably free projective modules of rank $\geq n$ are free. Bass' Cancellation Theorem [Bass, V.3.5], which is used to prove 1.3 and 2.3 below, actually states that $sr(R) \leq d+1$ if R is a d-dimensional commutative noetherian ring, or more generally if $\mathrm{Max}(R)$ is a finite union of spaces of dimension $\leq d$.

(c) Show that $sr(R) = 1$ for every artinian ring R. Conclude that all stably free projective R-modules are free over artinian rings.

(d) Show that if I is an ideal of R then $sr(R) \geq sr(R/I)$.

(e) (Veldkamp) If $sr(R) = n$ for some n, show that R satisfies the invariant basis property (IBP). *Hint:* Consider an isomorphism $B\colon R^N \cong R^{N+n}$, and apply (S_n) to convert B into a matrix of the form $\left(\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right)$.

1.6. (Ojanguren-Sridharan) Let D be a division ring which is not a field. Choose $\alpha, \beta \in D$ such that $\alpha\beta - \beta\alpha \neq 0$, and show that $\sigma = (x+\alpha, y+\beta)$ is a unimodular row over $R = D[x, y]$. Let $P = \ker(\sigma)$ be the associated rank 1 stably free module; $P \oplus R \cong R^2$. Prove that P is not a free $D[x, y]$-module, using these steps:

(i) If $P \cong R^n$, show that $n = 1$. Thus we may suppose that $P \cong R$ with $1 \in R$ corresponding to a vector $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$ with $r, s \in R$.

(ii) Show that P contains a vector $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with $f = c_1 x + c_2 y + c_3 xy + c_4 y^2$ and $g = d_1 x + d_2 y + d_3 xy + d_4 x^2$, $(c_i, d_i \in D)$.

(iii) Show that P cannot contain any vector $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with f and g linear polynomials in x and y. Conclude that the vector in (i) must be quadratic, and may be taken to be of the form given in (ii).

(iv) Show that P contains a vector $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with $f = \gamma_0 + \gamma_1 y + y^2$, $g = \delta_0 + \delta_1 x - \alpha y - xy$ and $\gamma_0 = \beta u^{-1} \beta u \neq 0$. This contradicts (iii), so we cannot have $P \cong R$.

1.7. For any ring R, let R^∞ be a fixed free R-module on a countably infinite basis. Then R^∞ is naturally a left module over the endomorphism ring $E = \mathrm{End}_R(R^\infty)$, and we can identify E with the ring of infinite column-finite matrices. If V is any R-module summand of R^∞, show that the right ideal $I = \{f \in E : f(R^\infty) \subseteq V\}$ is a projective E-module. Conclude that $E \cong E^2 \cong E^3 \cong \cdots$ as right E-modules, and that $P \oplus E \cong E$ for every f.g. projective E-module P. Is $E \cong E^\infty$?

1.8. *Cone Ring.* For any ring R, the endomorphism ring $\mathrm{End}_R(R^\infty)$ of the previous exercise contains a smaller ring, namely the subring $C(R)$ consisting of row-and-column finite matrices. The ring $C(R)$ is called the *cone ring* of R. Show that $C(R) \cong C(R) \oplus C(R)$ as right $C(R)$-modules.

1.9. To see why our notion of stably free module involves only finitely generated free modules, let R^∞ be the infinitely generated free module of exercise 1.7. Prove that if $P \oplus R^m \cong R^\infty$ then $P \cong R^\infty$. *Hint:* The image of R^m is contained in some $R^n \subseteq R^\infty$. Writing $R^\infty \cong R^n \oplus F$ and $Q = P \cap R^n$, show that $P \cong Q \oplus F$ and $F \cong F \oplus R^m$. This trick is a version of the Eilenberg Swindle of §2.

1.10. *Excision for GL_n.* If I is a ring without unit, let $\mathbb{Z} \oplus I$ be the canonical augmented ring with underlying abelian group $\mathbb{Z} \oplus I$. Let $GL_n(I)$ denote the kernel of the map $GL_n(\mathbb{Z} \oplus I) \to GL_n(\mathbb{Z})$, and let $M_n(I)$ denote the matrices with entries in I. If $g \in GL_n(I)$ then clearly $g - 1_n \in M_n(I)$.

(i) Characterize the set of all $x \in M_n(I)$ such that $1 + x \in GL_n(I)$.

(ii) If I is an ideal in a ring R, show that $GL_n(I)$ is the kernel of $GL_n(R) \to GL_n(R/I)$, and so is independent of the choice of R.

(iii) If $x = (x_{ij})$ is any nilpotent matrix in $M_n(I)$, such as a strictly upper triangular matrix, show that $1_n + x \in GL_n(I)$.

1.11. (Whitehead) If $g \in GL_n(R)$, verify the following identity in $GL_{2n}(R)$:

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Conclude that if $S \to R$ is a ring surjection then there is a matrix $h \in GL_{2n}(S)$ mapping to the block diagonal matrix with entries g, g^{-1} displayed above.

1.12. *Radical Ideals.* A 2-sided ideal I in R is called a *radical ideal* if $1 + x$ is a unit of R for every $x \in I$, *i.e.*, if $(\forall x \in I)(\exists y \in I)(x + y + xy = 0)$. Every ring has a unique largest radical ideal, called the *Jacobson radical* of R; it is the intersection of the maximal left ideals of R.

(i) Show that every nil ideal is a radical ideal. (A *nil ideal* is an ideal in which every element is nilpotent.)

(ii) A ring R is *local* if it has a unique maximal 2-sided ideal $\mathfrak{m}$, and $R/\mathfrak{m}$ is a field or division ring. Show that $\mathfrak{m}$ is the Jacobson radical of R.

(iii) If I is a radical ideal, show that $M_n(I)$ is a radical ideal of $M_n(R)$ for every n. *Hint:* Use elementary row operations to diagonalize any matrix which is congruent to 1_n modulo I.

(iv) If I is a radical ideal, show that $GL_n(R) \to GL_n(R/I)$ is surjective for each n. That is, there is a short exact sequence of groups:

$$1 \to GL_n(I) \to GL_n(R) \to GL_n(R/I) \to 1.$$

(v) If I is a radical ideal, show that $sr(R) = sr(R/I)$, where sr is the stable range of Exercise 1.5. Conclude that $sr(R) = 1$ for every local ring R.

1.13. A *von Neumann regular ring* is a ring R such that for every $r \in R$ there is an $x \in R$ such that $r = rxr$. It is called *unit-regular* if for every $r \in R$ there is a *unit* $x \in R$ such that $r = rxr$. If R is von Neumann regular, show that:

(a) R is unit-regular $\iff$ R has stable range 1 (in the sense of Exercise 1.5);

(b) If R is unit-regular then R satisfies condition (III) of Exercise 1.2. (The converse does not hold; see Example 5.10 of [Gdearl].)

A *rank function* on R is a set map $\rho\colon R \to [0,1]$ such that: (i) $\rho(0) = 0$ and $\rho(1) = 1$; (ii) $\rho(x) > 0$ if $x \neq 0$; (iii) $\rho(xy) \leq \rho(x), \rho(y)$; and (iv) $\rho(e+f) = \rho(e) + \rho(f)$ if e, f are orthogonal idempotents in A. Goodearl and Handelman proved (18.4 of [Gdearl]) that if R is a simple von Neumann ring then:

(III) holds $\iff$ R has a rank function.

(c) Let F be a field or division ring. Show that the matrix ring $M_n(F)$ is unit-regular, and that $\rho_n(g) = \text{rank}(g)/n$ is a rank function on $M_n(F)$. Then show that the ring $\text{End}_F(F^\infty)$ is von Neumann regular but not unit-regular.

(d) Consider the union R of the matrix rings $M_{n!}(F)$, where we embed $M_{n!}(F)$ in $M_{(n+1)!}(F) \cong M_{n!}(F) \otimes M_{n+1}(F)$ as $M_{n!} \otimes 1$. Show that R is a simple von Neumann regular ring, and that the union of the ρ_n of (c) gives a rank function $\rho\colon R \to [0,1]$ with image $\mathbb{Q} \cap [0,1]$.

(e) Show that a commutative ring R is von Neumann regular if and only if it is reduced and has Krull dimension 0. These rings are called *absolutely flat rings* by Bourbaki, because every R-module is flat. Use Exercise 1.12 to conclude that every commutative 0-dimensional ring R has stable range 1 (and is unit-regular).

§2. Projective Modules

Definition 2.1. An R-module P is called *projective* if there exists a module Q so that the direct sum $P \oplus Q$ is free. This is equivalent to saying that P satisfies the *projective lifting property:* For every surjection $s\colon M \to N$ of R-modules and every map $g\colon P \to N$ there exists a map $f\colon P \to M$ so that $g = sf$.

$$\begin{array}{ccccc} & & P & & \\ & {}^{\exists f}\swarrow & \downarrow g & & \\ M & \xrightarrow{s} & N & \to & 0 \end{array}$$

To see that these are equivalent, first observe that free modules satisfy this lifting property; in this case f is determined by lifting the image of a basis. To see that all projective modules satisfy the lifting property, extend g to a map from a free module $P \oplus Q$ to N and lift that. Conversely, suppose that P satisfies the projective lifting property. Choose a surjection $\pi\colon F \to P$ with F a free module; the lifting property splits π, yielding $F \cong P \oplus \ker(\pi)$.

If P is a projective module, then P is generated by n elements iff there is a decomposition $P \oplus Q \cong R^n$. Indeed, the generators give a surjection $\pi\colon R^n \to P$, and the lifting property yields the decomposition.

We will focus most of our attention on the category $\mathbf{P}(R)$ of finitely generated projective R-modules *(f.g. projective modules* for short); the morphisms are the R-module maps. Since the direct sum of projectives is projective, $\mathbf{P}(R)$ is an additive category. We may regard $\mathbf{P}$ as a covariant functor on rings, since if $R \to S$ is a ring map then there is an additive functor $\mathbf{P}(R) \to \mathbf{P}(S)$ sending P to $P \otimes_R S$.

HOM AND $\otimes$. If P is a projective R-module, then it is well-known that $P \otimes_R -$ is an exact functor on the category of (left) R-modules, and that $\mathrm{Hom}_R(P, -)$ is an exact functors on the category of (right) R-modules. (See [WHomo] for example.) That is, any exact sequence $0 \to L \to M \to N \to 0$ of R-modules yields exact sequences

$$0 \to P \otimes L \to P \otimes M \to P \otimes N \to 0$$

and

$$0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0.$$

EXAMPLES 2.1.1. Of course free modules and stably free modules are projective.

(1) If F is a field (or a division ring) then every F-module (vector space) is free, but this is not so for all rings.
(2) Consider the matrix ring $R = M_n(F)$. The R-module V of Example 1.1.1 is projective but not free, because $\mathrm{length}(V) = 1 < n = \mathrm{length}(R)$.
(3) *Componentwise free modules.* Another type of projective module arises for rings of the form $R = R_1 \times R_2$; both $P = R_1 \times 0$ and $Q = 0 \times R_2$ are projective but cannot be free because the element $e = (0, 1) \in R$ satisfies $Pe = 0$ yet $R^n e \neq 0$. We say that a module M is *componentwise free* if there is a decomposition $R = R_1 \times \cdots \times R_c$ and integers n_i such that $M \cong R_1^{n_1} \times \cdots \times R_c^{n_c}$. It is easy to see that all componentwise free modules are projective.
(4) *Topological Examples.* Other examples of nonfree projective modules come from topology, and will be discussed more in section 4 below. Consider the ring $R = C^0(X)$ of continuous functions $X \to \mathbb{R}$ on a paracompact topological space X. If $\eta: E \to X$ is a vector bundle then by Ex. 4.8 the set $\Gamma(E) = \{s: X \to E : \eta s = 1_X\}$ of continuous sections of η forms a projective R-module. For example, if T^n is the trivial bundle $\mathbb{R}^n \times X \to X$ then $\Gamma(T^n) = R^n$. I claim that if E is a nontrivial vector bundle then $\Gamma(E)$ cannot be a free R-module. To see this, observe that if $\Gamma(E)$ were free then the sections $\{s_1, ..., s_n\}$ in a basis would define a bundle map $f: T^n \to E$ such that $\Gamma(T^n) = \Gamma(E)$. Since the kernel and cokernel bundles of f have no nonzero sections they must vanish, and f is an isomorphism.

   When X is compact, the category $\mathbf{P}(R)$ of f.g. projective $C^0(X)$-modules is actually equivalent to the category of vector bundles over X; this result is called *Swan's Theorem.* (See Ex. 4.9 for a proof.)

IDEMPOTENTS 2.1.2. An element e of a ring R is called *idempotent* if $e^2 = e$. If $e \in R$ is idempotent then $P = eR$ is projective because $R = eR \oplus (1 - e)R$. Conversely, given any decomposition $R \cong P \oplus Q$, there are unique elements $e \in P$, $f \in Q$ such that $1 = e + f$ in R. By inspection, e and $f = 1 - e$ are idempotent. Thus idempotent elements of R are in 1-1 correspondence with decompositions $R \cong P \oplus Q$.

If $e \neq 0, 1$ and R is commutative then $P = eR$ cannot be free, because $P(1-e) = 0$ but $R(1 - e) \neq 0$. The same is true for noetherian rings by Ex.1.4, but obviously cannot be true for rings such that $R \cong R \oplus R$; see Ex.1.2 (III).

Every finitely generated projective R-module arises from an idempotent element in a matrix ring $M_n(R)$. To see this, note that if $P \oplus Q = R^n$ then the projection-inclusion $R^n \to P \to R^n$ is an idempotent element e of $M_n(R)$. By inspection, the image $e(R^n)$ of e is P. The study of projective modules via idempotent elements can be useful, especially for rings of operators on a Banach space.

If R is a Principal Ideal Domain (PID), such as $\mathbb{Z}$ or $F[x]$, then all projective R-modules are free. This follows from the Structure Theorem for modules over a PID (even for infinitely generated projectives).

Not many other classes of rings have all (f.g.) projective modules free. A famous theorem of Quillen and Suslin states that if R is a polynomial ring (or a Laurent polynomial ring) over a field or a PID then all projective R-modules are free; a good reference for this is the book [Lam]. In particular, if G is a free abelian group then the group ring $\mathbb{Z}[G]$ is the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}, ..., z, z^{-1}]$, and has all projectives free. In contrast, if G is a nonabelian torsion-free nilpotent group, Artamanov proved in [Art] that there are always projective $\mathbb{Z}[G]$-modules P which are stably free but not free: $P \oplus \mathbb{Z}[G] \cong (\mathbb{Z}[G])^2$.

It is an open problem to determine whether or not all projective $\mathbb{Z}[G]$-modules are stably free when G is a finitely presented torsion-free group. Some partial results and other examples are given in [Lam].

For our purposes, local rings form the most important class of rings with all projectives free. A ring R is called a *local ring* if R has a unique maximal (2-sided) ideal $\mathfrak{m}$, and $R/\mathfrak{m}$ is either a field or a division ring.

LEMMA 2.2. *If R is a local ring then every finitely generated projective R-module P is free. In fact $P \cong R^p$, where $p = \dim_{R/\mathfrak{m}}(P/\mathfrak{m}P)$.*

PROOF. We first observe that every element $u \in R - \mathfrak{m}$ is a unit of R, *i.e.*, $uv = vu = 1$ for some v. Indeed, by multiplying by a representative for the inverse of $\bar{u} \in R/\mathfrak{m}$ we may assume that $u \in 1 + \mathfrak{m}$. Since $\mathfrak{m}$ is the Jacobson radical of R, any element of $1 + \mathfrak{m}$ must be a unit of R.

Suppose that $P \oplus Q \cong R^n$. As vector spaces over $F = R/\mathfrak{m}$, $P/\mathfrak{m}P \cong F^p$ and $Q/\mathfrak{m}Q \cong F^q$ for some p and q. Since $F^p \oplus F^q \cong F^n$, $p + q = n$. Choose elements $\{e_1, ..., e_p\}$ of P and $\{e'_1, ..., e'_q\}$ of Q mapping to bases of $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$. The e_i and e'_j determine a homomorphism $R^p \oplus R^q \to P \oplus Q \cong R^n$, which may be represented by a square matrix $(r_{ij}) \in M_n(R)$ whose reduction $(\bar{r}_{ij}) \in M_n(F)$ is invertible. But every such matrix (r_{ij}) is invertible over R by Exercise 1.12. Therefore $\{e_1, ..., e_p, e'_1, ..., e'_q\}$ is a basis for $P \oplus Q$, and from this it follows that P is free on basis $\{e_1, ..., e_p\}$.

REMARK 2.2.1. Even infinitely generated projective R-modules are free when R is local. See [Kap58].

COROLLARY 2.2.2. *If $\mathfrak{p}$ is a prime ideal of a commutative ring R and P is a f.g. projective R-module, then the localization $P_\mathfrak{p}$ is isomorphic to $(R_\mathfrak{p})^n$ for some*

$n \geq 0$. *Moreover, there is an* $s \in R - \mathfrak{p}$ *such that the localization of* P *away from* s *is free:*

$$(P[\frac{1}{s}]) \cong (R[\frac{1}{s}])^n.$$

In particular, $P_{\mathfrak{p}'} \cong (R_{\mathfrak{p}'})^n$ *for every other prime ideal* $\mathfrak{p}'$ *of* R *not containing* s.

PROOF. If $P \oplus Q = R^m$ then $P_{\mathfrak{p}} \oplus Q_{\mathfrak{p}} \cong (R_{\mathfrak{p}})^m$, so $P_{\mathfrak{p}}$ is a f.g. projective $R_{\mathfrak{p}}$-module. Since $R_{\mathfrak{p}}$ is a local ring, $P_{\mathfrak{p}}$ is free by 2.2. Now every element of $P_{\mathfrak{p}}$ is of the form p/s for some $p \in P$ and $s \in R-\mathfrak{p}$. By clearing denominators, we may find an R-module homomorphism $f\colon R^n \to P$ which becomes an isomorphism upon localizing at $\mathfrak{p}$. As coker(f) is a finitely generated R-module which vanishes upon localization, it is annihilated by some $s \in R-\mathfrak{p}$. For this s, the map $f[\frac{1}{s}]\colon (R[\frac{1}{s}])^n \to P[\frac{1}{s}]$ is onto. Since $P[\frac{1}{s}]$ is projective, $(R[\frac{1}{s}])^n$ is the direct sum of $P[\frac{1}{s}]$ and a f.g. $R[\frac{1}{s}]$-module M with $M_{\mathfrak{p}} = 0$. Since M is annihilated by some $t \in R - \mathfrak{p}$ we have

$$f[\frac{1}{st}]: \quad (R[\frac{1}{st}])^n \xrightarrow{\cong} P[\frac{1}{st}].$$

Suppose that there is a ring homomorphism $f\colon R \to F$ from R to a field or a division ring F. If M is any R-module (projective or not) then the *rank of* M *at* f is the integer $\dim_F(M \otimes_R F)$. However, the rank depends upon f, as the example $R = F \times F$, $M = F \times 0$ shows. When R is commutative, every such homomorphism factors through the field $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of R, so we may consider rank(M) as a function on the set Spec(R) of prime ideals in R.

Recall that the set Spec(R) of prime ideals of R has the natural structure of a topological space in which the basic open sets are

$$D(s) = \{\mathfrak{p} \in \mathrm{Spec}(R) : s \notin \mathfrak{p}\} \cong \mathrm{Spec}(R[\frac{1}{s}]) \quad \text{for } s \in R.$$

DEFINITION 2.2.3 (RANK). Let R be a commutative ring. The *rank* of a f.g. R-module M at a prime ideal $\mathfrak{p}$ of R is $\mathrm{rank}_{\mathfrak{p}}(M) = \dim_{k(\mathfrak{p})} M \otimes_R k(\mathfrak{p})$. Since $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \cong k(\mathfrak{p})^{\mathrm{rank}_{\mathfrak{p}}(M)}$, $\mathrm{rank}_{\mathfrak{p}}(M)$ is the minimal number of generators of $M_{\mathfrak{p}}$.

If P is a f.g. projective R-module then $\mathrm{rank}(P)\colon \mathfrak{p} \mapsto \mathrm{rank}_{\mathfrak{p}}(P)$ is a *continuous* function from the topological space Spec(R) to the discrete topological space $\mathbb{N} \subset \mathbb{Z}$, as we see from Corollary 2.2.2. In this way, we shall view rank(P) as an element of the two sets $[\mathrm{Spec}(R), \mathbb{N}]$ and $[\mathrm{Spec}(R), \mathbb{Z}]$ of continuous maps from Spec(R) to $\mathbb{N}$ and to $\mathbb{Z}$, respectively.

We say that P has *constant rank* n if $n = \mathrm{rank}_{\mathfrak{p}}(P)$ is independent of $\mathfrak{p}$. If Spec(R) is topologically connected, every continuous function Spec$(R) \to \mathbb{N}$ must be constant, so every f.g. projective R-module has constant rank. For example, suppose that R is an integral domain with field of fractions F; then Spec(R) is connected, and every f.g. projective R-module P has constant rank: $\mathrm{rank}(P) = \dim_F(P \otimes_R F)$.

If a module M is not projective, rank(M) need not be a continuous function on Spec(R), as the example $R = \mathbb{Z}$, $M = \mathbb{Z}/p$ shows.

Componentwise free modules 2.2.4. Every continuous $f\colon \mathrm{Spec}(R) \to \mathbb{N}$ induces a decomposition of $\mathrm{Spec}(R)$ into the disjoint union of closed subspaces $f^{-1}(n)$. In fact, f takes only finitely many values (say $n_1, ..., n_c$), and it is possible to write R as $R_1 \times ... \times R_c$ such that $f^{-1}(n_i)$ is homeomorphic to $\mathrm{Spec}(R_i)$. (See Ex. 2.4.) Given such a function f, form the componentwise free R-module:

$$R^f = R_1^{n_1} \times ... \times R_c^{n_c}.$$

Clearly R^f has constant rank n_i at every prime in $\mathrm{Spec}(R_i)$ and $\mathrm{rank}(R^f) = f$. For $n \geq \max\{n_i\}$, $R^f \oplus R^{n-f} = R^n$, so R^f is a f.g. projective R-module. Hence continuous functions $\mathrm{Spec}(R) \to \mathbb{N}$ are in 1-1 correspondence with componentwise free modules.

The following variation allows us to focus on projective modules of constant rank in many arguments. Suppose that P is a f.g. projective R-module, so that $\mathrm{rank}(P)$ is a continuous function. Let $R \cong R_1 \times \cdots \times R_c$ be the corresponding decomposition of R. Then each component $P_i = P \otimes_R R_i$ of P is a projective R_i-module of constant rank and there is an R-module isomorphism $P \cong P_1 \times \cdots \times P_c$.

The next theorem allows us to further restrict our attention to projective modules of rank $\leq \dim(R)$. Its proof may be found in [Bass, IV]. We say that two R-modules M, M' are *stably isomorphic* if $M \oplus R^m \cong M' \oplus R^m$ for some $m \geq 0$.

Bass-Serre Cancellation Theorem 2.3. *Let R be a commutative noetherian ring of Krull dimension d, and let P be a projective R-module of constant rank $n > d$.*
(a) (Serre) $P \cong P_0 \oplus R^{n-d}$ for some projective R-module P_0 of constant rank d.
(b) (Bass) If P is stably isomorphic to P' then $P \cong P'$.
(c) (Bass) For all M, M', if $P \oplus M$ is stably isomorphic to M' then $P \oplus M \cong M'$.

Remark 2.3.1. If P is a projective module whose rank is not constant, then $P \cong P_1 \times \cdots \times P_c$ for some decomposition $R \cong R_1 \times \cdots \times R_c$. (See Ex. 2.4.) In this case, we can apply the results in 2.3 to each P_i individually. The reader is invited to phrase 2.3 in this generality.

Locally Free Modules 2.4. Let R be commutative. An R-module M is called *locally free* if for every prime ideal $\mathfrak{p}$ of R there is an $s \in R - \mathfrak{p}$ so that $M[\frac{1}{s}]$ is a free module. We saw in Corollary 2.2.2 that f.g. projective R-modules are locally free. In fact, the following are equivalent:

(1) M is a finitely generated projective R-module;
(2) M is a locally free R-module of finite rank (*i.e.*, $\mathrm{rank}_{\mathfrak{p}}(M) < \infty$ for all $\mathfrak{p}$);
(3) M is a finitely presented R-module, and for every prime $\mathfrak{p}$ of R:

$$M_{\mathfrak{p}} \text{ is a free } R_{\mathfrak{p}}\text{-module.}$$

Proof. The implication (2) $\Rightarrow$ (3) follows from the theory of descent; nowadays we would say that M is coherent (locally finitely presented), hence finitely presented. (See [Hart, II].) To see that (3) $\Rightarrow$ (1), note that finite presentation gives an exact sequence

$$R^m \to R^n \xrightarrow{\varepsilon} M \to 0.$$

We claim that the map $\varepsilon^*\colon \mathrm{Hom}_R(M, R^n) \to \mathrm{Hom}_R(M, M)$ is onto. To see this, recall that being onto is a local property; locally $\varepsilon^*_{\mathfrak{p}}$ is $\mathrm{Hom}(M_{\mathfrak{p}}, R^n_{\mathfrak{p}}) \to \mathrm{Hom}(M_{\mathfrak{p}}, M_{\mathfrak{p}})$. This is a split surjection because $M_{\mathfrak{p}}$ is projective and $\varepsilon_{\mathfrak{p}}\colon R^n_{\mathfrak{p}} \to M_{\mathfrak{p}}$ is a split surjection. If $s\colon M \to R^n$ is such that $\varepsilon^*(s) = \varepsilon s$ is id_M, then s makes M a direct summand of R^n, and M is a f.g. projective module.

Open Patching Data 2.5. It is sometimes useful to be able to build projective modules by patching free modules. The following data suffices Suppose that $s_1, ..., s_c \in R$ form a unimodular row, *i.e.*, $s_1 R + \cdots + s_c R = R$. Then $\mathrm{Spec}(R)$ is covered by the open sets $D(s_i) \cong \mathrm{Spec}(R[\frac{1}{s_i}])$. Suppose we are given $g_{ij} \in GL_n(R[\frac{1}{s_i s_j}])$ with $g_{ii} = 1$ and $g_{ij} g_{jk} = g_{ik}$ in $GL_n(R[\frac{1}{s_i s_j s_k}])$ for every i, j, k. Then

$$P = \{(x_1, ..., x_c) \in \prod_{i=1}^{c} (R[\frac{1}{s_i}])^n : g_{ij}(x_j) = x_i \text{ for all } \ i, j\}$$

is a f.g. projective R-module by 2.4, because each $P[\frac{1}{s_i}]$ is isomorphic to $R[\frac{1}{s_i}]^n$.

Milnor Squares 2.6. Another type of patching arises from an ideal I in R and a ring map $f\colon R \to S$ such that I is mapped isomorphically onto an ideal of S, which we also call I. In this case R is the "pullback" of S and R/I :

$$R = \{(\bar{r}, s) \in (R/I) \times S : \bar{f}(\bar{r}) \equiv s \text{ modulo } I\};$$

the square

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I \end{array}$$

is called a *Milnor square*, because their importance for in patching was emphasized by J. Milnor in [Milnor].

One special kind of Milnor square is the *conductor square*. This arises when R is commutative and S is a finite extension of R (S is often the integral closure of R). The ideal I is chosen to be the *conductor ideal*, *i.e.*, the largest ideal of S contained in R, which is just $I = \{x \in R : xS \subset R\} = \mathrm{ann}_R(S/R)$. If S is reduced then I cannot lie in any minimal prime of R or S, so the rings R/I and S/I have lower Krull dimension.

Given a Milnor square, we can construct an R-module $M = (M_1, g, M_2)$ from the following "descent data": an S-module M_1, an R/I-module M_2 and a S/I-module isomorphism $g\colon M_2 \otimes S/I \cong M_1/IM_1$. In fact M is the kernel of the map

$$M_1 \times M_2 \to M_1/IM_1, \qquad (m_1, m_2) \mapsto \bar{m}_1 - g(\bar{f}(m_2)).$$

We call M the R-module *obtained by patching* M_1 and M_2 together along g.

An important special case is when we patch S^n and $(R/I)^n$ together along a matrix $g \in GL_n(R/I)$. For example, R is obtained by patching R and R/I together along g=1. We will return to this point when we study $K_1(R)$ and $K_0(R)$.

Here is Milnor's result.

MILNOR PATCHING THEOREM 2.7. *In a Milnor square,*

(1) *If P is obtained by patching together a f.g. projective S-module P_1 and a f.g. projective R/I-module P_2, then P is a f.g. projective R-module;*
(2) $P \otimes_R S \cong P_1$ *and* $P/IP \cong P_2$;
(3) *Every f.g. projective R-module arises in this way;*
(4) *If P is obtained by patching free modules along $g \in GL_n(S/I)$, and Q is obtained by patching free modules along g^{-1}, then $P \oplus Q \cong R^{2n}$.*

We shall prove part (3) here; the rest of the proof will be described in Exercise 2.8. If M is any R-module, the Milnor square gives a natural map from M to the R-module M' obtained by patching $M_1 = M \otimes_R S$ and $M_2 = M \otimes_R (R/I) = M/IM$ along the canonical isomorphism

$$(M/IM) \otimes_{R/I} (S/I) \cong M \otimes_R (S/I) \cong (M \otimes_R S)/I(M \otimes_R S).$$

Tensoring M with $0 \to R \to (R/I) \oplus S \to S/I \to 0$ yields an exact sequence

$$\mathrm{Tor}_1^R(M, S/I) \to M \to M' \to 0,$$

so in general M' is just a quotient of M. However, if M is projective, the Tor-term is zero and $M \cong M'$. Thus every projective R-module may be obtained by patching, as (3) asserts.

REMARK 2.7.1. Other examples of patching may be found in [Landsbg].

EILENBERG SWINDLE 2.8. The following "swindle," discovered by Eilenberg, explains why we restrict our attention to finitely generated projective modules. Let R^∞ be an infinitely generated free module. If $P \oplus Q = R^n$, then

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \cong R^\infty.$$

Moreover $R^\infty \cong R^\infty \oplus R^\infty$, and if $P \oplus R^m \cong R^\infty$ then $P \cong R^\infty$ (see Ex. 1.9). Here are a few more facts about infinitely generated projective modules:

• (Bass) If R is noetherian, every infinitely generated projective module is free;
• (Kaplansky) Every infinitely generated projective module is the direct sum of countably generated projective modules;
• (Kaplansky) There are infinitely generated projectives P whose rank is finite but $\mathrm{rank}(P)$ is not continuous on $\mathrm{Spec}(R)$. (See Ex. 2.13)

EXERCISES

2.1 *Radical ideals.* Let I be a radical ideal in R (Exercise 1.12). If P_1, P_2 are f.g. projective R-modules such that $P_1/IP_1 \cong P_2/IP_2$, show that $P_1 \cong P_2$. *Hint:* Modify the proof of 2.2, observing that $\mathrm{Hom}(P, Q) \to \mathrm{Hom}(P/I, Q/I)$ is onto.

2.2 *Idempotent lifting.* Let I be a nilpotent ideal, or more generally an ideal that is *complete* in the sense that every Cauchy sequence $\sum_{n=1}^\infty x_n$ with $x_n \in I^n$ converges to a unique element of I. Show that there is a bijection between the isomorphism classes of f.g. projective R-modules and the isomorphism classes of f.g. projective R/I-modules. To do this, use Ex. 2.1 and proceed in two stages:

(i) Show that every idempotent $\bar{e} \in R/I$ is the image of an idempotent $e \in R$, and that any other idempotent lift is ueu^{-1} for some $u \in 1+I$. *Hint:* it suffices to suppose that $I^2 = 0$ (consider the tower of rings R/I^n). If r is a lift of $\bar{e}$, consider elements of the form $e = r + rxr + (1-r)y(1-r)$ and $(1+xe)e(1+xe)^{-1}$.

(ii) By applying (i) to $M_n(R)$, show that every f.g. projective R/I-module is of the form P/IP for some f.g. projective R-module P.

2.3 Let e, e_1 be idempotents in $M_n(R)$ defining projective modules P and P_1. If $e_1 = geg^{-1}$ for some $g \in GL_n(R)$, show that $P \cong P_1$. Conversely, if $P \cong P_1$ show that for some $g \in GL_{2n}(R)$:

$$\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} = g \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} g^{-1}.$$

2.4 *Rank.* If R is a commutative ring and $f\colon \mathrm{Spec}(R) \to \mathbb{Z}$ is a continuous function, show that we can write $R = R_1 \times \cdots \times R_c$ in such a way that $\mathrm{Spec}(R)$ is the disjoint union of the $\mathrm{Spec}(R_i)$, and f is constant on each of the components $\mathrm{Spec}(R_i)$ of R. To do this, proceed as follows.

(i) Show that $\mathrm{Spec}(R)$ is quasi-compact and conclude that f takes on only finitely many values, say $n_1, ..., n_c$. Each $V_i = f^{-1}(n_i)$ is a closed and open subset of $\mathrm{Spec}(R)$ because $\mathbb{Z}$ is discrete.

(ii) It suffices to suppose that R is reduced, *i.e.*, has no non-zero nilpotent elements. To see this, let $\mathfrak{N}$ be the ideal of all nilpotent elements in R, so $R/\mathfrak{N}$ is reduced. Since $\mathrm{Spec}(R) \cong \mathrm{Spec}(R/\mathfrak{N})$, we may apply idempotent lifting (Ex. 2.2).

(iii) Let I_i be the ideal defining V_i, i.e., $I_i = \cap\{\mathfrak{p} : \mathfrak{p} \in V_i\}$. If R is reduced, show that $I_1 + \cdots + I_c = R$ and that for every $i \neq j$ $I_i \cap I_j = \emptyset$. Conclude using the Chinese Remainder Theorem, which says that $R \cong \prod R_i$.

2.5 Show that the following are equivalent for every commutative ring R:

(1) $\mathrm{Spec}(R)$ is topologically connected
(2) Every f.g. projective R-module has constant rank
(3) R has no idempotent elements except 0 and 1.

2.6 *Dual Module.* If P is a f.g. projective R-module, the *dual module* is $\check{P} = \mathrm{Hom}_R(P, R)$. Show that $\check{P}$ is a f.g. projective R^{op}-module, where R^{op} denotes R with multiplication reversed. Now suppose that R is commutative, so that $R = R^{op}$. Show that $\mathrm{rank}(P) = \mathrm{rank}(\check{P})$ as functions from $\mathrm{Spec}(R)$ to $\mathbb{Z}$.

2.7 *Tensor Product.* Let P and Q be projective modules over a commutative ring R. Show that the tensor product $P \otimes_R Q$ is also a projective R-module, and is finitely generated if P and Q are. Finally, show that

$$\mathrm{rank}(P \otimes_R Q) = \mathrm{rank}(P) \cdot \mathrm{rank}(Q).$$

2.8 *Milnor Patching.* In this exercise we prove the Milnor Patching Theorem 2.7, that any R-module obtained by patching f.g. projective modules over S and R/I in a Milnor square is a f.g. projective R-module. Prove the following:

(i) If $g \in GL_n(S/I)$ is the image of a matrix in either $GL_n(S)$ or $GL_n(R/I)$, the the patched module $P = (S^n, g, (R/I)^n)$ is a free R-module.

(ii) Show that $(P_1, g, P_2) \oplus (Q_1, h, Q_2) \cong (P_1 \oplus Q_1, \left(\begin{smallmatrix} g & 0 \\ 0 & h \end{smallmatrix}\right), P_2 \oplus Q_2)$.

(iii) If $g \in GL_n(S/I)$, let M be the module obtained by patching S^{2n} and $(R/I)^{2n}$ together along the matrix

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \in GL_{2n}(S/I).$$

Use Ex. 1.11 to prove that $M \cong R^{2n}$. This establishes Theorem 2.7, part (4).

(iv) Given $P_1 \oplus Q_1 \cong S^n$, $P_2 \oplus Q_2 \cong (R/I)^n$ and isomorphisms $P_1/IP_1 \cong P_2 \otimes S/I$, $Q_1/IQ_1 \cong Q_2 \otimes S/I$, let P and Q be the R-modules obtained by patching the P_i and Q_i together. By (ii), $P \oplus Q$ is obtained by patching S^n and $(R/I)^n$ together along some $g \in GL_n(S/I)$. Use (iii) to show that P and Q are f.g. projective.

(v) If $P_1 \oplus Q_1 \cong S^m$ and $P_2 \oplus Q_2 \cong (R/I)^n$, and $g\colon P_1/IP_1 \cong P_2 \otimes S/I$, show that $(Q_1 \oplus S^n) \otimes S/I$ is isomorphic to $(R/I^m \oplus Q_2) \otimes S/I$. By (iv), this proves that (P_1, g, P_2) is f.g. projective, establishing part (1) of Theorem 2.7.

(vi) Prove part (2) of Theorem 2.7 by analyzing the above steps.

2.9 Consider a Milnor square (2.6). Let P_1, Q_1 be f.g. projective S-modules, and P_2, Q_2 be f.g. projective R/I-modules such that there are isomorphisms $g\colon P_2 \otimes S/I \cong P_1/IP_1$ and $h\colon Q_2 \otimes S/I \cong Q_1/IQ_1$.

(i) If $f\colon Q_2 \otimes S/I \cong P_1/IP_1$, show that $(P_1, g, P_2) \oplus (Q_1, h, Q_2)$ is isomorphic to $(Q_1, gf^{-1}h, P_2) \oplus (P_1, f, Q_2)$. *Hint:* Use Ex. 2.8 and the decomposition

$$\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} gf^{-1}h & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} h^{-1}f & 0 \\ 0 & f^{-1}h \end{pmatrix}.$$

(ii) Conclude that $(S^n, g, R/I^n) \oplus (S^n, h, R/I^n) \cong (S^n, gh, R/I^n) \oplus R^n$.

2.10 Suppose P, Q are modules over a commutative ring R such that $P \otimes Q \cong R^n$ for some $n \neq 0$. Show that P and Q are finitely generated projective R-modules. *Hint:* Find a generating set $\{p_i \otimes q_i | i = 1, ..., m\}$ for $P \otimes Q$; the $p_i \otimes q_j \otimes p_k$ generate $P \otimes Q \otimes P$. Show that $\{p_i\}$ define a split surjection $R^m \to P$.

2.11 Let M be a finitely generated module over a commutative ring R. Show that the following are equivalent for every n:

(1) M is a f.g. projective module of constant rank n

(2) $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$ for every prime ideal $\mathfrak{p}$ of R.

Conclude that in 2.4 we may add:

(4) M is finitely generated, $M_{\mathfrak{p}}$ is free for every prime ideal $\mathfrak{p}$ of R, and rank(M) is a continuous function on Spec(R).

2.12 If $f\colon R \to S$ is a homomorphism of commutative rings, there is a continuous map $f^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ sending $\mathfrak{p}$ to $f^{-1}(\mathfrak{p})$. If P is a f.g. projective R-module, show that rank$(P \otimes_R S)$ is the composition of f^* and rank(P). In particular, show that if P has constant rank n, then so does $P \otimes_R S$.

2.13 (Kaplansky) Here is an example of an infinitely generated projective module whose rank is not continuous. Let R be the ring of continuous functions

$f\colon[0,1]\to\mathbb{R}$ on the unit interval and I the ideal of all functions f which vanish on some neighborhood $[0,\varepsilon)$ of 0. Show that I is a projective R-module, yet $\operatorname{rank}(I)\colon\operatorname{Spec}(R)\to\{0,1\}$ is not continuous, so I is neither finitely generated nor free. We remark that every f.g. projective R-module is free; this follows from Swan's Theorem, since every vector bundle on $[0,1]$ is trivial (by 4.6.1 below).

Hint: Show that the functions $f_n=\max\{0,t-\frac{1}{n}\}$ generate I, and construct a splitting to the map $R^\infty\to I$. To see that $\operatorname{rank}(I)$ is not continuous, consider the rank of I at the primes $\mathfrak{m}_t=\{f\in R: f(t)=0\}$, $0\le t\le 1$.

§3. The Picard Group of a commutative ring

An *algebraic line bundle* L over a commutative ring R is just a f.g. projective R-module of constant rank 1. The name comes from the fact that if R is the ring of continuous functions on a compact space X, then a topological line bundle (vector bundle which is locally $\mathbb{R}\times X\to X$) corresponds to an algebraic line bundle by Swan's Theorem (see example 2.1.1(4) or Ex. 4.9 below).

The tensor product $L\otimes_R M\cong M\otimes_R L$ of line bundles is again a line bundle (by Ex. 2.7), and $L\otimes_R R\cong L$ for all L. Thus up to isomorphism the tensor product is a commutative associative operation on line bundles, with identity element R.

Lemma 3.1. *If L is a line bundle, then the dual module $\check{L}=\operatorname{Hom}_R(L,R)$ is also a line bundle, and $\check{L}\otimes_R L\cong R$.*

Proof. Since $\operatorname{rank}(\check{L})=\operatorname{rank}(L)=1$ by Ex. 2.6, $\check{L}$ is a line bundle. Consider the evaluation map $\check{L}\otimes_R L\to R$ sending $f\otimes x$ to $f(x)$. If $L\cong R$, this map is clearly an isomorphism. Therefore for every prime ideal $\mathfrak{p}$ the localization

$$(\check{L}\otimes_R L)_{\mathfrak{p}}=(L_{\mathfrak{p}})\check{}\otimes_{R_{\mathfrak{p}}}L_{\mathfrak{p}}\to R_{\mathfrak{p}}$$

is an isomorphism. Since being an isomorphism is a local property of an R-module homomorphism, the evaluation map must be an isomorphism.

Definition. the *Picard group* $\operatorname{Pic}(R)$ of a commutative ring R is the set of isomorphism classes of line bundles over R. As we have seen, the tensor product $\otimes_R$ endows $\operatorname{Pic}(R)$ with the structure of an abelian group, the identity element being $[R]$ and the inverse being $L^{-1}=\check{L}$.

Proposition 3.2. Pic *is a functor from commutative rings to abelian groups. That is, if $R\to S$ is a ring homomorphism then* $\operatorname{Pic}(R)\to\operatorname{Pic}(S)$ *is a homomorphism sending L to $L\otimes_R S$.*

Proof. If L is a line bundle over R, then $L\otimes_R S$ is a line bundle over S (see Ex. 2.12), so $\otimes_R S$ maps $\operatorname{Pic}(R)$ to $\operatorname{Pic}(S)$. The natural isomorphism $(L\otimes_R M)\otimes_R S\cong(L\otimes_R S)\otimes_S(M\otimes_R S)$, valid for all R-modules L and M, shows that $\otimes_R S$ is a group homomorphism.

Lemma 3.3. *If L is a line bundle, then* $\operatorname{End}_R(L)\cong R$.

Proof. Multiplication by elements in R yields a map from R to $\operatorname{End}_R(L)$. As it is locally an isomorphism, it must be an isomorphism.

Determinant line bundle of a projective module

If M is any module over a commutative ring R and $k \geq 0$, the k^{th} *exterior product* $\wedge^k M$ is the quotient of the k-fold tensor $M \otimes \cdots \otimes M$ by the submodule generated by terms $m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for some $i \neq j$. By convention, $\wedge^0 M = R$ and $\wedge^1 M = M$. The following facts are classical; see [B-AC, ch.2].

(i) $\wedge^k(R^n)$ is the free module of rank $\binom{n}{k}$ generated by terms $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $1 \leq i_1 \leq \cdots \leq i_k \leq n$. In particular, $\wedge^n(R^n) \cong R$ on $e_1 \wedge \cdots \wedge e_n$.

(ii) If $R \to S$ is a ring map, there is a natural isomorphism $(\wedge^k M) \otimes_R S \cong \wedge^k(M \otimes_R S)$, the first $\wedge^k$ being taken over R and the second being taken over S. In particular, $\text{rank}(\wedge^k M) = \binom{\text{rank}\, M}{k}$ as functions from $\text{Spec}(R)$ to $\mathbb{N}$.

(iii) *(Sum Formula)* If $M = P \oplus Q$, there is a natural isomorphism

$$\wedge^k(P \oplus Q) \cong \bigoplus_{i=1}^{k} (\wedge^i P) \otimes (\wedge^{k-i} Q).$$

If P is a projective module of constant rank n, then $\wedge^k P$ is a f.g. projective module of constant rank $\binom{n}{k}$, because $\wedge^k P$ is locally free: if $P[\frac{1}{s}] \cong (R[\frac{1}{s}])^n$ then $(\wedge^k P)[\frac{1}{s}] \cong (R[\frac{1}{s}])^{\binom{n}{k}}$. In particular, $\wedge^n P$ is a line bundle, and $\wedge^k P = 0$ for $k > n$. We write $\det(P)$ for $\wedge^n P$, and call it the *determinant line bundle* of P.

If the rank of a projective module P is not constant, we define the determinant line bundle $\det(P)$ componentwise, using the following recipe. From §2 we find a decomposition $R \cong R_1 \times \cdots \times R_c$ so that $P \cong P_1 \times \cdots \times P_c$ and each P_i has constant rank n_i as an R_i-module. We then define $\det(P)$ to be $(\wedge^{n_1} P_1) \times \cdots \times (\wedge^{n_c} P_c)$; clearly $\det(P)$ is a line bundle on R. If P has constant rank n, this agrees with our above definition: $\det(P) = \wedge^n P$.

As the name suggests, the determinant line bundle is related to the usual determinant of a matrix. An $n \times n$ matrix g is just an endomorphism of R^n, so it induces an endomorphism $\wedge^n g$ of $\wedge^n R^n \cong R$. By inspection, $\wedge^n g$ is multiplication by $\det(g)$.

Using the determinant line bundle, we can also take the determinant of an endomorphism g of a f.g. projective R-module P. By the naturality of $\wedge^n$, g induces an endomorphism $\det(g)$ of $\det(P)$. By Lemma 3.3, $\det(g)$ is an element of R, acting by multiplication; we call $\det(g)$ the *determinant* of the endomorphism g.

Here is an application of the det construction. Let L, L' be stably isomorphic line bundles. That is, $P = L \oplus R^n \cong L' \oplus R^n$ for some n. The Sum Formula (iii) shows that $\det(P) = L$, and $\det(P) = L'$, so $L \cong L'$. Taking $L' = R$, this shows that R is the only stably free line bundle. It also gives the following slight improvement upon the Cancellation Theorem 2.3 for 1-dimensional rings:

PROPOSITION 3.4. *Let R be a commutative noetherian 1-dimensional ring. Then all f.g. projective R-modules are completely classified by their rank and determinant. In particular, every f.g. projective R-module P of rank ≥ 1 is isomorphic to $L \oplus R^f$, where $L = \det(P)$ and $f = \text{rank}(P) - 1$.*

Invertible Ideals

When R is a commutative integral domain (=*domain*), we can give a particularly nice interpretation of $\mathrm{Pic}(R)$, using the following concepts. Let F be the field of fractions of R; a *fractional ideal* is a nonzero R-submodule I of F such that $I \subseteq fR$ for some $f \in F$. If I and J are fractional ideals then their product $IJ = \{\sum x_i y_i : x_i \in I, y_i \in J\}$ is also a fractional ideal, and the set $\mathrm{Frac}(R)$ of fractional ideals becomes an abelian monoid with identity element R. A fractional ideal I is called *invertible* if $IJ = R$ for some other fractional ideal J; invertible ideals are sometimes called *Cartier divisors.* The set of invertible ideals is therefore an abelian group, and one writes $\mathrm{Cart}(R)$ or $\mathrm{Pic}(R, F)$ for this group.

If $f \in F^\times$, the fractional ideal $\mathrm{div}(f) = fR$ is invertible because $(fR)(f^{-1}R) = R$; invertible ideals of this form are called *principal divisors.* Since $(fR)(gR) = (fg)R$, the function div: $F^\times \to \mathrm{Cart}(R)$ is a group homomorphism.

This all fits into the following overall picture (see Ex. 3.7 for a generalization).

PROPOSITION 3.5. *If R is a commutative integral domain, every invertible ideal is a line bundle, and every line bundle is isomorphic to an invertible ideal. If I and J are fractional ideals, and I is invertible, then $I \otimes_R J \cong IJ$. Finally, there is an exact sequence of abelian groups:*

$$1 \to R^\times \to F^\times \xrightarrow{div} \mathrm{Cart}(R) \to \mathrm{Pic}(R) \to 0.$$

PROOF. If I and J are invertible ideals such that $IJ \subseteq R$, then we can interpret elements of J as homomorphisms $I \to R$. If $IJ = R$ then we can find $x_i \in I$ and $y_i \in J$ so that $x_1y_1 + ... + x_ny_n = 1$. The $\{x_i\}$ assemble to give a map $R^n \to I$ and the $\{y_i\}$ assemble to give a map $I \to R^n$. The composite $I \to R^n \to I$ is the identity, because it sends $r \in I$ to $\sum x_i y_i r = r$. Thus I is a summand of R^n, *i.e.*, I is a f.g. projective module. As R is an integral domain and $I \subseteq F$, $\mathrm{rank}(I)$ is the constant $\dim_F(I \otimes_R F) = \dim_F(F) = 1$. Hence I is a line bundle.

This construction gives a set map $\mathrm{Cart}(R) \to \mathrm{Pic}(R)$; to show that it is a group homomorphism, it suffices to show that $I \otimes_R J \cong IJ$ for invertible ideals. Suppose that I is a submodule of F which is also a line bundle over R. As I is projective, $I \otimes_R -$ is an exact functor. Thus if J is an R-submodule of F then $I \otimes_R J$ is a submodule of $I \otimes_R F$. The map $I \otimes_R F \to F$ given by multiplication in F is an isomorphism because I is locally free and F is a field. Therefore the composite

$$I \otimes_R J \subseteq I \otimes_R F \xrightarrow{\text{multiply}} F$$

sends $\sum x_i \otimes y_i$ to $\sum x_i y_i$. Hence $I \otimes_R J$ is isomorphic to its image $IJ \subseteq F$. This proves the third assertion.

The kernel of $\mathrm{Cart}(R) \to \mathrm{Pic}(R)$ is the set of invertible ideals I having an isomorphism $I \cong R$. If $f \in I$ corresponds to $1 \in R$ under such an isomorphism then $I \cong fR = \mathrm{div}(f)$. This proves exactness of the sequence at $\mathrm{Cart}(R)$.

Clearly the units $R^\times$ of R inject into $F^\times$. If $f \in F^\times$ then $fR = R$ iff $f \in R$ and f is in no proper ideal, *i.e.*, iff $f \in R^\times$. This proves exactness at $R^\times$ and $F^\times$.

Finally, we have to show that every line bundle L is isomorphic to an invertible ideal of R. Since $\mathrm{rank}(L) = 1$, there is an isomorphism $L \otimes_R F \cong F$. This gives an injection $L \cong L \otimes_R R \subset L \otimes_R F \cong F$, *i.e.*, an isomorphism of L with a fractional ideal I. Choosing an isomorphism $\check{L} \cong J$, Lemma 3.1 yields

$$R \cong L \otimes_R \check{L} \cong I \otimes_R J \cong IJ.$$

Hence $IJ = fR$ for some $f \in F^\times$, and $I(f^{-1}J) = R$, so I is invertible.

Dedekind domains

Historically, the most important applications of the Picard group have been for Dedekind domains. A *Dedekind domain* is a commutative integral domain which is noetherian, integrally closed and has Krull dimension 1.

There are many equivalent definitions of Dedekind domain in the literature. Here is another: an integral domain R is Dedekind iff every fractional ideal of R is invertible. In a Dedekind domain every nonzero ideal (and fractional ideal) can be written uniquely as a product of prime ideals $\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$. Therefore $\mathrm{Cart}(R)$ is the free abelian group on the set of (nonzero) prime ideals of R, and $\mathrm{Pic}(R)$ is the set of isomorphism classes of (actual) ideals of R.

Another property of Dedekind domains is that every f.g. torsionfree R-module M is projective. To prove this fact we use induction on $\mathrm{rank}_0(M) = \dim_F(M \otimes F)$, the case $\mathrm{rank}_0(M) = 0$ being trivial. Set $\mathrm{rank}_0(M) = n+1$. As M is torsionfree, it is a submodule of $M \otimes F \cong F^{n+1}$. The image of M under any coordinate projection $F^{n+1} \to F$ is a fractional ideal I_0. As I_0 is invertible, the projective lifting property for I_0 shows that $M \cong M' \oplus I_0$ with $\mathrm{rank}_0(M') = n$. By induction, $M \cong I_0 \oplus \cdots \oplus I_n$ is a sum of ideals. By Propositions 3.4 and 3.5, $M \cong I \oplus R^n$ for the invertible ideal $I = \det(M) = I_0 \cdots I_n$.

Examples. Here are some particularly interesting classes of Dedekind domains.

- A *principal ideal domain* (or PID) is a domain R in which every ideal is rR for some $r \in R$. Clearly, these are just the Dedekind domains with $\mathrm{Pic}(R) = 0$. Examples of PID's include $\mathbb{Z}$ and polynomial rings $k[x]$ over a field k.
- A *discrete valuation domain* (or DVR) is a local Dedekind domain. By Lemma 2.2, a DVR is a PID R with a unique maximal ideal $M = \pi R$. Fixing π, it isn't hard to see that every ideal of R is of the form $\pi^i R$ for some $i \geq 0$. Consequently every fractional ideal of R can be written as $\pi^i R$ for a unique $i \in \mathbb{Z}$. By Proposition 3.5, $F^\times \cong R^\times \times \{\pi^i\}$. There is a (discrete) valuation ν on the field of fractions $F : \nu(f)$ is that integer i such that $fR \cong \pi^i R$.

  Examples of DVR's include the p-adic integers $\hat{\mathbb{Z}}_p$, the power series ring $k[[x]]$ over a field k, and localizations $\mathbb{Z}_p$ of $\mathbb{Z}$.
- Let F be a number field, *i.e.*, a finite field extension of $\mathbb{Q}$. An *algebraic integer* of F is an element which is integral over $\mathbb{Z}$, *i.e.*, a root of a monic polynomial $x^n + a_1 x^{n-1} + \cdots + a_n$ with integer coefficients ($a_i \in \mathbb{Z}$). The set $\mathcal{O}_F$ of all algebraic integers of F is a ring—it is the integral closure of $\mathbb{Z}$ in F. A famous result in ring theory asserts that $\mathcal{O}_F$ is a Dedekind domain with field of fractions F. It follows that $\mathcal{O}_F$ is a lattice in F, *i.e.*, a free abelian group of rank $\dim_{\mathbb{Q}}(F)$.

  In Number Theory, $\mathrm{Pic}(\mathcal{O}_F)$ is called the *ideal class group* of the number field F. A fundamental theorem states that $\mathrm{Pic}(\mathcal{O}_F)$ is always a finite group, but the precise structure of the ideal class group is only known for special number fields of small dimension. For example, if $\xi_p = e^{2\pi i/p}$ then $\mathbb{Z}[\xi_p]$ is the ring of algebraic integers of $\mathbb{Q}(\xi_p)$, and the class group is zero if and only if $p \leq 19$; $\mathrm{Pic}(\mathbb{Z}[\xi_{23}])$ is $\mathbb{Z}/3$. More details may be found in books on number theory, such as [BSh].

• If C is a smooth affine curve over a field k, then the coordinate ring R of C is a Dedekind domain. One way to construct a smooth affine curve is to start with a smooth projective curve $\bar{C}$. If $\{p_0, ..., p_n\}$ is any nonempty set of points on $\bar{C}$, the Riemann-Roch theorem implies that $C = \bar{C} - \{p_0, ..., p_n\}$ is a smooth affine curve.

If k is algebraically closed, $\mathrm{Pic}(R)$ is a divisible abelian group. Indeed, the points of the Jacobian variety $J(\bar{C})$ form a divisible abelian group, and $\mathrm{Pic}(R)$ is the quotient of $J(\bar{C})$ by the subgroup generated by the classes of the prime ideals of R corresponding to $p_1, ..., p_n$.

This is best seen when $k = \mathbb{C}$, because smooth projective curves over $\mathbb{C}$ are the same as compact Riemann surfaces. If $\bar{C}$ is a compact Riemann surface of genus g, then as an abelian group the points of the Jacobian $J(\bar{C})$ form the divisible group $(\mathbb{R}/\mathbb{Z})^{2g}$. In particular, when $C = \bar{C} - \{p_0\}$ then $\mathrm{Pic}(R) \cong J(\bar{C}) \cong (\mathbb{R}/\mathbb{Z})^{2g}$.

For example, $R = \mathbb{C}[x, y]/(y^2 - x(x-1)(x-\beta))$ is a Dedekind domain with $\mathrm{Pic}(R) \cong (\mathbb{R}/\mathbb{Z})^2$ if $\beta \neq 0, 1$. Indeed, R is the coordinate ring of a smooth affine curve C obtained by removing one point from an elliptic curve (= a projective curve of genus $g = 1$).

The Weil Divisor Class group

Let R be an integrally closed domain (= *normal domain*) with field of fractions F. If R is a noetherian normal domain, it is well-known that:

(i) $R_\mathfrak{p}$ is a discrete valuation ring (DVR) for all height 1 prime ideals $\mathfrak{p}$;

(ii) $R = \cap R_\mathfrak{p}$, the intersection being over all height 1 primes $\mathfrak{p}$ of R, each $R_\mathfrak{p}$ being a subring of F;

(iii) Every $r \neq 0$ in R is contained in only finitely many height 1 primes $\mathfrak{p}$.

An integral domain R satisfying (i), (ii) and (iii) is called a *Krull domain.*

Krull domains are integrally closed because every DVR $R_\mathfrak{p}$ is integrally closed. For a Krull domain R, the group $D(R)$ of *Weil divisors* is the free abelian group on the height 1 prime ideals of R. An *effective* Weil divisor is a divisor $D = \sum n_i[\mathfrak{p}_i]$ with all the $n_i \geq 0$.

We remark that effective divisors correspond to "divisorial" ideals of R, D corresponding to the intersection $\cap \mathfrak{p}_i^{(n_i)}$ of the symbolic powers of the $\mathfrak{p}_i$.

If $\mathfrak{p}$ is a height 1 prime of R, the $\mathfrak{p}$-adic valuation $\nu_\mathfrak{p}(I)$ of an invertible ideal I is defined to be that integer ν such that $I_\mathfrak{p} = \mathfrak{p}^\nu R_\mathfrak{p}$. By (iii), $v_\mathfrak{p}(I) \neq 0$ for only finitely many $\mathfrak{p}$, so $\nu(I) = \sum \nu_\mathfrak{p}(I)[\mathfrak{p}]$ is a well-defined element of $D(R)$. By 3.5, this gives a group homomorphism:

$$\nu\colon \quad \mathrm{Cart}(R) \to D(R).$$

If I is invertible, $\nu(I)$ is effective iff $I \subseteq R$. To see this, observe that $\nu(I)$ is effective $\iff I_\mathfrak{p} \subseteq R_\mathfrak{p}$ for all $\mathfrak{p} \iff I \subseteq \cap I_\mathfrak{p} \subseteq \cap R_\mathfrak{p} = R$. It follows that ν is an injection, because if both $\nu(I)$ and $\nu(I^{-1})$ are effective then I and I^{-1} are ideals with product R; this can only happen if $I = R$.

The *divisor class group* $Cl(R)$ of R is defined to be the quotient of $D(R)$ by the subgroup of all $\nu(fR)$, $f \in F^\times$. This yields a map $\mathrm{Pic}(R) \to Cl(R)$ which is evidently an injection. Summarizing, we have proven:

Proposition 3.6. *Let R be a Krull domain. Then* $\mathrm{Pic}(R)$ *is a subgroup of the class group* $Cl(R)$*, and there is a commutative diagram with exact rows:*

$$\begin{array}{cccccccccc} 1\to & R^* & \to & F^* & \stackrel{\mathrm{div}}{\to} & \mathrm{Cart}(R) & \to & \mathrm{Pic}(R) & \to 0 \\ & \downarrow = & & \downarrow = & & \bigcap \nu & & \bigcap & \\ 1\to & R^* & \to & F^* & \stackrel{\mathrm{div}}{\to} & D(R) & \to & Cl(R) & \to 0. \end{array}$$

Remark 3.6.1. Both the Picard group and the divisor class group of a Krull domain R are invariant under polynomial and Laurent polynomial extensions. That is, $\mathrm{Pic}(R) = \mathrm{Pic}(R[t]) = \mathrm{Pic}(R[t, t^{-1}])$ and $Cl(R) = Cl(R[t]) = Cl(R[t, t^{-1}])$. Most of this assertion is proven in [B-AC, ch.7,§1]; the $\mathrm{Pic}[t, t^{-1}]$ assertion is proven in [BM, 5.10].

Recall that an integral domain R is called *factorial*, or a *Unique Factorization Domain* (UFD) if every nonzero element $r \in R$ is either a unit or a product of prime elements. (This implies that the product is unique up to order and primes differing by a unit). It is not hard to see that UFD's are Krull domains; the following interpretation in terms of the class group is taken from [Matsu, §20].

Theorem 3.7. *Let R be a Krull domain. Then R is a UFD $\iff Cl(R) = 0$.*

Definition. A noetherian ring R is called *regular* if every R-module M has a finite resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$ with the P_i projective. Every localization $S^{-1}R$ of a regular ring R is also a regular ring, because $S^{-1}R$-modules are also R-modules, and a localization of an R-resolution is an $S^{-1}R$-resolution.

Now suppose that $(R, \mathfrak{m})$ is a regular local ring. It is well-known [Matsu, §14,19] that R is a noetherian, integrally closed domain (hence Krull), and that if $s \in \mathfrak{m} - \mathfrak{m}^2$ then sR is a prime ideal.

Theorem 3.8. *Every regular local ring is a UFD.*

Proof. We proceed by induction on $\dim(R)$. If $\dim(R) = 0$ then R is a field; if $\dim(R) = 1$ then R is a DVR, hence a UFD. Otherwise, choose $s \in \mathfrak{m} - \mathfrak{m}^2$. Since sR is prime, Ex. 3.8(b) yields $Cl(R) \cong Cl(R[\frac{1}{s}])$. Hence it suffices to show that $S = R[\frac{1}{s}]$ is a UFD. Let $\mathfrak{P}$ be a height 1 prime of S; we have to show that $\mathfrak{P}$ is a principal ideal. For every prime ideal $\mathfrak{Q}$ of S, $S_{\mathfrak{Q}}$ is a regular local ring of smaller dimension than R, so by induction $S_{\mathfrak{Q}}$ is a UFD. Hence $\mathfrak{P}_{\mathfrak{Q}}$ is principal: $xS_{\mathfrak{Q}}$ for some $x \in S$. By 2.4, $\mathfrak{P}$ is projective, hence invertible. Let $\mathfrak{p}$ be the prime ideal of R such that $\mathfrak{P} = \mathfrak{p}[\frac{1}{s}]$ and choose an R-resolution $0 \to P_n \to \cdots \to P_0 \to \mathfrak{p} \to 0$ of $\mathfrak{p}$ by f.g. projective R-modules P_i. Since R is local, the P_i are free. Since $\mathfrak{P}$ is projective, the localized sequence $0 \to P_n[\frac{1}{s}] \to \cdots \to P_0[\frac{1}{s}] \to \mathfrak{P} \to 0$ splits. Letting E (resp. F) denote the direct sum of the odd (resp. even) $P_i[\frac{1}{s}]$, we have $\mathfrak{P} \oplus E \cong F$. Since stably free line bundles are free, $\mathfrak{P}$ is free. That is, $\mathfrak{P} = xS$ for some $x \in \mathfrak{P}$, as desired.

Corollary 3.8.1. *If R is a regular domain, then* $\mathrm{Cart}(R) = D(R)$*, and hence*

$$\mathrm{Pic}(R) = Cl(R).$$

PROOF. We have to show that every height 1 prime ideal $\mathfrak{P}$ of R is invertible. For every prime ideal $\mathfrak{p}$ of R we have $\mathfrak{P}_\mathfrak{p} \cong R_\mathfrak{p}$ in the UFD $R_\mathfrak{p}$. By 2.4 and 3.5, $\mathfrak{P}$ is an invertible ideal.

REMARK 3.8.2. A ring is called *locally factorial* if $R_\mathfrak{p}$ is factorial for every prime ideal $\mathfrak{p}$ of R. For example, regular rings are locally factorial by 3.8. The proof of Cor. 3.8.1 shows that if R is a locally factorial Krull domain then $\mathrm{Pic}(R) = Cl(R)$.

Non-normal rings

The above discussion should make it clear that the Picard group of a normal domain is a classical object, even if it is hard to compute in practice. If R isn't normal, we can get a handle on $\mathrm{Pic}(R)$ using the techniques of the rest of this section. For example, the next lemma allows us to restrict attention to reduced noetherian rings with finite normalization, because the quotient R_{red} of any commutative ring R by its *nilradical* (the ideal of nilpotent elements) is a reduced ring, and every commutative ring is the filtered union of its finitely generated subrings–rings having these properties.

LEMMA 3.9. (1) $\mathrm{Pic}(R) = \mathrm{Pic}(R_{red})$.

(2) Pic *commutes with filtered direct limits of rings. In particular, if R is the filtered union of subrings R_α, then* $\mathrm{Pic}(R) \cong \varinjlim \mathrm{Pic}(R_\alpha)$.

PROOF. Part (1) is an instance of idempotent lifting (Ex. 2.2). To prove (2), recall from 2.5 that a line bundle L over R may be given by patching data: a unimodular row $(s_1, ..., s_c)$ and units g_{ij} over the $R[\frac{1}{s_is_j}]$. If R is the filtered direct limit of rings R_α, this finite amount of data defines a line bundle L_α over one of the R_α, and we have $L = L_\alpha \otimes_{R_\alpha} R$. If L_α and L'_α become isomorphic over R, the isomorphism is defined over some R_β, *i.e.*, L and L' become isomorphic over R_β.

If R is reduced noetherian, its normalization S is a finite product of normal domains S_i. We would like to describe $\mathrm{Pic}(R)$ in terms of the more classical group $\mathrm{Pic}(S) = \prod \mathrm{Pic}(S_i)$, using the conductor square of 2.6. For this it is convenient to assume that S is finite over R, an assumption which is always true for rings of finite type over a field.

More generally, suppose that we are given a Milnor square (2.6):

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar f} & S/I. \end{array}$$

Given a unit β of S/I, the Milnor Patching Theorem 2.7 constructs a f.g. projective R-module $L_\beta = (S, \beta, R/I)$ with $L_\beta \otimes_R S \cong S$ and $L_\beta / IL_\beta \cong R/I$. In fact L_β is a line bundle, because $\mathrm{rank}(L_\beta) = 1$; every map from R to a field factors through either R/I or S (for every prime ideal $\mathfrak{p}$ of R either $I \subseteq \mathfrak{p}$ or $R_\mathfrak{p} \cong S_\mathfrak{p}$). By Ex. 2.9, $L_\alpha \oplus L_\beta \cong L_{\alpha\beta} \oplus R$; applying $\wedge^2$ yields $L_\alpha \otimes_R L_\beta \cong L_{\alpha\beta}$. Hence the formula $\partial(\beta) = [L_\beta]$ yields a group homomorphism

$$\partial\colon (S/I)^\times \to \mathrm{Pic}(R).$$

THEOREM 3.10 (UNITS-PIC SEQUENCE). *Given a Milnor square, the following sequence is exact. Here Δ denotes the diagonal map and $\pm$ denotes the difference map sending $(s,\bar r)$ to $\bar s f(\bar r)^{-1}$, resp. (L',L) to $L'\otimes_S S/I\otimes_{R/I} L^{-1}$.*

$$1\to R^\times \xrightarrow{\Delta} S^\times\times(R/I)^\times \xrightarrow{\pm} (S/I)^\times \xrightarrow{\partial} \mathrm{Pic}(R) \xrightarrow{\Delta} \mathrm{Pic}(S)\times\mathrm{Pic}(R/I) \xrightarrow{\pm} \mathrm{Pic}(S/I)$$

PROOF. Since R is the pullback of S and R/I, exactness at the first two places is clear. Milnor Patching 2.7 immediately yields exactness at the last two places, leaving only the question of exactness at $(S/I)^\times$. Given $s\in S^\times$ and $\bar r\in(R/I)^\times$, set $\beta=\pm(s,\bar r)=\bar s f(\bar r)^{-1}$, where $\bar s$ denotes the reduction of s modulo I. By inspection, $\lambda=(s,\bar r)\in L_\beta\subset S\times R/I$, and every element of L_β is a multiple of λ. It follows that $L_\beta\cong R$. Conversely, suppose given $\beta\in(S/I)^\times$ with $L_\beta\cong R$. If $\lambda=(s,\bar r)$ is a generator of L_β we claim that s and $\bar r$ are units, which implies that $\beta=\bar s f(\bar r)^{-1}$ and finishes the proof. If $s'\in S$ maps to $\beta\in S/I$ then $(s',1)\in L_\beta$; since $(s',1)=(xs,x\bar r)$ for some $x\in R$ this implies that $\bar r\in(R/I)^\times$. If $t\in S$ maps to $f(\bar r)^{-1}\beta^{-1}\in S/I$ then $st\equiv 1$ modulo I. Now $I\subset sR$ because $I\times 0\subset L_\beta$, so $st=1+sx$ for some $x\in R$. But then $s(t-x)=1$, so $s\in S^\times$ as claimed.

EXAMPLE 3.10.1. (Cusp). Let k be a field and let R be $k[x,y]/(x^3=y^2)$, the coordinate ring of the cusp in the plane. Setting $x=t^2$, $y=t^3$ makes R isomorphic to the subring $k[t^2,t^3]$ of $S=k[t]$. The conductor ideal from S to R is $I=t^2S$, so we get a conductor square with $R/I=k$ and $S/I=k[t]/(t^2)$. Now $\mathrm{Pic}(k[t])=0$ and $(S/I)^\times\cong k^\times\times k$ with $\alpha\in k$ corresponding to $(1+\alpha t)\in(S/I)^\times$. Hence $\mathrm{Pic}(R)\cong k$. A little algebra shows that a nonzero $\alpha\in k$ corresponds to the invertible prime ideal $\mathfrak p=(1-\alpha^2x,x-\alpha y)R$ corresponding to the point $(x,y)=(\alpha^{-2},\alpha^{-3})$ on the cusp.

EXAMPLE 3.10.2. (Node). Let R be $k[x,y]/(y^2=x^2+x^3)$, the coordinate ring of the node in the plane over a field k with $\mathrm{char}(k)\neq 2$. Setting $x=t^2-1$ and $y=tx$ makes R isomorphic to a subring of $S=k[t]$ with conductor ideal $I=xS$. We get a conductor square with $R/I=k$ and $S/I\cong k\times k$. Since $(S/I)^\times\cong k^\times\times k^\times$ we see that $\mathrm{Pic}(R)\cong k^\times$. A little algebra shows that $\alpha\in k^\times$ corresponds to the invertible prime ideal $\mathfrak p$ corresponding to the point $(x,y)=\left(\frac{4\alpha}{(\alpha-1)^2},\frac{4\alpha(\alpha+1)}{(\alpha-1)^3}\right)$ on the node corresponding to $t=\left(\frac{1+\alpha}{1-\alpha}\right)$.

Seminormal rings

A reduced commutative ring R is called *seminormal* if whenever $x,y\in R$ satisfy $x^3=y^2$ there is an $s\in R$ with $s^2=x$, $s^3=y$. If R is an integral domain, there is an equivalent definition: R is seminormal if every s in the field of fractions satisfying $s^2,s^3\in R$ belongs to R. Normal domains are clearly seminormal; the node (3.10.2) is not normal ($t^2=1+x$), but it is seminormal (see Ex. 3.13). Arbitrary products of seminormal rings are also seminormal, because s may be found slotwise. The cusp (3.10.1) is the universal example of a reduced ring which is not seminormal.

Our interest in seminormal rings lies in the following theorem, first proven by C. Traverso for geometric rings and extended by several authors. For normal domains, it follows from Remark 3.6.1 above. Our formulation is taken from [Swan80].

THEOREM 3.11. *(Traverso). The following are equivalent for a commutative ring R:*

(1) R_{red} *is seminormal;*
(2) $\mathrm{Pic}(R) = \mathrm{Pic}(R[t])$;
(3) $\mathrm{Pic}(R) = \mathrm{Pic}(R[t_1, ..., t_n])$ *for all n.*

REMARK 3.11.1. If R is seminormal, it follows that $R[t]$ is also seminormal. By Ex. 3.11, $R[t, t^{-1}]$ and the local rings $R_{\mathfrak{p}}$ are also seminormal. However, the $\mathrm{Pic}[t, t^{-1}]$ analogue of Theorem 3.11 fails. For example, if R is the node (3.10.2) then $\mathrm{Pic}(R[t, t^{-1}]) \cong \mathrm{Pic}(R) \times \mathbb{Z}$. For more details, see [Weib91].

To prove Traverso's theorem, we shall need the following standard result about units of polynomial rings.

LEMMA 3.12. *Let R be a commutative ring with nilradical* $\mathfrak{N}$. *If* $r_0 + r_1 t + \cdots + r_n t^n$ *is a unit of* $R[t]$ *then* $r_0 \in R^\times$ *and* $r_1, ..., r_n$ *are nilpotent. Consequently, if* $NU(R)$ *denotes the subgroup* $1 + t\mathfrak{N}[t]$ *of* $R[t]^\times$ *then:*

(1) $R[t]^\times = R^\times \times NU(R)$;
(2) *If R is reduced then* $R^\times = R[t]^\times$;
(3) *Suppose that R is an algebra over a field k. If char(k)* $= p$, $NU(R)$ *is a p-group. If char(k)* $= 0$, $NU(R)$ *is a uniquely divisible abelian group (= a* $\mathbb{Q}$*-module).*

PROOF OF TRAVERSO'S THEOREM. We refer the reader to Swan's paper for the proof that (1) implies (2) and (3). By Lemma 3.9, we may suppose that R is reduced but not seminormal. Choose $x, y \in R$ with $x^3 = y^2$ such that no $s \in R$ satisfies $s^2 = x, s^3 = y$. Then the reduced ring $S = R[s]/(s^2 - x, s^3 - y)_{red}$ is strictly larger than R. Since $I = xS$ is an ideal of both R and S, we have Milnor squares

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R[t] & \xrightarrow{f} & S[t] \\ \downarrow & & \downarrow \\ R/I[t] & \xrightarrow{\bar{f}} & S/I[t]. \end{array}$$

The Units-Pic sequence 3.10 of the first square is a direct summand of the Units-Pic sequence for the second square. Using Lemma 3.12, we obtain the exact quotient sequence

$$0 \to NU(R/I) \to NU(S/I) \xrightarrow{\partial} \frac{\mathrm{Pic}(R[t])}{\mathrm{Pic}(R)}.$$

By construction, $s \notin R$ and $\bar{s} \notin R/I$. Hence $\partial(1 + \bar{s}t)$ is a nonzero element of the quotient $\mathrm{Pic}(R[t])/\mathrm{Pic}(R)$. Therefore if R isn't seminormal we have $\mathrm{Pic}(R) \neq \mathrm{Pic}(R[t])$, which is the (3) $\Rightarrow$ (2) $\Rightarrow$ (1) half of Traverso's theorem.

EXERCISES

In these exercises, R is always a commutative ring.

3.1 Show that the following are equivalent for every R-module L:

(a) There is a R-module M such that $L \otimes M \cong R$.
(b) L is an algebraic line bundle.
(c) L is a finitely generated R-module and $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R.

Hint: Use Exercises 2.10 and 2.11.

3.2 Show that the tensor product $P \otimes_R Q$ of two line bundles may be described using "Open Patching" 2.5 as follows. Find $s_1, ..., s_r \in R$ forming a unimodular row, such that P (resp. Q) is obtained by patching the $R[\frac{1}{s_i}]$ by units g_{ij} (resp. h_{ij}) in $R[\frac{1}{s_i s_j}]^\times$. Then $P \otimes_R Q$ is obtained by patching the $R[\frac{1}{s_i}]$ using the units $f_{ij} = g_{ij} h_{ij}$).

3.3 Let P be a locally free R-module, obtained by patching free modules of rank n by $g_{ij} \in GL_n(R[\frac{1}{s_i s_j}])$. Show that $\det(P)$ is the line bundle obtained by patching free modules of rank 1 by the units $\det(g_{ij}) \in (R[\frac{1}{s_i s_j}])^\times$.

3.4 Let P and Q be f.g. projective modules of constant ranks m and n respectively. Show that there is a natural isomorphism $(\det P)^{\otimes n} \otimes (\det Q)^{\otimes m} \to \det(P \otimes Q)$. *Hint:* Send $(p_{11} \wedge \cdots \otimes \cdots \wedge p_{mn}) \otimes (q_{11} \wedge \cdots \otimes \cdots \wedge q_{mn})$ to $(p_{11} \wedge q_{11}) \wedge \cdots \wedge (p_{mn} \wedge q_{mn})$. Then show that this map is locally an isomorphism.

3.5 If an ideal $I \subseteq R$ is a projective R-module and $J \subseteq R$ is any other ideal, show that $I \otimes_R J$ is isomorphic to the ideal IJ of R.

3.6 *Excision for* Pic. If I is a commutative ring without unit, let $\mathrm{Pic}(I)$ denote the kernel of the canonical map $\mathrm{Pic}(\mathbb{Z} \oplus I) \to \mathrm{Pic}(\mathbb{Z})$. Write $I^\times$ for the group $GL_1(I)$ of Ex. 1.10. Show that if I is an ideal of R then there is an exact sequence:

$$1 \to I^\times \to R^\times \to (R/I)^\times \xrightarrow{\partial} \mathrm{Pic}(I) \to \mathrm{Pic}(R) \to \mathrm{Pic}(R/I).$$

3.7 (Roberts-Singh) This exercise generalizes Proposition 3.5. Let $R \subseteq S$ be an inclusion of commutative rings. An R-submodule I of S is called an *invertible R-ideal of S* if $IJ = R$ for some other R-submodule J of S.

(i) If $I \subseteq S$ is an invertible R-ideal of S, show that I is finitely generated over R, and that $IS = S$.

(ii) Show that the invertible R-ideals of S form an abelian group $\mathrm{Pic}(R, S)$ under multiplication.

(iii) Show that every invertible R-ideal of S is a line bundle over R. *Hint:* use Ex. 3.5 to compute its rank. Conversely, if I is a line bundle over R contained in S and $IS = S$, then I is an R-ideal.

(iv) Show that there is a natural exact sequence:

$$1 \to R^\times \to S^\times \xrightarrow{\mathrm{div}} \mathrm{Pic}(R, S) \to \mathrm{Pic}(R) \to \mathrm{Pic}(S).$$

3.8 *Relative Class groups.* Suppose that R is a Krull domain and that $R_S = S^{-1}R$ for some multiplicatively closed set S in R. Let $D(R, R_S)$ denote the free abelian group on the height 1 primes $\mathfrak{p}$ of R such that $\mathfrak{p} \cap S \neq \phi$. Since $D(R_S)$ is free on the remaining height 1 primes of R, $D(R) = D(R, R_S) \oplus D(R_S)$.

(a) Show that the group $\mathrm{Pic}(R, R_S)$ of Ex. 3.7 is a subgroup of $D(R, R_S)$, and that there is an exact sequence compatible with Ex. 3.7

$$1 \to R^\times \to R_S^\times \to D(R, R_S) \to Cl(R) \to Cl(R_S) \to 0.$$

(b) Suppose that sR is a prime ideal of R. Prove that $(R[\frac{1}{s}])^\times \cong R^\times \times \mathbb{Z}^n$ and that $Cl(R) \cong Cl(R[\frac{1}{s}])$.

(c) Suppose that every height 1 prime $\mathfrak{p}$ of R with $\mathfrak{p}\cap S \neq \phi$ is an invertible ideal. Show that $\mathrm{Pic}(R, R_S) = D(R, R_S)$ and that $\mathrm{Pic}(R) \to \mathrm{Pic}(R_S)$ is onto. (This always happens if R is a regular ring, or if the local rings R_M are unique factorization domains for every maximal ideal M of R with $M \cap S \neq \emptyset$.)

3.9 Suppose that we are given a Milnor square with $R \subseteq S$. If $\bar{s} \in (S/I)^\times$ is the image of a nonzerodivisor $s \in S$, show that $\partial(\bar{s}) \in \mathrm{Pic}(R)$ is the class of the ideal $(sS) \cap R$.

3.10 Let R be a 1-dimensional noetherian ring with finite normalization S, and let I be the conductor ideal from S to R. Show that for every maximal ideal $\mathfrak{p}$ of R, $\mathfrak{p}$ is a line bundle $\iff I \not\subseteq \mathfrak{p}$. Using Ex. 3.9, show that these $\mathfrak{p}$ generate $\mathrm{Pic}(R)$.

3.11 If R is seminormal, show that every localization $S^{-1}R$ is seminormal.

3.12 *Seminormality is a local property.* Show that the following are equivalent:
(a) R is seminormal;
(b) $R_\mathfrak{m}$ is seminormal for every maximal ideal $\mathfrak{m}$ of R;
(c) $R_\mathfrak{p}$ is seminormal for every prime ideal $\mathfrak{p}$ of R.

3.13 If R is a pullback of a diagram of seminormal rings, show that R is seminormal. This shows that the node (3.10.2) is seminormal.

3.14 *Normal rings.* A ring R is called *normal* if each local ring $R_\mathfrak{p}$ is an integrally closed domain. If R and R' are normal rings, so is the product $R \times R'$. Show that normal domains are normal rings, and that every reduced 0-dimensional ring is normal. Then show that every normal ring is seminormal.

3.15 *Seminormalization.* Show that every reduced commutative ring R has an extension $R \subseteq {}^+R$ with ${}^+R$ seminormal, satisfying the following universal property: if S is seminormal, then every ring map $R \to S$ has a unique extension ${}^+R \to S$. The extension ${}^+R$ is unique up to isomorphism, and is called the *seminormalization* of R. *Hint:* First show that it suffices to construct the seminormalization of a noetherian ring R whose normalization S is finite. In that case, construct the seminormalization as a subring of S, using the observation that if $x^3 = y^2$ for $x, y \in R$, there is an $s \in S$ with $s^2 = x$, $s^3 = y$.

3.16 An extension $R \subset R'$ is called *subintegral* if $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a bijection, and the residue fields $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ are isomorphic. Show that the seminormalization $R \subset {}^+R$ of the previous exercise is a subintegral extension.

3.17 Let R be a commutative ring with nilradical $\mathfrak{N}$.
(a) Show that the subgroup $1 + \mathfrak{N}[t, t^{-1}]$ of $R[t, t^{-1}]^\times$ is the product of the three groups $1 + \mathfrak{N}$, $N_tU(R) = 1 + t\mathfrak{N}[t]$, and $N_{t^{-1}}U(R) = 1 + t^{-1}\mathfrak{N}[t^{-1}]$.
(b) Show that there is a homomorphism $t\colon [\mathrm{Spec}(R), \mathbb{Z}] \to R[t, t^{-1}]^\times$ sending f to the unit t^f of $R[t, t^{-1}]$ which is t^n on the factor R_i of R where $f = n$. Here R_i is given by 2.2.4 and Ex. 2.4.
(c) Show that there is a natural decomposition

$$R[t, t^{-1}]^\times \cong R^\times \times N_tU(R) \times N_{t^{-1}}U(R) \times [\mathrm{Spec}(R), \mathbb{Z}],$$

or equivalently, that there is a split exact sequence:

$$1 \to R^\times \to R[t]^\times \times R[t^{-1}]^\times \to R[t, t^{-1}]^\times \to [\mathrm{Spec}(R), \mathbb{Z}] \to 0.$$

3.18 Show that the following sequence is exact:

$$1 \to \mathrm{Pic}(R) \to \mathrm{Pic}(R[t]) \times \mathrm{Pic}(R[t^{-1}]) \to \mathrm{Pic}(R[t,t^{-1}]).$$

Hint: If R is finitely generated, construct a diagram whose rows are Units-Pic sequences 3.10, and whose first column is the naturally split sequence of Ex. 3.17.

3.19 (NPic) Let $\mathrm{NPic}(R)$ denote the cokernel of the natural map $\mathrm{Pic}(R) \to \mathrm{Pic}(R[t])$. Show that $\mathrm{Pic}(R[t]) \cong \mathrm{Pic}(R) \times \mathrm{NPic}(R)$, and that $\mathrm{NPic}(R) = 0$ iff R_{red} is a seminormal ring. If R is an algebra over a field k, prove that:

(a) If $\mathrm{char}(k) = p > 0$ then $\mathrm{NPic}(R)$ is a p-group;

(b) If $\mathrm{char}(k) = 0$ then $\mathrm{NPic}(R)$ is a uniquely divisible abelian group.

To do this, first reduce to the case when R is finitely generated, and proceed by induction on $\dim(R)$ using conductor squares.

§4. Topological Vector Bundles and Chern Classes

Because so much of the theory of projective modules is based on analogy with the theory of topological vector bundles, it is instructive to review the main aspects of the structure of vector bundles. Details and further information may be found in [CC], [Atiyah] or [Huse]. We will work with vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

Let X be a topological space. A *family of vector spaces* over X is a topological space E, together with a continuous map $\eta: E \to X$ and a finite dimensional vector space structure (over $\mathbb{R}$ or $\mathbb{C}$) on each fiber $E_x = \eta^{-1}(x)$, $x \in X$. We require the vector space structure on E_x to be compatible with the topology on E. By a *homomorphism* from one family $\eta: E \to X$ to another family $\varphi: F \to X$ we mean a continuous map $f: E \to F$ with $\eta = \varphi f$, such that each induced map $f_x: E_x \to F_x$ is a linear map of vector spaces. There is an evident category of families of vector spaces over X and their homomorphisms.

For example, if V is an n-dimensional vector space, the projection from $T^n = X \times V$ to X forms a "constant" family of vector spaces. We call such a family, and any family isomorphic to it, a *trivial vector bundle* over X.

If $Y \subseteq X$, we write $E|Y$ for the restriction $\eta^{-1}(Y)$ of E to Y; the restriction $\eta|Y: E|Y \to Y$ of η makes $E|Y$ into a family of vector spaces over Y. More generally, if $f: Y \to X$ is any continuous map then we can construct an induced family $f^*(\xi): f^*E \to Y$ as follows. The space f^*E is the subspace of $Y \times E$ consisting of all pairs (y, e) such that $f(y) = \xi(e)$, and $f^*E \to Y$ is the restriction of the projection map. Since the fiber of f^*E at $y \in Y$ is $E_{f(y)}$, f^*E is a family of vector spaces over Y.

A *vector bundle* over X is a family of vector spaces $\eta: E \to X$ such that every point $x \in X$ has a neighborhood U such that $\eta|U: E|U \to U$ is trivial. A vector bundle is also called a *locally trivial* family of vector spaces.

The most historically important example of a vector bundle is the tangent bundle $TX \to X$ of a smooth manifold X. Another famous example is the *Möbius bundle* E over S^1; E is the open Möbius strip and $E_x \cong \mathbb{R}$ for each $x \in S^1$.

Suppose that $f: X \to Y$ is continuous. If $E \to X$ is a vector bundle, then the induced family $f^*E \to Y$ is a vector bundle on Y. To see this, note that if E is trivial over a neighborhood U of $f(y)$ then f^*E is trivial over $f^{-1}(U)$.

The symbol $\mathbf{VB}(X)$ denotes the category of vector bundles and homomorphisms. If emphasis on the field is needed, we write $\mathbf{VB}_{\mathbb{R}}(X)$ or $\mathbf{VB}_{\mathbb{C}}(X)$. The induced bundle construction gives rise to an additive functor f^* from $\mathbf{VB}(X)$ to $\mathbf{VB}(Y)$.

The *Whitney sum* $E \oplus F$ of two vector bundles $\eta: E \to X$ and $\varphi: F \to X$ is the family of all the vector spaces $E_x \oplus F_x$, topologized as a subspace of $E \times F$. Since E and F are locally trivial, so is $E \oplus F$; hence $E \oplus F$ is a vector bundle. By inspection, the Whitney sum is the product in the category $\mathbf{VB}(X)$. Since there is a natural notion of the sum $f+g$ of two homomorphisms $f, g: E \to F$, this makes $\mathbf{VB}(X)$ into an additive category with Whitney sum the direct sum operation.

A *sub-bundle* of a vector bundle $\eta: E \to X$ is a subspace F of E which is a vector bundle under the induced structure. That is, each fiber F_x is a vector subspace of E_x and the family $F \to X$ is locally trivial. The *quotient bundle* E/F is the union of all the vector spaces E_x/F_x, given the quotient topology. Since F is locally a Whitney direct summand in E, we see that E/F is locally trivial, hence a vector bundle. This gives a "short exact sequence" of vector bundles in $\mathbf{VB}(X)$:

$$0 \to F \to E \to E/F \to 0.$$

A vector bundle $E \to X$ is said to be of *finite type* if there is a finite covering $U_1, \dots, U_n$ of X such that each each $E|U_i$ is a trivial bundle. Every bundle over a compact space X must be of finite type; the same is true if X is a finite-dimensional connected CW complex [Huse, §3.5], or more generally if there is an integer n such that every open cover of X has a refinement $\mathcal{V}$ such that no point of X is contained in more that n elements of $\mathcal{V}$. We will see in Exercise 4.15 that the canonical line bundle on infinite dimensional projective space $\mathbb{P}^\infty$ is an example of a vector bundle which is *not* of finite type.

Riemannian Metrics. Let $E \to X$ be a real vector bundle. A *Riemannian metric* on E is a family of inner products $\beta_x: E_x \times E_x \to \mathbb{R}$, $x \in X$, which varies continuously with x (in the sense that β is a continuous function on the Whitney sum $E \oplus E$). The notion of *Hermitian metric* on a complex vector bundle is defined similarly. A fundamental result [Huse, 3.5.5 and 3.9.5] states that every vector bundle over a paracompact space X has a Riemannian (or Hermitian) metric.

Dimension of vector bundles

If E is a vector bundle over X then $\dim(E_x)$ is a locally constant function on X with values in $\mathbb{N} = \{0, 1, ...\}$. Hence $\dim(E)$ is a continuous function from X to the discrete topological space $\mathbb{N}$; it is the analogue of the rank of a projective module. An *n-dimensional vector bundle* is a bundle E such that $\dim(E) = n$ is constant; 1-dimensional vector bundles are frequently called *line bundles*. The Möbius bundle is an example of a nontrivial line bundle.

A vector bundle E is called *componentwise trivial* if we can write X as a disjoint union of components X_i in such a way that each $E|X_i$ is trivial. Isomorphism classes of componentwise trivial bundles are in 1-1 correspondence with the set $[X, \mathbb{N}]$ of all continuous maps from X to $\mathbb{N}$. To see this, note that any continuous map $f: X \to \mathbb{N}$ induces a decomposition of X into components $X_i = f^{-1}(i)$. Given such an f, let T^f denote the disjoint union

$$T^f = \coprod_{i \in \mathbb{N}} X_i \times F^i, \qquad F = \mathbb{R} \text{ or } \mathbb{C}.$$

The projection $T^f \to \amalg X_i = X$ makes T^f into a componentwise trivial vector bundle with $\dim(T^f) = f$. Conversely, if E is componentwise trivial, then $E \cong T^{\dim(E)}$. Note that $T^f \oplus T^g \cong T^{f+g}$. Thus if f is bounded then by choosing $g = n - f$ we can make T^f into a summand of the trivial bundle T^n.

The following result, which we cite from [Huse, 3.5.8 and 3.9.6], illustrates some of the similarities between $\mathbf{VB}(X)$ and the category of f.g. projective modules. It is proven using the Riemannian (or Hermitian) metric on E: $F_x^\perp$ is the subspace of E_x perpendicular to F_x.

Subbundle Theorem 4.1. *Let $E \to X$ be a vector bundle on a paracompact topological space X. Then:*

(1) *If F is a sub-bundle of E, there is a sub-bundle $F^\perp$ such that $E \cong F \oplus F^\perp$.*
(2) *E has finite type if and only if E is isomorphic to a sub-bundle of a trivial bundle. That is, if and only if there is another bundle F such that $E \oplus F$ is trivial.*

Corollary 4.1.1. *Suppose that X is compact, or that X is a finite-dimensional CW complex with finitely many components. Then every vector bundle over X is a Whitney direct summand of a trivial bundle.*

Example 4.1.2. If X is a smooth d-dimensional manifold, its tangent bundle $TX \to X$ is a d-dimensional real vector bundle. Embedding X in $\mathbb{R}^n$ allows us to form the *normal bundle* $NX \to X$; N_xX is the orthogonal complement of T_xX in $\mathbb{R}^n$. Clearly $TX \oplus NX$ is the trivial n-dimensional vector bundle $X \times \mathbb{R}^n \to X$ over X.

Example 4.1.3. Consider the canonical line bundle E_1 on projective n-space; a point x of $\mathbb{P}^n$ corresponds to a line L_x in $n+1$-space, and the fiber of E_1 at x is just L_x. In fact, E_1 is a subbundle of the trivial bundle T^{n+1}. Letting F_x be the n-dimensional hyperplane perpendicular to L_x, the family of vector spaces F forms a vector bundle such that $E_1 \oplus F = T^{n+1}$.

Example 4.1.4. (Global sections) A *global section* of a vector bundle $\eta\colon E \to X$ is a continuous map $s\colon X \to E$ such that $\eta s = 1_X$. It is *nowhere zero* if $s(x) \neq 0$ for all $x \in X$. Every global section s determines a map from the trivial line bundle T^1 to E; if s is nowhere zero then the image is a line subbundle of E. If X is paracompact the Subbundle Theorem determines a splitting $E \cong F \oplus T^1$.

Patching vector bundles

4.2 One technique for creating vector bundles uses transition functions. The idea is to patch together a collection of vector bundles which are defined on subspaces of X. A related technique is the clutching construction discussed in 4.7 below.

Let $\eta\colon E \to X$ be an n-dimensional vector bundle on X over the field F (F is $\mathbb{R}$ or $\mathbb{C}$). Since E is locally trivial, we can find an open covering $\{U_i\}$ of X, and isomorphisms $h_i : U_i \times F^n \cong E|U_i$. If $U_i \cap U_j \neq \phi$, the isomorphism

$$h_i^{-1}h_j : (U_i \cap U_j) \times F^n \cong \eta|U_i \cap U_j \cong (U_i \cap U_j) \times F^n$$

sends $(x, v) \in (U_i \cap U_j) \times F^n$ to $(x, g_{ij}(x)(v))$ for some $g_{ij}(x) \in GL_n(F)$.

Conversely, suppose we are given maps $g_{ij}: U_i \cap U_j \to GL_n(F)$ such that $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. On the disjoint union of the $U_i \times F^n$, form the equivalence relation $\sim$ which is generated by the relation that $(x, v) \in U_j \times F^n$ and $(x, g_{ij}(x)(v)) \in U_i \times F^n$ are equivalent for every $x \in U_i \cap U_j$. Let E denote the quotient space $(\coprod U_i \times F^n)/\sim$. It is not hard to see that there is an induced map $\eta: E \to X$ making E into a vector bundle over X.

We call E the vector bundle *obtained by patching* via the transition functions g_{ij}; this patching construction is the geometric motivation for open patching of projective modules in 2.5.

Construction 4.2.1. (Tensor product). Let E and F be vector bundles over X. There is a vector bundle $E \otimes F$ over X whose fiber over $x \in X$ is the vector space tensor product $E_x \otimes F_x$, and $\dim(E \otimes F) = \dim(E)\dim(F)$.

To construct $E \otimes F$, we first suppose that E and F are trivial bundles, *i.e.*, $E = X \times V$ and $F = X \times W$ for vector spaces V, W. In this case we let $E \otimes F$ be the trivial bundle $X \times (V \otimes W)$. In the general case, we proceed as follows. Restricting to a component of X on which $\dim(E)$ and $\dim(F)$ are constant, we may assume that E and F have constant ranks m and n respectively. Choose a covering $\{U_i\}$ and transition maps g_{ij}, g'_{ij} defining E and F by patching. Identifying $M_m(F) \otimes M_n(F)$ with $M_{mn}(F)$ gives a map $GL_m(F) \times GL_n(F) \to GL_{mn}(F)$, and the elements $g_{ij} \otimes g'_{ij}$ give transition maps from $U_i \cap U_j$ to $GL_{mn}(F)$. The last assertion comes from the classical vector space formula $\dim(E_x \otimes F_x) = \dim(E_x)\dim(F_x)$.

Construction 4.2.2. (Determinant bundle). For every n-dimensional vector bundle E, there is an associated "determinant" line bundle $\det(E) = \wedge^n E$ whose fibers are the 1-dimensional vector spaces $\wedge^n(E_x)$. In fact, $\det(E)$ is a line bundle obtained by patching, the transition functions for $\det(E)$ being the determinants $\det(g_{ij})$ of the transition functions g_{ij} of E. More generally, if E is any vector bundle then this construction may be performed componentwise to form a line bundle $\det(E) = \wedge^{\dim(E)} E$. As in §3, if L is a line bundle and $E = L \oplus T^f$, then $\det(E) = L$, so E uniquely determines L. Taking E trivial, this shows that line bundles cannot be stably trivial.

Orthogonal and unitary structure groups 4.2.3. We say that an n-dimensional vector bundle $E \to X$ has *structure group* O_n or U_n if the transition functions g_{ij} map $U_i \cap U_j$ to the subgroup O_n of $GL_n(\mathbb{R})$ or the subgroup U_n of $GL_n(\mathbb{C})$. If X is paracompact, this can always be arranged, because then E has a (Riemannian or Hermitian) metric. Indeed, it is easy to continuously modify the isomorphisms $h_i: U_i \times F^n \to E|U_i$ so that on each fiber the map $F^n \cong E_x$ is an isometry. But then the fiber isomorphisms $g_{ij}(x)$ are isometries, and so belong to O_n or U_n. Using the same continuous modification trick, any vector bundle isomorphism between vector bundles with a metric gives rise to a metric-preserving isomorphism. If X is paracompact, this implies that $\mathbf{VB}_n(X)$ is also the set of equivalence classes of vector bundles with structure group O_n or U_n.

The following pair of results forms the historical motivation for the Bass-Serre Cancellation Theorem 2.3. Their proofs may be found in [Huse, 8.1].

Real Cancellation Theorem 4.3. *Suppose X is a d-dimensional CW complex, and that $\eta: E \to X$ is an n-dimensional real vector bundle with $n > d$. Then*

(i) $E \cong E_0 \oplus T^{n-d}$ *for some d-dimensional vector bundle* E_0
(ii) If F *is another bundle and* $E \oplus T^k \cong F \oplus T^k$, *then* $E \cong F$.

COROLLARY 4.3.1. *Over a 1-dimensional CW complex, every real vector bundle* E *of rank* ≥ 1 *is isomorphic to* $L \oplus T^f$, *where* $L = \det(E)$ *and* $f = \dim(E) - 1$.

COMPLEX CANCELLATION THEOREM 4.4. *Suppose* X *is a d-dimensional CW complex, and that* $\eta: E \to X$ *is a complex vector bundle with* $\dim(E) \geq d/2$.
(i) $E \cong E_0 \oplus T^k$ *for some vector bundle* E_0 *of dimension* $\leq d/2$
(ii) If F *is another bundle and* $E \oplus T^k \cong F \oplus T^k$, *then* $E \cong F$.

COROLLARY 4.4.1. *Let* X *be a CW complex of dimension* ≤ 3. *Every complex vector bundle* E *of rank* ≥ 1 *is isomorphic to* $L \oplus T^f$, *where* $L = \det(E)$ *and* $f = \dim(E) - 1$.

Vector bundles are somewhat more tractable than projective modules, as the following result shows. Its proof may be found in [Huse, 3.4.7].

HOMOTOPY INVARIANCE THEOREM 4.5. *If* $f, g: Y \to X$ *are homotopic maps and* Y *is paracompact, then* $f^*E \cong g^*E$ *for every vector bundle* E *over* X.

COROLLARY 4.6. *If* X *and* Y *are homotopy equivalent paracompact spaces, there is a 1-1 correspondence between isomorphism classes of vector bundles on* X *and* Y.

APPLICATION 4.6.1. If Y is a contractible paracompact space then every vector bundle over Y is trivial.

CLUTCHING CONSTRUCTION 4.7. Here is an analogue for vector bundles of Milnor Patching 2.7 for projective modules. Suppose that X is a paracompact space, expressed as the union of two closed subspaces X_1 and X_2, with $X_1 \cap X_2 = A$. Given vector bundles $E_i \to X_i$ and an isomorphism $g: E_1|A \to E_2|A$, we form a vector bundle $E = E_1 \cup_g E_2$ over X as follows. As a topological space E is the quotient of the disjoint union $(E_1 \coprod E_2)$ by the equivalence relation indentifying $e_1 \in E_1|A$ with $g(e_1) \in E_2|A$. Clearly the natural projection $\eta: E \to X$ makes E a family of vector spaces, and $E|X_i \cong E_i$. Moreover, E is locally trivial over X (see [Atiyah, p.21]; paracompactness is needed to extend g off of A). The isomorphism $g: E_1|A \cong E_2|A$ is called the *clutching map* of the construction. As with Milnor patching, every vector bundle over X arises by this clutching construction. A new feature, however, is homotopy invariance: if f, g are homotopic clutching isomorphisms $E_1|A \cong E_2|A$, then $E_1 \cup_f E_2$ and $E_1 \cup_g E_2$ are isomorphic vector bundles over X.

PROPOSITION 4.8. *Let* SX *denote the suspension of a paracompact space* X. *A choice of basepoint for* X *yields a 1-1 correspondence between the set* $\mathbf{VB}_n(SX)$ *of isomorphism classes of n-dimensional (real, resp. complex) vector bundles over* SX *and the set of based homotopy classes of maps*

$$[X, O_n]_*, \qquad \text{resp.} \quad [X, U_n]_*,$$

from X *to the orthogonal group* O_n, *resp. to the unitary group* U_n.

SKETCH OF PROOF. SX is the union of two contractible cones C_1 and C_2 whose intersection is X. As every vector bundle on the cones C_i is trivial, every vector bundle on SX is obtained from an isomorphism of trivial bundles over X via the clutching construction. Such an isomorphism is given by a homotopy class of maps from X to GL_n, or equivalently to the deformation retract O_n, resp. U_n of GL_n. The indeterminacy in the resulting map from $[X, GL_n]$ to classes of vector bundles is eliminated by insisting that the basepoint of X map to $1 \in GL_n$.

Vector Bundles on Spheres

Proposition 4.8 allows us to use homotopy theory to determine the vector bundles on the sphere S^d, because S^d is the suspension of S^{d-1}. Hence n-dimensional (real, resp. complex) bundles on S^d are in 1-1 correspondence with elements of $\pi_{d-1}(O_n)$ and $\pi_{d-1}(U_n)$, respectively. For example, every line bundle over S^d is trivial if $d \geq 3$, because the appropriate homotopy groups of $O_1 \cong S^0$ and $U_1 \cong S^1$ vanish. The classical calculation of the homotopy groups of O_n and U_n (see [Huse, 7.12]) yields the following facts:

(4.9.1) On S^1, there are $|\pi_0(O_n)| = 2$ real vector bundles of dimension n for all $n \geq 1$. The nontrivial line bundle on S^1 is the Möbius bundle. The Whitney sum of the Möbius bundle with trivial bundles yields all the other nontrivial bundles. Since $|\pi_0(U_n)| = 1$ for all n, every complex vector bundle on S^1 is trivial.

(4.9.2) On S^2, the situation is more complicated. Since $\pi_1(O_1) = 0$ there are no nontrivial real line bundles on S^2. There are infinitely many 2-dimensional real vector bundles on S^2 (indexed by the degree d of their clutching functions), because $\pi_1(O_2) = \mathbb{Z}$. However, there is only one nontrivial n-dimensional real vector bundle for each $n \geq 3$, because $\pi_1(O_n) = \mathbb{Z}/2$. A real 2-dimensional bundle E is stably trivial (and $E \oplus T \cong T^3$) iff the degree d is even. The tangent bundle of S^2 has degree $d = 2$.

There are infinitely many complex line bundles L_d on S^2, indexed by the degree d (in $\pi_1(U_1) = \mathbb{Z}$) of their clutching function. The Complex Cancellation theorem (4.4) states that every other complex vector bundle on S^2 is isomorphic to a Whitney sum $L_d \oplus T^n$, and that all the $L_d \oplus T^n$ are distinct.

(4.9.3) Every vector bundle on S^3 is trivial. This is a consequence of the classical result that $\pi_2(G) = 0$ for every compact Lie group G, such as $G = O_n$ or U_n.

(4.9.4) As noted above, every line bundle on S^4 is trivial. S^4 carries infinitely many distinct n-dimensional vector bundles for $n \geq 5$ over $\mathbb{R}$, and for $n \geq 2$ over $\mathbb{C}$, because $\pi_3(O_n) = \mathbb{Z}$ for $n \geq 5$ and $\pi_3(U_n) = \mathbb{Z}$ for $n \geq 2$. In the intermediate range, we have $\pi_3(O_2) = 0$, $\pi_3(O_3) = \mathbb{Z}$ and $\pi_3(O_4) = \mathbb{Z} \oplus \mathbb{Z}$. Every 5-dimensional real bundle comes from a unique 3-dimensional bundle but every 4-dimensional real bundle on S^4 is stably isomorphic to infinitely many other distinct 4-dimensional vector bundles.

(4.9.5) For $d \geq 3$ there are no 2-dimensional real vector bundles on S^d, because the appropriate homotopy groups of $O_2 \cong S^1 \times \mathbb{Z}/2$ vanish. This vanishing phenomenon doesn't persist though; if $d \geq 5$ the 2-dimensional complex bundles, as well as the 3-dimensional real bundles on S^d, correspond to elements of $\pi_{d-1}(O_3) \cong \pi_{d-1}(U_2) \cong \pi_{d-1}(S^3)$. This is a finite group which is rarely trivial.

Classifying Vector Bundles

One feature present in the theory of vector bundles, yet absent in the theory of projective modules, is the classification of vector bundles using Grassmannians.

If V is any finite-dimensional vector space, the set $G_n(V)$ of all n-dimensional linear subspaces of V is a smooth manifold, called the *Grassmann manifold* of n-planes in V. If $V \subset W$, then $G_n(V)$ is naturally a submanifold of $G_n(W)$. The *infinite Grassmannian* G_n is the union of the $G_n(V)$ as V ranges over all finite-dimensional subspaces of a fixed infinite-dimensional vector space ($\mathbb{R}^\infty$ or $\mathbb{C}^\infty$); thus G_n is an infinite-dimensional CW complex. For example, if $n = 1$ then G_1 is either $\mathbb{RP}^\infty$ or $\mathbb{CP}^\infty$, depending on whether the vector spaces are over $\mathbb{R}$ or $\mathbb{C}$.

There is a canonical n-dimensional vector bundle $E_n(V)$ over each $G_n(V)$, whose fibre over each $x \in G_n(V)$ is the linear subspace of V corresponding to x. To topologize this family of vector spaces, and see that it is a vector bundle, we define $E_n(V)$ to be the sub-bundle of the trivial bundle $G_n(V) \times V \to G_n(V)$ having the prescribed fibers. For $n = 1$ this is just the canonical line bundle on projective space described in Example 4.1.3.

The union (as V varies) of the $E_n(V)$ yields an n-dimensional vector bundle $E_n \to G_n$, called the *n-dimensional classifying bundle* because of the following theorem (see [Huse, 3.7.2]).

Classification Theorem 4.10. *Let X be a paracompact space. Then the set $\mathbf{VB}_n(X)$ of isomorphism classes of n-dimensional vector bundles over X is in 1-1 correspondence with the set $[X, G_n]$ of homotopy classes of maps from X into G_n:*

$$\mathbf{VB}_n(X) \cong [X, G_n].$$

In more detail, every n-dimensional vector bundle $\eta: E \to X$ is isomorphic to $f^*(E_n)$ for some map $f: X \to G_n$, and E determines f up to homotopy.

Remark 4.10.1. (Classifying Spaces) The Classification Theorem 4.10 states that the contravariant functor $\mathbf{VB}_n$ is "representable" by the infinite Grassmannian G_n. Because X is paracompact we may assume (by 4.2.3) that all vector bundles have structure group O_n, resp. U_n. For this reason, the infinite Grassmannian G_n is called the *classifying space* of O_n, resp. U_n (depending on the choice of $\mathbb{R}$ or $\mathbb{C}$). It is the custom to write BO_n, resp. BU_n, for the Grassmannian G_n (or any space homotopy equivalent to it) over $\mathbb{R}$, resp. over $\mathbb{C}$.

In fact, there are homotopy equivalences $\Omega(BO_n) \simeq O_n$ and $\Omega(BU_n) \simeq U_n$. We can deduce this from 4.8 and 4.10: for any paracompact space X we have $[X, O_n]_* \cong \mathbf{VB}_n(SX) \cong [SX, BO_n] \cong [X, \Omega(BO_n)]_*$, and $[X, U_n]_* \cong [X, \Omega(BU_n)]_*$ similarly. Taking X to be O_n and $\Omega(BO_n)$, resp. U_n and $\Omega(BU_n)$, yields the homotopy equivalences.

It is well-known that there are canonical isomorphisms $[X, \mathbb{RP}^\infty] \cong H^1(X; \mathbb{Z}/2)$ and $[X, \mathbb{CP}^\infty] \cong H^2(X; \mathbb{Z})$ respectively. Therefore the case $n = 1$ may be reformulated as follows.

Classification Theorem for Line Bundles 4.11. *If X is paracompact, there are natural isomorphisms:*

$$w_1 : \mathbf{VB}_{1,\mathbb{R}}(X) = \{\textit{real line bundles on } X\} \cong H^1(X; \mathbb{Z}/2)$$

$$c_1 : \mathbf{VB}_{1,\mathbb{C}}(X) = \{\textit{complex line bundles on } X\} \cong H^2(X;\mathbb{Z}).$$

Remark 4.11.1. Since $H^1(X)$ and $H^2(X)$ are abelian groups, it follows that the set $\mathbf{VB}_1(X)$ of isomorphism classes of line bundles is an abelian group. We can understand this group structure in a more elementary way, as follows. The tensor product $E \otimes F$ of line bundles is again a line bundle by 4.2.1, and $\otimes$ is the product in the group $\mathbf{VB}_1(X)$. The inverse of E in this group is the dual bundle $\check{E}$ of Ex. 4.3, because $\check{E} \otimes E$ is a trivial line bundle (see Ex. 4.4).

Riemann Surfaces 4.11.2. Here is a complete classification of complex vector bundles on a Riemann surface X. Recall that a Riemann surface is a compact 2-dimensional oriented manifold; the orientation gives a canonical isomorphism $H^2(X;\mathbb{Z}) = \mathbb{Z}$. If $\mathcal{L}$ is a complex line bundle, the *degree* of $\mathcal{L}$ is that integer d such that $c_1(\mathcal{L}) = d$. By Theorem 4.11, there is a unique complex line bundle $\mathcal{O}(d)$ of each degree on X. By Corollary 4.4.1, every complex vector bundle of rank r on X is isomorphic to $\mathcal{O}(d) \oplus T^{r-1}$ for some d. Therefore complex vector bundles on a Riemann surface are completely classified by their rank and degree.

For example, the tangent bundle $\mathcal{T}_X$ of a Riemann surface X has the structure of a complex line bundle, because every Riemann surface has the structure of a 1-dimensional complex manifold. The Riemann-Roch Theorem states that $\mathcal{T}_X$ has degree $2-2g$, where g is the *genus* of X. (Riemann surfaces are completely classified by their genus $g \geq 0$, a Riemann surface of genus g being a surface with g "handles.")

In contrast, there are 2^{2g} distinct real line bundles on X, because $H^1(X;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{2g}$. The Real Cancellation Theorem 4.3 shows that every real vector bundle is the sum of a trivial bundle and a bundle of dimension ≤ 2, but there are infinitely many 2-dimensional bundles over X. For example, the complex line bundles $\mathcal{O}(d)$ all give distinct 2-dimensional real vector bundles on X; they are distinguished by an invariant called the *Euler class* (see [CC]).

Characteristic Classes

By Theorem 4.11, the determinant line bundle $\det(E)$ of a vector bundle E yields a cohomology class: if E is a real vector bundle, it is the first Stiefel-Whitney class $w_1(E)$ in $H^1(X;\mathbb{Z}/2)$; if E is a complex vector bundle, it is the first Chern class $c_1(E)$ in $H^2(X,\mathbb{Z})$. These classes fit into a more general theory of characteristic classes, which are constructed and described in the book [CC]. Here is an axiomatic description of these classes.

Axioms for Stiefel-Whitney classes 4.12. The *Stiefel–Whitney classes* of a real vector bundle E over X are elements $w_i(E) \in H^i(X;\mathbb{Z}/2)$, which satisfy the following axioms. By convention $w_0(E) = 1$.

(SW1) (Dimension) If $i > \dim(E)$ then $w_i(E) = 0$.

(SW2) (Naturality) If $f: Y \to X$ is continuous then $f^*: H^i(X;\mathbb{Z}/2) \to H^i(Y;\mathbb{Z}/2)$ sends $w_i(E)$ to $w_i(f^*E)$. If E and E' are isomorphic bundles then $w_i(E) = w_i(E')$.

(SW3) (Whitney sum formula) If E and F are bundles, then in the graded cohomology ring $H^*(X;\mathbb{Z}/2)$ we have:

$$w_n(E \oplus F) = \sum w_i(E) w_{n-i}(F) = w_n(E) + w_{n-1}(E) w_1(F) + \cdots + w_n(F).$$

(SW4) (Normalization) For the canonical line bundle E_1 over $\mathbb{RP}^\infty$, $w_1(E_1)$ is the unique nonzero element of $H^1(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

The axioms (SW2) and (SW4), together with the Classification Theorem 4.10, show that w_1 classifies real line bundles in the sense that it gives the isomorphism $\mathbf{VB}_1(X) \cong H^1(X;\mathbb{Z}/2)$ of Theorem 4.11. The fact that $w_1(E) = w_1(\det E)$ is a consequence of the "Splitting Principle" for vector bundles, and is left to the exercises.

Since trivial bundles are induced from the map $X \to \{*\}$, it follows from (SW1) and (SW2) that $w_i(T^n) = 0$ for every trivial bundle T^n (and $i \neq 0$). The same is true for componentwise trivial bundles; see Ex. 4.2. From (SW3) it follows that $w_i(E \oplus T^n) = w_i(E)$ for every bundle E and every trivial bundle T^n.

The *total Stiefel-Whitney class* $w(E)$ of E is defined to be the formal sum

$$w(E) = 1 + w_1(E) + \cdots + w_i(E) + ...$$

in the complete cohomology ring $\hat{H}^*(X;\mathbb{Z}/2) = \prod_i H^i(X;\mathbb{Z}/2)$, which consists of all formal infinite series $a_0+a_1+\dots$ with $a_i \in H^i(X;\mathbb{Z}/2)$. With this formalism, the Whitney sum formula becomes a product formula: $w(E \oplus F) = w(E)w(F)$. Now the collection U of all formal sums $1 + a_1 + \cdots$ in $\hat{H}^*(X;\mathbb{Z}/2)$ forms an abelian group under multiplication (the group of units of $\hat{H}^*(X;\mathbb{Z}/2)$ if X is connected). Therefore if $E \oplus F$ is trivial we can compute $w(F)$ via the formula $w(F) = w(E)^{-1}$.

For example, consider the canonical line bundle $E_1(\mathbb{R}^n)$ over $\mathbb{RP}^n$. By axiom (SW4) we have $w(E_1) = 1+x$ in the ring $H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{F}_2[x]/(x^{n+1})$. We saw in Example 4.1.3 that there is an n-dimensional vector bundle F with $F \oplus E_1 = T^{n+1}$. Using the Whitney Sum formula (SW3), we compute that $w(F) = 1+x+\cdots+x^n$. Thus $w_i(F) = x^i$ for $i \leq n$ and $w_i(F) = 0$ for $i > n$.

Stiefel-Whitney classes were named for E. Stiefel and H. Whitney, who discovered the w_i independently in 1935, and used them to study the tangent bundle of a smooth manifold.

Axioms for Chern classes 4.13. If E is a complex vector bundle over X, the *Chern classes* of E are certain elements $c_i(E) \in H^{2i}(X;\mathbb{Z})$, with $c_0(E) = 1$. They satisfy the following axioms. Note that the natural inclusion of $S^2 \cong \mathbb{CP}^1$ in $\mathbb{CP}^\infty$ induces a canonical isomorphism $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$.

(C1) (Dimension) If $i > \dim(E)$ then $c_i(E) = 0$

(C2) (Naturality) If $f\colon Y \to X$ is continuous then $f^*\colon H^{2i}(X;\mathbb{Z}) \to H^{2i}(Y;\mathbb{Z})$ sends $c_i(E)$ to $c_i(f^*E)$. If $E \cong E'$ then $c_i(E) = c_i(E')$.

(C3) (Whitney sum formula) If E and F are bundles then

$$c_n(E \oplus F) = \sum c_i(E)c_{n-i}(F) = c_n(E) + c_{n-1}(E)c_1(F) + \cdots + c_n(F).$$

(C4) (Normalization) For the canonical line bundle E_1 over $\mathbb{CP}^\infty$, $c_1(E_1)$ is the canonical generator x of $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}$.

Axioms (C2) and (C4) and the Classification Theorem 4.10 imply that the first Chern class c_1 classifies complex line bundles; it gives the isomorphism $\mathbf{VB}_1(X) \cong H^2(X;\mathbb{Z})$ of Theorem 4.11. The identity $c_1(E) = c_1(\det E)$ is left to the exercises.

The total *Chern class* $c(E)$ of E is defined to be the formal sum

$$c(E) = 1 + c_1(E) + \cdots + c_i(E) + \cdots .$$

in the complete cohomology ring $\hat{H}^*(X;\mathbb{Z}) = \prod_i H^i(X;\mathbb{Z})$. With this formalism, the Whitney sum formula becomes $c(E \oplus F) = c(E)c(F)$. As with Stiefel-Whitney classes, axioms (C1) and (C2) imply that for a trivial bundle T^n we have $c_i(T^n) = 0$ $(i \neq 0)$, and axiom (C3) implies that for all E

$$c_i(E \oplus T^n) = c_i(E).$$

For example, consider the canonical line bundle $E_1(\mathbb{C}^n)$ over $\mathbb{CP}^n$. By axiom (C4), $c(E_1) = 1 + x$ in the truncated polynomial ring $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$. We saw in Example 4.1.3 that there is a canonical n-dimensional vector bundle F with $F \oplus E_1 = T^{n+1}$. Using the Whitney Sum Formula (C3), we compute that $c(F) = \sum (-1)^i x^i$. Thus $c_i(F) = (-1)^i x^i$ for all $i \leq n$.

Chern classes are named for S.-S. Chern, who discovered them in 1946 while studying L. Pontrjagin's 1942 construction of cohomology classes $p_i(E) \in H^{4i}(X;\mathbb{Z})$ associated to a real vector bundle E. In fact, $p_i(E)$ is $(-1)^i c_{2i}(E \otimes \mathbb{C})$, where $E \otimes \mathbb{C}$ is the complexification of E (see Ex. 4.5). However, the Whitney sum formula for Pontrjagin classes only holds up to elements of order 2 in $H^{4n}(X;\mathbb{Z})$; see Ex. 4.13.

EXERCISES

4.1 Let $\eta\colon E \to X$ and $\varphi\colon F \to X$ be two vector bundles, and form the induced bundle $\eta^* F$ over E. Show that the Whitney sum $E \oplus F \to X$ is $\eta^* F$, considered as a bundle over X by the map $\eta^* F \to E \to X$.

4.2 Show that all of the uncountably many vector bundles on the discrete space $X = \mathbb{N}$ are componentwise trivial. Let $T^{\mathbb{N}} \to \mathbb{N}$ be the bundle with $\dim(T^{\mathbb{N}}_n) = n$ for all n. Show that every componentwise trivial vector bundle $T^f \to Y$ over every space Y is isomorphic to $f^* T^{\mathbb{N}}$. Use this to show that the Stiefel-Whitney and Chern classes vanish for componentwise trivial vector bundles.

4.3 If E and F are vector bundles over X, show that there are vector bundles $\mathrm{Hom}(E,F)$, $\check{E}$ and $\wedge^k E$ over X whose fibers are, respectively: $\mathrm{Hom}(E_x,F_x)$, the dual space $(\check{E}_x)$ and the exterior product $\wedge^k(E_x)$. Then show that there are natural isomorphisms $(E \oplus F)\check{} \cong \check{E} \oplus \check{F}$, $\check{E} \otimes F \cong \mathrm{Hom}(E,F)$, $\wedge^1 E \cong E$ and

$$\wedge^k(E \oplus F) \cong \wedge^k E \oplus (\wedge^{k-1} E \otimes F) \oplus \cdots \oplus (\wedge^i E \otimes \wedge^{k-i} F) \oplus \cdots \oplus \wedge^k F.$$

4.4 Show that the global sections of the bundle $\mathrm{Hom}(E,F)$ of Ex. 4.3 are in 1-1 correspondence with vector bundle maps $E \to F$. (Cf. 4.1.4.) If E is a line bundle, show that the vector bundle $\check{E} \otimes E \cong \mathrm{Hom}(E,E)$ is trivial.

4.5 *Complexification.* Let $E \to X$ be a real vector bundle. Show that there is a complex vector bundle $E_{\mathbb{C}} \to X$ with fibers $E_x \otimes_{\mathbb{R}} \mathbb{C}$ and that there is a natural isomorphism $(E \oplus F)_{\mathbb{C}} \cong (E_{\mathbb{C}}) \oplus (F_{\mathbb{C}})$. Then show that $E_{\mathbb{C}} \to X$, considered as a real vector bundle, is isomorphic to the Whitney sum $E \oplus E$.

4.6 *Complex conjugate bundle.* If $F \to X$ is a complex vector bundle, given by transition functions g_{ij}, let $\bar{F}$ denote the complex vector bundle obtained by using the complex conjugates $\bar{g}_{ij}$ for transition functions; $\bar{F}$ is called the *complex conjugate bundle* of F. Show that F and $\bar{F}$ are isomorphic as real vector bundles, and that the complexification $F_{\mathbb{C}} \to X$ of Ex. 4.5 is isomorphic to the Whitney sum $F \oplus \bar{F}$. If $F = E_{\mathbb{C}}$ for some real bundle E, show that $F \cong \bar{F}$. Finally, show that for every complex line bundle L on X we have $\bar{L} \cong \check{L}$.

4.7 Use the formula $\bar{L} \cong \check{L}$ of Ex. 4.6 to show that $c_1(\bar{E}) = -c_1(E)$ in $H^2(X;\mathbb{Z})$ for every complex vector bundle E.

4.8 *Global sections.* If $\eta: E \to X$ is a vector bundle, let $\Gamma(E)$ denote the set of all global sections of E (see 4.1.4). Show that $\Gamma(E)$ is a module over the ring $C^0(X)$ of continuous functions on X (taking values in $\mathbb{R}$ or $\mathbb{C}$). If E is an n-dimensional trivial bundle, show that $\Gamma(E)$ is a free $C^0(X)$-module of rank n.

Conclude that if X is paracompact then $\Gamma(E)$ is a locally free $C^0(X)$-module in the sense of 2.4, and that $\Gamma(E)$ is a finitely generated projective module if X is compact or if E is of finite type. This is the easy half of Swan's theorem; the rest is given in the next exercise.

4.9 *Swan's Theorem.* Let X be a compact space, and write R for $C^0(X)$. Show that the functor Γ of the previous exercise is a functor from $\mathbf{VB}(X)$ to the category $\mathbf{P}(R)$ of f.g. projective modules, and that the homomorphisms

$$\Gamma\colon \mathrm{Hom}_{\mathbf{VB}(X)}(E,F) \to \mathrm{Hom}_{\mathbf{P}(R)}(\Gamma(E),\Gamma(F)) \qquad (*)$$

are isomorphisms. This proves Swan's Theorem, that Γ is an equivalence of categories $\mathbf{VB}(X) \approx \mathbf{P}(C^0(X))$. *Hint:* First show that $(*)$ holds when E and F are trivial bundles, and then use Corollary 4.1.1.

4.10 *Projective and Flag bundles.* If $E \to X$ is a vector bundle, consider the subspace $E_0 = E - X$ of E, where X lies in E as the zero section. The units $\mathbb{R}^\times$ (or $\mathbb{C}^\times$) act fiberwise on E_0, and the quotient space $\mathbb{P}(E)$ obtained by dividing out by this action is called the *projective bundle* associated to E. If $p\colon \mathbb{P}(E) \to X$ is the projection, the fibers $p^{-1}(x)$ are projective spaces.

(a) Show that there is a line sub-bundle L of p^*E over $\mathbb{P}(E)$. Use the Subbundle Theorem to conclude that $p^*E \cong E' \oplus L$.

Now suppose that $E \to X$ is an n-dimensional vector bundle, and let $\mathbb{F}(E)$ be the *flag space* $f\colon \mathbb{F}(E) \to X$ obtained by iterating the construction

$$\cdots \to \mathbb{P}(E'') \to \mathbb{P}(E') \to \mathbb{P}(E) \to X.$$

(b) Show that the bundle $f^*E \to \mathbb{F}(E)$ is a direct sum $L_1 \oplus \cdots \oplus L_n$ of line bundles.

4.11 If E is a direct sum $L_1 \oplus \cdots \oplus L_n$ of line bundles, show that $\det(E) \cong L_1 \otimes \cdots \otimes L_n$. Then use the Whitney Sum formula to show that $w_1(E) = w_1(\det(E))$, resp. $c_1(E) = c_1(\det(E))$. Prove that every $w_i(E)$, resp. $c_i(E)$ is the i^{th} elementary symmetric function of the n cohomology classes $\{w_1(L_i)\}$, resp. $\{c_1(L_i)\}$.

4.12 *Splitting Principle.* Write $H^i(X)$ for $H^i(X;\mathbb{Z}/2)$ or $H^{2i}(X;\mathbb{Z})$, depending on whether our base field is $\mathbb{R}$ or $\mathbb{C}$, and let $p\colon \mathbb{F}(E) \to X$ be the flag bundle of a vector bundle E over X (see Ex. 4.10). Prove that $p^*\colon H^i(X) \to H^i(\mathbb{F}(E))$ is an injection. Then use Ex. 4.11 to show that the characteristic classes $w_i(E)$ or $c_i(E)$ in $H^i(X)$ may be calculated inside $H^i(\mathbb{F}(E))$. *Hint:* for a trivial bundle this follows easily from the Künneth formula.

4.13 *Pontrjagin classes.* In this exercise we assume the results of Ex. 4.6 on the conjugate bundle $\bar{F}$ of a complex bundle F. Use the Splitting Principle to show that $c_i(\bar{F}) = (-1)^i c_i(F)$. Then prove the following:

(i) The Pontrjagin classes $p_n(F)$ of F (considered as a real bundle) are

$$p_n(F) = c_n(F)^2 + 2\sum_{i=1}^{n-1}(-1)^i c_{n-i}(F)c_{n+i}(F) + (-1)^n 2c_{2n}(F).$$

(ii) If $F = E \otimes \mathbb{C}$ for some real bundle E, the odd Chern classes $c_1(F), c_3(F), \dots$ all have order 2 in $H^*(X;\mathbb{Z})$.

(iii) The Whitney sum formula for Pontrjagin classes holds modulo 2:

$$p_n(E \oplus E') - \sum p_i(E)p_{n-i}(E') \text{ has order 2 in } H^{4n}(X;\mathbb{Z}).$$

4.14 *Disk with double origin.* This exercise shows that the classification theorems 4.10 and 4.11 fail for locally compact spaces which aren't Hausdorff. Let D denote the closed unit disk in $\mathbb{R}^2$. The *disk with double origin* is the non-Hausdorff space X obtained from the disjoint union of two copies of D by identifying together the common subsets $D - \{0\}$. Show that $[X, BO_n] = [X, BU_n] = 0$ for all n, but that $\mathbf{VB}_{2,\mathbb{R}}(X) \cong \mathbb{Z}$, $\mathbf{VB}_{n,\mathbb{R}}(X) \cong \mathbb{Z}/2$ for $n \geq 3$ and $\mathbf{VB}_{n,\mathbb{C}}(X) \cong \mathbb{Z} \cong H^2(X;\mathbb{Z})$ for all $n \geq 1$.

4.15 Show that the canonical line bundles E_1 over $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ do not have finite type. *Hint:* Use characteristic classes and the Subbundle Theorem, or II.3.7.2.

4.16 Consider the suspension SX of a paracompact space X. Show that every vector bundle E over SX has finite type. *Hint:* If $\dim(E) = n$, use 4.8 and Ex. 1.11 to construct a bundle E' such that $E \oplus E' \cong T^{2n}$.

§5. Algebraic Vector Bundles

Modern Algebraic Geometry studies sheaves of modules over schemes. This generalizes modules over commutative rings, and has many features in common with the topological vector bundles that we considered in the last section. In this section we discuss the main aspects of the structure of algebraic vector bundles.

We will assume the reader has some rudimentary knowledge of the language of schemes, in order to get to the main points quickly. Here is a glossary of the basic concepts; details for most things may be found in [Hart], but the ultimate source is [EGA].

A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$; it is a *locally ringed space* if each $\mathcal{O}_X(U)$ is a commutative ring, and if for every $x \in X$ the stalk ring $\mathcal{O}_{X,x} = \varinjlim_{x\in U} \mathcal{O}_X(U)$ is a local ring. By definition, an *affine scheme* is a locally ringed space isomorphic to $(\mathrm{Spec}(R), \tilde{R})$ for some commutative ring R (where $\tilde{R}$ is the canonical structure sheaf), and a *scheme* is a ringed space $(X, \mathcal{O}_X)$ which can be covered by open sets U_i such that each $(U_i, \mathcal{O}_X|U_i)$ is an affine scheme.

An $\mathcal{O}_X$*-module* is a sheaf $\mathcal{F}$ on X such that (i) for each open $U \subseteq X$ the set $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and (ii) if $V \subset U$ then the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures. A morphism $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a sheaf map such that each $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. The category $\mathcal{O}_X$-**mod** of all $\mathcal{O}_X$-modules is an abelian category.

A *global section* of an $\mathcal{O}_X$-module $\mathcal{F}$ is an element e_i of $\mathcal{F}(X)$. We say that $\mathcal{F}$ is *generated by global sections* if there is a set $\{e_i\}_{i\in I}$ of global sections of $\mathcal{F}$ whose restrictions $e_i|_U$ generate $\mathcal{F}(U)$ as an $\mathcal{O}_X(U)$-module for every open $U \subseteq X$. We can reinterpret these definitions as follows. Giving a global section e of $\mathcal{F}$ is equivalent to giving a morphism $\mathcal{O}_X \to \mathcal{F}$ of $\mathcal{O}_X$-modules, and to say that $\mathcal{F}$ is generated by the global sections $\{e_i\}$ is equivalent to saying that the corresponding morphism $\oplus_{i\in I} \mathcal{O}_X \to \mathcal{F}$ is a surjection.

Free modules. We say that $\mathcal{F}$ is a *free* $\mathcal{O}_X$*-module* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. A set $\{e_i\} \subset \mathcal{F}(X)$ is called a *basis* of $\mathcal{F}$ if the restrictions $e_i|_U$ form a basis of each $\mathcal{F}(U)$, *i.e.*, if the e_i provide an explicit isomorphism $\oplus\mathcal{O}_X \cong \mathcal{F}$.

The rank of a free $\mathcal{O}_X$-module $\mathcal{F}$ is not well-defined over all ringed spaces. For example, if X is a 1-point space then $\mathcal{O}_X$ is just a ring R and an $\mathcal{O}_X$-module is just an R-module, so our remarks in §1 about the invariant basis property (IBP) apply. There is no difficulty in defining the rank of a free $\mathcal{O}_X$-module when $(X, \mathcal{O}_X)$ is a scheme, or a locally ringed space, or even more generally when any of the rings $\mathcal{O}_X(U)$ satisfy the IBP. We shall avoid these difficulties by assuming henceforth that $(X, \mathcal{O}_X)$ is a locally ringed space.

We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_U$-module. The *rank* of a locally free module $\mathcal{F}$ is defined at each point x of X: $\mathrm{rank}_x(\mathcal{F})$ is the rank of the free $\mathcal{O}_U$-module $\mathcal{F}|_U$, where U is a neighborhood of x on which $\mathcal{F}|_U$ is free. Since the function $x \mapsto \mathrm{rank}_x(\mathcal{F})$ is locally constant, $\mathrm{rank}(\mathcal{F})$ is a continuous function on X. In particular, if X is a connected space then every locally free module has constant rank.

DEFINITION 5.1 (VECTOR BUNDLES). A *vector bundle* over a ringed space X is a locally free $\mathcal{O}_X$-module whose rank is finite at every point. We will write $\mathbf{VB}(X)$ or $\mathbf{VB}(X, \mathcal{O}_X)$ for the category of vector bundles on $(X, \mathcal{O}_X)$; the morphisms in $\mathbf{VB}(X)$ are just morphisms of $\mathcal{O}_X$-modules. Since the direct sum of locally free modules is locally free, $\mathbf{VB}(X)$ is an additive category.

A *line bundle* $\mathcal{L}$ is a locally free module of constant rank 1. A line bundle is also called an *invertible sheaf* because as we shall see in 5.3 there is another sheaf $\mathcal{L}'$ such that $\mathcal{L} \otimes \mathcal{L}' = \mathcal{O}_X$.

These notions are the analogues for ringed spaces of f.g. projective modules and algebraic line bundles, as can be seen from the discussion in 2.4 and §3. However, the analogy breaks down if X is not locally ringed; in effect locally projective modules need not be locally free.

EXAMPLE 5.1.1. (Topological spaces). Fix a topological space X. Then $X_{top} = (X, \mathcal{O}_{top})$ is a locally ringed space, where $\mathcal{O}_{top}$ is the sheaf of ($\mathbb{R}$ or $\mathbb{C}$-valued) continuous functions on X: $\mathcal{O}_{top}(U) = C^0(U)$ for all $U \subseteq X$. The following constructions give an equivalence between the category $\mathbf{VB}(X_{top})$ of vector bundles over the ringed space X_{top} and the category $\mathbf{VB}(X)$ of (real or complex) topological vector bundles over X in the sense of §4. Thus our notation is consistent with the notation of §4.

If $\eta\colon E \to X$ is a topological vector bundle, then the sheaf $\mathcal{E}$ of continuous sections of E is defined by $\mathcal{E}(U) = \{s\colon U \to E : \eta s = 1_U\}$. By Ex. 4.8 we know that $\mathcal{E}$ is a locally free $\mathcal{O}_{top}$-module. Conversely, given a locally free $\mathcal{O}_{top}$-module $\mathcal{E}$, choose a cover $\{U_i\}$ and bases for the free $\mathcal{O}_{top}$-modules $\mathcal{E}|U_i$; the base-change isomorphisms over the $U_i \cap U_j$ are elements g_{ij} of $GL_n(C^0(U_i \cap U_j))$. Interpreting the g_{ij} as maps $U_i \cap U_j \to GL_n(\mathbb{C})$, they become transition functions for a topological vector bundle $E \to X$ in the sense of 4.2.

EXAMPLE 5.1.2 (AFFINE SCHEMES). Suppose $X = \operatorname{Spec}(R)$. Every R-module M yields an $\mathcal{O}_X$-module $\tilde{M}$, and $\tilde{R} = \mathcal{O}_X$. Hence every free $\mathcal{O}_X$-module arises as $\tilde{M}$ for a free R-module M. The $\mathcal{O}_X$-module $\mathcal{F} = \tilde{P}$ associated to a f.g. projective R-module P is locally free by 2.4, and the two rank functions agree: $\operatorname{rank}(P) = \operatorname{rank}(\mathcal{F})$. Conversely, if $\mathcal{F}$ is locally free $\mathcal{O}_X$-module, it can be made trivial on a covering by open sets of the form $U_i = D(s_i)$, *i.e.*, there are free modules M_i such that $\mathcal{F}|_{U_i} = \tilde{M}_i$. The isomorphisms between the restrictions of $\tilde{M}_i$ and $\tilde{M}_j$ to $U_i \cap U_j$ amount to open patching data defining a projective R-module P as in 2.5. In fact it is not hard to see that $\mathcal{F} \cong \tilde{P}$. Thus vector bundles on $\operatorname{Spec}(R)$ are in 1-1 correspondence with f.g. projective R-modules. And it is no accident that the notion of an algebraic line bundle over a ring R in §3 corresponds exactly to the notion of a line bundle over the ringed space $(\operatorname{Spec}(R), \tilde{R})$.

More is true: the categories $\mathbf{VB}(X)$ and $\mathbf{P}(R)$ are equivalent when $X = \operatorname{Spec}(R)$. To see this, recall that an $\mathcal{O}_X$-module is called *quasicoherent* if it is isomorphic to some $\tilde{M}$ ([Hart, II.5.4]). The above correspondence shows that every vector bundle is quasicoherent. It turns out that the category $\mathcal{O}_X$-$\mathbf{mod}_{qcoh}$ of quasicoherent $\mathcal{O}_X$-modules is equivalent to the category R-$\mathbf{mod}$ of all R-modules (see [Hart, II.5.5]). Since the subcategories $\mathbf{VB}(\operatorname{Spec} R)$ and $\mathbf{P}(R)$ correspond, they are equivalent.

DEFINITION (COHERENT MODULES). Suppose that X is any scheme. We say that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasicoherent* if X may be covered by affine opens $U_i = \mathrm{Spec}(R_i)$ such that each $\mathcal{F}|U_i$ is $\tilde{M}_i$ for an R_i-module M_i. (If X is affine, this agrees with the definition of quasicoherent in Example 5.1.2 by [Hart, II.5.4].) We say that $\mathcal{F}$ is *coherent* if moreover each M_i is a finitely presented R_i-module.

If X is affine then $\mathcal{F} = \tilde{M}$ is coherent iff M is a finitely presented R-module by [EGA, I(1.4.3)]. In particular, if R is noetherian then "coherent" is just a synonym for "finitely generated." If X is a noetherian scheme, our definition of coherent module agrees with [Hart] and [EGA]. For general schemes, our definition is slightly stronger than in [Hart], and slightly weaker than in [EGA, 0_I(5.3.1)]; $\mathcal{O}_X$ is always coherent in our sense, but not in the sense of [EGA].

The equivalent conditions for locally free modules in 2.4 translate into:

LEMMA 5.1.3. *For every scheme X and $\mathcal{O}_X$-module $\mathcal{F}$, the following conditions are equivalent:*

(1) *$\mathcal{F}$ is a vector bundle (i.e., is locally free of finite rank);*
(2) *$\mathcal{F}$ is quasicoherent and the stalks $\mathcal{F}_x$ are free $\mathcal{O}_{X,x}$-modules of finite rank;*
(3) *$\mathcal{F}$ is coherent and the stalks $\mathcal{F}_x$ are free $\mathcal{O}_{X,x}$-modules;*
(4) *For every affine open $U = \mathrm{Spec}(R)$ in X, $\mathcal{F}|_U$ is a f.g. projective R-module.*

EXAMPLE 5.1.4. (Analytic spaces). Analytic spaces form another family of locally ringed spaces. To define them, one proceeds as follows. On the topological space $\mathbb{C}^n$, the subsheaf $\mathcal{O}_{an}$ of $\mathcal{O}_{top}$ consisting of analytic functions makes $(\mathbb{C}^n, \mathcal{O}_{an})$ into a locally ringed space. A *basic analytic set* in an open subset U of $\mathbb{C}^n$ is the zero locus V of a finite number of holomorphic functions, made into a locally ringed space $(V, \mathcal{O}_{V,an})$ as follows. If $\mathcal{I}_V$ is the subsheaf of $\mathcal{O}_{U,an}$ consisting of functions vanishing on V, the quotient sheaf $\mathcal{O}_{V,an} = \mathcal{O}_{U,an}/\mathcal{I}_V$ is supported on V, and is a subsheaf of the sheaf $\mathcal{O}_{V,top}$. By definition, an *analytic space* $X_{an} = (X, \mathcal{O}_{an})$ is a ringed space which is locally isomorphic to a basic analytic set. A good reference for analytic spaces is [GA]; the original source is Serre's [GAGA].

Let X_{an} be an analytic space. For clarity, a vector bundle over X_{an} (in the sense of Definition 5.1) is sometimes called an *analytic vector bundle*. There is also a notion of coherence on an analytic space: an $\mathcal{O}_{an}$-module $\mathcal{F}$ is called *coherent* if it is locally finitely presented in the sense that in a neighborhood U of any point it is presented as a cokernel:

$$\mathcal{O}_{U,an}^n \to \mathcal{O}_{U,an}^m \to \mathcal{F}|_U \to 0.$$

One special class of analytic spaces is the class of *Stein spaces*. It is known that analytic vector bundles are the same as topological vector bundles over a Stein space. For example, any analytic subspace of $\mathbb{C}^n$ is a Stein space. See [GR].

Morphisms of ringed spaces

Here are two basic ways to construct new ringed spaces and morphisms:

(1) If $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras, $(X, \mathcal{A})$ is a ringed space;
(2) If $f\colon Y \to X$ is a continuous map and $(Y, \mathcal{O}_Y)$ is a ringed space, the direct image sheaf $f_*\mathcal{O}_Y$ is a sheaf of rings on X, so $(X, f_*\mathcal{O}_Y)$ is a ringed space.

A *morphism of ringed spaces* $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a continuous map $f\colon Y \to X$ together with a map $f^{\#}\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves of rings on X. In case (1) there is a morphism $i\colon (X, \mathcal{A}) \to (X, \mathcal{O}_X)$; in case (2) the morphism is $(Y, \mathcal{O}_Y) \to (X, f_*\mathcal{O}_Y)$; in general, every morphism factors as $(Y, \mathcal{O}_Y) \to (X, f_*\mathcal{O}_Y) \to (X, \mathcal{O}_X)$.

A morphism of ringed spaces $f\colon X \to Y$ between two locally ringed spaces is a *morphism of locally ringed spaces* if in addition for each point $y \in Y$ the map of stalk rings $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ sends the maximal ideal $\mathfrak{m}_{f(y)}$ into the maximal ideal $\mathfrak{m}_y$. A *morphism of schemes* is a morphism of locally ringed spaces $f\colon Y \to X$ between schemes.

If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then the direct image sheaf $f_*\mathcal{F}$ is an $f_*\mathcal{O}_Y$-module, and hence also an $\mathcal{O}_X$-module. Thus f_* is a functor from $\mathcal{O}_Y$-modules to $\mathcal{O}_X$-modules, making $\mathcal{O}_X$-**mod** covariant in X. If $\mathcal{F}$ is a vector bundle over Y then $f_*\mathcal{F}$ is a vector bundle over $(X, f_*\mathcal{O}_Y)$. However, $f_*\mathcal{F}$ will not be a vector bundle over $(X, \mathcal{O}_X)$ unless $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module of finite rank, which rarely occurs.

If $f\colon Y \to X$ is a *proper* morphism between noetherian schemes then Serre's "Theorem B" states that if $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module then the direct image $f_*\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. (See [EGA, III(3.2.2)] or [Hart, III.5.2 and II.5.19].)

Example 5.2.1 (Projective Schemes). When Y is a projective scheme over a field k, the structural map $\pi\colon Y \to \operatorname{Spec}(k)$ is proper. In this case the direct image $\pi_*\mathcal{F} = H^0(Y, \mathcal{F})$ is a finite-dimensional vector space over k. Indeed, every coherent k-module is finitely generated. Not surprisingly, $\dim_k H^0(Y, \mathcal{F})$ gives an important invariant for coherent modules (and vector bundles) over projective schemes.

The functor f_* has a left adjoint f^* (from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules):

$$\operatorname{Hom}_{\mathcal{O}_Y}(f^*\mathcal{E}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, f_*\mathcal{F})$$

for every $\mathcal{O}_X$-module $\mathcal{E}$ and $\mathcal{O}_Y$-module $\mathcal{F}$. The explicit construction is given in [Hart, II.5], and shows that f^* sends free $\mathcal{O}_X$-modules to free $\mathcal{O}_Y$-modules, with $f^*\mathcal{O}_X \cong \mathcal{O}_Y$. If $i\colon U \subset X$ is the inclusion of an open subset then $i^*\mathcal{E}$ is just $\mathcal{E}|U$; it follows that if $\mathcal{E}|U$ is free then $(f^*\mathcal{E})|f^{-1}(U)$ is free. Thus f^* sends locally free $\mathcal{O}_X$-modules to locally free $\mathcal{O}_Y$-modules, and yields a functor $f^*\colon \mathbf{VB}(X) \to \mathbf{VB}(Y)$, making $\mathbf{VB}(X)$ contravariant in the ringed space X.

Example 5.2.2. If R and S are commutative rings then ring maps $f^{\#}\colon R \to S$ are in 1-1 correspondence with morphisms $f\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ of ringed spaces. The direct image functor f_* corresponds to the forgetful functor from S-modules to R-modules, and the functor f^* corresponds to the functor $\otimes_R S$ from R-modules to S-modules.

Associated analytic and topological bundles 5.2.3. Suppose that X is a scheme of finite type over $\mathbb{C}$, such as a subvariety of $\mathbb{P}^n_{\mathbb{C}}$ or $\mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x_1, ..., x_n])$. The closed points $X(\mathbb{C})$ of X have the natural structure of an analytic space; in particular it is a locally compact topological space. Indeed, $X(\mathbb{C})$ is covered by open sets $U(\mathbb{C})$ homeomorphic to analytic subspaces of $\mathbb{A}^n(\mathbb{C})$, and $\mathbb{A}^n(\mathbb{C}) \cong \mathbb{C}^n$. Note that if X is a projective variety then $X(\mathbb{C})$ is compact, because it is a closed subspace of the compact space $\mathbb{P}^n(\mathbb{C}) \cong \mathbb{CP}^n$.

Considering $X(\mathbb{C})$ as topological and analytic ringed spaces as in Examples 5.1.1 and 5.1.4, the evident continuous map $\tau\colon X(\mathbb{C}) \to X$ induces morphisms of ringed spaces $X(\mathbb{C})_{top} \to X(\mathbb{C})_{an} \to X$. This yields functors from $\mathbf{VB}(X, \mathcal{O}_X)$ to $\mathbf{VB}(X(\mathbb{C})_{an})$, and from $\mathbf{VB}(X_{an})$ to $\mathbf{VB}(X(\mathbb{C})_{top}) \cong \mathbf{VB}_{\mathbb{C}}(X(\mathbb{C}))$. Thus every vector bundle $\mathcal{E}$ over the scheme X has an associated analytic vector bundle $\mathcal{E}_{an}$, as well as an associated complex vector bundle $\tau^*\mathcal{E}$ over $X(\mathbb{C})$. In particular, every vector bundle $\mathcal{E}$ on X has topological Chern classes $c_i(\mathcal{E}) = c_i(\tau^*\mathcal{E})$ in the group $H^{2i}(X(\mathbb{C}); \mathbb{Z})$.

The main theorem of [GAGA] is that if X is a projective algebraic variety over $\mathbb{C}$ then it has there is an equivalence between the categories of coherent modules over X and over X_{an}. In particular, the categories of vector bundles $\mathbf{VB}(X)$ and $\mathbf{VB}(X_{an})$ are equivalent.

A similar situation arises if X is a scheme of finite type over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the closed points of X with residue field $\mathbb{R}$; it too is a locally compact space. We consider $X(\mathbb{R})$ as a ringed space, using $\mathbb{R}$-valued functions as in Example 5.1.1. There is a morphism of ringed spaces $\tau\colon X(\mathbb{R}) \to X$, and the functor τ^* sends $\mathbf{VB}(X)$ to $\mathbf{VB}_{\mathbb{R}}(X(\mathbb{R}))$. That is, every vector bundle $\mathcal{F}$ over X has an associated real vector bundle $\tau^*\mathcal{F}$ over $X(\mathbb{R})$; in particular, every vector bundle $\mathcal{F}$ over X has Stiefel-Whitney classes $w_i(\mathcal{F}) = w_i(\tau^*\mathcal{F}) \in H^i(X(\mathbb{R}); \mathbb{Z}/2)$.

Patching and Operations 5.3. Just as we built up projective modules by patching in 2.5, we can obtain a locally free sheaf $\mathcal{F}$ by patching (or *glueing*) locally free sheaves $\mathcal{F}_i$ of $\mathcal{O}_{U_i}$-modules via isomorphisms g_{ij} between $\mathcal{F}_j|U_i \cap U_j$ and $\mathcal{F}_i|U_i \cap U_j$, as long as $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ for all i, j, k.

The patching process allows us to take any natural operation on free modules and extend it to locally free modules. For example, if $\mathcal{O}_X$ is commutative we can construct tensor products $\mathcal{F} \otimes \mathcal{G}$, Hom-modules $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, dual modules $\check{\mathcal{F}}$ and exterior products $\wedge^i\mathcal{F}$ using $P \otimes_R Q$, $\mathrm{Hom}_R(P, Q)$, $\check{P}$ and $\wedge^i P$. If $\mathcal{F}$ and $\mathcal{G}$ are vector bundles, then so are $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, $\check{\mathcal{F}}$ and $\wedge^i\mathcal{F}$. All of the natural isomorphisms such as $\check{\mathcal{F}} \otimes \mathcal{G} \cong \mathcal{H}om(\mathcal{F}, \mathcal{G})$ hold for locally free modules, because a sheaf map is an isomorphism if it is locally an isomorphism.

The Picard group and determinant bundles

If $(X, \mathcal{O}_X)$ is a commutative ringed space, the set $\mathrm{Pic}(X)$ of isomorphism classes of line bundles forms a group, called the *Picard group* of X. To see this, we modify the proof in §3: the dual $\check{\mathcal{L}}$ of a line bundle $\mathcal{L}$ is again a line bundle and $\check{\mathcal{L}} \otimes \mathcal{L} \cong \mathcal{O}_X$ because by Lemma 3.1 this is true locally. Note that if X is $\mathrm{Spec}(R)$, we recover the definition of §3: $\mathrm{Pic}(\mathrm{Spec}(R)) = \mathrm{Pic}(R)$.

If $\mathcal{F}$ is locally free of rank n, then $\det(\mathcal{F}) = \wedge^n(\mathcal{F})$ is a line bundle. Operating componentwise as in §3, every locally free $\mathcal{O}_X$-module $\mathcal{F}$ has an associated determinant line bundle $\det(\mathcal{F})$. The natural map $\det(\mathcal{F}) \otimes \det(\mathcal{G}) \to \det(\mathcal{F} \oplus \mathcal{G})$ is an isomorphism because this is true locally by the Sum Formula in §3 (see Ex. 5.4 for a generalization). Thus det is a useful invariant of a locally free $\mathcal{O}_X$-module. We will discuss $\mathrm{Pic}(X)$ in terms of divisors at the end of this section.

Projective schemes

If X is a projective variety, maps between vector bundles are most easily described using graded modules. Following [Hart, II.2], this trick works more generally if X is $\mathrm{Proj}(S)$ for a commutative graded ring $S = S_0 \oplus S_1 \oplus \cdots$. By definition, the scheme

$\mathrm{Proj}(S)$ is the union of the affine open sets $D_+(f) = \mathrm{Spec}\, S_{(f)}$, where $f \in S_n$ ($n \geq 1$) and $S_{(f)}$ is the degree 0 subring of the $\mathbb{Z}$-graded ring $S[\frac{1}{f}]$. To cover $\mathrm{Proj}(S)$, it suffices to use $D_+(f)$ for a family of f's generating the ideal $S_+ = S_1 \oplus S_2 \oplus \cdots$ of S. For example, *projective n-space* over R is $\mathbb{P}^n_R = \mathrm{Proj}(R[X_0, ..., X_n])$; it is covered by the $D_+(X_i)$ and if $x_j = X_j/X_i$ then $D_+(X_i) = \mathrm{Spec}(R[x_1, ..., x_n])$.

If $M = \oplus_{i\in\mathbb{Z}} M_i$ is a graded S-module, there is an associated $\mathcal{O}_X$-module $\tilde{M}$ on $X = \mathrm{Proj}(S)$. The restriction of $\tilde{M}$ to $D_+(f)$ is the sheaf associated to $M_{(f)}$, the $S_{(f)}$-module which constitutes the degree 0 submodule of $M[\frac{1}{f}]$; more details of the construction of $\tilde{M}$ are given in [Hart, II.5.11]. Clearly $\tilde{S} = \mathcal{O}_X$. The functor $M \mapsto \tilde{M}$ is exact, and has the property that $\tilde{M} = 0$ whenever $M_i = 0$ for large i.

Example 5.3.1 (Twisting Line Bundles). The most important example of this construction is when M is $S(n)$, the module S regraded so that the degree i part is S_{n+i}; the associated sheaf $\tilde{S}(n)$ is written as $\mathcal{O}_X(n)$. If $f \in S_1$ then $S(n)_{(f)} \cong S_{(f)}$, so if S is generated by S_1 as an S_0-algebra then $\mathcal{O}_X(n)$ is a line bundle on $X = \mathrm{Proj}(S)$; it is called the n^{th} *twisting line bundle*. If $\mathcal{F}$ is any $\mathcal{O}_X$-module, we write $\mathcal{F}(n)$ for $\mathcal{F} \otimes \mathcal{O}_X(n)$, and call it "$\mathcal{F}$ twisted n times."

We will usually assume that S is generated by S_1 as an S_0-algebra, so that the $\mathcal{O}_X(n)$ are line bundles. This hypothesis ensures that every quasicoherent $\mathcal{O}_X$-module has the form $\tilde{M}$ for some M ([Hart, II.5.15]). It also ensures that the canonical maps $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \to (M \otimes_S N)\tilde{}$ are isomorphisms, so if $\mathcal{F} = \tilde{M}$ then $\mathcal{F}(n)$ is the $\mathcal{O}_X$-module associated to $M(n) = M \otimes_S S(n)$. Since $S(m) \otimes S(n) \cong S(m+n)$ we have the formula

$$\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n).$$

Thus there is a homomorphism from $\mathbb{Z}$ to $\mathrm{Pic}(X)$ sending n to $\mathcal{O}_X(n)$. Operating componentwise, the same formula yields a homomorphism $[X, \mathbb{Z}] \to \mathrm{Pic}(X)$.

Here is another application of twisting line bundles. An element $x \in M_n$ gives rise to a graded map $S(-n) \to M$ and hence a sheaf map $\mathcal{O}_X(-n) \to \tilde{M}$. Taking the direct sum over a generating set for M, we see that for every quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ there is a surjection from a locally free module $\oplus \mathcal{O}_X(-n_i)$ onto $\mathcal{F}$. In contrast, there is a surjection from a free $\mathcal{O}_X$-module onto $\mathcal{F}$ iff $\mathcal{F}$ can be generated by global sections, which is not always the case.

If P is a graded f.g. projective S-module, the $\mathcal{O}_X$-module $\tilde{P}$ is a vector bundle over $\mathrm{Proj}(S)$. To see this, suppose the generators of P lie in degrees $n_1, ..., n_r$ and set $F = S(-n_1) \oplus \cdots \oplus S(-n_r)$. The kernel Q of the surjection $F \to P$ is a graded S-module, and that the projective lifting property implies that $P \oplus Q \cong F$. Hence $\tilde{P} \oplus \tilde{Q}$ is the direct sum $\tilde{F}$ of the line bundles $\mathcal{O}_X(-n_i)$, proving that $\tilde{P}$ is a vector bundle.

Example 5.4 (No projective vector bundles). Consider the projective line $\mathbb{P}^1_R = \mathrm{Proj}(S)$, $S = R[x, y]$. Associated to the "Koszul" exact sequence of graded S-modules

$$0 \to S(-2) \xrightarrow{(y,-x)} S(-1) \oplus S(-1) \xrightarrow{(x,y)} S \to R \to 0 \tag{5.4.1}$$

is the short exact sequence of vector bundles over $\mathbb{P}^1_R$:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to 0. \tag{5.4.2}$$

The sequence (5.4.2) cannot split, because there are no nonzero maps from $\mathcal{O}_{\mathbb{P}^1}$ to $\mathcal{O}_{\mathbb{P}^1}(-1)$ (see Ex. 5.2). This shows that the projective lifting property of §2 fails for the free module $\mathcal{O}_{\mathbb{P}^1}$. In fact, the projective lifting property fails for every vector bundle over $\mathbb{P}^1_R$; the category of $\mathcal{O}_{\mathbb{P}^1}$-modules has no "projective objects." This failure is the single biggest difference between the study of projective modules over rings and vector bundles over schemes.

The strict analogue of the Cancellation Theorem 2.3 does not hold for projective schemes. To see this, we cite the following result from [Atiy56]. A vector bundle is called *indecomposable* if it cannot be written as the sum of two proper sub-bundles. For example, every line bundle is indecomposable.

Krull-Schmidt Theorem 5.5 (Atiyah). *Let X be a projective scheme over a field k. Then the Krull-Schmidt theorem holds for vector bundles over X. That is, every vector bundle over X can be written uniquely (up to reordering) as a direct sum of indecomposable vector bundles.*

In particular, the direct sums of line bundles $\mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_r)$ are all distinct whenever $\dim(X) \neq 0$, *because then all the $\mathcal{O}_X(n_i)$ are distinct.*

Example 5.5.1. If X is a smooth projective curve over $\mathbb{C}$, then the associated topological space $X(\mathbb{C})$ is a Riemann surface. We saw in 4.11.2 that every topological line bundle on $X(\mathbb{C})$ is completely determined by its topological degree, and that every topological vector bundle is completely determined by its rank and degree. Now it is not hard to show that the twisting line bundle $\mathcal{O}_X(d)$ has degree d. Hence every topological vector bundle $\mathcal{E}$ of rank r and degree d is isomorphic to the direct sum $\mathcal{O}_X(d) \oplus T^{r-1}$. Moreover, the topological degree of a line bundle agrees with the usual algebraic degree one encounters in Algebraic Geometry.

The Krull-Schmidt Theorem shows that for each $r \geq 2$ and $d \in \mathbb{Z}$ there are infinitely many vector bundles over X with rank r and degree d. Indeed, there are infinitely many ways to choose integers $d_1, \dots, d_r$ so that $\sum d_i = d$, and these choices yield the vector bundles $\mathcal{O}_X(d_1) \oplus \cdots \oplus \mathcal{O}_X(d_r)$, which are all distinct with rank r and degree d.

For $X = \mathbb{P}^1_k$, the only indecomposable vector bundles are the line bundles $\mathcal{O}(n)$. This is a theorem of A. Grothendieck, proven in [Groth57]. Using the Krull-Schmidt Theorem, we obtain the following classification.

Theorem 5.6 (Classification of Vector Bundles over $\mathbb{P}^1_k$). *Let k be an algebraically closed field. Every vector bundle $\mathcal{F}$ over $X = \mathbb{P}^1_k$ is a direct sum of the line bundles $\mathcal{O}_X(n)$ in a unique way. That is, $\mathcal{F}$ determines a finite decreasing family of integers $n_1 \geq \cdots \geq n_r$ such that*

$$\mathcal{F} \cong \mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_r).$$

The classification over other spaces is much more complicated than it is for $\mathbb{P}^1$. The following example is taken from [Atiy57]. Atiyah's result holds over any algebraically closed field k, but we shall state it for $k = \mathbb{C}$ because we have not yet introduced the notion on the degree of a line bundle. (Using the Riemann-Roch theorem, we could define the degree of a line bundle $\mathcal{L}$ over an elliptic curve as the integer $\dim H^0(X, \mathcal{L}(n)) - n$ for $n >> 0$.)

Classification of vector bundles over elliptic curves 5.7. Let X be a smooth elliptic curve over $\mathbb{C}$. Every vector bundle $\mathcal{E}$ over X has two integer invariants: its rank, and its *degree*, which we saw in 5.5.1 is just the Chern class $c_1(E) \in H^2(X(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}$ of the associated topological vector bundle over the Riemann surface $X(\mathbb{C})$ of genus 1, defined in 5.2.3. Let $\mathbf{VB}^{ind}_{r,d}(X)$ denote the set of isomorphism classes of *indecomposable* vector bundles over X having rank r and degree d. Then for all $r \geq 1$ and $d \in \mathbb{Z}$:

(1) All the vector bundles in the set $\mathbf{VB}^{ind}_{r,d}(X)$ yield the same topological vector bundle E over $X(\mathbb{C})$. This follows from Example 5.5.1.
(2) There is a natural identification of each $\mathbf{VB}^{ind}_{r,d}(X)$ with the set $X(\mathbb{C})$; in particular, there are uncountably many indecomposable vector bundles of rank r and degree d.
(3) Tensoring with the twisting bundle $\mathcal{O}_X(d)$ induces a bijection between $\mathbf{VB}^{ind}_{r,0}(X)$ and $\mathbf{VB}^{ind}_{r,d}(X)$.
(4) The r^{th} exterior power $\wedge^r$ maps $\mathbf{VB}^{ind}_{r,d}(X)$ onto $\mathbf{VB}^{ind}_{1,d}(X)$. This map is a bijection iff r and d are relatively prime. If $(r,d) = h$ then for each line bundle $\mathcal{L}$ of degree d there are h^2 vector bundles $\mathcal{E}$ with rank r and determinant $\mathcal{L}$.

Construction 5.8 (Projective bundles). If $\mathcal{E}$ is an vector bundle over a scheme X, we can form a *projective space bundle* $\mathbb{P}(\mathcal{E})$, which is a scheme equipped with a map $\pi: \mathbb{P}(\mathcal{E}) \to X$ and a canonical line bundle $\mathcal{O}(1)$. To do this, we first construct $\mathbb{P}(\mathcal{E})$ when X is affine, and then glue the resulting schemes together.

If M is any module over a commutative ring R, the i^{th} *symmetric product* $Sym^i M$ is the quotient of the i-fold tensor product $M \otimes \cdots \otimes M$ by the permutation action of the symmetric group, identifying $m_1 \otimes \cdots \otimes m_i$ with $m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(i)}$ for every permutation σ. The obvious concatenation product $(Sym^i M) \otimes_R (Sym^j M) \to Sym^{i+j} M$ makes $Sym(M) = \oplus Sym^i(M)$ into a graded commutative R-algebra, called the *symmetric algebra* of M. As an example, note that if $M = R^n$ then $Sym(M)$ is the polynomial ring $R[x_1, ..., x_n]$. This construction is natural in R: if $R \to S$ is a ring homomorphism, then $Sym(M) \otimes_R S \cong Sym(M \otimes_R S)$.

If E is a f.g. projective R-module, let $\mathbb{P}(E)$ denote the scheme $\mathrm{Proj}(Sym(E))$. This scheme comes equipped with a map $\pi: \mathbb{P}(E) \to \mathrm{Spec}(R)$ and a canonical line bundle $\mathcal{O}(1)$; the scheme $\mathbb{P}(E)$ with this data is called the *projective space bundle* associated to E. If $E = R^n$, then $\mathbb{P}(E)$ is just the projective space $\mathbb{P}^{n-1}_R$. In general, the fact that E is locally free implies that $\mathrm{Spec}(R)$ is covered by open sets $D(s) = \mathrm{Spec}(R[\frac{1}{s}])$ on which E is free. If $E[\frac{1}{s}]$ is free of rank n then the restriction of $\mathbb{P}(E)$ to $D(s)$ is

$$\mathbb{P}(E[\frac{1}{s}]) \cong \mathrm{Proj}(R[\frac{1}{s}][x_1, ..., x_n]) = \mathbb{P}^{n-1}_{D(s)}.$$

Hence $\mathbb{P}(E)$ is locally just a projective space over $\mathrm{Spec}(R)$. The vector bundles $\mathcal{O}(1)$ and $\pi^*\tilde{E}$ on $\mathbb{P}(E)$ are the sheaves associated to the graded S-modules $S(1)$ and $E \otimes_R S$, where S is $Sym(E)$. The concatenation $E \otimes Sym^j(E) \to Sym^{1+j}(E)$ yields an exact sequence of graded modules,

$$0 \to E_1 \to E \otimes_R S \to S(1) \to R(-1) \to 0 \qquad (5.8.1)$$

hence a natural short exact sequence of $\mathbb{P}(E)$-modules

$$0 \to \mathcal{E}_1 \to \pi^* \tilde{E} \to \mathcal{O}(1) \to 0. \tag{5.8.2}$$

Since $\pi^* \tilde{E}$ and $\mathcal{O}(1)$ are locally free, $\mathcal{E}_1$ is locally free and $\operatorname{rank}(\mathcal{E}_1) = \operatorname{rank}(E) - 1$. For example, if $E = R^2$ then $\mathbb{P}(E)$ is $\mathbb{P}^1_R$ and $\mathcal{E}_1$ is $\mathcal{O}(-1)$ because (5.8.1) is the sequence (5.4.1) tensored with $S(1)$. That is, (5.8.2) is just:

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(+1) \to 0.$$

Having constructed $\mathbb{P}(E)$ over affine schemes, we now suppose that $\mathcal{E}$ is a vector bundle over any scheme X. We can cover X by affine open sets U and construct the projective bundles $\mathbb{P}(\mathcal{E}|U)$ over each U. By naturality of the construction of $\mathbb{P}(\mathcal{E}|U)$, the restrictions of $\mathbb{P}(\mathcal{E}|U)$ and $\mathbb{P}(\mathcal{E}|V)$ to $U \cap V$ may be identified with each other. Thus we can glue the $\mathbb{P}(\mathcal{E}|U)$ together to obtain a projective space bundle $\mathbb{P}(\mathcal{E})$ over X; a patching process similar to that in 5.3 yields a canonical line bundle $\mathcal{O}(1)$ over $\mathbb{P}(\mathcal{E})$.

By naturality of $E \otimes_R Sym(E) \to Sym(E)(1)$, we have a natural short exact sequence of vector bundles on $\mathbb{P}(\mathcal{E})$, which is locally the sequence (5.8.2):

$$0 \to \mathcal{E}_1 \to \pi^* \mathcal{E} \to \mathcal{O}(1) \to 0. \tag{5.8.3}$$

Let ρ denote the projective space bundle $\mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E})$ and let $\mathcal{E}_2$ denote the kernel of $\rho^*\mathcal{E}_1 \to \mathcal{O}(1)$. Then $(\pi\rho)^*\mathcal{E}$ has a filtration $\mathcal{E}_2 \subset \rho^*\mathcal{E}_1 \subset (\rho\pi)^*\mathcal{E}$ with filtration quotients $\mathcal{O}(1)$ and $\rho^*\mathcal{O}(1)$. This yields a projective space bundle $\mathbb{P}(\mathcal{E}_2) \to \mathbb{P}(\mathcal{E}_1)$. As long as $\mathcal{E}_i$ has rank ≥ 2 we can iterate this construction, forming a new projective space bundle $\mathbb{P}(\mathcal{E}_i)$ and a vector bundle $\mathcal{E}_{i+1}$. If $\operatorname{rank} \mathcal{E} = r$, $\mathcal{E}_{r-1}$ will be a line bundle. We write $\mathbb{F}(\mathcal{E})$ for $\mathbb{P}(\mathcal{E}_{r-2})$, and call it the *flag bundle* of $\mathcal{E}$. We may summarize the results of this construction as follows.

THEOREM 5.9 (SPLITTING PRINCIPLE). *Given a vector bundle $\mathcal{E}$ of rank r on a scheme X, there exists a morphism $f: \mathbb{F}(\mathcal{E}) \to X$ such that $f^*\mathcal{E}$ has a filtration*

$$f^*\mathcal{E} = \mathcal{E}'_0 \supset \mathcal{E}'_1 \supset \cdots \supset \mathcal{E}'_r = 0$$

by sub-vector bundles whose successive quotients $\mathcal{E}'_i/\mathcal{E}'_{i+1}$ are all line bundles.

Cohomological classification of vector bundles

The formation of vector bundles via the patching process in 5.3 may be codified into a classification of rank n vector bundles via a Čech cohomology set $\check{H}^1(X, GL_n(\mathcal{O}_X))$ which is associated to the sheaf of groups $\mathcal{G} = GL_n(\mathcal{O}_X)$. This cohomology set is defined more generally for any sheaf of groups $\mathcal{G}$ as follows. A *Čech 1-cocycle* for an open cover $\mathcal{U} = \{U_i\}$ of X is a family of elements g_{ij} in $\mathcal{G}(U_i \cap U_j)$ such that $g_{ij} = 1$ and $g_{ij}g_{jk} = g_{ik}$ for all i, j, k. Two 1-cocycles $\{g_{ij}\}$ and $\{h_{ij}\}$ are said to be *equivalent* if there are $f_i \in \mathcal{G}(U_i)$ such that $h_{ij} = f_i g_{ij} f_j^{-1}$. The equivalence classes of 1-cocycles form the set $\check{H}^1(\mathcal{U}, \mathcal{G})$. If $\mathcal{V}$ is a refinement of a cover $\mathcal{U}$, there is a set map from $\check{H}^1(\mathcal{U}, \mathcal{G})$ to $\check{H}^1(\mathcal{V}, \mathcal{G})$. The cohomology set $\check{H}^1(X, \mathcal{G})$ is defined to be the direct limit of the $\check{H}^1(\mathcal{U}, \mathcal{G})$ as $\mathcal{U}$ ranges over the system of all open covers of X.

We saw in 5.3 that every rank n vector bundle arises from patching, using a 1-cocycle for $\mathcal{G} = GL_n(\mathcal{O}_X)$. It isn't hard to see that equivalent cocycles give isomorphic vector bundles. From this, we deduce the following result.

Classification Theorem 5.10. *For every ringed space X, the set $\mathbf{VB}_n(X)$ of isomorphism classes of vector bundles of rank n over X is in 1-1 correspondence with the cohomology set $\check{H}^1(X, GL_n(\mathcal{O}_X))$:*

$$\mathbf{VB}_n(X) \cong \check{H}^1(X, GL_n(\mathcal{O}_X)).$$

When $\mathcal{G}$ is an abelian sheaf of groups, such as $\mathcal{O}_X^* = GL_1(\mathcal{O}_X)$, it is known that the Čech set $\check{H}^1(X, \mathcal{G})$ agrees with the usual sheaf cohomology group $H^1(X, \mathcal{G})$ (see Ex. III.4.4 of [Hart]). In particular, each $\check{H}^1(X, \mathcal{G})$ is an abelian group. A little work, detailed in [EGA, $0_I(5.6.3)$]) establishes:

Corollary 5.10.1. *For every locally ringed space X the isomorphism of Theorem 5.10 is a group isomorphism:*

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*).$$

As an application, suppose that X is the union of two open sets V_1 and V_2. Write $U(X)$ for the group $H^0(X, \mathcal{O}_X^*) = \mathcal{O}_X^*(X)$ of global units on X. The cohomology Mayer-Vietoris sequence translates to the following exact sequence.

$$\begin{aligned} 1 \to U(X) \to U(V_1) \times U(V_2) \to U(V_1 \cap V_2) \xrightarrow{\partial} \\ \xrightarrow{\partial} \mathrm{Pic}(X) \to \mathrm{Pic}(V_1) \times \mathrm{Pic}(V_2) \to \mathrm{Pic}(V_1 \cap V_2). \end{aligned} \tag{5.10.2}$$

To illustrate how this sequence works, consider the standard covering of $\mathbb{P}^1_R$ by $\mathrm{Spec}(R[t])$ and $\mathrm{Spec}(R[t^{-1}])$. Their intersection is $\mathrm{Spec}(R[t, t^{-1}])$. Comparing (5.10.2) with the sequences of Ex. 3.17 and Ex. 3.18 yields

Theorem 5.11. *For any commutative ring R,*

$$U(\mathbb{P}^1_R) = U(R) = R^* \text{ and } \quad \mathrm{Pic}(\mathbb{P}^1_R) \cong \mathrm{Pic}(R) \times [\mathrm{Spec}(R), \mathbb{Z}].$$

As in 5.3.1, the continuous function $n\colon \mathrm{Spec}(R) \to \mathbb{Z}$ corresponds to the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1_R$ obtained by patching $R[t]$ and $R[t^{-1}]$ together via $t^n \in R[t, t^{-1}]^*$.

Here is an application of Corollary 5.10.1 to nonreduced schemes. Suppose that $\mathcal{I}$ is a sheaf of nilpotent ideals, and let X_0 denote the ringed space $(X, \mathcal{O}_X/\mathcal{I})$. Writing $\mathcal{I}^*$ for the sheaf $GL_1(\mathcal{I})$ of Ex. 1.10, we have an exact sequence of sheaves of abelian groups:

$$1 \to \mathcal{I}^* \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 1.$$

The resulting long exact cohomology sequence starts with global units:

$$U(X) \to U(X_0) \to H^1(X, \mathcal{I}^*) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_0) \to H^2(X, \mathcal{I}^*) \cdots . \tag{5.11.1}$$

Thus $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$ may not be an isomorphism, as it is in the affine case (Lemma 3.9).

Invertible ideal sheaves

Suppose that X is an integral scheme, *i.e.*, that each $\mathcal{O}_X(U)$ is an integral domain. The function field $K(X)$ of X is the common quotient field of the integral domains $\mathcal{O}_X(U)$. Following the discussion in §3, we use $\mathcal{K}$ to denote the constant sheaf $U \mapsto K(X)$ and consider $\mathcal{O}_X$-submodules of $\mathcal{K}$. Those that lie in some $f\mathcal{O}_X$ we call *fractional*; a fractional ideal $\mathcal{I}$ is called *invertible* if $\mathcal{I}\mathcal{J} = \mathcal{O}_X$ for some $\mathcal{J}$. As in Proposition 3.5, invertible ideals are line bundles and $\mathcal{I} \otimes \mathcal{J} \cong \mathcal{I}\mathcal{J}$. The set $\mathrm{Cart}(X)$ of invertible ideals in $\mathcal{K}$ is therefore an abelian group.

Proposition 5.12. *If X is an integral scheme, there is an exact sequence*

$$1 \to U(X) \to K(X)^* \to \mathrm{Cart}(X) \to \mathrm{Pic}(X) \to 1. \tag{5.12.1}$$

Proof. The proof of 3.5 goes through to prove everything except that every line bundle $\mathcal{L}$ on X is isomorphic to an invertible ideal. On any affine open set U we have $(\mathcal{L}\otimes\mathcal{K})|U \cong \mathcal{K}|U$, a constant sheaf on U. This implies that $\mathcal{L}\otimes\mathcal{K} \cong \mathcal{K}$, because over an irreducible scheme like X any locally constant sheaf must be constant. Thus the natural inclusion of $\mathcal{L}$ in $\mathcal{L}\otimes\mathcal{K}$ expresses $\mathcal{L}$ as an $\mathcal{O}_X$-submodule of $\mathcal{K}$, and the rest of the proof of 3.5 goes through.

Here is another way to understand $\mathrm{Cart}(X)$. Let $\mathcal{K}^*$ denote the constant sheaf of units of $\mathcal{K}$; it contains the sheaf $\mathcal{O}_X^*$. Associated to the exact sequence

$$1 \to \mathcal{O}_X^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}_X^* \to 1$$

is a long exact cohomology sequence. Since X is irreducible and $\mathcal{K}^*$ is constant, we have $H^0(X,\mathcal{K}^*) = K(X)^*$ and $H^1(X,\mathcal{K}^*) = 0$. Since $U(X) = H^0(X,\mathcal{O}_X^*)$ we get the exact sequence

$$1 \to U(X) \to K(X)^* \to H^0(X,\mathcal{K}^*/\mathcal{O}_X^*) \to \mathrm{Pic}(X) \to 1. \tag{5.12.2}$$

Motivated by this sequence, we use the term *Cartier divisor* for a global section of the sheaf $\mathcal{K}^*/\mathcal{O}_X^*$. A Cartier divisor can be described by giving an open cover $\{U_i\}$ of X and $f_i \in K(X)^*$ such that f_i/f_j is in $\mathcal{O}_X^*(U_i \cap U_j)$ for each i and j.

Lemma 5.12.3. *Over every integral scheme X, there is a 1-1 correspondence between Cartier divisors on X and invertible ideal sheaves. Under this identification the sequences (5.12.1) and (5.12.2) are the same.*

Proof. If $\mathcal{I} \subset \mathcal{K}$ is an invertible ideal, there is a cover $\{U_i\}$ on which it is trivial, *i.e.*, $\mathcal{I}|U_i \cong \mathcal{O}_{U_i}$. Choosing $f_i \in \mathcal{I}(U_i) \subseteq K(X)$ generating $\mathcal{I}|U_i$ gives a Cartier divisor. This gives a set map $\mathrm{Cart}(X) \to H^0(X,\mathcal{K}^*/\mathcal{O}_X^*)$; it is easily seen to be a group homomorphism compatible with the map from $K(X)^*$, and with the map to $\mathrm{Pic}(X) = H^1(X,\mathcal{O}_X^*)$. This gives a map between the sequences (5.12.1) and (5.12.2); the 5-lemma implies that $\mathrm{Cart}(X) \cong H^0(X,\mathcal{K}^*/\mathcal{O}_X^*)$.

Variation 5.12.4. Let D be a Cartier divisor, represented by $\{(U_i,f_i)\}$. Historically, the invertible ideal sheaf associated to D is the subsheaf $\mathcal{L}(D)$ of $\mathcal{K}$ defined by letting $\mathcal{L}(D)|U_i$ be the submodule of $K(X)$ generated by f_i^{-1}. Since f_i/f_j is a unit on $U_i \cap U_j$, these patch to yield an invertible ideal. If $\mathcal{I}$ is invertible and D is the Cartier divisor attached to $\mathcal{I}$ by (5.12.3), then $\mathcal{L}(D)$ is $\mathcal{I}^{-1}$. Under the correspondence $D \leftrightarrow \mathcal{L}(D)$ the sequences (5.12.1) and (5.12.2) differ by a minus sign.

For example if $X = \mathbb{P}^1_R$, let D be the Cartier divisor given by t^n on $\mathrm{Spec}(R[t])$ and 1 on $\mathrm{Spec}(R[t^{-1}])$. The correspondence of Lemma 5.12.3 sends D to $\mathcal{O}(n)$, but $\mathcal{L}(D) \cong \mathcal{O}(-n)$.

Weil divisors

There is a notion of Weil divisor corresponding to that for rings (see 3.6). We say that a scheme X is *normal* if all the local rings $\mathcal{O}_{X,x}$ are normal domains (if X is affine this is the definition of Ex. 3.14), and *Krull* if it is integral, separated and has an affine cover $\{\mathrm{Spec}(R_i)\}$ with the R_i Krull domains. For example, if X is noetherian, integral and separated, then X is Krull iff it is normal.

A *prime divisor* on X is a closed integral subscheme Y of codimension 1; this is the analogue of a height 1 prime ideal. A *Weil divisor* is an element of the free abelian group $D(X)$ on the set of prime divisors of X; we call a Weil divisor $D = \sum n_i Y_i$ *effective* if all the $n_i \geq 0$.

Let $K(X)$ be the function field of X. Every prime divisor Y yields a discrete valuation on $K(X)$, because the local ring $\mathcal{O}_{X,y}$ at the generic point y of Y is a DVR. Conversely, each discrete valuation on $K(X)$ determines a unique prime divisor on X, because X is separated [Hart, Ex. II(4.5)]. Having made these observations, the discussion in §3 applies to yield group homomorphisms $\nu\colon K(X)^* \to D(X)$ and $\nu\colon \mathrm{Cart}(X) \to D(X)$. We define the *divisor class group* $Cl(X)$ to be the quotient of $D(X)$ by the subgroup of all Weil divisors $\nu(f)$, $f \in K(X)^*$. The proof of Proposition 3.6 establishes the following result.

PROPOSITION 5.13. *Let X be Krull. Then* $\mathrm{Pic}(X)$ *is a subgroup of the divisor class group $Cl(X)$, and there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
1 \to & U(X) & \to & K(X)^* & \to & \mathrm{Cart}(X) & \to & \mathrm{Pic}(X) & \to 1 \\
& \downarrow & & \downarrow & & \cap\, \nu & & \cap & \\
1 \to & U(X) & \to & K(X)^* & \to & D(X) & \to & Cl(X) & \to 1.
\end{array}$$

A scheme X is called *regular* (resp. *locally factorial*) if the local rings $\mathcal{O}_{X,x}$ are all regular local rings (resp. UFD's). By (3.8), regular schemes are locally factorial. Suppose that X is locally factorial and Krull. If $\mathcal{I}_Y$ is the ideal of a prime divisor Y and $U = \mathrm{Spec}(R)$ is an affine open subset of X, $\mathcal{I}_Y|U$ is invertible by Corollary 3.8.1. Since $\nu(\mathcal{I}_Y) = Y$, this proves that $\nu : \mathrm{Cart}(X) \to D(X)$ is onto. Inspecting the diagram of Proposition 5.13, we have:

PROPOSITION 5.14. *Let X be an integral, separated and locally factorial scheme. Then*

$$\mathrm{Cart}(X) \cong D(X) \quad \textit{and} \quad \mathrm{Pic}(X) \cong Cl(X).$$

EXAMPLE 5.14.1. ([Hart, II(6.4)]). If X is the projective space $\mathbb{P}^n_k$ over a field k, then $\mathrm{Pic}(\mathbb{P}^n_k) \cong Cl(\mathbb{P}^n_k) \cong \mathbb{Z}$. By Theorem 5.11, $\mathrm{Pic}(\mathbb{P}^n)$ is generated by $\mathcal{O}(1)$. The class group $Cl(\mathbb{P}^n)$ is generated by the class of a hyperplane H, whose corresponding ideal sheaf $\mathcal{I}_H$ is isomorphic to $\mathcal{O}(1)$. If Y is a hypersurface defined by a homogeneous polynomial of degree d, we say $\deg(Y) = d$; $Y \sim dH$ in $D(\mathbb{P}^n)$.

The *degree* of a Weil divisor $D = \sum n_i Y_i$ is defined to be $\sum n_i \deg(Y_i)$; the degree function $D(\mathbb{P}^n) \to \mathbb{Z}$ induces the isomorphism $Cl(\mathbb{P}^n) \cong \mathbb{Z}$. We remark that when $k = \mathbb{C}$ the degree of a Weil divisor agrees with the topological degree of the associated line bundle in $H^2(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}$, defined by the first Chern class as in Example 4.11.2.

Blowing Up 5.14.2. Let X be a smooth variety over an algebraically closed field, and let Y be a smooth subvariety of codimension ≥ 2. If the ideal sheaf of Y is $\mathcal{I}$, then $\mathcal{I}/\mathcal{I}^2$ is a vector bundle on Y. The *blowing up of X along Y* is a nonsingular variety $\tilde{X}$, containing a prime divisor $\tilde{Y} \cong \mathbb{P}(\mathcal{I}/\mathcal{I}^2)$, together with a map $\pi: \tilde{X} \to X$ such that $\pi^{-1}(Y) = \tilde{Y}$ and $\tilde{X} - \tilde{Y} \cong X - Y$ (see [Hart, II.7]). For example, the blowing up of a smooth surface X at a point x is a smooth surface $\tilde{X}$, and the smooth curve $\tilde{Y} \cong \mathbb{P}^1$ is called the *exceptional divisor*.

The maps $\pi^*: \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ and $\mathbb{Z} \to \mathrm{Pic}(\tilde{X})$ sending n to $n[\tilde{Y}]$ give rise to an isomorphism (see [Hart, Ex. II.8.5 or V.3.2]):

$$\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}.$$

Example 5.14.3. Consider the rational ruled surface S in $\mathbb{P}^1 \times \mathbb{P}^2$, defined by $X_iY_j = X_jY_i$ $(i,j = 1,2)$, and the smooth quadric surface Q in $\mathbb{P}^3$, defined by $xy = zw$. Now S is obtained by blowing up $\mathbb{P}^2_k$ at a point [Hart, V.2.11.5], while Q is obtained from $\mathbb{P}^2_k$ by first blowing up two points, and then blowing down the line between them [Hart, Ex. V.4.1]. Thus $\mathrm{Pic}(S) = Cl(S)$ and $\mathrm{Pic}(Q) = Cl(Q)$ are both isomorphic to $\mathbb{Z} \times \mathbb{Z}$. For both surfaces, divisors are classified by a pair (a, b) of integers (see [Hart, II.6.6.1]).

EXERCISES

5.1 Give an example of a ringed space $(X, \mathcal{O}_X)$ such that the rank of $\mathcal{O}_X(X)$ is well-defined, but such that the rank of $\mathcal{O}_X(U)$ is not well-defined for any proper open $U \subseteq X$.

5.2 Show that the global sections of the vector bundle $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ are in 1-1 correspondence with vector bundle maps $\mathcal{E} \to \mathcal{F}$. Conclude that there is a non-zero map $\mathcal{O}(m) \to \mathcal{O}(n)$ over $\mathbb{P}^1_R$ only if $m \leq n$.

5.3 *Projection Formula.* If $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, show that there is a natural isomorphism $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}$.

5.4 Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of locally free sheaves. Show that each $\wedge^n \mathcal{F}$ has a finite filtration

$$\wedge^n \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^{n+1} = 0$$

with successive quotients $F^i/F^{i+1} \cong (\wedge^i \mathcal{E}) \otimes (\wedge^{n-i} \mathcal{G})$. In particular, show that $\det(\mathcal{F}) \cong \det(\mathcal{E}) \otimes \det(\mathcal{G})$.

5.5 Let S be a graded ring generated by S_1 and set $X = \mathrm{Proj}(S)$. Show that $\mathcal{O}_X(n)\check{} \cong \mathcal{O}_X(-n)$ and $\mathcal{H}om(\mathcal{O}_X(m), \mathcal{O}_X(n)) \cong \mathcal{O}_X(n-m)$.

5.6 *Serre's "Theorem A."* Suppose that X is $\mathrm{Proj}(S)$ for a graded ring S which is finitely generated as an S_0-algebra by S_1. Recall from 5.3.1 (or [Hart, II.5.15]) that every quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is isomorphic to $\tilde{M}$ for some graded S-module M. In fact, we can take M_n to be $H^0(X, \mathcal{F}(n))$.

(a) If M is generated by M_0 and the M_i with $i < 0$, show that the sheaf $\tilde{M}$ is generated by global sections. *Hint:* consider $M_0 \oplus M_1 \oplus \cdots$.

(b) By (a), $\mathcal{O}_X(n)$ is generated by global sections if $n \geq 0$. Is the converse true?
(c) If M is a finitely generated S-module, show that $\tilde{M}(n)$ is generated by global sections for all large n (*i.e.*, for all $n \geq n_0$ for some n_0).
(d) If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, show that $\mathcal{F}(n)$ is generated by global sections for all large n. This result is known as Serre's "Theorem A," and it implies that $\mathcal{O}_X(1)$ is an *ample line bundle* in the sense of [EGA, II(4.5.5)].

5.7 Let X be a d-dimensional quasi-projective variety, *i.e.*, a locally closed integral subscheme of some $\mathbb{P}^n_k$, where k is an algebraically closed field.
(a) Suppose that $\mathcal{E}$ is a vector bundle generated by global sections. If $\operatorname{rank}(\mathcal{E}) > d$, Bertini's Theorem implies that $\mathcal{E}$ has a global section s such that $s_x \notin \mathfrak{m}_x\mathcal{E}_x$ for each $x \in X$. Establish the analogue of the Serre Cancellation Theorem 2.3(a), that there is a short exact sequence of vector bundles

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{E} \to \mathcal{F} \to 0.$$

(b) Now suppose that X is a curve. Show that every vector bundle $\mathcal{E}$ is a successive extension of invertible sheaves in the sense that there is a filtration of $\mathcal{E}$

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_r = 0.$$

by sub-bundles such that each $\mathcal{E}_i/\mathcal{E}_{i+1}$ is a line bundle. *Hint*: by Ex. 5.6(d), $\mathcal{E}(n)$ is generated by global sections for large n.

5.8 *Complex analytic spaces.* Recall from Example 5.1.4 that a *complex analytic space* is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to a basic analytic subset of $\mathbb{C}^n$.
(a) Use Example 5.2.3 to show that every analytic vector bundle on $\mathbb{C}^n$ is free, *i.e.*, $\mathcal{O}^r_{an}$ for some r. What about $\mathbb{C}^n - 0$?
(b) Let X be the complex affine node defined by the equation $y^2 = x^3 - x^2$. We saw in 3.10.2 that $\operatorname{Pic}(X) \cong \mathbb{C}^\times$. Use (4.9.1) to show that $\operatorname{Pic}(X(\mathbb{C})_{an}) = 0$.
(c) (Serre) Let X be the scheme $\operatorname{Spec}(\mathbb{C}[x,y]) - \{0\}$, 0 being the origin. Using the affine cover of X by $D(x)$ and $D(y)$, show that $\operatorname{Pic}(X) = 0$ but $\operatorname{Pic}(X_{an}) \neq 0$.

5.9 *Picard Variety.* Let X be a scheme over $\mathbb{C}$ and $X_{an} = X(\mathbb{C})_{an}$ the associated complex analytic space of Example 5.1.4. There is an exact sequence of sheaves of abelian groups on the topological space $X(\mathbb{C})$ underlying X_{an}:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_{X_{an}} \xrightarrow{\exp} \mathcal{O}^*_{X_{an}} \to 0, \tag{$*$}$$

where $\mathbb{Z}$ is the constant sheaf on $X(\mathbb{C})$.
(a) Show that the Chern class $c_1 \colon \operatorname{Pic}(X_{an}) \to H^2(X(\mathbb{C})_{top}; \mathbb{Z})$ of Example 5.2.3 is naturally isomorphic to the composite map

$$\operatorname{Pic}(X_{an}) \cong H^1(X_{an}, \mathcal{O}^*_{X_{an}}) \cong H^1(X(\mathbb{C})_{top}; \mathcal{O}^*_{X_{an}}) \xrightarrow{\partial} H^2(X(\mathbb{C})_{top}; \mathbb{Z})$$

coming from Corollary 5.10.1, the map $X_{an} \to X(\mathbb{C})_{top}$ of Example 5.1.4, and boundary map of ($*$).

Now suppose that X is projective. The image of $\operatorname{Pic}(X) \cong \operatorname{Pic}(X_{an})$ in $H^2(X(\mathbb{C}); \mathbb{Z})$ is called the *Néron-Severi group* $NS(X)$ and the kernel of $\operatorname{Pic}(X) \to NS(X)$

is written as $\mathrm{Pic}^0(X)$. Since $H^2(X(\mathbb{C});\mathbb{Z})$ is a finitely generated abelian group, so is $NS(X)$. It turns out that $H^1(X(\mathbb{C}),\mathcal{O}_{X_{an}}) \cong \mathbb{C}^n$ for some n, and that $H^1(X(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}^{2n}$ is a lattice in $H^1(X,\mathcal{O}_{X_{an}})$.

(b) Show that $\mathrm{Pic}^0(X)$ is isomorphic to $H^1(X,\mathcal{O}_X)/H^1(X(\mathbb{C});\mathbb{Z})$. Thus $\mathrm{Pic}^0(X)$ is a complex analytic torus; in fact it is the set of closed points of an abelian variety, called the *Picard variety* of X.

5.10 If E and F are f.g. projective R-modules, show that their projective bundles $\mathbb{P}(E)$ and $\mathbb{P}(F)$ are isomorphic as schemes over R if and only if $E \cong F \otimes_R L$ for some line bundle L on R.

5.11 Let X be a Krull scheme and Z an irreducible closed subset with complement U. Define a map $\rho\colon Cl(X) \to Cl(U)$ of class groups by sending the Weil divisor $\sum n_i Y_i$ to $\sum n_i(Y_i \cap U)$, ignoring terms n_iY_i for which $Y_i \cap U = \phi$. (Cf. Ex. 3.8.)

(a) If Z has codimension ≥ 2, show that $\rho\colon Cl(X) \cong Cl(U)$.

(b) If Z has codimension 1, show that there is an exact sequence

$$\mathbb{Z} \xrightarrow{[Z]} Cl(X) \xrightarrow{\rho} Cl(U) \to 0.$$

(c) If X is a smooth projective curve over a field k and $x \in X$ is a closed point, the complement $U = X - \{x\}$ is affine. Show that the map $\mathbb{Z} \xrightarrow{[x]} \mathrm{Pic}(X) = Cl(X)$ in part (b) is injective. If $k(x) = k$, conclude that $\mathrm{Pic}(X) \cong \mathrm{Pic}(U) \times \mathbb{Z}$. What happens if $k(x) \neq k$?

CHAPTER II

THE GROTHENDIECK GROUP K_0

There are several ways to construct the "Grothendieck group" of a mathematical object. We begin with the group completion version, because it has been the most historically important. After giving the applications to rings and topological spaces, we discuss λ-operations in §4. In sections 6 and 7 we describe the Grothendieck group of an "exact category," and apply it to the K-theory of schemes in §8. This construction is generalized to the Grothendieck group of a "Waldhausen category" in §9.

§1. The Group Completion of a monoid

Both $K_0(R)$ and $K^0(X)$ are formed by taking the group completion of an abelian monoid—the monoid $\mathbf{P}(R)$ of f.g. projective R-modules and the monoid $\mathbf{VB}(X)$ of vector bundles over X, respectively. We begin with a description of this construction.

Recall that an *abelian monoid* is a set M together with an associative, commutative operation $+$ and an "additive" identity element 0. A monoid map $f\colon M \to N$ is a set map such that $f(0) = 0$ and $f(m+m') = f(m) + f(m')$. The most famous example of an abelian monoid is $\mathbb{N} = \{0, 1, 2, ...\}$, the natural numbers with additive identity zero. If A is an abelian group then not only is A an abelian monoid, but so is any additively closed subset of A containing 0.

The *group completion* of an abelian monoid M is an abelian group $M^{-1}M$, together with a monoid map $[\ ]\colon M \to M^{-1}M$ which is universal in the sense that, for every abelian group A and every monoid map $\alpha\colon M \to A$, there is a unique abelian group homomorphism $\tilde{\alpha}\colon M^{-1}M \to A$ such that $\tilde{\alpha}([m]) = \alpha(m)$ for all $m \in M$.

For example, the group completion of $\mathbb{N}$ is $\mathbb{Z}$. If A is an abelian group then clearly $A^{-1}A = A$; if M is a submonoid of A (additively closed subset containing 0), then $M^{-1}M$ is the subgroup of A generated by M.

Every abelian monoid M has a group completion. One way to contruct it is to form the free abelian group $F(M)$ on symbols $[m]$, $m \in M$, and then factor out by the subgroup $R(M)$ generated by the relations $[m+n] - [m] - [n]$. By universality, if $M \to N$ is a monoid map, the map $M \to N \to N^{-1}N$ extends uniquely to a homomorphism from $M^{-1}M$ to $N^{-1}N$. Thus group completion is a functor from abelian monoids to abelian groups. A little decoding shows that in fact it is left adjoint to the forgetful functor, because of the natural isomorphism

$$\mathrm{Hom}_{\mathrm{monoids}}(M, A) \cong \mathrm{Hom}_{\mathrm{groups}}(M^{-1}M, A).$$

PROPOSITION 1.1. *Let M be an abelian monoid. Then:*

(a) *Every element of $M^{-1}M$ is of the form $[m]-[n]$ for some $m,n \in M$;*
(b) *If $m,n \in M$ then $[m]=[n]$ in $M^{-1}M$ iff $m+p=n+p$ for some $p \in M$;*
(c) *The monoid map $M\times M \to M^{-1}M$ sending (m,n) to $[m]-[n]$ is surjective.*
(d) *Hence $M^{-1}M$ is the set-theoretic quotient of $M \times M$ by the equivalence relation generated by $(m,n) \sim (m+p, n+p)$.*

PROOF. Every element of a free abelian group is a difference of sums of generators, and in $F(M)$ we have $([m_1]+[m_2]+\cdots) \equiv [m_1+m_2+\cdots]$ modulo $R(M)$. Hence every element of $M^{-1}M$ is a difference of generators. This establishes (a) and (c). For (b), suppose that $[m]-[n]=0$ in $M^{-1}M$. Then in the free abelian group $F(M)$ we have

$$[m]-[n]=\sum\left([a_i+b_i]-[a_i]-[b_i]\right)-\sum\left([c_j+d_j]-[c_j]-[d_j]\right).$$

Translating negative terms to the other side yields the following equation:

$$(*) \qquad [m]+\sum([a_i]+[b_i])+\sum[c_j+d_j]=[n]+\sum[a_i+b_i]+\sum([c_j]+[d_j]).$$

Now in a free abelian group two sums of generators $\sum[x_i]$ and $\sum[y_j]$ can only be equal if they have the same number of terms, and the generators differ by a permutation σ in the sense that $y_i = x_{\sigma(i)}$. Hence the generators on the left and right of $(*)$ differ only by a permutation. This means that in M the sum of the terms on the left and right of $(*)$ are the same, *i.e.*,

$$m+\sum(a_i+b_i)+\sum(c_j+d_j)=n+\sum(a_i+b_i)+\sum(c_j+d_j)$$

in M. This yields (b), and part (d) follows from (a) and (b).

The two corollaries below are immediate from Proposition 1.1, given the following definitions. A *cancellation monoid* is an abelian monoid M such that for all $m,n,p \in M$, $m+p=n+p$ implies $m=n$. A submonoid L of an abelian monoid M is called *cofinal* if for every $m \in M$ there is an $m' \in M$ so that $m+m' \in L$.

COROLLARY 1.2. *M injects into $M^{-1}M$ if and only if M is a cancellation monoid.*

COROLLARY 1.3. *If L is cofinal in an abelian monoid M, then:*

(a) *$L^{-1}L$ is a subgroup of $M^{-1}M$;*
(b) *Every element of $M^{-1}M$ is of the form $[m]-[\ell]$ for some $m \in M$, $\ell \in L$;*
(c) *If $[m]=[m']$ in $M^{-1}M$ then $m+\ell=m'+\ell$ for some $\ell \in L$.*

A *semiring* is an abelian monoid $(M,+)$, together with an associative product $\cdot$ which distributes over $+$, and a 2-sided multiplicative identity element 1. That is, a semiring satisfies all the axioms for a ring except for the existence of subtraction. The prototype semiring is $\mathbb{N}$.

The group completion $M^{-1}M$ (with respect to $+$) of a semiring M is a ring, the product on $M^{-1}M$ being extended from the product on M using 1.1. If $M \to N$ is a semiring map, then the induced map $M^{-1}M \to N^{-1}N$ is a ring homomorphism. Hence group completion is also a functor from semirings to rings, and from commutative semirings to commutative rings.

Example 1.4. Let X be a topological space. The set $[X,\mathbb{N}]$ of continuous maps $X\to\mathbb{N}$ is a semiring under pointwise $+$ and $\cdot$. The group completion of $[X,\mathbb{N}]$ is the ring $[X,\mathbb{Z}]$ of all continuous maps $X\to\mathbb{Z}$.

If X is (quasi-)compact, $[X,\mathbb{Z}]$ is a free abelian group. Indeed, $[X,\mathbb{Z}]$ is a subgroup of the group S of all bounded set functions from X to $\mathbb{Z}$, and S is a free abelian group (S is a "Specker group"; see [Fuchs]).

Example 1.5 (Burnside Ring). Let G be a finite group. The set M of (isomorphism classes of) finite G-sets is an abelian monoid under disjoint union, '0' being the empty set $\emptyset$. Suppose there are c distinct G-orbits. Since every G-set is a disjoint union of orbits, M is the free abelian monoid $\mathbb{N}^c$, a basis of M being the classes of the c distinct orbits of G. Each orbit is isomorphic to a coset G/H, where H is the stabilizer of an element, and $G/H\cong G/H'$ iff H and H' are conjugate subgroups of G, so c is the number of conjugacy classes of subgroups of G. Therefore the group completion $A(G)$ of M is the free abelian group $\mathbb{Z}^c$, a basis being the set of all c coset spaces $[G/H]$.

The direct product of two G-sets is again a G-set, so M is a semiring with '1' the 1-element G-set. Therefore $A(G)$ is a commutative ring; it is called the *Burnside ring* of G. The forgetful functor from G-sets to sets induces a map $M\to\mathbb{N}$ and hence an augmentation map $\varepsilon\colon A(G)\to\mathbb{Z}$. For example, if G is cyclic of prime order p, then $A(G)$ is the ring $\mathbb{Z}[x]/(x^2=px)$ and $x=[G]$ has $\varepsilon(x)=p$.

Example 1.6. (Representation ring). Let G be a finite group. The set $Rep_{\mathbb{C}}(G)$ of finite-dimensional representations $\rho\colon G\to GL_n\mathbb{C}$ (up to isomorphism) is an abelian monoid under $\oplus$. By Maschke's Theorem, $\mathbb{C}G$ is semisimple and $Rep_{\mathbb{C}}(G)\cong\mathbb{N}^r$, where r is the number of conjugacy classes of elements of G. Therefore the group completion $R(G)$ of $Rep_{\mathbb{C}}(G)$ is isomorphic to $\mathbb{Z}^r$ as an abelian group.

The tensor product $V\otimes_{\mathbb{C}}W$ of two representations is also a representation, so $Rep_{\mathbb{C}}(G)$ is a semiring (the element 1 is the 1-dimensional trivial representation). Therefore $R(G)$ is a commutative ring; it is called the *Representation ring* of G. For example, if G is cyclic of prime order p then $R(G)$ is isomorphic to the group ring $\mathbb{Z}[G]$, a subring of $\mathbb{Q}[G]=\mathbb{Q}\times\mathbb{Q}(\zeta)$, $\zeta^p=1$.

Every representation is determined by its character $\chi\colon G\to\mathbb{C}$, and irreducible representations have linearly independent characters. Therefore $R(G)$ is isomorphic to the ring of all complex characters $\chi\colon G\to\mathbb{C}$, a subring of $Map(G,\mathbb{C})$.

Definition. A (connected) *partially ordered abelian group* (A,P) is an abelian group A, together with a submonoid P of A which generates A (so $A=P^{-1}P$) and $P\cap(-P)=\{0\}$. This structure induces a translation-invariant partial ordering $\geq$ on A: $a\geq b$ if $a-b\in P$. Conversely, given a translation-invariant partial order on A, let P be $\{a\in A: a\geq 0\}$. If $a,b\geq 0$ then $a+b\geq a\geq 0$, so P is a submonoid of A. If P generates A then (A,P) is a partially ordered abelian group.

If M is an abelian monoid, $M^{-1}M$ need not be partially ordered (by the image of M), because we may have $[a]+[b]=0$ for $a,b\in M$. However, interesting examples are often partially ordered. For example, the Burnside ring $A(G)$ and Representation ring $R(G)$ are partially ordered (by G-sets and representations).

When it exists, the ordering on $M^{-1}M$ is an extra piece of structure. For example, $\mathbb{Z}^r$ is the group completion of both $\mathbb{N}^r$ and $M=\{0\}\cup\{(n_1,...,n_r)\in\mathbb{N}^r: n_1,...,n_r>0\}$. However, the two partially ordered structures on $\mathbb{Z}^r$ are different.

EXERCISES

1.1 Show that the group completion of a non-abelian monoid M is a group $\widehat{M}$, together with a monoid map $M \to \widehat{M}$ which is universal for maps from M to groups. Show that every monoid has a group completion in this sense, and that if M is abelian $\widehat{M} = M^{-1}M$. If M is the free monoid on a set X, show that the group completion of M is the free group on the set X.

1.2 If $M = M_1 \times M_2$, show that $M^{-1}M$ is the product group $(M_1^{-1}M_1) \times (M_2^{-1}M_2)$.

1.3 If M is the filtered colimit of abelian monoids M_α, show that $M^{-1}M$ is the filtered colimit of the abelian groups $M_\alpha^{-1}M_\alpha$.

1.4 *Mayer-Vietoris for group completions.* Suppose that a sequence $L \to M_1 \times M_2 \to N$ of abelian monoids is "exact" in the sense that whenever $m_1 \in M_1$ and $m_2 \in M_2$ agree in N then m_1 and m_2 are the image of a common $\ell \in L$. If L is cofinal in both M_1 and M_2, show that there is an exact sequence $L^{-1}L \to (M_1^{-1}M_1) \oplus (M_2^{-1}M_2) \to N^{-1}N$, where the first map is the diagonal inclusion and the second map is the difference map $(m_1, m_2) \mapsto \bar{m}_1 - \bar{m}_2$.

1.5 Classify all abelian monoids which are quotients of $\mathbb{N} = \{0, 1, \dots\}$ and show that they are all finite. How many quotient monoids $M = \mathbb{N}/\sim$ of $\mathbb{N}$ have m elements and group completion $\widehat{M} = \mathbb{Z}/n\mathbb{Z}$?

§2. K_0 of a ring

Let R be a ring. The set $\mathbf{P}(R)$ of isomorphism classes of f.g. projective R-modules, together with direct sum $\oplus$ and identity 0, forms an abelian monoid. The *Grothendieck group of* R, $K_0(R)$, is the group completion $\mathbf{P}^{-1}\mathbf{P}$ of $\mathbf{P}(R)$.

When R is commutative, $K_0(R)$ is a commutative ring with $1 = [R]$, because the monoid $\mathbf{P}(R)$ is a commutative semiring with product $\otimes_R$. This follows from the following facts: $\otimes$ distributes over $\oplus$; $P \otimes_R Q \cong Q \otimes_R P$ and $P \otimes_R R \cong P$; if P, Q are f.g. projective modules then so is $P \otimes_R Q$ (by Ex. I.2.7).

For example, let F be a field or division ring. Then the abelian monoid $\mathbf{P}(F)$ is isomorphic to $\mathbb{N} = \{0, 1, 2, \dots\}$, so $K_0(F) = \mathbb{Z}$. The same argument applies to show that $K_0(R) = \mathbb{Z}$ for every local ring R by (I.2.2), and also for every PID (by the Structure Theorem for modules over a PID). In particular, $K_0(\mathbb{Z}) = \mathbb{Z}$.

The *Eilenberg Swindle* I.2.8 shows why we restrict to finitely generated projectives. If we included R^∞, then the formula $P \oplus R^\infty \cong R^\infty$ would imply that $[P] = 0$ for every f.g. projective R-module, and therefore that $K_0(R) = 0$.

K_0 is a functor from rings to abelian groups, and from commutative rings to commutative rings. To see this, suppose that $R \to S$ is a ring homomorphism. The functor $\otimes_R S \colon \mathbf{P}(R) \to \mathbf{P}(S)$ (sending P to $P \otimes_R S$) yields a monoid map $\mathbf{P}(R) \to \mathbf{P}(S)$, hence a group homomorphism $K_0(R) \to K_0(S)$. If R, S are commutative rings then $\otimes_R S \colon K_0(R) \to K_0(S)$ is a ring homomorphism, because $\otimes_S \colon \mathbf{P}(R) \to \mathbf{P}(S)$ is a semiring map:

$$(P \otimes_R Q) \otimes_R S \cong (P \otimes_R S) \otimes_S (Q \otimes_R S).$$

The free modules play a special role in understanding $K_0(R)$ because they are cofinal in $\mathbf{P}(R)$. By Corollary 1.3 every element of $K_0(R)$ can be written as $[P] - [R^n]$ for some P and n. Moreover, $[P] = [Q]$ in $K_0(R)$ iff P, Q are stably isomorphic: $P \oplus R^m \cong Q \oplus R^m$ for some m. In particular, $[P] = [R^n]$ iff P is stably free. The monoid L of isomorphism classes of free modules is $\mathbb{N}$ iff R satisfies the Invariant Basis Property of Chapter 1, §1. This yields the following information about $K_0(R)$.

LEMMA 2.1. *The monoid map $\mathbb{N} \to \mathbf{P}(R)$ sending n to R^n induces a group homomorphism $\mathbb{Z} \to K_0(R)$. We have:*

(1) *$\mathbb{Z} \to K_0(R)$ is injective iff R satisfies the Invariant Basis Property (IBP);*
(2) *Suppose that R satisfies the IBP (e.g., R is commutative). Then*

$$K_0(R) \cong \mathbb{Z} \qquad \text{iff every f.g. projective } R\text{-module is stably free.}$$

EXAMPLE 2.1.1. Suppose that R is commutative, or more generally that there is a ring map $R \to F$ to a field F. In this case $\mathbb{Z}$ is a direct summand of $K_0(R)$, because the map $K_0(R) \to K_0(F) \cong \mathbb{Z}$ takes [R] to 1. A ring with $K_0(R) = Q$ is given in Exercise 2.12 below.

EXAMPLE 2.1.2 (SIMPLE RINGS). Consider the matrix ring $R = M_n(F)$ over a field F. We saw in Example I.1.1 that every R-module is projective (because it is a sum of copies of the projective module $V \cong F^n$), and that length is an invariant of finitely generated R-modules. Thus *length* is an abelian group isomorphism $K_0(M_n(F)) \xrightarrow{\cong} \mathbb{Z}$ sending $[V]$ to 1. Since R has length n, the subgroup of $K_0(R) \cong \mathbb{Z}$ generated by the free modules has index n. In particular, the inclusion $\mathbb{Z} \subset K_0(R)$ of Lemma 2.1 does not split.

EXAMPLE 2.1.3. (Karoubi) We say a ring R is *flasque* if there is an R-bimodule M, f.g. projective as a right module, and a bimodule isomorphism $\theta : R \oplus M \cong M$. If R is flasque then $K_0(R) = 0$. This is because for every P we have a natural isomorphism $P \oplus (P \otimes_R M) \cong P \otimes_R (R \oplus M) \cong (P \otimes_R M)$.

If R is flasque and the underlying right R-module structure on M is R, we say that R is an *infinite sum ring*. The right module isomorphism $R^2 \cong R$ underlying θ makes R a direct sum ring (Ex. I.1.7). The Cone Rings of Ex. I.1.8, and the rings $\mathrm{End}_R(R^\infty)$ of Ex. I.1.7, are examples of infinite sum rings, and hence flasque rings; see exercise 2.15.

If $R = R_1 \times R_2$ then $\mathbf{P}(R) \cong \mathbf{P}(R_1) \times \mathbf{P}(R_2)$. As in Example 1.2, this implies that $K_0(R) \cong K_0(R_1) \times K_0(R_2)$. Thus K_0 may be computed componentwise.

EXAMPLE 2.1.4 (SEMISIMPLE RINGS). Let R be a semisimple ring, with simple modules $V_1, ..., V_r$ (see Ex. I.1.1). Schur's Lemma states that each $D_i = \mathrm{Hom}_R(V_i, V_i)$ is a division ring; the Artin-Wedderburn Theorem states that

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $\dim_{D_i}(V_i) = n_i$. By (2.1.2), $K_0(R) \cong \prod K_0(M_{n_i}(D_i)) \cong \mathbb{Z}^r$.

Another way to see that $K_0(R) \cong \mathbb{Z}^r$ is to use the fact that $\mathbf{P}(R) \cong \mathbb{N}^r$: the Krull-Schmidt Theorem states that every f.g. (projective) module M is $V_1^{\ell_1} \times \cdots \times V_r^{\ell_r}$ for well-defined integers $\ell_1, ..., \ell_r$.

EXAMPLE 2.1.5 (VON NEUMANN REGULAR RINGS). A ring R is said to be *von Neumann regular* if for every $r \in R$ there is an $x \in R$ such that $rxr = r$. Since $rxrx = rx$, the element $e = rx$ is idempotent, and the ideal $rR = eR$ is a projective module. In fact, every finitely generated right ideal of R is of the form eR for some idempotent, and these form a lattice. Declaring $e \simeq e'$ if $eR = e'R$, the equivalence classes of idempotents in R form a lattice: $(e_1 \wedge e_2)$ and $(e_1 \vee e_2)$ are defined to be the idempotents generating $e_1R + e_2R$ and $e_1R \cap e_2R$, respectively. Kaplansky proved in [Kap58] that every projective R-module is a direct sum of the modules eR. It follows that $K_0(R)$ is determined by the lattice of idempotents (modulo $\simeq$) in R. We will see several examples of von Neumann regular rings in the exercises.

Many von Neumann regular rings do not satisfy the (IBP), the ring $End_F(F^\infty)$ of Ex. I.1.7 being a case in point.

We call a ring R *unit-regular* if for every $r \in R$ there is a unit $x \in R$ such that $rxr = rx$. Every unit-regular ring is Von Neumann regular, has stable range 1, and satisfies the (IBP) (Ex. I.1.13). In particular, $\mathbb{Z} \subseteq K_0(R)$. It is unknown whether or not for every simple unit-regular ring R the group $K_0(R)$ is *strictly unperforated*, meaning that whenever $x \in K_0(R)$ and $nx = [Q]$ for some Q, then $x = [P]$ for some P. Goodearl [Gdrl1] has given examples of simple unit-regular rings R in which the group $K_0(R)$ is strictly unperforated, but has torsion.

An example of a von Neumann regular ring R having the IBP and stable range 2, and $K_0(R) = \mathbb{Z} \oplus \mathbb{Z}/n$ is given in [MM82].

Suppose that R is the direct limit of a filtered system $\{R_i\}$ of rings. Then every f.g. projective R-module is of the form $P_i \otimes_{R_i} R$ for some i and some f.g. projective R_i-module P_i. Any isomorphism $P_i \otimes_{R_i} R \cong P_i' \otimes_{R_i} R$ may be expressed using finitely many elements of R, and hence $P_i \otimes_{R_i} R_j \cong P'_i \otimes_{R_i} R_j$ for some j. That is, $\mathbf{P}(R)$ is the filtered colimit of the $\mathbf{P}(R_i)$. By Ex. 1.3 we have

$$K_0(R) \cong \varinjlim K_0(R_i).$$

This observation is useful when studying $K_0(R)$ of a commutative ring R, because R is the direct limit of its finitely generated subrings. As finitely generated commutative rings are noetherian with finite normalization, properties of $K_0(R)$ may be deduced from properties of K_0 of these nice subrings. If R is integrally closed we may restrict to finitely generated normal subrings, so $K_0(R)$ is determined by K_0 of noetherian integrally closed domains.

Here is another useful reduction; it follows immediately from the observation that if I is nilpotent (or complete) then idempotent lifting (Ex. I.2.2) yields a monoid isomorphism $\mathbf{P}(R) \cong \mathbf{P}(R/I)$. Recall that an ideal I is said to be *complete* if every Cauchy sequence $\sum_{n=1}^{\infty} x_n$ with $x_n \in I^n$ converges to a unique element of I.

LEMMA 2.2. *If I is a nilpotent ideal of R, or more generally a complete ideal, then*

$$K_0(R) \cong K_0(R/I).$$

In particular, if R is commutative then $K_0(R) \cong K_0(R_{red})$.

EXAMPLE 2.2.1 (0-DIMENSIONAL COMMUTATIVE RINGS). Let R be a commutative ring. It is elementary that R_{red} is Artinian iff $\mathrm{Spec}(R)$ is finite and discrete. More generally, it is known (see Ex. I.1.13) that the following are equivalent:

(i) R_{red} is a commutative von Neumann regular ring (2.1.5);
(ii) R has Krull dimension 0;
(iii) $X = \mathrm{Spec}(R)$ is compact, Hausdorff and totally disconnected. (For example, to see that a commutative von Neumann regular R must be reduced, observe that if $r^2 = 0$ then $r = rxr = 0$.)

When R is a commutative von Neumann regular ring, the modules eR are componentwise free; Kaplansky's result states that every projective module is componentwise free. By I.2, the monoid $\mathbf{P}(R)$ is just $[X, \mathbb{N}]$, $X = \mathrm{Spec}(R)$. By (1.4) this yields $K_0(R) = [X, \mathbb{Z}]$. By Lemma 2.2, this proves

PIERCE'S THEOREM 2.2.2. *For every 0-dimensional commutative ring R:*

$$K_0(R) = [\mathrm{Spec}(R), \mathbb{Z}].$$

EXAMPLE 2.2.3 (K_0 DOES NOT COMMUTE WITH INFINITE PRODUCTS). Let R be an infinite product of fields $\prod F_i$. R is von Neumann regular, so $X = \mathrm{Spec}(R)$ is an uncountable totally disconnected compact Hausdorff space. By Pierce's Theorem, $K_0(R) \cong [X, \mathbb{Z}]$. This is contained in but not equal to the product $\prod K_0(F_i) \cong \prod \mathbb{Z}$.

Rank and H_0

DEFINITION. When R is commutative, we write $H_0(R)$ for $[\mathrm{Spec}(R), \mathbb{Z}]$, the ring of all continuous maps from $\mathrm{Spec}(R)$ to $\mathbb{Z}$. Since $\mathrm{Spec}(R)$ is quasi-compact, we know by (1.4) that $H_0(R)$ is always a free abelian group. If R is a noetherian ring, then $\mathrm{Spec}(R)$ has only finitely many (say c) components, and $H_0(R) \cong \mathbb{Z}^c$. If R is a domain, or more generally if $\mathrm{Spec}(R)$ is connected, then $H_0(R) = \mathbb{Z}$.

$H_0(R)$ is a subring of $K_0(R)$. To see this, consider the submonoid L of $\mathbf{P}(R)$ consisting of componentwise free modules R^f. Not only is L cofinal in $\mathbf{P}(R)$, but $L \to \mathbf{P}(R)$ is a semiring map: $R^f \otimes R^g \cong R^{fg}$; by (1.3), $L^{-1}L$ is a subring of $K_0(R)$. Finally, L is isomorphic to $[\mathrm{Spec}(R), \mathbb{N}]$, so as in (1.4) we have $L^{-1}L \cong H_0(R)$. For example, Pierce's theorem (2.2.2) states that if $\dim(R) = 0$ then $K_0(R) \cong H_0(R)$.

Recall from I.2 that the rank of a projective module gives a map from $\mathbf{P}(R)$ to $[\mathrm{Spec}(R), \mathbb{N}]$. Since $\mathrm{rank}(P \oplus Q) = \mathrm{rank}(P) + \mathrm{rank}(Q)$ and $\mathrm{rank}(P \otimes Q) = \mathrm{rank}(P)\,\mathrm{rank}(Q)$ (by Ex. I.2.7), this is a semiring map. As such it induces a ring map

$$\mathrm{rank}\colon K_0(R) \to H_0(R).$$

Since $\mathrm{rank}(R^f) = f$ for every componentwise free module, the composition $H_0(R) \subset K_0(R) \to H_0(R)$ is the identity. Thus $H_0(R)$ is a direct summand of $K_0(R)$.

DEFINITION 2.3. The ideal $\widetilde{K}_0(R)$ of the ring $K_0(R)$ is defined as the kernel of the rank map. By the above remarks, there is a natural decomposition

$$K_0(R) \cong H_0(R) \oplus \widetilde{K}_0(R).$$

We will see later (in §4, §6) that $\widetilde{K}_0(R)$ is a nil ideal. Since $H_0(R)$ is visibly a reduced ring, $\widetilde{K}_0(R)$ is the nilradical of $K_0(R)$.

LEMMA 2.3.1. *If R is commutative, let $\mathbf{P}_n(R)$ denote the subset of $\mathbf{P}(R)$ consisting of projective modules of constant rank n. There is a map $\mathbf{P}_n(R) \to K_0(R)$ sending P to $[P]-[R^n]$. This map is compatible with the stabilization map $\mathbf{P}_n(R) \to \mathbf{P}_{n+1}(R)$ sending P to $P \oplus R$, and the induced map is an isomorphism:*

$$\varinjlim \mathbf{P}_n(R) \cong \widetilde{K}_0(R).$$

PROOF. This follows easily from (1.3).

COROLLARY 2.3.2. *Let R be a commutative noetherian ring of Krull dimension d — or more generally any commutative ring of stable range $d+1$ (Ex. I.1.5). For every $n > d$ the above maps are isomorphisms: $\mathbf{P}_n(R) \cong \widetilde{K}_0(R)$.*

PROOF. If P and Q are f.g. projective modules of rank $> d$, then by Bass Cancellation (I.2.3b) we may conclude that

$$[P] = [Q] \text{ in } K_0(R) \quad \text{iff} \quad P \cong Q.$$

Here is another interpretation of $\widetilde{K}_0(R)$: it is it is the intersection of the kernels of $K_0(R) \to K_0(F)$ over all maps $R \to F$, F a field. This follows from naturality of rank and the observation that $\widetilde{K}_0(F) = 0$ for every field F.

This motivates the following definition for a noncommutative ring R: let $\widetilde{K}_0(R)$ denote the intersection of the kernels of $K_0(R) \to K_0(S)$ over all maps $R \to S$, where S is a simple artinian ring. If no such map $R \to S$ exists, we set $\widetilde{K}_0(R) = K_0(R)$. We define $H_0(R)$ to be the quotient of $K_0(R)$ by $\widetilde{K}_0(R)$. When R is commutative, this agrees with the above definitions of H_0 and $\widetilde{K}_0$, because the maximal commutative subrings of a simple ring S are fields.

$H_0(R)$ is a torsionfree abelian group for every ring R. To see this, note that there is a set X of maps $R \to S_x$ through which every other $R \to S'$ factors. Since each $K_0(S_x) \to K_0(S')$ is an isomorphism, $\widetilde{K}_0(R)$ is the intersection of the kernels of the maps $K_0(R) \to K_0(S_x)$, $x \in X$. Hence $H_0(R)$ is the image of $K_0(R)$ in the torsionfree group $\prod_{x\in X} K_0(S_x) \cong \prod_x \mathbb{Z} \cong Map(X, \mathbb{Z})$.

Example 2.4 (Whitehead group Wh_0). If R is the group ring $\mathbb{Z}G$ of a group G, the (zero-th) *Whitehead group* $Wh_0(G)$ is the quotient of $K_0(\mathbb{Z}G)$ by the subgroup $K_0(\mathbb{Z}) = \mathbb{Z}$. The augmentation map $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$ sending G to 1 induces a decomposition $K_0(\mathbb{Z}G) \cong \mathbb{Z} \oplus Wh_0(G)$, and clearly $\widetilde{K}_0(\mathbb{Z}G) \subseteq Wh_0(G)$. It follows from a theorem of Swan ([Bass, XI(5.2)]) that if G is finite then $\widetilde{K}_0(\mathbb{Z}G) = Wh_0(G)$ and $H_0(\mathbb{Z}G) = \mathbb{Z}$. I do not know whether or not $\widetilde{K}_0(\mathbb{Z}G) = Wh_0(G)$ for every group.

The group $Wh_0(G)$ arose in topology via the following result of C.T.C. Wall. We say that a CW complex X is *dominated* by a complex K if there is a map $f\colon K \to X$ having a right homotopy inverse; this says that X is a retract of K in the homotopy category.

Theorem 2.4.1 (Wall Finiteness Obstruction). *Suppose that X is dominated by a finite CW complex, with fundamental group $G = \pi_1(X)$. This data determines an element $w(X)$ of $Wh_0(G)$ such that $w(X) = 0$ iff X is homotopy equivalent to a finite CW complex.*

Hattori-Stallings trace map

For any associative ring R, let $[R, R]$ denote the subgroup of R generated by the elements $[r, s] = rs - sr$, $r, s \in R$.

For each n, the trace of an $n \times n$ matrix provides an additive map from $M_n(R)$ to $R/[R, R]$ invariant under conjugation; the inclusion of $M_n(R)$ in $M_{n+1}(R)$ via $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is compatible with the trace map. It is not hard to show that the trace $M_n(R) \to R/[R, R]$ induces an isomorphism:

$$M_n(R)/[M_n(R), M_n(R)] \cong R/[R, R].$$

If P is a f.g. projective, choosing an isomorphism $P \oplus Q \cong R^n$ yields an idempotent e in $M_n(R)$ such that $P = e(R^n)$ and $Aut(P) = eM_n(R)e$. By Ex. I.2.3, any other choice yields an e_1 which is conjugate to e in some larger $M_m(R)$. Therefore the trace of an automorphism of P is a well-defined element of $R/[R, R]$, independent of the choice of e. This gives the *trace map* $Aut(P) \to R/[R, R]$. In particular, the trace of the identity map of P is the trace of e; we call it the *trace* of P.

If P' is represented by an idempotent matrix f then $P \oplus P'$ is represented by the idempotent matrix $\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right)$ so the trace of $P \oplus P'$ is trace(P)+trace(P'). Therefore the trace is an additive map on the monoid $\mathbf{P}(R)$. The map $K_0(R) \to R/[R,R]$ induced by universality is called the *Hattori-Stallings trace map*, after the two individuals who first studied it.

When R is commutative, we can provide a direct description of the ring map $H_0(R) \to R$ obtained by restricting the trace map to the subring $H_0(R)$ of $K_0(R)$. Any continuous map $f\colon \mathrm{Spec}(R) \to \mathbb{Z}$ induces a decomposition $R = R_1 \times \cdots \times R_c$ by Ex. I.2.4; the coordinate idempotents $e_1, ..., e_c$ are elements of R. Since $\mathrm{trace}(e_iR)$ is e_i, it follows immediately that $\mathrm{trace}(f)$ is $\sum f(i)e_i$. The identity $\mathrm{trace}(fg) = \mathrm{trace}(f)\mathrm{trace}(g)$ which follows immediately from this formula shows that trace is a ring map.

Proposition 2.5. *If R is commutative then the Hattori-Stallings trace factors as*

$$K_0(R) \xrightarrow{\text{rank}} H_0(R) \to R.$$

Proof. The product over all $\mathfrak{p}$ in $\mathrm{Spec}(R)$ yields the commutative diagram:

$$\begin{tikzcd} K_0(R) \arrow[r] \arrow[d, "\text{trace}"'] & \prod K_0(R_{\mathfrak{p}}) \arrow[d, "\text{trace}"] \\ R \arrow[r, "\text{diagonal}", "\text{inclusion}"'] & \prod R_{\mathfrak{p}}. \end{tikzcd}$$

The kernel of the top arrow is $\widetilde{K}_0(R)$, so the left arrow factors as claimed.

Example 2.5.1 (Group rings). Let k be a commutative ring, and suppose that R is the group ring kG of a group G. If g and h are congugate elements of G then $h - g \in [R,R]$ because $xgx^{-1} - g = [xg, x^{-1}]$. From this it is not hard to see that $R/[R,R]$ is isomorphic to the free k-module $\oplus k[g]$ on the set $G/\!\sim$ of conjugacy classes of elements of G. We write

$$\mathrm{trace}(P) = \sum r_P(g)[g].$$

The coefficients $r_P(g)$ of $\mathrm{trace}(P)$ are therefore functions on the set $G/\!\sim$ for each P.

If G is finite, then any f.g. projective kG-module P is also a projective k-module, and we may also form the trace map $Aut_k(P) \to k$ and hence the "character" $\chi_P\colon G \to k$ by the formula $\chi_P(g) = \mathrm{trace}(g)$. Hattori proved that if $Z_G(g)$ denotes the centralizer of $g \in G$ then Hattori's formula holds:

$$\chi_P(g) = |Z_G(g)|\; r_P(g^{-1}). \tag{2.5.2}$$

Corollary 2.5.3. *If G is a finite group, the ring $\mathbb{Z}G$ has no idempotents except 0 and 1.*

Proof. Let e be an idempotent element of $\mathbb{Z}G$. $\chi_P(1)$ is the rank of the $\mathbb{Z}$-module $P = e\mathbb{Z}G$, which must be less than the rank $|G|$ of $\mathbb{Z}G$. Since $r_P(1) \in \mathbb{Z}$, this contradicts Hattori's formula $\chi_P(1) = |G|\, r_P(1)$.

Bass has conjectured that for every group G and every f.g. projective $\mathbb{Z}G$-module P we have $r_P(g) = 0$ for $g \neq 1$ and $r_P(1) = \mathrm{rank}_{\mathbb{Z}}(P \otimes_{\mathbb{Z}G} \mathbb{Z})$. For G finite, this follows from Hattori's formula and Swan's theorem (cited in 2.3.2) that $\widetilde{K}_0 = Wh_0$. See [Bass76].

EXAMPLE 2.5.4. Suppose that F is a field of characteristic 0 and that G is a finite group with c conjugacy classes, so that $FG/[FG,FG] \cong F^c$. By Maschke's theorem, FG is a product of simple F-algebras: $S_1 \times \cdots \times S_c$ so $FG/[FG,FG]$ is F^c. By (2.1.4) $K_0(FG) \cong \mathbb{Z}^c$. Hattori's formula (and some classical representation theory) shows that the trace map from $K_0(FG)$ to $FG/[FG,FG]$ is isomorphic to the natural inclusion of $\mathbb{Z}^c$ in F^c.

Determinant

Suppose now that R is a commutative ring. Recall from I.3 that the determinant of a fin. gen. projective module P is an element of the Picard group $\mathrm{Pic}(R)$.

PROPOSITION 2.6. *The determinant induces a surjective group homomorphism*

$$\det\colon K_0(R) \to \mathrm{Pic}(R)$$

PROOF. By the universal property of K_0, it suffices to show that $\det(P \oplus Q) \cong \det(P) \otimes_R \det(Q)$. We may assume that P and Q have constant rank m and n, respectively. Then $\wedge^{m+n}(P \oplus Q)$ is the sum over all i, j such that $i + j = m + n$ of $(\wedge^i Q) \otimes (\wedge^j P)$. If $i > m$ or $j > n$ we have $\wedge^i P = 0$ or $\wedge^j Q = 0$, respectively. Hence $\wedge^{m+n}(P \oplus Q) = (\wedge^m P) \otimes (\wedge^n Q)$, as asserted.

DEFINITION 2.6.1. Let $SK_0(R)$ denote the subset of $K_0(R)$ consisting of the classes $x = [P] - [R^m]$, where P has constant rank m and $\wedge^m P \cong R$. This is the kernel of $\det\colon \widetilde{K}_0(R) \to \mathrm{Pic}(R)$, by Lemma 2.3.1 and Proposition 2.6.

$SK_0(R)$ is an ideal of $K_0(R)$. To see this, we use Ex. I.3.4: if $x = [P] - [R^n]$ and Q has rank n then $\det(x \cdot Q) = (\det P)^{\otimes n} (\det Q)^{\otimes m} (\det Q)^{\otimes -m} = R$.

COROLLARY 2.6.2. *For every commutative ring R there is a surjective ring homomorphism with kernel $SK_0(R)$:*

$$\mathrm{rank} \oplus \det\colon K_0(R) \to H_0(R) \oplus \mathrm{Pic}(R)$$

COROLLARY 2.6.3. *If R is a 1-dimensional commutative noetherian ring, then the classification of f.g. projective R-modules in I.3.4 induces an isomorphism:*

$$K_0(R) \cong H_0(R) \oplus \mathrm{Pic}(R).$$

Morita Equivalence

We say that two rings R and S are *Morita equivalent* if **mod**-R and **mod**-S are equivalent as abelian categories. That is, if there exist additive functors T and U

$$\textbf{mod}\text{-}R \quad \underset{U}{\overset{T}{\rightleftarrows}} \quad \textbf{mod}\text{-}S$$

such that $UT \cong id_R$ and $TU \cong id_S$. Set $P = T(R)$ and $Q = U(S)$; P is an R-S bimodule and Q is a S-R bimodule via the maps $R = \mathrm{End}_R(R) \xrightarrow{T} \mathrm{End}_S(P)$ and $S = \mathrm{End}_S(S) \xrightarrow{U} \mathrm{End}_R(Q)$. Both $UT(R) \cong P \otimes_S Q \cong R$ and $TU(S) \cong Q \otimes_R P \cong S$ are bimodule isomorphisms. The following result is taken from [Bass, II.3].

Structure Theorem for Morita Equivalence 2.7. *If R and S are Morita equivalent, and P, Q are as above, then:*

(a) *P and Q are f.g. projective, both as R-modules and as S-modules;*
(b) *$End_S(P) \cong R \cong End_S(Q)^{op}$ and $End_R(Q) \cong S \cong End_R(P)^{op}$;*
(c) *P and Q are dual S-modules: $P \cong \mathrm{Hom}_S(Q, S)$ and $Q \cong \mathrm{Hom}_S(P, S)$;*
(d) *$T(M) \cong M \otimes_R P$ and $U(N) \cong N \otimes_S Q$ for every M and N;*
(e) *P is a "faithful" S-module in the sense that the functor $\mathrm{Hom}_S(P, -)$ from **mod**-S to abelian groups is a faithful functor. (If S is commutative then P is faithful iff $\mathrm{rank}(P) \geq 1$.) Similarly, Q is a "faithful" R-module.*

Since P and Q are f.g. projective, the Morita functors T and U also induce an equivalence between the categories $\mathbf{P}(R)$ and $\mathbf{P}(S)$. This implies the following:

Corollary 2.7.1. *If R and S are Morita equivalent then $K_0(R) \cong K_0(S)$.*

Example 2.7.2. $R = M_n(S)$ is always Morita equivalent to S; P is the bimodule S^n of "column vectors" and Q is the bimodule $(S^n)^t$ of "row vectors." More generally suppose that P is a "faithful" f.g. projective S-module. Then $R = \mathrm{End}_S(P)$ is Morita equivalent to S, the bimodules being P and $Q = \mathrm{Hom}_S(P, S)$. By 2.7.1, we see that $K_0(S) \cong K_0(M_n(S))$.

Additive Functors 2.8. Any R-S bimodule P which is f.g. projective as a right S-module, induces an additive (hence exact) functor $T(M) = M \otimes_R P$ from $\mathbf{P}(R)$ to $\mathbf{P}(S)$, and therefore induces a map $K_0(R) \to K_0(S)$. If all we want is an additive functor T from $\mathbf{P}(R)$ to $\mathbf{P}(S)$, we do not need the full strength of Morita equivalence. Given T, set $P = T(R)$. By additivity we have $T(R^n) = P^n \cong R^n \otimes_R P$; from this it is not hard to see that $T(M) \cong M \otimes_R P$ for every f.g. projective M, and that T is isomorphic to $-\otimes_R P$. See Ex. 2.14 for more details.

A bimodule map (resp., isomorphism) $P \to P'$ induces an additive natural transformation (resp., isomorphism) $T \to T'$. This is the case, for example, with the bimodule isomorphism $R \oplus M \cong M$ defining a flasque ring (2.1.3).

Example 2.8.1 (Basechange and Transfer maps). Suppose that $f: R \to S$ is a ring map. Then S is an R–S bimodule, and it represents the *basechange functor* $f^*: K_0(R) \to K_0(S)$ sending P to $P \otimes_R S$. If in addition S is f.g. projective as a right R-module then there is a forgetful functor from $\mathbf{P}(S)$ to $\mathbf{P}(R)$; it is represented by S as a S–R bimodule because it sends Q to $Q \otimes_S S$. The induced map $f_*: K_0(S) \to K_0(R)$ is called the *transfer map*. We will return to this point in 7.8 below, explaining why we have selected the contravariant notation f^* and f_*.

Mayer-Vietoris sequences

For any ring R with unit, we can include $GL_n(R)$ in $GL_{n+1}(R)$ as the matrices $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. The group $GL(R)$ is the union of the groups $GL_n(R)$. Now suppose we are given a Milnor square of rings, as in I.2:

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I \end{array}$$

Define $\partial_n\colon GL_n(S/I) \to K_0(R)$ by Milnor patching: $\partial_n(g)$ is $[P]-[R^n]$, where P is the projective R-module obtained by patching free modules along g as in (I.2.6). The formulas of Ex. I.2.9 imply that $\partial_n(g) = \partial_{n+1}\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\partial_n(g)+\partial_n(h) = \partial_n(gh)$. Therefore the $\{\partial_n\}$ assemble to give a group homomorphism ∂ from $GL(S/I)$ to $K_0(R)$. The following result now follows from (I.2.6) and Ex. 1.4.

Theorem 2.9 (Mayer-Vietoris). *Given a Milnor square as above, the sequence*

$$GL(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I)$$

is exact. The image of ∂ is the double coset space

$$GL(S)\backslash GL(S/I)/GL(R/I) = GL(S/I)/\sim$$

where $x \sim gxh$ for $x \in GL(S/I), g \in GL(S)$ and $h \in GL(R/I)$.

Example 2.9.1. If R is the coordinate ring of the node over a field F (I.3.10.2) then $K_0(R) \cong \mathbb{Z} \oplus F^\times$. If R is the coordinate ring of the cusp over F (I.3.10.1) then $K_0(R) \cong \mathbb{Z} \oplus F$. Indeed, the coordinate rings of the node and the cusp are 1-dimensional noetherian rings, so 2.6.3 reduces the Mayer-Vietoris sequence to the Units-Pic sequence I.3.10.

We conclude with a useful construction, anticipating several later developments.

Definition 2.10. Let $T : \mathbf{P}(R) \to \mathbf{P}(S)$ be an additive functor, such as the basechange or transfer of 2.8.1. $\mathbf{P}(T)$ is the category whose objects are triples (P, α, Q), where $P, Q \in \mathbf{P}(R)$ and $\alpha : T(P) \to T(Q)$ is an isomorphism. A morphism $(P, \alpha, Q) \to (P', \alpha', Q')$ is a pair of R-module maps $p : P \to P'$, $q : Q \to Q'$ such that $\alpha' T(p) = T(q)\alpha$. An *exact sequence* in $\mathbf{P}(T)$ is a sequence

$$(*) \qquad 0 \to (P', \alpha', Q') \to (P, \alpha, Q) \to (P'', \alpha'', Q'') \to 0$$

whose underlying sequences $0 \to P' \to P \to P'' \to 0$ and $0 \to Q' \to Q \to Q'' \to 0$ are exact. We define $K_0(T)$ to be the abelian group with generators the objects of $\mathbf{P}(T)$ and relations:

(a) $[(P, \alpha, Q)] = [(P', \alpha', Q')] + [(P'', \alpha'', Q'')]$ for every exact sequence $(*)$;
(b) $[(P_1, \alpha, P_2)] + [(P_2, \beta, P_3)] = [(P_1, \beta\alpha, P_3)]$.

If T is the basechange f^*, we write $K_0(f)$ for $K_0(T)$.

It is easy to see that there is a map $K_0(T) \to K_0(R)$ sending $[(P, \alpha, Q)]$ to $[P] - [Q]$. If T is a basechange functor f^* associated to $f : R \to S$, or more generally if the $T(R^n)$ are cofinal in $\mathbf{P}(S)$, then there is an exact sequence:

$$(2.10.1) \qquad GL(S) \xrightarrow{\partial} K_0(T) \to K_0(R) \to K_0(S).$$

The construction of ∂ and verification of exactness is not hard, but lengthy enough to relegate to exercise 2.17. If $f : R \to R/I$ then $K_0(f^*)$ is the group $K_0(I)$ of Ex. 2.4; see Ex. 2.4(e).

EXERCISES

2.1 Let R be a commutative ring. If A is an R-algebra, show that the functor $\otimes_R\colon \mathbf{P}(A)\times\mathbf{P}(R)\to\mathbf{P}(A)$ yields a map $K_0(A)\otimes_{\mathbb{Z}} K_0(R)\to K_0(A)$ making $K_0(A)$ into a $K_0(R)$-module. If $A\to B$ is an algebra map, show that $K_0(A)\to K_0(B)$ is a $K_0(R)$-module homomorphism.

2.2 *Projection Formula.* Let R be a commutative ring, and A an R-algebra which as an R-module is f.g. projective of rank n. By Ex. 2.1, $K_0(A)$ is a $K_0(R)$-module, and the basechange map $f^*\colon K_0(R)\to K_0(A)$ is a module homomorphism. We shall write $x\cdot f^*y$ for the product in $K_0(A)$ of $x\in K_0(A)$ and $y\in K_0(R)$; this is an abuse of notation when A is noncommutative.

(a) Show that the transfer map $f_*\colon K_0(A)\to K_0(R)$ of Example 2.8.1 is a $K_0(R)$-module homomorphism, *i.e.*, that the *projection formula* holds:

$$f_*(x\cdot f^*y)=f_*(x)\cdot y \qquad \text{for every } \ x\in K_0(A),\ y\in K_0(R).$$

(b) Show that both compositions f^*f_* and f_*f^* are multiplication by $[A]$.

(c) Show that the kernels of f^*f_* and f_*f^* are annihilated by a power of n.

2.3 *Excision for K_0.* If I is an ideal in a ring R, form the augmented ring $R\oplus I$ and let $K_0(I)=K_0(R,I)$ denote the kernel of $K_0(R\oplus I)\to K_0(R)$.

(a) If $R\to S$ is a ring map sending I isomorphically onto an ideal of S, show that $K_0(R,I)\cong K_0(S,I)$. Thus $K_0(I)$ is independent of R. *Hint.* Show that $GL(S)/GL(S\oplus I)=1$.

(b) If $I\cap J=0$, show that $K_0(I+J)\cong K_0(I)\oplus K_0(J)$.

(c) *Ideal sequence.* Show that there is an exact sequence

$$GL(R)\to GL(R/I)\xrightarrow{\partial} K_0(I)\to K_0(R)\to K_0(R/I).$$

(d) If R is commutative, use Ex. I.3.6 to show that there is a commutative diagram with exact rows, the vertical maps being determinants:

$$\begin{array}{ccccccccc}
GL(R) & \longrightarrow & GL(R/I) & \xrightarrow{\partial} & K_0(I) & \longrightarrow & K_0(R) & \longrightarrow & K_0(R/I)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
R^\times & \longrightarrow & (R/I)^\times & \xrightarrow{\partial} & \mathrm{Pic}(I) & \longrightarrow & \mathrm{Pic}(R) & \longrightarrow & \mathrm{Pic}(R/I).
\end{array}$$

2.4 K_0I. If I is a ring without unit, we define $K_0(I)$ as follows. Let R be a ring with unit acting upon I, form the augmented ring $R\oplus I$, and let $K_0(I)$ be the kernel of $K_0(R\oplus I)\to K_0(R)$. Thus $K_0(R\oplus I)\cong K_0(R)\oplus K_0(I)$ by definition.

(a) If I has a unit, show that $R\oplus I\cong R\times I$ as rings with unit. Since $K_0(R\times I)=K_0(R)\times K_0(I)$, this shows that the definition of Ex. 2.3 agrees with the usual definition of $K_0(I)$.

(b) Show that a map $I\to J$ of rings without unit induces a map $K_0(I)\to K_0(J)$

(c) Let $M_\infty(R)$ denote the union $\cup M_n(R)$ of the matrix groups, where $M_n(R)$ is included in $M_{n+1}(R)$ as the matrices $\left(\begin{smallmatrix} g & 0\\ 0 & 0\end{smallmatrix}\right)$. $M_\infty(R)$ is a ring without unit. Show that the inclusion of $R=M_1(R)$ in $M_\infty(R)$ induces an isomorphism

$$K_0(R)\cong K_0(M_\infty(R)).$$

(d) If F is a field, show that $R=F\oplus M_\infty(F)$ is a von Neumann regular ring. Then show that $H_0(R)=\mathbb{Z}$ and $K_0(R)\cong\mathbb{Z}\oplus\mathbb{Z}$.

(e) If $f : R \to R/I$, show that $K_0(I)$ is the group $K_0(f^*)$ of 2.10. *Hint:* Use $f_0 : R \oplus I \to R$ and Ex. 2.3(c).

2.5 *Radical ideals.* Let I be a radical ideal in a ring R (see Ex. I.1.12, I.2.1).

(a) Show that $K_0(I) = 0$, and that $K_0(R) \to K_0(R/I)$ is an injection.

(b) If I is a complete ideal, $K_0(R) \cong K_0(R/I)$ by Lemma 2.2. If R is a semilocal but not local domain, show that $K_0(R) \to K_0(R/I)$ is not an isomorphism.

2.6 *Semilocal rings.* A ring R is called *semilocal* if R/J is semisimple for some radical ideal J. Show that if R is semilocal then $K_0(R) \cong \mathbb{Z}^n$ for some $n > 0$.

2.7 Show that if $f : R \to S$ is a map of commutative rings, then:

$$\ker(f) \text{ contains no idempotents}(\neq 0) \Leftrightarrow H_0(R) \to H_0(S) \text{ is an injection.}$$

Conclude that $H_0(R) = H_0(R[t]) = H_0(R[t, t^{-1}])$.

2.8 Consider the following conditions on a ring R (cf. Ex. I.1.2):

(IBP) R satisfies the Invariant Basis Property (IBP);

(PO) $K_0(R)$ is a partially ordered abelian group (see §1);

(III) For all n, if $R^n \cong R^n \oplus P$ then $P = 0$.

Show that $(III) \Rightarrow (PO) \Rightarrow (IBP)$. This implies that $K_0(R)$ is a partially ordered abelian group if R is either commutative or noetherian. (See Ex. I.1.4.)

2.9 *Rim squares.* Let G be a cyclic group of prime order p, and $\zeta = e^{2\pi i/p}$ a primitive p^{th} root of unity. Show that the map $\mathbb{Z}G \to \mathbb{Z}[\zeta]$ sending a generator of G to ζ induces an isomorphism $K_0(\mathbb{Z}G) \cong K_0(\mathbb{Z}[\zeta])$ and hence $Wh_0(G) \cong \mathrm{Pic}(\mathbb{Z}[\zeta])$. *Hint:* Form a Milnor square with $\mathbb{Z}G/I = \mathbb{Z}$, $\mathbb{Z}[\zeta]/I = \mathbb{F}_p$, and consider the cyclotomic units $u = \frac{\zeta^i - 1}{\zeta - 1}$, $1 \le i < p$.

2.10 Let R be a commutative ring. Prove that

(a) If $\mathrm{rank}(x) > 0$ for some $x \in K_0(R)$, then there is an $n > 0$ and a fin. gen. projective module P so that $nx = [P]$. (This says that the partially ordered group $K_0(R)$ is "unperforated" in the sense of [Gdearl].)

(b) If P, Q are f.g. projectives such that $[P] = [Q]$ in $K_0(R)$, then there is an $n > 0$ such that $P \oplus \cdots \oplus P \cong Q \oplus \cdots \oplus Q$ (n copies of P, n copies of Q).

Hint: First assume that R is noetherian of Krull dimension $d < \infty$, and use Bass-Serre Cancellation. In the general case, write R as a direct limit.

2.11 A (normalized) *dimension function* for a von Neumann regular ring R is a group homomorphism $d : K_0(R) \to \mathbb{R}$ so that $d(R^n) = n$ and $d(P) > 0$ for every nonzero f.g. projective P.

(a) Show that whenever $P \subseteq Q$ any dimension function must have $d(P) \le d(Q)$

(b) If R has a dimension function, show that the formula $\rho(r) = d(rR)$ defines a rank function $\rho\colon R \to [0, 1]$ in the sense of Ex. I.1.13. Then show that this gives a 1-1 correspondence between rank functions on R and dimension functions on $K_0(R)$.

2.12 Let R be the union of the matrix rings $M_{n!}(F)$ constructed in Ex. I.1.13. Show that the inclusion $\mathbb{Z} \subset K_0(R)$ extends to an isomorphism $K_0(R) \cong \mathbb{Q}$.

2.13 Let R be the infinite product of the matrix rings $M_i(\mathbb{C})$, $i = 1, 2, ...$

(a) Show that every f.g. projective R-module P is componentwise trivial in the sense that $P \cong \prod P_i$, the P_i being f.g. projective $M_i(\mathbb{C})$-modules.

(b) Show that the map from $K_0(R)$ to the group $\prod K_0(M_i(\mathbb{C})) = \prod \mathbb{Z}$ of infinite sequences $(n_1, n_2, ...)$ of integers is an injection, and that $K_0(R) = H_0(R)$ is isomorphic to the group of bounded sequences.

(c) Show that $K_0(R)$ is not a free abelian group, even though it is torsionfree. *Hint:* Consider the subgroup S of sequences $(n_1, ...)$ such that the power of 2 dividing n_i approaches ∞ as $i \to \infty$; show that S is uncountable but that $S/2S$ is countable.

2.14 *Bivariant* K_0. If R and R' are rings, let $Rep(R, R')$ denote the set of isomorphism classes of R–R' bimodules M such that M is finitely generated projective as a right R'-module. Each M gives a functor $\otimes_R M$ from $\mathbf{P}(R)$ to $\mathbf{P}(R')$ sending P to $P \otimes_R M$. This induces a monoid map $\mathbf{P}(R) \to \mathbf{P}(R')$ and hence a homomorphism from $K_0(R)$ to $K_0(R')$. For example, if $f \colon R \to R'$ is a ring homomorphism and R' is considered as an element of $Rep(R, R')$, we obtain the map $\otimes_R R'$. Show that:

(a) Every additive functor $\mathbf{P}(R) \to \mathbf{P}(R')$ is induced from an M in $Rep(R, R')$;

(b) If $K_0(R, R')$ denotes the group completion of $Rep(R, R')$, then $M \otimes_{R'} N$ induces a bilinear map from $K_0(R, R') \otimes K_0(R', R'')$ to $K_0(R, R'')$;

(c) $K_0(\mathbb{Z}, R)$ is $K_0(R)$, and if $M \in Rep(R, R')$ then the map $\otimes_R M : K_0(R) \to K_0(R')$ is induced from the product of (b).

(d) If R and R' are Morita equivalent, and P is the R-R' bimodule giving the isomorphism $\mathbf{mod}\text{-}R \cong \mathbf{mod}\text{-}R'$, the class of P in $K_0(R, R')$ gives the Morita isomorphism $K_0(R) \cong K_0(R')$.

2.15 In this exercise, we connect the definition 2.1.3 of infinite sum ring with a more elementary description due to Wagoner. If R is a direct sum ring, the isomorphism $R^2 \cong R$ induces a ring homomorphism $\oplus : R \times R \subset \mathrm{End}_R(R^2) \cong \mathrm{End}_R(R) = R$.

(a) Suppose that R is an infinite sum ring with bimodule M, and write $r \mapsto r^\infty$ for the ring homomorphism $R \to \mathrm{End}_R(M) \cong R$ arising from the left action of R on the right R-module M. Show that $r \oplus r^\infty = r^\infty$ for all $r \in R$.

(b) Conversely, suppose that R is a direct sum ring, and that $R \xrightarrow{\infty} R$ is a ring map so that $r \oplus r^\infty = r^\infty$ for all $r \in R$. Show that R is an infinite sum ring.

(c) (Wagoner) Show that the Cone Rings of Ex. I.1.8, and the rings $\mathrm{End}_R(R^\infty)$ of Ex. I.1.7, are infinite sum rings. *Hint:* $R^\infty \cong \coprod_{i=1}^\infty R^\infty$, so a version of the Eilenberg Swindle I.2.8 applies.

2.16 For any ring R, let J be the (nonunital) subring of $E = \mathrm{End}_R(R^\infty)$ of all f such that $f(R^\infty)$ is finitely generated (Ex. I.1.7). Show that $M_\infty(R) \subset J_n$ induces an isomorphism $K_0(R) \cong K_0(J)$. *Hint:* For the projection $e_n : R^\infty \to R^n$, $J_n = e_n E$ maps onto $M_n(R) = e_n E e_n$ with nilpotent kernel. But $J = \cup J_n$.

2.17 This exercise shows that there is an exact sequence (2.10.1) when T is cofinal.

(a) Show that $[(P, \alpha, Q)] + [(Q, -\alpha^{-1}, P)] = 0$ and $[(P, T(\gamma), Q)] = 0$ in $K_0(T)$.

(b) Show that every element of $K_0(T)$ has the form $[(P, \alpha, R^n)]$.

(c) Use cofinality and the maps $\partial(\alpha) = [(R^n, \alpha, R^n)]$ of 2.10(b), from $\mathrm{Aut}(TR^n)$ to $K_0(T)$, to show that there is a homomorphism $K_1(S) \to K_0(T)$.

(d) Use (a), (b) and (c) to show that (2.10.1) is exact at $K_0(T)$.

(e) Show that (2.10.1) is exact at $K_0(R)$.

§3. $K(X)$, $KO(X)$ and $KU(X)$ of a topological space

Let X be a paracompact topological space. The sets $\mathbf{VB}_{\mathbb{R}}(X)$ and $\mathbf{VB}_{\mathbb{C}}(X)$ of isomorphism classes of real and complex vector bundles over X are abelian monoids under Whitney sum. By Construction I.4.2, they are commutative semirings under $\otimes$. Hence the group completions $KO(X)$ of $\mathbf{VB}_{\mathbb{R}}(X)$ and $KU(X)$ of $\mathbf{VB}_{\mathbb{C}}(X)$ are commutative rings with identity $1 = [T^1]$. If the choice of $\mathbb{R}$ or $\mathbb{C}$ is understood, we will just write $K(X)$ for simplicity.

Similarly, the set $\mathbf{VB}_{\mathbb{H}}(X)$ is an abelian monoid under $\oplus$, and we write $KSp(X)$ for its group completion. Although it has no natural ring structure, the construction of Ex. I.4.18 endows $KSp(X)$ with the structure of a module over the ring $KO(X)$.

For example if $*$ denotes a 1-point space then $K(*) = \mathbb{Z}$. If X is contractible, then $KO(X) = KU(X) = \mathbb{Z}$ by I.4.6.1. More generally, $K(X) \cong K(Y)$ whenever X and Y are homotopy equivalent by I.4.6.

The functor $K(X)$ is contravariant in X. Indeed, if $f\colon Y \to X$ is continuous, the induced bundle construction $E \mapsto f^*E$ yields a monoid map $f^*\colon \mathbf{VB}(X) \to \mathbf{VB}(Y)$ and hence a ring homomorphism $f^*\colon K(X) \to K(Y)$. By the Homotopy Invariance Theorem I.4.5, the map f^* depends only upon the homotopy class of f in $[Y, X]$.

For example, the universal map $X \to *$ induces a ring map from $\mathbb{Z} = K(*)$ into $K(X)$, sending $n > 0$ to the class of the trivial bundle T^n over X. If $X \neq \emptyset$ then any point of X yields a map $* \to X$ splitting the universal map $X \to *$. Thus the subring $\mathbb{Z}$ is a direct summand of $K(X)$ when $X \neq \emptyset$. (But if $X = \emptyset$ then $K(\emptyset) = 0$.) For the rest of this section, we will assume $X \neq \emptyset$ in order to avoid awkward hypotheses.

The trivial vector bundles T^n and the componentwise trivial vector bundles T^f form sub-semirings of $\mathbf{VB}(X)$, naturally isomorphic to $\mathbb{N}$ and $[X, \mathbb{N}]$, respectively. When X is compact, the semirings $\mathbb{N}$ and $[X, \mathbb{N}]$ are cofinal in $\mathbf{VB}(X)$ by the Subbundle Theorem I.4.1, so by Corollary 1.3 we have subrings

$$\mathbb{Z} \subset [X, \mathbb{Z}] \subset K(X).$$

More generally, it follows from Construction I.4.2 that $\dim\colon \mathbf{VB}(X) \to [X, \mathbb{N}]$ is a semiring map splitting the inclusion $[X, \mathbb{N}] \subset \mathbf{VB}(X)$. Passing to Group Completions, we get a natural ring map

$$\dim\colon K(X) \to [X, \mathbb{Z}]$$

splitting the inclusion of $[X, \mathbb{Z}]$ in $K(X)$.

The kernel of dim will be written as $\widetilde{K}(X)$, or as $\widetilde{KO}(X)$ or $\widetilde{KU}(X)$ if we wish to emphasize the choice of $\mathbb{R}$ or $\mathbb{C}$. Thus $\widetilde{K}(X)$ is an ideal in $K(X)$, and there is a natural decomposition

$$K(X) \cong \widetilde{K}(X) \oplus [X, \mathbb{Z}].$$

Warning. If X is not connected, our group $\widetilde{K}(X)$ differs slightly from the notation in the literature. However, most applications will involve connected spaces, where the notation is the same. This will be clarified by Theorem 3.2 below.

Consider the set map $\mathbf{VB}_n(X) \to \widetilde{K}(X)$ sending E to $[E] - n$. This map is compatible with the stabilization map $\mathbf{VB}_n(X) \to \mathbf{VB}_{n+1}(X)$ sending E to $E \oplus T$, giving a map

$$\varinjlim \mathbf{VB}_n(X) \to \widetilde{K}(X). \tag{3.1.0}$$

We can interpret this in terms of maps between the infinite Grassmannian spaces G_n ($= BO_n$, BU_n or BSp_n) as follows. Recall from the Classification Theorem I.4.10 that the set $\mathbf{VB}_n(X)$ is isomorphic to the set $[X, G_n]$ of homotopy classes of maps. Adding a trivial bundle T to the universal bundle E_n over G_n gives a vector bundle over G_n, so again by the Classification Theorem there is a map $i_n: G_n \to G_{n+1}$ such that $E_n \oplus T \cong i_n^*(E_{n+1})$. By Cellular Approximation there is no harm in assuming i_n is cellular. Using I.4.10.1, the map $\Omega i_n: \Omega G_n \to \Omega G_{n+1}$ is homotopic to the standard inclusion $O_n \hookrightarrow O_{n+1}$ (resp. $U_n \hookrightarrow U_{n+1}$ or $Sp_n \hookrightarrow Sp_{n+1}$), which sends an $n \times n$ matrix g to the $n+1 \times n+1$ matrix $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. By construction, the resulting map $i_n: [X, G_n] \to [X, G_{n+1}]$ corresponds to the stabilization map. The direct limit $\varinjlim [X, G_n]$ is then in 1-1 correspondence with the direct limit $\varinjlim \mathbf{VB}_n(X)$ of (3.1.0).

Stabilization Theorem 3.1. *Let X be either a compact space or a finite dimensional connected CW complex. Then the map (3.1.0) induces an isomorphism $\widetilde{K}(X) \cong \varinjlim \mathbf{VB}(X) \cong \varinjlim [X, G_n]$. In particular,*

$$\widetilde{KO}(X) \cong \varinjlim [X, BO_n], \quad \widetilde{KU}(X) \cong \varinjlim [X, BU_n] \text{ and } \widetilde{KSp}(X) \cong \varinjlim [X, BSp_n].$$

Proof. We argue as in Lemma 2.3.1. Since the monoid of (componentwise) trivial vector bundles T^f is cofinal in $\mathbf{VB}(X)$ (I.4.1), we see from Corollary 1.3 that every element of $\widetilde{K}(X)$ is represented by an element $[E] - n$ of some $\mathbf{VB}_n(X)$, and if $[E] - n = [F] - n$ then $E \oplus T^\ell \cong F \oplus T^\ell$ in some $\mathbf{VB}_{n+\ell}(X)$. Thus $\widetilde{K}(X) \cong \varinjlim \mathbf{VB}_n(X)$, as claimed.

Examples 3.1.1 (Spheres). From I(4.9) we see that $KO(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ but $KU(S^1) = \mathbb{Z}$, $KO(S^2) = \mathbb{Z} \oplus \mathbb{Z}/2$ but $KU(S^2) = \mathbb{Z} \oplus \mathbb{Z}$, $KO(S^3) = KU(S^3) = \mathbb{Z}$ and $KO(S^4) \cong KU(S^4) = \mathbb{Z} \oplus \mathbb{Z}$.

By Prop. I.4.8, the n-dimensional ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) vector bundles on S^d are classified by the homotopy groups $\pi_{d-1}(O_n)$, $\pi_{d-1}(U_n)$ and $\pi_{d-1}(Sp_n)$, respectively. By the Stabilization Theorem, $\widetilde{KO}(S^d) = \lim_{n\to\infty} \pi_{d-1}(O_n)$ and $\widetilde{KU}(S^d) = \lim_{n\to\infty} \pi_{d-1}(U_n)$.

Now Bott Periodicity says that the homotopy groups groups of O_n, U_n and Sp_n stabilize for $n \gg 0$. Moreover, if $n \geq d/2$ then $\pi_{d-1}(U_n)$ is 0 for d odd and $\mathbb{Z}$ for d even. Thus $KU(S^d) = \mathbb{Z} \oplus \widetilde{KU}(S^d)$ is periodic of order 2 in $d > 0$: the ideal $\widetilde{KU}(S^d)$ is 0 for d odd and $\mathbb{Z}$ for d even, $d \neq 0$.

Similarly, the $\pi_{d-1}(O_n)$ and $\pi_{d-1}(Sp_n)$ stabilize for $n \geq d$ and $n \geq d/4$; both are periodic of order 8. Thus $KO(S^d) = \mathbb{Z} \oplus \widetilde{KO}(S^d)$ and $KSp(S^d) = \mathbb{Z} \oplus \widetilde{KSp}(S^d)$ are periodic of order 8 in $d > 0$, with the groups $\widetilde{KO}(S^d) = \pi_{d-1}(O)$ and $\widetilde{KSp}(S^d) = \pi_{d-1}(Sp)$ being tabulated in the following table.

d (mod 8)	1	2	3	4	5	6	7	8
$\widetilde{KO}(S^d)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\widetilde{KSp}(S^d)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

Both of the ideals $\widetilde{KO}(S^d)$ and $\widetilde{KU}(S^d)$ are of square zero.

REMARK 3.1.2. The complexification maps $\mathbb{Z} \cong \widetilde{KO}(S^{4k}) \to \widetilde{KU}(S^{4k}) \cong \mathbb{Z}$ are multiplication by 2 if k is odd, and by 1 if k is even. Similarly, the maps $\mathbb{Z} \cong \widetilde{KU}(S^{4k}) \to \widetilde{KSp}(S^{4k}) \cong \mathbb{Z}$ are multiplication by 2 if k is odd, and by 1 if k is even. These calculations are taken from [MT] IV.5.12 and IV.6.1.

Let BO (resp. BU, BSp) denote the direct limit of the Grassmannians Grass_n. As noted after (3.1.0) and in I.4.10.1, the notation reflects the fact that $\Omega\,\mathrm{Grass}_n$ is O_n (resp. U_n, Sp_n), and the maps in the direct limit correspond to the standard inclusions, so that we have $\Omega BO \simeq O = \bigcup O_n$, $\Omega BU \simeq U = \bigcup U_n$ and $\Omega BSp \simeq Sp = \bigcup Sp_n$.

THEOREM 3.2. *For every compact space X:*

$$KO(X) \cong [X, \mathbb{Z} \times BO] \quad \textit{and} \quad \widetilde{KO}(X) \cong [X, BO];$$

$$KU(X) \cong [X, \mathbb{Z} \times BU] \quad \textit{and} \quad \widetilde{KU}(X) \cong [X, BU];$$

$$KSp(X) \cong [X, \mathbb{Z} \times BSp] \quad \textit{and} \quad \widetilde{KSp}(X) \cong [X, BSp].$$

In particular, the homotopy groups $\pi_n(BO) = \widetilde{KO}(S^n)$, $\pi_n(BU) = \widetilde{KU}(S^n)$ and $\pi_n(BSp) = \widetilde{KSp}(S^n)$ are periodic and given in Example 3.1.1.

PROOF. If X is compact then we have $[X, BO] = \varinjlim [X, BO_n]$ and similarly for $[X, BU]$ and $[X, BSp]$. The result now follows from Theorem 3.1 for connected X. For non-connected compact spaces, we only need to show that the maps $[X, BO] \to \widetilde{KO}(X)$, $[X, BU] \to \widetilde{KU}(X)$ and $[X, BSp] \to \widetilde{KSp}(X)$ of Theorem 3.1 are still isomorphisms.

Since X is compact, every continuous map $X \to \mathbb{Z}$ is bounded. Hence the rank of every vector bundle E is bounded, say $\operatorname{rank} E \le n$ for some $n \in \mathbb{N}$. If $f = n - \operatorname{rank} E$ then $F = E \oplus T^f$ has constant rank n, and $[E] - \operatorname{rank} E = [F] - n$. Hence every element of $\widetilde{K}(X)$ comes from some $\mathbf{VB}_n(X)$.

To see that these maps are injective, suppose that $E, F \in \mathbf{VB}_n(X)$ are such that $[E] - n = [F] - n$. By (1.3) we have $E \oplus T^f = F \oplus T^f$ in $\mathbf{VB}_{n+f}(X)$ for some $f \in [X, \mathbb{N}]$. If $f \le p$, $p \in \mathbb{N}$, then adding T^{p-f} yields $E \oplus T^p = F \oplus T^p$. Hence E and F agree in $\mathbf{VB}_{n+p}(X)$.

DEFINITION 3.2.1. For every paracompact X we write $KO^0(X)$ for $[X, \mathbb{Z} \times BO]$, $KU^0(X)$ for $[X, \mathbb{Z} \times BU]$ and $KSp^0(X)$ for $[X, \mathbb{Z} \times BSp]$. By Theorem 3.2, we have $KO^0(X) \cong KO(X)$, $KU^0(X) \cong KU(X)$ and $KSp^0(X) \cong KSp(X)$ for every compact X. Similarly, we shall write $\widetilde{KO}^0(X)$, $\widetilde{KU}^0(X)$ and $\widetilde{KSp}^0(X)$ for $[X, BO]$, $[X, BU]$ and $[X, BSp]$. When the choice of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ is clear, we will just write $K^0(X)$ and $\widetilde{K}^0(X)$.

If Y is a subcomplex of X, we define relative groups $K^0(X,Y) = K^0(X/Y)/\mathbb{Z}$ and $\widetilde{K}^0(X,Y) = \widetilde{K}^0(X/Y)$.

When X is paracompact but not compact, $\tilde{K}^0(X)$ and $\tilde{K}(X)$ are connected by stabilization and the map (3.1.0):

$$\widetilde{KO}(X) \leftarrow \varinjlim \mathbf{VB}(X) \cong \varinjlim [X, BO_n] \to [X, BO] = \widetilde{KO}^0(X)$$

and similarly for $\widetilde{KU}(X)$ and $\widetilde{KSp}(X)$. We will see in Example 3.7.2 and Ex. 3.2 that the left map need not be an isomorphism. Here is an example showing that the right map need not be an isomorphism either.

Example 3.2.2. (McGibbon) Let X be the infinite bouquet of odd-dimensional spheres $S^3 \vee S^5 \vee S^7 \vee \cdots$. By homotopy theory, there is a map $f\colon X \to BO_3$ whose restriction to S^{2p+1} is essential of order p for each odd prime p. If E denotes the 3-dimensional vector bundle f^*E_3 on X, then the class of f in $\varinjlim[X, BO_n]$ corresponds to $[E] - 3 \in KO(X)$. In fact, since X is a suspension, we have $\varinjlim[X, BO_n] \cong \widetilde{KO}(X)$ by Ex. 3.8.

Each $(n+3)$-dimensional vector bundle $E \oplus T^n$ is nontrivial, since its restriction to S^{2p+1} is nontrivial whenever $2p > n+3$ (again by homotopy theory). Hence $[E] - 3$ is a nontrivial element of $\widetilde{KO}(X)$. However, the corresponding element in $\widetilde{KO}^0(X) = [X, BO]$ is zero, because the homotopy groups of BO have no odd torsion.

Proposition 3.3. *If Y is a subcomplex of a CW complex X, the following sequences are exact:*

$$\widetilde{K}^0(X/Y) \to \widetilde{K}^0(X) \to \widetilde{K}^0(Y),$$
$$K^0(X,Y) \to K^0(X) \to K^0(Y).$$

Proof. Since $Y \subset X$ is a cofibration, we have an exact sequence $[X/Y, B] \to [X, B] \to [Y, B]$ for every connected space B; see III(6.3) in [Wh]. This yields the first sequence (B is BO, BU or BSp). The second follows from this and the classical exact sequence $\widetilde{H}^0(X/Y;\mathbb{Z}) \to H^0(X;\mathbb{Z}) \to H^0(Y;\mathbb{Z})$.

Change of structure field 3.4. If X is any space, the monoid (semiring) map $\mathbf{VB}_{\mathbb{R}}(X) \to \mathbf{VB}_{\mathbb{C}}(X)$ sending $[E]$ to $[E \otimes \mathbb{C}]$ (see Ex. I.4.5) extends by universality to a ring homomorphism $KO(X) \to KU(X)$. For example, $KO(S^{8n}) \to KU(S^{8n})$ is an isomorphism but $\widetilde{KO}(S^{8n+4}) \cong \mathbb{Z}$ embeds in $\widetilde{KU}(S^{8n+4}) \cong \mathbb{Z}$ as a subgroup of index 2.

Similarly, the forgetful map $\mathbf{VB}_{\mathbb{C}}(X) \to \mathbf{VB}_{\mathbb{R}}(X)$ extends to a group homomorphism $KU(X) \to KO(X)$. As $\dim_{\mathbb{R}}(V) = 2 \cdot \dim_{\mathbb{C}}(V)$, the summand $[X, \mathbb{Z}]$ of $KU(X)$ embeds as $2[X, \mathbb{Z}]$ in the summand $[X, \mathbb{Z}]$ of $KO(X)$. Since $E \otimes \mathbb{C} \cong E \oplus E$ as real vector bundles (by Ex. I.4.5), the composition $KO(X) \to KU(X) \to KO(X)$ is multiplication by 2. The composition in the other direction is more complicated; see Exercise 3.1. For example, it is the zero map on $\widetilde{KU}(S^{8n+4}) \cong \mathbb{Z}$ but is multiplication by 2 on $\widetilde{KU}(S^{8n}) \cong \mathbb{Z}$.

There are analogous maps $KU(X) \to KSp(X)$ and $KSp(X) \to KU(X)$, whose properties we leave to the exercises.

Higher Topological K-theory

Once we have a representable functor like K^0, standard techniques in infinite loop space theory all us to expand it into a generalized cohomology theory. Rather than get distracted by infinite loop spaces now, we choose to adopt a rather pedestrian approach, ignoring the groups K^n for $n > 0$. For this we use the suspensions S^nX of X, which are all connected paracompact spaces.

DEFINITION 3.5. For each integer $n > 0$, we define $\widetilde{KO}^{-n}(X)$ and $KO^{-n}(X)$ by:

$$\widetilde{KO}^{-n}(X) = \widetilde{KO}^0(S^nX) = [S^nX, BO]; \quad KO^{-n}(X) = \widetilde{KO}^{-n}(X) \oplus \widetilde{KO}(S^n).$$

Replacing 'O' by 'U' yields definitions $\widetilde{KU}^{-n}(X) = \widetilde{KU}^0(S^nX) = [S^nX, BU]$ and $KU^{-n}(X) = \widetilde{KU}^{-n}(X) \oplus \widetilde{KU}(S^n)$; replacing '$O$' by '$Sp$' yields definitions for $\widetilde{KSp}^{-n}(X)$ and $KSp^{-n}(X)$. When the choice of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ is clear, we shall drop the 'O,' 'U' and 'Sp,', writing simply $\widetilde{K}^{-n}(X)$ and $K^{-n}(X)$.

We shall also define relative groups as follows. If Y is a subcomplex of X, and $n > 0$, we set $K^{-n}(X,Y) = \widetilde{K}^{-n}(X/Y)$.

BASED MAPS 3.5.1. Note that our definitions do not assume X to have a basepoint. If X has a nondegenerate basepoint and Y is an H-space with homotopy inverse (such as BO, BU or BSp), then the group $[X, Y]$ is isomorphic to the group $\pi_0(Y) \times [X,Y]_*$, where the second term denotes homotopy classes of *based* maps from X to Y; see pp. 100 and 119 of [Wh]. For such spaces X we can interpret the formulas for $KO^{-n}(X)$, $KU^{-n}(X)$ and $KSp^{-n}(X)$ in terms of based maps, as is done in [Atiyah, p.68].

If X_* denotes the disjoint union of X and a basepoint $*$, then we have the usual formula for an unreduced cohomology theory: $K^{-n}(X) = \widetilde{K}(S^n(X_*))$. This easily leads (see Ex. 3.11) to the formulas for $n \geq 1$:

$$KO^{-n}(X) \cong [X, \Omega^n BO], \; KU^{-n}(X) \cong [X, \Omega^n BU] \text{ and } KSp^{-n}(X) \cong [X, \Omega^n BSp].$$

THEOREM 3.6. *If Y is a subcomplex of a CW complex X, we have the exact sequences (infinite to the left):*

$$\cdots \to \widetilde{K}^{-2}(Y) \to \widetilde{K}^{-1}(X/Y) \to \widetilde{K}^{-1}(X) \to \widetilde{K}^{-1}(Y) \to \widetilde{K}^0(X/Y) \to \widetilde{K}^0(X) \to \widetilde{K}^0(Y),$$

$$\cdots \to K^{-2}(Y) \to K^{-1}(X,Y) \to K^{-1}(X) \to K^{-1}(Y) \to K^0(X,Y) \to K^0(X) \to K^0(Y).$$

PROOF. Exactness at $K^0(X)$ was proven in Proposition 3.3. The mapping cone cone(i) of $i\colon Y \subset X$ is homotopy equivalent to X/Y, and $j\colon X \subset$ cone(i) induces cone$(i)/X \simeq SY$. This gives exactness at $K^0(X,Y)$. Similarly, cone$(j) \simeq SY$ and cone$(j)/$cone$(i) \simeq SX$, giving exactness at $K^{-1}(Y)$. The long exact sequences follows by replacing $Y \subset X$ by $SY \subset SX$.

Characteristic Classes 3.7. The total Stiefel-Whitney class $w(E)$ of a real vector bundle E was defined in Ch. I, §4. By (SW3) it satisfies the product formula: $w(E \oplus F) = w(E)w(F)$. Therefore if we interpret $w(E)$ as an element of the abelian group U of all formal sums $1 + a_1 + \cdots$ in $\hat{H}^*(X;\mathbb{Z}/2)$ we get a get a group homomorphism $w: KO(X) \to U$. It follows that each Stiefel-Whitney class induces a well-defined set map $w_i: KO(X) \to H^i(X;\mathbb{Z}/2)$. In fact, since w vanishes on each componentwise trivial bundle T^f it follows that $w([E] - [T^f]) = w(E)$. Hence each Stiefel-Whitney class w_i factors through the projection $KO(X) \to \widetilde{KO}(X)$.

Similarly, the total Chern class $c(E) = 1 + c_1(E) + \cdots$ satisfies $c(E \oplus F) = c(E)c(F)$, so we may think of it as a group homomorphism from $KU(X)$ to the abelian group U of all formal sums $1 + a_2 + a_4 + \cdots$ in $\hat{H}^*(X;\mathbb{Z})$. It follows that the Chern classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ of a complex vector bundle define set maps $c_i: KU(X) \longrightarrow H^{2i}(X;\mathbb{Z})$. Again, since c vanishes on componentwise trivial bundles, each Chern class c_i factors through the projection $KU(X) \to \widetilde{KU}(X)$.

Example 3.7.1. For even spheres the Chern class $c_n: \widetilde{KU}(S^{2n}) \to H^{2n}(S^n;\mathbb{Z})$ is an isomorphism. We will return to this point in Ex. 3.6 and in §4.

Example 3.7.2. The map $\varinjlim[\mathbb{R}\mathbb{P}^\infty, BO_n] \to \widetilde{KO}(\mathbb{R}\mathbb{P}^\infty)$ of (3.1.0) cannot be onto. To see this, consider the element $\eta = 1 - [E_1]$ of $\widetilde{KO}(\mathbb{R}\mathbb{P}^\infty)$, where E_1 is the canonical line bundle. Since $w(-\eta) = w(E_1) = 1 + x$ we have $w(\eta) = (1+x)^{-1} = \sum_{i=0}^\infty x^i$, and $w_i(\eta) \neq 0$ for every $i \geq 0$. Axiom (SW1) implies that η cannot equal $[F] - \dim(F)$ for any bundle F.

Similarly, $\varinjlim[\mathbb{C}\mathbb{P}^\infty, BU_n] \to \widetilde{KU}(\mathbb{C}\mathbb{P}^\infty)$ cannot be onto; the argument is similar, again using the canonical line bundle: $c_i(1 - [E_1]) \neq 0$ for every $i \geq 0$.

EXERCISES

3.1 Let X be a topological space. Show that there is an involution of $\mathbf{VB}_{\mathbb{C}}(X)$ sending $[E]$ to the complex conjugate bundle $[\bar{E}]$ of Ex. I.4.6. The corresponding involution c on $KU(X)$ can be nontrivial; use I(4.9.2) to show that c is multiplication by -1 on $\widetilde{KU}(S^2) \cong \mathbb{Z}$. (By Bott periodicity, this implies that c is multiplication by $(-1)^k$ on $\widetilde{KU}(S^2) \cong \mathbb{Z}$.) Finally, show that the composite $KU(X) \to KO(X) \to KU(X)$ is the map $1 + c$ sending $[E]$ to $[E] + [\bar{E}]$.

3.2 If $\amalg X_i$ is the disjoint union of spaces X_i, show that $K(\amalg X_i) \cong \prod K(X_i)$. Then construct a space X such that the map $\varinjlim \mathbf{VB}_n(X) \to \widetilde{K}(X)$ of (3.1.0) is not onto.

3.3 *External products.* Show that there is a bilinear map $K(X_1) \otimes K(X_2) \to K(X_1 \times X_2)$ for every X_1 and X_2, sending $[E_1] \otimes [E_2]$ to $[\pi_1^*(E_1) \otimes \pi_2^*(E_2)]$, where $pi_i: X_1 \times X_2 \to X_i$ is the projection. Then show that if $X_1 = X_2 = X$ the composition with the diagonal map $\Delta^*: K(X \times X) \to K(X)$ yields the usual product in the ring $K(X)$, sending $[E_1] \otimes [E_2]$ to $[E_1 \otimes E_2]$.

3.4 Recall that the smash product $X \wedge Y$ of two based spaces is the quotient $X \times Y / X \vee Y$, where $X \vee Y$ is the union of $X \times \{*\}$ and $\{*\} \times Y$. Show that

$$\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y).$$

3.5 Show that $KU^{-2}(*) \otimes KU^{-n}(X) \to KU^{-n-2}(X)$ induces a "periodicity" isomorphism $\beta: KU^{-n}(X) \xrightarrow{\sim} KU^{-n-2}(X)$ for all n. *Hint:* $S^2 \wedge S^n X \simeq S^{n+2} X$.

3.6 Let X be a finite CW complex with only even-dimensional cells, such as $\mathbb{CP}^n$. Show that $KU(X)$ is a free abelian group on the set of cells of X, and that $KU(SX) = \mathbb{Z}$, so that $KU^{-1}(X) = 0$. Then use Example 3.7.1 to show that the total Chern class injects the group $\widetilde{KU}(X)$ into $\prod H^{2i}(X;\mathbb{Z})$.

3.7 *Chern character for* $\mathbb{CP}^n$. Let E_1 be the canonical line bundle on $\mathbb{CP}^n$, and let x denote the class $[E_1]-1$ in $KU(\mathbb{CP}^n)$. Use Chern classes and the previous exercise to show that $\{1, [E_1], [E_1 \otimes E_1], \dots, [E_1^{\otimes n}]\}$, and hence $\{1, x, x^2, \dots, x^n\}$, forms a basis of the free abelian group $KU(\mathbb{CP}^n)$. Then show that $x^{n+1} = 0$, so that the ring $KU(\mathbb{CP}^n)$ is isomorphic to $\mathbb{Z}[x]/(x^{n+1})$. We will see in Ex. 4.11 below that the Chern character ch maps the ring $KU(\mathbb{CP}^n)$ isomorphically onto the subring $\mathbb{Z}[t]/(t^{n+1})$ of $H^*(\mathbb{CP}^n;\mathbb{Q})$ generated by $t = e^{c_1(x)} - 1$.

3.8 Consider the suspension $X = SY$ of a paracompact space Y. Use Ex. I.4.16 to show that $\varinjlim [X, BO_n] \cong \widetilde{KO}(X)$.

3.9 If X is a finite CW complex, show by induction on the number of cells that both $KO(X)$ and $KU(X)$ are finitely generated abelian groups.

3.10 Show that $KU(\mathbb{RP}^{2n}) = KU(\mathbb{RP}^{2n+1}) = \mathbb{Z} \oplus \mathbb{Z}/2^n$. *Hint:* Try the total Stiefel-Whitney class, using 3.3.

3.11 Let X be a compact space with a nondegenerate basepoint. Show that $KO^{-n}(X) \cong [X, \Omega^n BO] \cong [X, \Omega^{n-1} O]$ and $KU^{-n}(X) \cong [X, \Omega^n BU] \cong [X, \Omega^{n-1} U]$ for all $n \geq 1$. In particular, $KU^{-1}(X) \cong [X, U]$ and $KO^{-1}(X) \cong [X, O]$.

3.12 Let X be a compact space with a nondegenerate basepoint. Show that the homotopy groups of the topological groups $GL(\mathbb{R}^X) = \mathrm{Hom}(X, GL(\mathbb{R}))$ and $GL(\mathbb{C}^X) = \mathrm{Hom}(X, GL(\mathbb{C}))$ are (for $n > 0$):

$$\pi_{n-1} GL(\mathbb{R}^X) = KO^{-n}(X) \quad \text{and} \quad \pi_{n-1} GL(\mathbb{C}^X) = KU^{-n}(X).$$

3.13 If $E \to X$ is a complex bundle, there is a quaternionic vector bundle $E_{\mathbb{H}} \to X$ with fibers $E_x \otimes_{\mathbb{C}} \mathbb{H}$, as in Ex. I.4.5; this induces the map $KU(X) \to KSp(X)$ mentioned in 3.4. Show that $E_{\mathbb{H}} \to X$, considered as a complex vector bundle, is isomorphic to the Whitney sum $E \oplus E$. Deduce that the composition $KU(X) \to KSp(X) \to KU(X)$ is multiplication by 2.

3.14 Show that $\mathbb{H} \otimes_{\mathbb{C}} \mathbb{H}$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$ as an $\mathbb{H}$-bimodule, on generators $1 \otimes 1 \pm j \otimes j$. This induces a natural isomorphism $V \otimes_{\mathbb{C}} \mathbb{H} \cong V \oplus V$ of vector spaces over $\mathbb{H}$. If $E \to X$ is a quaternionic vector bundle, with underlying complex bundle $uE \to X$, show that there is a natural isomorphism $(uE)_{\mathbb{H}} \cong E \oplus E$. Conclude that the composition $KSp(X) \to KU(X) \to KSp(X)$ is multiplication by 2.

3.15 Let $\bar{E}$ be the complex conjugate bundle of a complex vector bundle $E \to X$; see Ex. I.4.6. Show that $\bar{E}_{\mathbb{H}} \cong E_{\mathbb{H}}$ as quaternionic vector bundles. This shows that $KU(X) \to KSp(X)$ commutes with the involution c of Ex. 3.1.

Using exercises 3.1 and 3.14, show that the composition $KSp(X) \to KO(X) \to KSp(X)$ is multiplication by 4.

§4. Lambda and Adams Operations

A commutative ring K is called a *λ-ring* if we are given a family of set operations $\lambda^k: K \to K$ for $k \geq 0$ such that for all $x, y \in K$:

- $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for all $x \in K$;
- $\lambda^k(x+y) = \sum_{i=0}^k \lambda^i(x)\lambda^{k-i}(y) = \lambda^k(x) + \lambda^{k-1}(x)\lambda^1 y + \cdots + \lambda^k(y)$.

This last condition is equivalent to the assertion that there is a group homomorphism λ_t from the additive group of K to the multiplicative group $W(K) = 1 + tK[[t]]$ given by the formula $\lambda_t(x) = \sum \lambda^k(x)t^k$.

Example 4.1.1 (Binomial Rings). . The integers $\mathbb{Z}$ and the rationals $\mathbb{Q}$ are λ-rings with $\lambda^k(n) = \binom{n}{k}$. If K is any $\mathbb{Q}$-algebra, we define $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ for $x \in K$ and $k \geq 1$; again the formula $\lambda^k(x) = \binom{x}{k}$ makes K into a λ-ring.

More generally, a *binomial ring* is a subring K of a $\mathbb{Q}$-algebra $K_{\mathbb{Q}}$ such that for all $x \in K$ and $k \geq 1$, $\binom{x}{k} \in K$. We make a binomial ring into a λ-ring by setting $\lambda^k(x) = \binom{x}{k}$. If K is a binomial ring then formally λ_t is given by the formula $\lambda_t(x) = (1+t)^x$. For example, if X is a topological space, then the ring $[X, \mathbb{Z}]$ is a λ-ring with $\lambda^k(f) = \binom{f}{k}$, the function sending x to $\binom{f(x)}{k}$.

The notion of λ-semiring is very useful in constructing λ-rings. Let M be a semiring (see §1); we know that the group completion $M^{-1}M$ of M is a ring. We call M a *λ–semiring* if it is equipped with operations $\lambda^k: M \to M$ such that $\lambda^0(x) = 1$, $\lambda^1(x) = x$ and $\lambda^k(x+y) = \sum \lambda^i(x)\lambda^{k-i}(y)$.

If M is a λ-semiring then the group completion $K = M^{-1}M$ is a λ-ring. To see this, note that sending $x \in M$ to the power series $\sum \lambda^k(x)t^k$ defines a monoid map $\lambda_t: M \to 1 + tK[[t]]$. By universality of K, this extends to a group homomorphism λ_t from K to $1 + tK[[t]]$, and the coefficients of $\lambda_t(x)$ define the operations $\lambda^k(x)$.

Example 4.1.2 (Algebraic K_0). Let R be a commutative ring and set $K = K_0(R)$. If P is a f.g. projective R-module, consider the formula $\lambda^k(P) = [\wedge^k P]$. The decomposition $\wedge^k(P \oplus Q) \cong \sum (\wedge^i P) \otimes (\wedge^{k-1} Q)$ given in ch.I, §3 shows that $\mathbf{P}(R)$ is a λ-semiring. Hence $K_0(R)$ is a λ-ring.

Since $\mathrm{rank}(\wedge^k P) = \binom{\mathrm{rank}\, P}{k}$, it follows that rank: $K_0(R) \to [\mathrm{Spec}(R), \mathbb{Z}]$ is a morphism of λ-rings, and hence that $\widetilde{K}_0(R)$ is a λ-ideal of $K_0(R)$.

Example 4.1.3 (Topological K^0). Let X be a topological space and let K be either $KO(X)$ or $KU(X)$. If $E \to X$ is a vector bundle, let $\lambda^k(E)$ be the exterior power bundle $\wedge^k E$ of Ex. I.4.3. The decomposition of $\wedge^k(E \oplus F)$ given in Ex. I.4.3 shows that the monoid $\mathbf{VB}(X)$ is a λ-semiring. Hence $KO(X)$ and $KU(X)$ are λ-rings, and $KO(X) \to KU(X)$ is a morphism of λ-rings.

Since $\dim(\wedge^k E) = \binom{\dim E}{k}$, it follows that $KO(X) \to [X, \mathbb{Z}]$ and $KU(X) \to [X, \mathbb{Z}]$ are λ-ring morphisms, and that $\widetilde{KO}(X)$ and $\widetilde{KU}(X)$ are λ-ideals.

Example 4.1.4 (Representation Ring). Let G be a finite group, and consider the complex representation ring $R(G)$ constructed in Example 1.6. $R(G)$ is the group completion of $Rep_{\mathbb{C}}(G)$, the semiring of finite dimensional representations of G; as an abelian group $R(G) \cong \mathbb{Z}^c$, where c is the number of conjugacy classes of elements in G. The exterior powers $\Lambda^i(V)$ of a representation V are also G-modules, and the decomposition of $\Lambda^k(V \oplus W)$ as complex vector spaces used in (4.1.2) shows that $Rep_{\mathbb{C}}(G)$ is a λ-semiring. Hence $R(G)$ is a λ-ring.

If $d = \dim_{\mathbb{C}}(V)$ then $\dim_{\mathbb{C}}(\Lambda^k V) = \binom{d}{k}$, so $\dim_{\mathbb{C}}$ is a λ-ring map from $R(G)$ to $\mathbb{Z}$. The kernel $\widetilde{R}(G)$ of this map is a λ-ideal of $R(G)$.

Example 4.1.5. Let X be a scheme, or more generally a locally ringed space (Ch. I, §5). We will define a ring $K_0(X)$ in §7 below, using the category $\mathbf{VB}(X)$. As an abelian group it is generated by the classes of vector bundles on X. We will see in §8 that the operations $\lambda^k[\mathcal{E}] = [\wedge^k \mathcal{E}]$ are well-defined on $K_0(X)$ and make it into a λ-ring. (The formula for $\lambda^k(x+y)$ will follow from Ex. I.5.4.)

Positive structures

Not every λ-ring is well-behaved. In order to avoid pathologies, we introduce a further condition, satisfied by the above examples: the λ-ring K must have a positive structure and satisfy the Splitting Principle.

Definition 4.2.1. By a *positive structure* on a λ-ring K we mean: 1) a λ-subring H^0 of K which is a binomial ring; 2) a λ-ring surjection $\varepsilon\colon K \to H^0$ which is the identity on H^0 (ε is called the *augmentation*); and 3) a subset $P \subset K$ (the *positive* elements), such that

(1) $\mathbb{N} = \{0, 1, 2, \cdots\}$ is contained in P.
(2) P is a λ-sub-semiring of K. That is, P is closed under addition, multiplication, and the operations λ^k.
(3) Every element of the kernel $\widetilde{K}$ of ε can be written as $p-q$ for some p, $q \in P$.
(4) If $p \in P$ then $\varepsilon(p) = n \in \mathbb{N}$. Moreover, $\lambda^i(p) = 0$ for $i > n$ and $\lambda^n(p)$ is a unit of K.

Condition (2) states that the group completion $P^{-1}P$ of P is a λ-subring of K; by (3) we have $P^{-1}P = \mathbb{Z} \oplus \widetilde{K}$. By (4), $\varepsilon(p) > 0$ for $p \neq 0$, so $P \cap (-P) = 0$; therefore $P^{-1}P$ is a partially ordered abelian group in the sense of §1. An element $\ell \in P$ with $\varepsilon(\ell) = 1$ is called a *line element*; by (4), $\lambda^1(\ell) = \ell$ and ℓ is a unit of K. That is, the line elements form a subgroup L of the units of K.

The λ-rings in examples (4.1.2)–(4.1.5) all have positive structures. The λ-ring $K_0(R)$ has a positive structure with

$$H^0 = H_0(R) = [\mathrm{Spec}(R), \mathbb{Z}] \quad \text{and} \quad P = \{[P] : \mathrm{rank}(P) \text{ is constant}\};$$

the line elements are the classes of line bundles, so $L = \mathrm{Pic}(R)$. Similarly, the λ-rings $KO(X)$ and $KU(X)$ have a positive structure in which H^0 is $H^0(X, \mathbb{Z}) = [X, \mathbb{Z}]$ and P is $\{[E] : \dim(E) \text{ is constant}\}$, as long as we restrict to compact spaces or spaces with $\pi_0(X)$ finite, so that (I.4.1.1) applies. Again, line elements are the classes of line bundles; for $KO(X)$ and $KU(X)$ we have $L = H^1(X; \mathbb{Z}/2)$ and $L = H^2(X; \mathbb{Z})$, respectively. For $R(G)$, the classes $[V]$ of representations V are the positive elements; H^0 is $\mathbb{Z}$, and L is the set of 1-dimensional representations of G. Finally, if X is a scheme (or locally ringed space) then in the positive structure on $K_0(X)$ we have $H^0 = H^0(X; \mathbb{Z})$ and P is $\{[\mathcal{E}] : \mathrm{rank}(\mathcal{E}) \text{ is constant}\}$; see I.5.1. The line bundles are again the line elements, so $L = \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X\times)$ by I.5.10.1.

There is a natural group homomorphism "det" from K to L, which vanishes on H^0. If $p \in P$ we define $\det(p) = \lambda^n(p)$, where $\varepsilon(p) = n$. The formula for $\lambda^n(p+q)$ and the vanishing of $\lambda^i(p)$ for $i > \varepsilon(p)$ imply that $\det\colon P \to L$ is a monoid map, *i.e.*, that $\det(p+q) = \det(p)\det(q)$. Thus det extends to a map from $P^{-1}P$ to L. As $\det(n) = \binom{n}{n} = 1$ for every $n \geq 0$, $\det(\mathbb{Z}) = 1$. By (iii), defining $\det(H^0) = 1$ extends det to a map from K to L. When K is $K_0(R)$ the map det was introduced in §2. For $KO(X)$, det is the first Stiefel-Whitney class; for $KU(X)$, det is the first Chern class.

Having described what we mean by a positive structure on K, we can now state the Splitting Principle.

Definition 4.2.2. The *Splitting Principle* states that for every positive element p in K there is a extension $K \subset K'$ (of λ-rings with positive structure) such that p is a sum of line elements in K'.

The Splitting Principle for $KO(X)$ and $KU(X)$ holds by Ex. 4.12. Using algebraic geometry, we will show in 8.7 that the Splitting Principle holds for $K_0(R)$ as well as K_0 of a scheme. The Splitting Principle also holds for $R(G)$; see [AT, 1.5]. The importance of the Splitting Principle lies in its relation to "special λ-rings," a notion we shall define after citing the following motivational result from [FL, ch.I].

Theorem 4.2.3. *If K is a λ-ring with a positive structure, the Splitting Principle holds iff K is a special λ-ring.*

In order to define special λ-ring, we need the following technical example:

Example 4.3 (Witt Vectors). For every commutative ring R, the abelian group $W(R) = 1 + tR[[t]]$ has the structure of a commutative ring, natural in R; $W(R)$ is called the ring of (big) *Witt vectors* of R. The multiplicative identity of the ring $W(R)$ is $(1-t)$, and multiplication $*$ is completely determined by naturality, formal factorization of elements of $W(R)$ as $f(t) = \prod_{i=1}^{\infty}(1 - r_i t^i)$ and the formula:

$$(1 - rt) * f(t) = f(rt).$$

It is not hard to see that there are "universal" polynomials P_n in $2n$ variables so that:

$$(\sum a_i t^i) * (\sum b_j t^j) = \sum c_n t^n, \text{ with } c_n = P_n(a_1, \dots, a_n; b_1, \dots, b_n).$$

Grothendieck observed that there are operations λ^k on $W(R)$ making it into a λ-ring; they are defined by naturality, formal factorization and the formula

$$\lambda^k(1 - rt) = 0 \text{ for all } k \geq 2.$$

Another way to put it is that there are universal polynomials $P_{n,k}$ such that:

$$\lambda^k(\sum a_i t^i) = \sum b_n t^n, \text{ with } b_n = P_{n,k}(a_1, ..., a_{nk}).$$

Definition 4.3.1. A *special λ-ring* is a λ-ring K such that the group homomorphism λ_t from K to $W(K)$ is a λ-homomorphism. In effect, a special λ-ring is a λ-ring K such that

- $\lambda^k(1) = 0$ for $k \neq 0, 1$
- $\lambda^k(xy)$ is $P_k(\lambda^1(x), ..., \lambda^k(x); \lambda^1(y), ..., \lambda^k(y))$, and
- $\lambda^n(\lambda^k(x)) = P_{n,k}(\lambda^1(x), ..., \lambda^{nk}(x))$.

EXAMPLE 4.3.2. The formula $\lambda^n(s_1) = s_n$ defines a special λ-ring structure on the polynomial ring $U = \mathbb{Z}[s_1, ..., s_n, ...]$; see [AT]. Clearly if x is any element in any special λ-ring K then the map $U \to K$ sending s_n to $\lambda^n(x)$ is a λ-homomorphism. The λ-ring U cannot have a positive structure by Theorem 4.6 below, since U has no nilpotent elements except 0.

Adams operations

For every augmented λ-ring K we can define the *Adams operations* $\psi^k\colon K \to K$ for $k \geq 0$ by setting $\psi^0(x) = \varepsilon(x)$, $\psi^1(x) = x$, $\psi^2(x) = x^2 - 2\lambda^2(x)$ and inductively

$$\psi^k(x) = \lambda^1(x)\psi^{k-1}(x) - \lambda^2(x)\psi^{k-2}(x) + \cdots + (-1)^k\lambda^{k-1}(x)\psi^1(x) + (-1)^{k-1}k\lambda^k(x).$$

From this inductive definition we immediately deduce three facts:

- if ℓ is a line element then $\psi^k(\ell) = \ell^k$;
- if I is a λ-ideal with $I^2 = 0$ then $\psi^k(x) = (-1)^{k-1}k\lambda^k(x)$ for all $x \in I$;
- For every binomial ring H we have $\psi^k = 1$. Indeed, the formal identity $x\sum_{i=0}^{k-1}(-1)^i\binom{x}{i} = (-1)^{k+1}k\binom{x}{k}$ shows that $\psi^k(x) = x$ for all $x \in H$.

The operations ψ^k are named after J.F. Adams, who first introduced them in 1962 in his study of vector fields on spheres.

Here is a slicker, more formal presentation of the Adams operations. Define $\psi^k(x)$ to be the coefficient of t^k in the power series:

$$\psi_t(x) = \sum \psi^k(x)t^k = \varepsilon(x) - t\frac{d}{dt}\log\lambda_{-t}(x).$$

The proof that this agrees with the inductive definition of $\psi^k(x)$ is an exercise in formal algebra, which we relegate to Exercise 4.6 below.

PROPOSITION 4.4. *Assume K satisfies the Splitting Principle. Each ψ^k is a ring endomorphism of K, and $\psi^j\psi^k = \psi^{jk}$ for all $j, k \geq 0$.*

PROOF. The logarithm in the definition of ψ_t implies that $\psi_t(x+y) = \psi_t(x) + \psi_t(y)$, so each ψ^k is additive. The Splitting Principle and the formula $\psi^k(\ell) = \ell^k$ for line elements yield the formulas $\psi^k(pq) = \psi^k(p)\psi^k(q)$ and $\psi^j(\psi^k(p)) = \psi^{jk}(p)$ for positive p. The extension of these formulas to K is clear.

EXAMPLE 4.4.1. Consider the λ-ring $KU(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}$ of 3.1.1. On $H^0 = \mathbb{Z}$, $\psi^k = 1$, but on $\widetilde{KU}(S^n) \cong \mathbb{Z}$, ψ^k is multiplication by $k^{n/2}$. (See [Atiyah, 3.2.2].)

EXAMPLE 4.4.2. Consider $KU(\mathbb{R}\mathbb{P}^{2n})$, which by Ex. 3.10 is $\mathbb{Z} \oplus \mathbb{Z}/2^n$. I claim that for all $x \in \widetilde{KU}(X)$:

$$\psi^k(x) = \begin{cases} x \text{ if } k \text{ is odd} \\ 0 \text{ if } k \text{ is even.} \end{cases}$$

To see this, note that $\widetilde{KU}(\mathbb{R}\mathbb{P}^{2n}) \cong \mathbb{Z}/2^n$ is additively generated by $(\ell - 1)$, where ℓ is the nonzero element of $L = H^2(\mathbb{R}\mathbb{P}^{2n}; \mathbb{Z}) = \mathbb{Z}/2$. Since $\ell^2 = 1$, we see that $\psi^k(\ell - 1) = (\ell^k - 1)$ is 0 if k is even and $(\ell - 1)$ if k is odd. The assertion follows.

γ-operations

Associated to the operations λ^k are the operations $\gamma^k\colon K \to K$. To construct them, note that if we set $s = t/(1-t)$ then $K[[t]] = K[[s]]$ and $t = s/(1+s)$. Therefore we can rewrite $\lambda_s(x) = \sum \lambda^i(x)s^i$ as a power series $\gamma_t(x) = \sum \gamma^k(x)t^k$ in t. By definition, $\gamma^k(x)$ is the coefficient of t^k in $\gamma_t(x)$. Since $\gamma_t(x) = \lambda_s(x)$ we have $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$. In particular $\gamma^0(x) = 1$, $\gamma^1(x) = x$ and $\gamma^k(x+y) = \sum \gamma^i(x)\gamma^{k-i}(y)$. That is, the γ-operations satisfy the axioms for a λ-ring structure on K. An elementary calculation, left to the reader, yields the useful identity:

Formula 4.5. $\gamma^k(x) = \lambda^k(x+k-1)$. This implies that $\gamma^2(x) = \lambda^2(x) + x$ and

$$\gamma^k(x) = \lambda^k(x+k-1) = \lambda^k(x) + \binom{k-1}{1}\lambda^{k-1}(x) + \cdots + \binom{k-1}{k-2}\lambda^2(x) + x.$$

EXAMPLE 4.5.1. If H is a binomial ring then for all $x \in H$ we have

$$\gamma^k(x) = \binom{x+k-1}{k} = (-1)^k\binom{-x}{k}.$$

EXAMPLE 4.5.2. $\gamma^k(1) = 1$ for all k. More generally, if ℓ is a line element then $\gamma^k(\ell) = \ell$ for all $k \geq 1$.

LEMMA 4.5.3. *If $p \in P$ is a positive element with $\varepsilon(p) = n$, then $\gamma^k(p-n) = 0$ for all $k > n$. In particular, if $\ell \in K$ is a line element then $\gamma^k(\ell - 1) = 0$ for every $k > 1$.*

PROOF. If $k > n$ then $q = p + (k-n-1)$ is a positive element with $\varepsilon(q) = k-1$. Thus $\gamma^k(p-n) = \lambda^k(q) = 0$.

If $x \in K$, the *γ-dimension* $\dim_\gamma(x)$ of x is defined to be the largest integer n for which $\gamma^n(x - \varepsilon(x)) \neq 0$, provided n exists. For example, $\dim_\gamma(h) = 0$ for every $h \in H^0$ and $\dim_\gamma(\ell) = 1$ for every line element ℓ (except $\ell = 1$ of course). By the above remarks if $p \in P$ and $n = \varepsilon(p)$ then $\dim_\gamma(p) = \dim_\gamma(p-n) \leq n$. The supremum of the $\dim_\gamma(x)$ for $x \in K$ is called the *γ–dimension* of K.

EXAMPLES 4.5.4. If R is a commutative noetherian ring, the Serre Cancellation I.2.4 states that every element of $\widetilde{K}_0(R)$ is represented by $[P] - n$, where P is a f.g. projective module of rank $< \dim(R)$. Hence $K_0(R)$ has γ-dimension at most $\dim(R)$.

Suppose that X is a CW complex with finite dimension d. The Real Cancellation Theorem I.4.3 allows us to use the same argument to deduce that $KO(X)$ has γ-dimension at most d; the Complex Cancellation Theorem I.4.4 shows that $KU(X)$ has γ-dimension at most $d/2$.

COROLLARY 4.5.5. *If K has a positive structure in which $\mathbb{N}$ is cofinal in P, then every element of $\widetilde{K}$ has finite γ-dimension.*

PROOF. Recall that "$\mathbb{N}$ is cofinal in P" means that for every p there is a p' so that $p + p' = n$ for some $n \in \mathbb{N}$. Therefore every $x \in \widetilde{K}$ can be written as $x = p - m$ for some $p \in P$ with $m = \varepsilon(p)$. By Lemma 4.5.3, $\dim_\gamma(x) \leq m$.

THEOREM 4.6. *If every element of K has finite γ-dimension (e.g., K has a positive structure in which $\mathbb{N}$ is cofinal in P), then $\widetilde{K}$ is a nil ideal. That is, every element of $\widetilde{K}$ is nilpotent.*

PROOF. Fix $x \in \widetilde{K}$, and set $m = \dim_\gamma(x)$, $n = \dim_\gamma(-x)$. Then both $\gamma_t(x) = 1+xt+\gamma^2(x)t^2+\cdots+\gamma^m(x)t^m$ and $\gamma_t(-x) = 1-xt+\cdots+\gamma^n(-x)t^n$ are polynomials in t. Since $\gamma_t(x)\gamma_t(-x) = \gamma_t(0) = 1$, the polynomials $\gamma_t(x)$ and $\gamma_t(-x)$ are units in the polynomial ring $K[t]$. By (I.3.12), the coefficients of these polynomials are nilpotent elements of K.

COROLLARY 4.6.1. *The ideal $\widetilde{K}_0(R)$ is the nilradical of $K_0(R)$ for every commutative ring R.*

If X is compact (or connected and paracompact) then $\widetilde{KO}(X)$ and $\widetilde{KU}(X)$ are the nilradicals of the rings $KO(X)$ and $KU(X)$, respectively.

EXAMPLE 4.6.2. The conclusion of Theorem 4.6 fails for the representation ring $R(G)$ of a cyclic group of order 2. If σ denotes the 1-dimensional sign representation, then $L = \{1, \sigma\}$ and $\widetilde{R}(G) \cong \mathbb{Z}$ is generated by $(\sigma - 1)$. Since $(\sigma - 1)^2 = (\sigma^2 - 2\sigma + 1) = (-2)(\sigma - 1)$, we see that $(\sigma - 1)$ is not nilpotent, and in fact that $\widetilde{R}(G)^n = (2^{n-1})\widetilde{R}(G)$ for every $n \geq 1$. The hypothesis of Corollary 4.5.5 fails here because σ cannot be a summand of a trivial representation. In fact $\dim_\gamma(1-\sigma) = \infty$, because $\gamma^n(1-\sigma) = (1-\sigma)^n = 2^{n-1}(1-\sigma)$ for all $n \geq 1$.

The γ-Filtration

The *γ-filtration* on K is a descending sequence of ideals:

$$K = F^0_\gamma K \supset F^1_\gamma K \supset \cdots \supset F^n_\gamma K \supset \cdots .$$

It starts with $F^0_\gamma K = K$ and $F^1_\gamma K = \widetilde{K}$ (the kernel of ε). The first quotient F^0_γ / F^1_γ is clearly $H^0 = K/\widetilde{K}$. For $n \geq 2$, $F^n_\gamma K$ is defined to be the ideal of K generated by the products $\gamma^{k_1}(x_1) \cdots \gamma^{k_m}(x_m)$ with $x_i \in \widetilde{K}$ and $\sum k_i \geq n$. In particular, $F^n_\gamma K$ contains $\gamma^k(x)$ for all $x \in \widetilde{K}$ and $k \geq n$.

It follows immediately from the definition that $F^i_\gamma F^j_\gamma \subseteq F^{i+j}_\gamma$. For $j = 1$, this implies that the quotients $F^i_\gamma K / F^{i+1}_\gamma K$ are H-modules. We will prove that the quotient F^1_γ / F^2_γ is L:

THEOREM 4.7. *If K satisfies the Splitting Principle, then the map $\ell \mapsto \ell - 1$ induces a group isomorphism, split by the map* det*:*

$$L \xrightarrow{\cong} F^1_\gamma K / F^2_\gamma K.$$

COROLLARY 4.7.1. *For every commutative ring R, the first two ideals in the γ-filtration of $K_0(R)$ are $F^1_\gamma = \widetilde{K}_0(R)$ and $F^2_\gamma = SK_0(R)$. (See 2.6.2.) In particular,*

$$F^0_\gamma / F^1_\gamma \cong H_0(R) \quad \text{and} \quad F^1_\gamma / F^2_\gamma \cong \operatorname{Pic}(R).$$

COROLLARY 4.7.2. *The first two quotients in the γ-filtration of $KO(X)$ are*

$$F^0_\gamma/F^1_\gamma \cong [X,\mathbb{Z}] \quad \textit{and} \quad F^1_\gamma/F^2_\gamma \cong H^1(X;\mathbb{Z}/2).$$

The first few quotients in the γ-filtration of $KU(X)$ are

$$F^0_\gamma/F^1_\gamma \cong [X,\mathbb{Z}] \textit{ and } \quad F^1_\gamma/F^2_\gamma \cong H^2(X;\mathbb{Z}).$$

For the proof of Theorem 4.7, we shall need the following consequence of the Splitting Principle. A proof of this principle may be found in [FL, III.1].

FILTERED SPLITTING PRINCIPLE. *Let K be a λ-ring satisfying the Splitting Principle, and let x be an element of $F^n_\gamma K$. Then there exists a λ-ring extension $K \subset K'$ such that $F^n_\gamma K = K \cap F^n_\gamma K'$, and x is an H-linear combination of products $(\ell_1 - 1)\cdots(\ell_m - 1)$, where the ℓ_i are line elements of K' and $m \geq n$.*

PROOF OF THEOREM 4.7. Since $(\ell_1 - 1)(\ell_2 - 1) \in F^2_\gamma K$, the map $\ell \mapsto \ell - 1$ is a homomorphism. If ℓ_1, ℓ_2, ℓ_3 are line elements of K,

$$\det((\ell_1 - 1)(\ell_2 - 1)\ell_3) = \det(\ell_1\ell_2\ell_3)\det(\ell_3)/det(\ell_1\ell_3)\det(\ell_2\ell_3) = 1.$$

By Ex. 4.3, the Filtered Splitting Principle implies that every element of $F^2_\gamma K$ can be written as a sum of terms $(\ell_1 - 1)(\ell_2 - 1)\ell_3$ in some extension K' of K. This shows that $\det(F^2_\gamma) = 1$, so det induces a map $\widetilde{K}/F^2_\gamma K \to L$. Now det is the inverse of the map $\ell \mapsto \ell - 1$ because for $p \in P$ the Splitting Principle shows that $p - \varepsilon(p) \equiv \det(p) - 1$ modulo $F^2_\gamma K$.

PROPOSITION 4.8. *If the γ-filtration on K is finite then $\widetilde{K}$ is a nilpotent ideal. If $\widetilde{K}$ is a nilpotent ideal which is finitely generated as an abelian group, then the γ-filtration on K is finite. That is, $F^N_\gamma K = 0$ for some N.*

PROOF. The first assertion follows from the fact that $\widetilde{K}^n \subset F^n_\gamma K$ for all n. If $\widetilde{K}$ is additively generated by $\{x_1, ..., x_s\}$, then there is an upper bound on the k for which $\gamma^k(x_i) \neq 0$; using the sum formula there is an upper bound n on the k for which γ^k is nonzero on $\widetilde{K}$. If $\widetilde{K}^m = 0$ then clearly we must have $F^{mn}_\gamma K = 0$.

EXAMPLE 4.8.1. If X is a finite CW complex, both $KO(X)$ and $KU(X)$ are finitely generated abelian groups by Ex. 3.9. Therefore they have finite γ-filtrations.

EXAMPLE 4.8.2. If R is a commutative noetherian ring of Krull dimension d, then $F^{d+1}_\gamma K_0(R) = 0$ by [FL, V.3.10], even though $K_0(R)$ may not be a finitely generated abelian group.

EXAMPLE 4.8.3. For the representation ring $R(G)$, G cyclic of order 2, we saw in Example 4.6.2 that $\widetilde{R}$ is not nilpotent. In fact $F^n_\gamma R(G) = \widetilde{R}^n = 2^{n-1}\widetilde{R} \neq 0$. An even worse example is the λ-ring $R_\mathbb{Q} = R(G) \otimes \mathbb{Q}$, because $F^n_\gamma R_\mathbb{Q} = \widetilde{R}_\mathbb{Q} \cong \mathbb{Q}$ for all $n \geq 1$.

REMARK 4.8.4. Fix x. It follows from the nilpotence of the $\gamma^k(x)$ that there is an integer N such that $x^N = 0$, and for every $k_1, \dots, k_n$ with $\sum k_i > N$ we have

$$\gamma^{k_1}(x)\gamma^{k_2}(x)\cdots\gamma^{k_n}(x) = 0.$$

The best general bound for such an N is $N = mn = \dim_\gamma(x)\dim_\gamma(-x)$.

PROPOSITION 4.9. *Let k, $n \geq 1$ be integers. If $x \in F_\gamma^n K$ then modulo $F_\gamma^{n+1}K$:*

$$\psi^k(x) \equiv k^n x; \qquad \textit{and} \qquad \lambda^k(x) \equiv (-1)^k k^{n-1} x.$$

PROOF. If ℓ is a line element then modulo $(\ell-1)^2$ we have

$$\psi^k(\ell - 1) = (\ell^{k-1} + ... + \ell + 1)(\ell - 1) \equiv k(\ell - 1).$$

Therefore if $\ell_1, \cdots, \ell_m$ are line elements and $m \geq n$ we have

$$\psi^k((\ell_1 - 1)\cdots(\ell_n - 1)) \equiv k^n(\ell_1 - 1)\cdots(\ell_n - 1) \text{ modulo } F_\gamma^{n+1}K.$$

The Filtered Splitting Principle implies that $\psi^k(x) \equiv k^n x$ modulo $F_\gamma^{n+1}K$ for every $x \in F_\gamma^n K$. For λ^k, we use the inductive definition of ψ^k to see that $k^n x = (-1)^{k-1} k \lambda^k(x)$ for every $x \in F_\gamma^n K$. The Filtered Splitting Principle allows us to consider the universal case $W = W_s$ of Exercise 4.4. Since there is no torsion in $F_\gamma^n W / F_\gamma^{n+1} W$, we can divide by k to obtain the formula $k^{n-1}x = (-1)^{k-1}\lambda^k(x)$.

THEOREM 4.10 (STRUCTURE OF $K \otimes \mathbb{Q}$). *Suppose that K has a positive structure in which every element has finite γ-dimension (e.g., if $\mathbb{N}$ is cofinal in P). Then:*

1. *The eigenvalues of ψ^k on $K_\mathbb{Q} = K \otimes \mathbb{Q}$ are a subset of $\{1, k, k^2, k^3, ...\}$ for each k;*
2. *The subspace $K_\mathbb{Q}^{(n)} = K_\mathbb{Q}^{(n,k)}$ of eigenvectors for $\psi^k = k^n$ is independent of k;*
3. *$K_\mathbb{Q}^{(n)}$ is isomorphic to $F_\gamma^n K_\mathbb{Q} / F_\gamma^{n+1} K_\mathbb{Q} \cong (F_\gamma^n K / F_\gamma^{n+1} K) \otimes \mathbb{Q}$;*
4. *$K_\mathbb{Q}^{(0)} \cong H^0 \otimes \mathbb{Q}$ and $K_\mathbb{Q}^{(1)} \cong L \otimes \mathbb{Q}$;*
5. *The ring $K \otimes \mathbb{Q}$ is isomorphic to the graded ring $K_\mathbb{Q}^{(0)} \oplus K_\mathbb{Q}^{(1)} \oplus \cdots \oplus K_\mathbb{Q}^{(n)} \oplus \cdots$.*

PROOF. For every positive p, consider the universal λ-ring $U_\mathbb{Q} = \mathbb{Q}[s_1, ...]$ of Example 4.3.2, and the map $U_\mathbb{Q} \to K_\mathbb{Q}$ sending s_1 to p and s_k to $\lambda^k(p)$. If $\varepsilon(p) = n$ then s_i maps to zero for $i > n$ and each $s_i - \binom{n}{i}$ maps to a nilpotent element by Theorem 4.6. The image A of this map is a λ-ring which is finite-dimensional over $\mathbb{Q}$, so A is an artinian ring. Clearly $F_\gamma^N A = 0$ for some large N. Consider the linear operation $\prod_{n=0}^N (\psi^k - k^n)$ on A; by Proposition 4.9 it is trivial on each $F_\gamma^n / F_\gamma^{n+1}$, so it must be zero. Therefore the characteristic polynomial of ψ^k on A divides $\Pi(t - k^n)$, and has distinct integer eigenvalues. This proves (1) and that $K_\mathbb{Q}$ is the direct sum of the eigenspaces $K_\mathbb{Q}^{(n,k)}$ for ψ^k. As ψ^k preserves products, Proposition 4.9 now implies (3) and (4). The rest is immediate from Theorem 4.7.

Chern class homomorphisms

The formalism in §3 for the Chern classes $c_i\colon KU(X) \to H^{2i}(X;\mathbb{Z})$ extends to the current setting. Suppose we are given a λ-ring K with a positive structure and a commutative graded ring $A = A^0 \oplus A^1 \oplus \cdots$. *Chern classes* on K with values in A are set maps $c_i\colon K \to A^i$ for $i \geq 0$ with $c_0(x) = 1$, satisfying the following axioms:

(CC0) The c_i send H^0 to zero (for $i \geq 1$): $c_i(h) = 0$ for every $h \in H^0$.

(CC1) (Dimension) $c_i(p) = 0$ whenever p is positive and $i \geq \varepsilon(p)$.

(CC2) (Sum Formula) For every x, y in K and every n:

$$c_n(x+y) = \sum_{i=0}^{n} c_i(x)c_{n-i}(y).$$

(CC3) (Normalization) $c_1 \colon L \to A^1$ is a group homomorphism. That is, for ℓ, ℓ':

$$c_1(\ell\ell') = c_1(\ell) + c_1(\ell').$$

The *total Chern class* of x is the element $c(x) = \sum c_i(x)$ of the completion $\hat{A} = \prod A^i$ of A. In terms of the total Chern class, (CC2) becomes the product formula

$$c(x+y) = c(x)c(y).$$

Example 4.11.1. The Stiefel-Whitney classes $w_i \colon KO(X) \to A^i = H^i(X; \mathbb{Z}/2)$ and the Chern classes $c_i \colon KU(X) \to A^i = H^{2i}(X; \mathbb{Z})$ are Chern classes in this sense.

Example 4.11.2. Associated to the γ-filtration on K we have the associated graded ring $Gr^{\bullet}_{\gamma}K$ with $Gr^i_{\gamma}K = F^i_{\gamma}/F^{i+1}_{\gamma}$. For a positive element p in K, define $c_i(p)$ to be $\gamma^i(p - \varepsilon(p))$ modulo F^{i+1}_{γ}. The multiplicative formula for γ_t implies that $c_i(p+q) = c_i(p) + c_i(q)$, so that the c_i extend to classes $c_i \colon K \to Gr^{\bullet}_{\gamma}K$. The total Chern class $c \colon K \to Gr^{\bullet}_{\gamma}K$ is a group homomorphism with torsion kernel and cokernel, because by Theorem 4.10 and Ex. 4.10 the induced map $c_n \colon K^{(n)}_{\mathbb{Q}} \to Gr^n_{\gamma}K_{\mathbb{Q}} \cong K^{(n)}_{\mathbb{Q}}$ is multiplication by $(-1)^n(n-1)!$.

The Splitting Principle implies the following Splitting Principle (see [FL, I.3.1]).

Chern Splitting Principle. *Given a finite set $\{p_i\}$ of positive elements of K, there is a λ-ring extension $K \subset K'$ in which each p_i splits as a sum of line elements, and a graded extension $A \subset A'$ such that the c_i extend to maps $c_i \colon K' \to (A')^i$ satisfying (CC1) and (CC2).*

The existence of "Chern roots" is an important consequence of this Splitting Principle. Suppose that $p \in K$ is positive, and that in an extension K' of K we can write $p = \ell_1 + \cdots + \ell_n$, $n = \varepsilon(p)$. The *Chern roots of* p are the elements $a_i = c_1(\ell_i)$ in $(A')^1$; they determine the $c_k(p)$ in A^k. Indeed, because $c(p)$ is the product of the $c(\ell_i) = 1 + a_i$, we see that $c_k(p)$ is the k^{th} elementary symmetric polynomial $\sigma_k(a_1, ..., a_n)$ of the a_i in the larger ring A'. In particular, the first Chern class is $c_1(p) = \sum a_i$ and the "top" Chern class is $c_n(p) = \prod a_i$.

A famous theorem of Isaac Newton states that every symmetric polynomial in n variables $t_1, ..., t_n$ is in fact a polynomial in the symmetric polynomials $\sigma_k = \sigma_k(t_1, ..., t_n)$, $k = 1, 2, \cdots$. Therefore every symmetric polynomial in the Chern roots of p is also a polynomial in the Chern classes $c_k(p)$, and as such belongs to the subring A of A'. Here is an elementary application of these ideas.

PROPOSITION 4.11.3. *Suppose that K satisfies the Splitting Principle. Then $c_n(\psi^k x) = k^n c_n(x)$ for all $x \in K$. That is, the following diagram commutes:*

$$\begin{array}{ccc} K & \xrightarrow{c_n} & A^n \\ \downarrow{\scriptstyle \psi^k} & & \downarrow{\scriptstyle k^n} \\ K & \xrightarrow{c_n} & A^n. \end{array}$$

COROLLARY 4.11.4. *If $\mathbb{Q} \subset A$ then c_n vanishes on $K_{\mathbb{Q}}^{(i)}$, $i \neq n$.*

Chern character

As an application of the notion of Chern roots, suppose given Chern classes $c_i\colon K \to A^i$, where for simplicity A is an algebra over $\mathbb{Q}$. If $p \in K$ is a positive element, with Chern roots a_i, define $ch(p)$ to be the formal expansion

$$ch(p) = \sum_{i=0}^{n} \exp(a_i) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=0}^{n} a_i^k \right)$$

of terms in A'. The k^{th} term $\frac{1}{k!} \sum a_i^k$ is symmetric in the Chern roots, so it is a polynomial in the Chern classes $c_1(p), ..., c_k(p)$ and hence belongs to A^k. Therefore $ch(p)$ is a formal expansion of terms in A, *i.e.*, an element of $\hat{A} = \prod A^k$. For example, if ℓ is a line element of K then $ch(\ell)$ is just $\exp(c_1(\ell))$. From the definition, it is immediate that $ch(p+q) = ch(p) + ch(q)$, so ch extends to a map from $P^{-1}P$ to $\hat{A}$. Since $ch(1) = 1$, this is compatible with the given map $H^0 \to A^0$, and so it defines a map $ch\colon K \to \hat{A}$, called the *Chern character* on K. The first few terms in the expansion of the Chern character are

$$ch(x) = \varepsilon(x) + c_1(x) + \frac{1}{2}[c_1(x)^2 - c_2(x)] + \frac{1}{6}[c_1(x)^3 - 3c_1(x)c_2(x) + 3c_3(x)] + \cdots$$

An inductive formula for the term in $ch(x)$ is given in Exercise 4.14.

PROPOSITION 4.12. *If $\mathbb{Q} \subset A$ then the Chern character is a ring homomorphism*

$$ch\colon K \to \hat{A}.$$

PROOF. By the Splitting Principle, it suffices to verify that $ch(pq) = ch(p)ch(q)$ when p and q are sums of line elements. Suppose that $p = \sum \ell_i$ and $q = \sum m_j$ have Chern roots $a_i = c_1(\ell_i)$ and $b_j = c_1(m_j)$, respectively. Since $pq = \sum \ell_i m_j$, the Chern roots of pq are the $c_1(\ell_i m_j) = c_1(\ell_i) + c_1(m_j) = a_i + b_j$. Hence

$$ch(pq) = \sum ch(\ell_i m_j) = \sum exp(a_i + b_j) = \sum exp(a_i)\exp(b_j) = ch(p)ch(q).$$

COROLLARY 4.12.1. *Suppose that K has a positive structure in which every $x \in K$ has finite γ-dimension (e.g., $\mathbb{N}$ is cofinal in P). Then the Chern character lands in A, and the induced map from $K_{\mathbb{Q}} = \oplus K_{\mathbb{Q}}^{(n)}$ to A is a graded ring map. That is, the n^{th} term $ch_n\colon K_{\mathbb{Q}} \to A^n$ vanishes on $K_{\mathbb{Q}}^{(i)}$ for $i \neq n$.*

EXAMPLE 4.12.2. The universal Chern character $ch\colon K_{\mathbb{Q}} \to K_{\mathbb{Q}}$ is the identity map. Indeed, by Ex. 4.10(b) and Ex. 4.14 we see that ch_n is the identity map on each $K_{\mathbb{Q}}^{(n)}$.

The following result was proven by M. Karoubi in [Kar63]. (See Exercise 4.11 for the proof when X is a finite CW complex.)

THEOREM 4.13. *If X is a compact topological space and $\check{H}$ denotes Čech cohomology, then the Chern character is an isomorphism of graded rings.*

$$ch\colon KU(X) \otimes \mathbb{Q} \cong \bigoplus \check{H}^{2i}(X; \mathbb{Q})$$

EXAMPLE 4.13.1 (SPHERES). For each even sphere, we know by Example 3.7.1 that c_n maps $\widetilde{KU}(S^{2n})$ isomorphically onto $H^{2n}(S^{2n}; \mathbb{Z}) = \mathbb{Z}$. The inductive formula for ch_n shows that in this case $ch(x) = \dim(x) + (-1)^n c_n(x)/(n-1)!$ for all $x \in KU(X)$. In this case it is easy to see directly that $ch\colon KU(S^{2n}) \otimes \mathbb{Q} \cong H^{2*}(S^{2n}; \mathbb{Q})$

EXERCISES

4.1 Show that in $K_0(R)$ or $K^0(X)$ we have

$$\lambda^k([P] - n) = \sum (-1)^i \binom{n+i-1}{i} [\wedge^{k-i} P].$$

4.2 For every group G and every commutative ring R, let $R_A(G)$ denote the group $K_0(RG, R)$ of Ex. 2.14, *i.e.*, the group completion of the monoid $Rep(RG, R)$ of all RG-modules which are f.g. projective as R-modules. Show that $R_A(G)$ is a λ-ring with a positive structure given by $Rep(RG, R)$. Then show that $R_A(G)$ satisfies the Splitting Principle.

4.3 Suppose that a λ-ring K is generated as an H-algebra by line elements. Show that $F_\gamma^n = \widetilde{K}^n$ for all n, so the γ-filtration is the adic filtration defined by the ideal $\widetilde{K}$. Then show that if K is any λ-ring satisfying the Splitting Principle every element x of $F_\gamma^n K$ can be written in an extension K' of K as a product

$$x = (\ell_1 - 1) \cdots (\ell_m - 1)$$

of line elements with $m \geq n$. In particular, show that every $x \in F_\gamma^2$ can be written as a sum of terms $(\ell_i - 1)(\ell_j - 1)\ell$ in K'.

4.4 *Universal special λ-ring.* Let W_s denote the Laurent polynomial ring $\mathbb{Z}[u_1, u_1^{-1}, ..., u_s, u_s^{-1}]$, and $\varepsilon\colon W_s \to \mathbb{Z}$ the augmentation defined by $\varepsilon(u_i) = 1$.

(a) Show that W_s is a λ-ring with a positive structure, the line elements being the monomials $u^\alpha = \prod u_i^{n_i}$. This implies that W_s is generated by the group $L \cong \mathbb{Z}^s$ of line elements, so by Exercise 4.3 $F_\gamma^n W_s$ is $\widetilde{W}^n$.

(b) Show that each $F_\gamma^n W / F_\gamma^{n+1} W$ is a torsionfree abelian group.

(c) If K is a special λ-ring show that any family $\{\ell_1, ..., \ell_s\}$ of line elements determines a λ-ring map $W_s \to K$ sending u_i to ℓ_i.

4.5 A line element ℓ is called *ample* for K if for every $x \in \widetilde{K}$ there is an integer $N = N(x)$ such that for every $n \geq N$ there is a positive element p_n so that $\ell^n x = p_n - \varepsilon(p_n)$. (The terminology comes from Algebraic Geometry; see 8.7.4 below.) If K has an ample line element, show that every element of $\widetilde{K}$ is nilpotent.

4.6 Verify that the inductive definition of ψ^k agrees with the ψ_t definition of ψ^k.

4.7 If p is prime, use the Splitting Principle to verify that $\psi^p(x) \equiv x^p$ modulo p for every $x \in K$.

4.8 *Adams e-invariant.* Suppose given a map $f\colon S^{2m-1} \to S^{2n}$. The mapping cone $C(f)$ fits into a cofibration sequence $S^{2n} \overset{i}{\rightarrowtail} C(f) \overset{j}{\twoheadrightarrow} S^{2m}$. Associated to this is the exact sequence:

$$0 \to \widetilde{KU}(S^{2m}) \xrightarrow{j^*} \widetilde{KU}(C) \xrightarrow{i^*} \widetilde{KU}(S^{2n}) \to 0.$$

Choose $x, y \in \widetilde{KU}(C)$ so that $i^*(x)$ generates $\widetilde{KU}(S^{2n}) \cong \mathbb{Z}$ and y is the image of a generator of $\widetilde{KU}(S^{2m}) \cong \mathbb{Z}$. Since j^* is a ring map, $y^2 = 0$.

(a) Show by applying ψ^k that $xy = 0$, and that if $m \neq 2n$ then $x^2 = 0$. (When $m = 2n$, x^2 defines the Hopf invariant of f; see the next exercise.)

(b) Show that $\psi^k(x) = k^n x + a_k y$ for appropriate integers a_k. Then show (for fixed x and y) that the rational number

$$e(f) = \frac{a_k}{k^m - k^n}$$

is independent of the choice of k.

(c) Show that a different choice of x only changes $e(f)$ by an integer, so that $e(f)$ is a well-defined element of $\mathbb{Q}/\mathbb{Z}$; $e(f)$ is called the *Adams e-invariant* of f.

(d) If f and f' are homotopic maps, it follows from the homotopy equivalence between $C(f)$ and $C(f')$ that $e(f) = e(f')$. By considering the mapping cone of $f \vee g$, show that the well-defined set map $e\colon \pi_{2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}$ is a group homomorphism. J.F. Adams used this e-invariant to detect an important cyclic subgroup of $\pi_{2m-1}(S^{2n})$, namely the "image of J."

4.9 *Hopf Invariant One.* Given a continuous map $f\colon S^{4n-1} \to S^{2n}$, define an integer $H(f)$ as follows. Let $C(f)$ be the mapping cone of f. As in the previous exercise, we have an exact sequence:

$$0 \to \widetilde{KU}(S^{4n}) \xrightarrow{j^*} \widetilde{KU}(C(f)) \xrightarrow{i^*} \widetilde{KU}(S^{2n}) \to 0.$$

Choose $x, y \in \widetilde{KU}(C(f))$ so that $i^*(x)$ generates $\widetilde{KU}(S^{2n}) \cong \mathbb{Z}$ and y is the image of a generator of $\widetilde{KU}(S^{4n}) \cong \mathbb{Z}$. Since $i^*(x^2) = 0$, we can write $x^2 = Hy$ for some integer H; this integer $H = H(f)$ is called the *Hopf invariant* of f.

(a) Show that $H(f)$ is well-defined, up to $\pm$ sign.

(b) If $H(f)$ is odd, show that n is 1, 2, or 4. *Hint:* Use Ex. 4.7 to show that the integer a_2 of the previous exercise is odd. Considering $e(f)$, show that 2^n divides $p^n - 1$ for every odd p.

It turns out that the classical "Hopf maps" $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$ all have Hopf invariant $H(f) = 1$. In contrast, for every even integer H there is a map $S^{4n-1} \to S^{2n}$ with Hopf invariant H.

4.10 *Operations.* A natural *operation* τ on λ-rings is a map $\tau\colon K \to K$ defined for every λ-ring K such that $f\tau = \tau f$ for every λ-ring map $f\colon K \to K'$. The operations λ_k, γ_k, and ψ_k are all natural operations on λ-ring.

(a) If K satisfies the Splitting Principle, generalize Proposition 4.9 to show that every natural operation τ preserves the γ-filtration of K and that there are integers $\omega_n = \omega_n(\tau)$, independent of K, such that for every $x \in F_\gamma^n K$

$$\tau(x) \equiv \omega_n x \text{ modulo } \; F_\gamma^{n+1} K.$$

(b) Show that for $\tau = \gamma^k$ and $x \in F_\gamma^n$ we have:

$$\gamma^k_{(x)} = \begin{cases} 0 \quad \text{if} \quad n < k \\ (-1)^{k-1}(k-1)! \text{ if } \quad n = k \\ \omega_n \neq 0 \quad \text{if} \quad n > k \end{cases}$$

(c) Show that $x_k \mapsto \lambda^k$ gives a ring map from the power series ring $\mathbb{Z}[[x_1, x_2, \cdots]]$ to the ring $\mathbb{C}$ of all natural operations on λ-rings. In fact this is a ring isomorphism; see [Atiyah, 3.1.7].

4.11 By Example 4.13.1, the Chern character $ch\colon KU(S^n) \otimes \mathbb{Q} \to H^{2*}(S^n; \mathbb{Q})$ is an isomorphism for every sphere S^n. Use this to show that $ch\colon KU(X) \otimes \mathbb{Q} \to H^{2*}(X; \mathbb{Q})$ is an isomorphism for every finite CW complex X.

4.12 Let K be a λ-ring. Given a K-module M, construct the ring $K \oplus M$ in which $M^2 = 0$. Given a sequence of K-linear endomorphisms φ_k of M with $\varphi_1(x) = x$, show that the formulae $\lambda^k(x) = \varphi_k(x)$ extend the λ-ring structure on K to a λ-ring structure on $K \oplus M$. Then show that $K \oplus M$ has a positive structure if K does, and that $K \oplus M$ satisfies the Splitting Principle whenever K does. (The elements in $1 + M$ are to be the new line elements.)

4.13 *Hirzebruch characters.* Suppose that A is an H^0-algebra and we fix a power series $\alpha(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \cdots$ in $A^0[[t]]$. Suppose given Chern classes $c_i\colon K \to A^i$. If $p \in K$ is a positive element, with Chern roots a_i, define $ch_\alpha(p)$ to be the formal expansion

$$ch_\alpha(p) = \sum_{i=0}^{n} \alpha(a_i) \sum_{k=0}^{\infty} \alpha_k \left(\sum_{i=0}^{n} a_i^k \right)$$

of terms in A'. Show that $ch_\alpha(p)$ belongs to the formal completion $\hat{A}$ of A, and that it defines a group homomorphism $ch_\alpha\colon K \to \hat{A}$. This map is called the *Hirzebruch character* for α.

4.14 Establish the following inductive formula for the n^{th} term ch_n in the Chern character:

$$ch_n - \frac{1}{n} c_1 ch_{n-1} + \cdots \pm \frac{1}{i!\binom{n}{i}} c_i ch_{n-i} + \cdots + \frac{(-1)^n}{(n-1)!} c_n = 0.$$

To do this, set $x = -t_i$ in the identity $\prod(x + a_i) = x^n + c_1 x^{n-1} + \cdots + c_n$.

§5. K_0 of a Symmetric Monoidal Category

The idea of group completion in §1 can be applied to more categories than just the categories $\mathbf{P}(R)$ in §2 and $\mathbf{VB}(X)$ in §3. It applies to any category with a "direct sum", or more generally any natural product $\square$ making the isomorphism classes of objects into an abelian monoid. This leads us to the notion of a symmetric monoidal category.

DEFINITION 5.1. A *symmetric monoidal category* is a category S, equipped with a functor $\square\colon S \times S \to S$, a distinguished object e and four basic natural isomorphisms:

$$e\square s \cong s, \quad s\square e \cong s, \quad s\square(t\square u) \cong (s\square t)\square u, \quad \text{and } s\square t \cong t\square s.$$

These basic isomorphisms must be "coherent" in the sense that two natural isomorphisms of products of $s_1, \dots, s_n$ built up from the four basic ones are the same whenever they have the same source and target. (We refer the reader to [Mac] for the technical details needed to make this definition of "coherent" precise.) Coherence permits us to write expressions without parentheses like $s_1\square\cdots\square s_n$ unambiguously (up to natural isomorphism).

EXAMPLE 5.1.1. Any category with a direct sum $\oplus$ is symmetric monoidal; this includes additive categories like $\mathbf{P}(R)$ and $\mathbf{VB}(X)$ as we have mentioned. More generally, a category with finite coproducts is symmetric monoidal with $s\square t = s\amalg t$. Dually, any category with finite products is symmetric monoidal with $s\square t = s \times t$.

DEFINITION 5.1.2 (K_0S). Suppose that the isomorphism classes of objects of S form a *set*, which we call S^{iso}. If S is symmetric monoidal, this set S^{iso} is an abelian monoid with product $\square$ and identity e. The group completion of this abelian monoid is called the *Grothendieck group* of S, and is written as $K_0^{\square}(S)$, or simply as $K_0(S)$ if $\square$ is understood.

From §1 we see that $K_0^{\square}(S)$ may be presented with one generator $[s]$ for each isomorphism class of objects, with relations that $[s\square t] = [s] + [t]$ for each s and t. From Proposition 1.1 we see that every element of $K_0^{\square}(S)$ may be written as a difference $[s] - [t]$ for some objects s and t.

EXAMPLES 5.2. (1) The category $\mathbf{P}(R)$ of f.g. projective modules over a ring R is symmetric monoidal under direct sum. Since the above definition is identical to that in §2, we see that we have $K_0(R) = K_0^{\oplus}(\mathbf{P}(R))$.

(2) Similarly, the category $\mathbf{VB}(X)$ of (real or complex) vector bundles over a topological space X is symmetric monoidal, with $\square$ being the Whitney sum $\oplus$. From the definition we see that we also have $K(X) = K_0^{\oplus}(\mathbf{VB}(X))$, or more explicitly:

$$KO(X) = K_0^{\oplus}(\mathbf{VB}_{\mathbb{R}}(X)), \qquad KU(X) = K_0^{\oplus}(\mathbf{VB}_{\mathbb{C}}(X)).$$

(3) Let $\mathbf{Sets}_f$ denote the category of finite sets. It has a coproduct, the disjoint sum $\amalg$, and it is not hard to see that $K_0^{\amalg}(\mathbf{Sets}_f) = \mathbb{Z}$. It also has a product $(\times)$, but since the empty set satisfies $\emptyset = \emptyset \times X$ for all X we have $K_0^{\times}(\mathbf{Sets}_f) = 0$.

(4) The category $\mathbf{Sets}_f^{\times}$ of nonempty finite sets has for its isomorphism classes the set $\mathbb{N}_{>0} = \{1, 2, \dots\}$ of positive integers, and the product of finite sets corresponds

to multiplication. Since the group completion of $(\mathbb{N}_{>0}, \times)$ is the multiplicative monoid $\mathbb{Q}\times_{>0}$ of positive rational numbers, we have $K_0^\times(\mathbf{Sets}\times_f) \cong \mathbb{Q}\times_{>0}$.

(5) If R is a commutative ring, let $\mathbf{Pic}(R)$ denote the category of algebraic line bundles L over R and their isomorphisms (§I.3). This is a symmetric monoidal category with $\Box = \otimes_R$, and the isomorphism classes of objects already form a group, so $K_0\mathbf{Pic}(R) = \mathrm{Pic}(R)$.

Cofinality

Let T be a full subcategory of a symmetric monoidal category S. If T contains e and is closed under finite products, then T is also symmetric monoidal. We say that T is *cofinal* in S if for every s in S there is an s' in S such that $s\Box s'$ is in T, *i.e.*, if the abelian monoid T^{iso} is cofinal in S^{iso} in th sense of §1. When this happens, we may restate Corollary 1.3 as follows.

Cofinality Theorem 5.3. *Let T be cofinal in a symmetric monoidal category S. Then (assuming S^{iso} is a set):*

(1) *$K_0(T)$ is a subgroup of $K_0(S)$;*
(2) *Every element of $K_0(S)$ is of the form $[s] - [t]$ for some s in S and t in T;*
(3) *If $[s] = [s']$ in $K_0(S)$ then $s\Box t = s'\Box t$ for some t in T.*

Example 5.4.1 (Free modules). Let R be a ring. The category $\mathbf{F}(R)$ of f.g. free R-modules is cofinal (for $\Box = \oplus$) in the category $\mathbf{P}(R)$ of f.g. projective modules. Hence $K_0\mathbf{F}(R)$ is a subgroup of $K_0(R)$. In fact $K_0\mathbf{F}(R)$ is is a cyclic abelian group, equal to $\mathbb{Z}$ whenever R satisfies the Invariant Basis Property. Moreover, the subgroup $K_0\mathbf{F}(R)$ of $K_0(R) = K_0\mathbf{P}(R)$ is the image of the map $\mathbb{Z} \to K_0(R)$ described in Lemma 2.1.

$K_0\mathbf{F}(R)$ is also cofinal in the smaller category $\mathbf{P}^{st.free}(R)$ of f.g. stably free modules. Since every stably free module P satisfies $P \oplus R^m \cong R^n$ for some m and n, the Cofinality Theorem yields $K_0\mathbf{F}(R) = K_0\mathbf{P}^{st.free}(R)$.

Example 5.4.2. Let R be a commutative ring. A f.g. projective R-module is called *faithfully projective* if its rank is never zero. The tensor product of faithfully projective modules is again faithfully projective by Ex. 2.7. Hence the category $\mathbf{FP}(R)$ of faithfully projective R-modules is a symmetric monoidal category under the tensor product $\otimes_R$. For example, if R is a field then the monoid $\mathbf{FP}^{\text{iso}}$ is the multiplicative monoid $(\mathbb{N}_{>0}, \times)$ of Example 5.2(4), so in this case we have $K_0^\otimes\mathbf{FP}(R) \cong \mathbb{Q}\times_{>0}$. We will describe the group $K_0^\otimes\mathbf{FP}(R)$ in the exercises below.

Example 5.4.3 (Brauer groups). Suppose first that F is a field, and let $\mathbf{Az}(F)$ denote the category of central simple F-algebras. This is a symmetric monoidal category with product $\otimes_F$, because if A and B are central simple then so is $A \otimes_F B$. The matrix rings $M_n(F)$ form a cofinal subcategory, with $M_m(F) \otimes_F M_n(F) \cong M_{mn}(F)$. From the previous example we see that the Grothendieck group of this subcategory is $\mathbb{Q}\times_{>0}$. The classical *Brauer group* $Br(F)$ of the field F is the quotient of $K_0\mathbf{Az}(F)$ by this subgroup. That is, $Br(F)$ is generated by classes $[A]$ of central simple algebras with two families of relations: $[A \otimes_F B] = [A] + [B]$ and $[M_n(F)] = 0$.

More generally, suppose that R is a commutative ring. Recall that an R-algebra A is is called an *Azumaya algebra* if there is another R-algebra B such that $A \otimes_R B \cong$

$M_n(R)$ for some n. The category $\mathbf{Az}(R)$ of Azumaya R-algebras is thus symmetric monoidal with product $\otimes_R$. If P is a faithfully projective R-module, $\mathrm{End}_R(P)$ is an Azumaya algebra. Since $\mathrm{End}_R(P \otimes_R P') \cong \mathrm{End}_R(P) \otimes_R \mathrm{End}_R(P')$, there is a monoidal functor End_R from $\mathbf{FP}(R)$ to $\mathbf{Az}(R)$, and a group homomorphism $K_0\mathbf{FP}(R) \to K_0\mathbf{Az}(R)$. The cokernel $Br(R)$ of this map is called the *Brauer group* of R. That is, $Br(R)$ is generated by classes $[A]$ of Azumaya algebras with two families of relations: $[A \otimes_R B] = [A] + [B]$ and $[\mathrm{End}_R(P)] = 0$.

Burnside Ring 5.4.4. Suppose that G is a finite group, and let G-$\mathbf{Sets}_{fin}$ denote the category of finite G-sets. It is a symmetric monoidal category under disjoint union. We saw in Example 1.5 that $K_0(G\text{-}\mathbf{Sets}_{fin})$ is the Burnside Ring $A(G) \cong \mathbb{Z}^c$, where c is the number of conjugacy classes of subgroups of G.

Representation ring 5.4.5. Similarly, the finite-dimensional complex representations of a finite group G form a category $\mathbf{Rep}_{\mathbb{C}}(G)$. It is symmetric monoidal under $\oplus$. We saw in Example 1.6 that $K_0\mathbf{Rep}_{\mathbb{C}}(G)$ is the representation ring $R(G)$ of G, which is a free abelian group on the classes $[V_1], ..., [V_r]$ of the irreducible representations of G.

G-bundles and equivariant K-theory

The following discussion is taken from the very readable book [Atiyah]. Suppose that G is a finite group and that X is a topological space on which G acts continuously. A (complex) vector bundle E over X is called a *G-vector bundle* if G acts continuously on E, the map $E \to X$ commutes with the action of G, and for each $g \in G$ and $x \in X$ the map $E_x \to E_{gx}$ is a vector space homomorphism. The category $\mathbf{VB}_G(X)$ of G-vector bundles over X is symmetric monoidal under the usual Whitney sum, and we write $K_G^0(X)$ for the Grothendieck group $K_0^{\oplus}\mathbf{VB}_G(X)$. For example, if X is a point then $\mathbf{VB}_G(X) = \mathbf{Rep}_{\mathbb{C}}(G)$, so we have $K_G^0(\text{point}) = R(G)$. More generally, if x is a fixed point of X, then $E \mapsto E_x$ defines a monoidal functor from $\mathbf{VB}_G(X)$ to $\mathbf{Rep}_{\mathbb{C}}(G)$, and hence a group map $K_G^0(X) \to R(G)$.

If G acts trivially on X, every vector bundle E on X can be considered as a G-bundle with trivial action, and the tensor product $E \otimes V$ with a representation V of G is another G-bundle. The following result is proven on p. 38 of [Atiyah].

Proposition 5.5 (Krull-Schmidt Theorem). *Let $V_1, ..., V_r$ be a complete set of irreducible G-modules, and suppose that G acts trivially on X. Then for every G-bundle F over X there are unique vector bundles $E_i = \mathrm{Hom}_G(V_i, F)$ so that*

$$F \cong (E_1 \otimes V_1) \oplus \cdots \oplus (E_r \otimes V_r).$$

Corollary 5.5.1. *If G acts trivially on X then $K_G^0(X) \cong KU(X) \otimes_{\mathbb{Z}} R(G)$.*

The Witt ring $W(F)$ of a field

5.6. Symmetric bilinear forms over a field F provide another classical application of the K_0 construction. The following discussion is largely taken from the pretty book [M-SBF], and the reader is encouraged to look there for the connections with other branches of mathematics.

A *symmetric inner product space* (V, B) is a finite dimensional vector space V, equipped with a nondegenerate symmetric bilinear form $B\colon V \otimes V \to F$. The category $\mathbf{SBil}(F)$ of symmetric inner product spaces and form-preserving maps is symmetric monoidal, where the operation $\square$ is the orthogonal sum $(V, B) \oplus (V', B')$, defined as the vector space $V \oplus V'$, equipped with the bilinear form $\beta(v \oplus v', w \oplus w') = B(v, w) + B'(v', w')$.

A crucial role is played by the *hyperbolic plane* H, which is $V = F^2$ equipped with the bilinear form B represented by the symmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. An inner product space is called *hyperbolic* if it is isometric to an orthogonal sum of hyperbolic planes.

Let $(V, B) \otimes (V', B')$ denote the tensor product $V \otimes V'$, equipped with the bilinear form $\beta(v \otimes v', w \otimes w') = B(v, w)B'(v', w')$; this is also a symmetric inner product space, and the isometry classes of inner product spaces forms a semiring under $\oplus$ and $\otimes$ (see Ex. 5.9). Thus $K_0\mathbf{SBil}(F)$ is a commutative ring with unit $1 = \langle 1 \rangle$, and the forgetful functor $\mathbf{SBil}(F) \to \mathbf{P}(F)$ sending (V, B) to V induces a ring augmentation $\varepsilon\colon K_0\mathbf{SBil}(F) \to K_0(F) \cong \mathbb{Z}$. We write $\hat{I}$ for the augmentation ideal of $K_0\mathbf{SBil}(F)$.

Example 5.6.1. For each $a \in F^\times$, we write $\langle a \rangle$ for the inner product space with $V = F$ and $B(v, w) = avw$. Clearly $\langle a \rangle \otimes \langle b \rangle \cong \langle ab \rangle$. Note that a change of basis $1 \mapsto b$ of F induces an isometry $\langle a \rangle \cong \langle ab^2 \rangle$ for every unit b, so the inner product space only determines a up to a square.

If $\mathrm{char}(F) \neq 2$, it is well known that every symmetric bilinear form is diagonalizable. Thus every symmetric inner product space is isometric to an orthogonal sum $\langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle$. For example, it is easy to see that $H \cong \langle 1 \rangle \oplus \langle -1 \rangle$. This also implies that $\hat{I}$ is additively generated by the elements $\langle a \rangle - 1$.

If $\mathrm{char}(F) = 2$, every symmetric inner product space is isomorphic to $\langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle \oplus N$, where N is hyperbolic; see [M-SBF, I.3]. In this case $\hat{I}$ has the extra generator $H - 2$.

If $\mathrm{char}(F) \neq 2$, there is a Cancellation Theorem due to Witt: if X, Y, Z are inner product spaces, then $X \oplus Y \cong X \oplus Z$ implies that $Y \cong Z$. For a proof, we refer the reader to [M-SBF]. We remark that cancellation fails if $\mathrm{char}(F) = 2$; see Ex. 5.10(d). The following definition is due to Knebusch.

Definition 5.6.2. Suppose that $\mathrm{char}(F) \neq 2$. The *Witt ring* $W(F)$ is defined to be the quotient of the ring $K_0\mathbf{SBil}(F)$ by the subgroup $\{nH\}$ generated by the hyperbolic plane H. This subgroup is an ideal by Ex. 5.10, so $W(F)$ is also a commutative ring.

Since the augmentation $K_0\mathbf{SBil}(F) \to \mathbb{Z}$ has $\varepsilon(H) = 2$, it induces an augmentation $\varepsilon\colon W(F) \to \mathbb{Z}/2$. We write I for the augmentation ideal $\ker(\varepsilon)$ of $W(F)$.

When $\mathrm{char}(F) = 2$, $W(F)$ is defined similarly, as the quotient of $K_0\mathbf{SBil}(F)$ by the subgroup of "split" spaces; see Ex. 5.10. In this case we have $2 = 0$ in the Witt ring $W(F)$, because the inner product space $\langle 1 \rangle \oplus \langle 1 \rangle$ is split (Ex. 5.10(d)).

When $\mathrm{char}(F) \neq 2$, the augmentation ideals of $K_0\mathbf{SBil}(F)$ and $W(F)$ are isomorphic: $\hat{I} \cong I$. This is because $\varepsilon(nH) = 2n$, so that $\{nH\} \cap \hat{I} = 0$ in $K_0\mathbf{SBil}(F)$.

Since $(V, B) + (V, -B) = 0$ in $W(F)$ by Ex. 5.10, every element of $W(F)$ is represented by an inner product space. In particular, I is additively generated by the classes $\langle a \rangle + \langle -1 \rangle$, even if $\mathrm{char}(F) = 2$. The powers I^n of I form a decreasing chain of ideals $W(F) \supset I \supset I^2 \supset \cdots$. We shall describe I/I^2 now, and return to this topic in chapter III, §7.

The discriminant of an inner product space (V, B) is a classical invariant with values in $F^\times/F^{\times 2}$, where $F^{\times 2}$ denotes $\{a^2 | a \in F^\times\}$. For each basis of V, there is a matrix M representing B, and the determinant of M is a unit of F. A change of basis replaces M by A^tMA, and $\det(A^tMA) = \det(M)\det(A)^2$, so $w_1(V, B) = \det(M)$ is a well defined element in $F^\times/F^{\times 2}$, called the *first Stiefel-Whitney class* of (V, B). Since $w_1(H) = -1$, we have to modify the definition slightly in order to get an invariant on the Witt ring.

DEFINITION 5.6.3. If $\dim(V) = r$, the *discriminant* of (V, B) is defined to be the element $d(V, B) = (-1)^{r(r-1)/2}\det(M)$ of $F^\times/F^{\times 2}$.

For example, we have $d(H) = d(1) = 1$ but $d(2) = -1$. It is easy to verify that the discriminant of $(V, B) \oplus (V', B')$ is $(-1)^{rr'}d(V, B)d(V', B')$, where $r = \dim(V)$ and $r' = \dim(V')$. In particular, (V, B) and $(V, B) \oplus H$ have the same discriminant. It follows that the discriminant is a well-defined map from $W(F)$ to $F^\times/F^{\times 2}$, and its restriction to I is additive.

THEOREM 5.6.4. *(Pfister) The discriminant induces an isomorphism between I/I^2 and $F^\times/F^{\times 2}$.*

PROOF. Since the discriminant of $\langle a\rangle \oplus \langle -1\rangle$ is a, the map $d\colon I \to F^\times/F^{\times 2}$ is onto. This homomorphism annihilates I^2 because I^2 is additively generated by products of the form

$$\big(\langle a\rangle - 1\big)\big(\langle b\rangle - 1\big) = \langle ab\rangle + \langle -a\rangle + \langle -b\rangle + 1,$$

and these have discriminant 1. Setting these products equal to zero, the identity $\langle a\rangle + \langle -a\rangle = 0$ yields the congruence

$$\big(\langle a\rangle - 1\big) + \big(\langle b\rangle - 1\big) \equiv \langle ab\rangle - 1 \qquad \text{mod } I^2. \tag{5.6.5}$$

Hence the formula $s(a) = \langle a\rangle - 1$ defines a surjective homomorphism $s\colon F^\times \to I/I^2$. Since $ds(a) = a$, it follows that s is an isomorphism with inverse induced by d.

COROLLARY 5.6.6. *$W(F)$ contains $\mathbb{Z}/2$ as a subring (i.e., $2 = 0$) if and only if -1 is a square in F.*

CLASSICAL EXAMPLES 5.6.7. If F is an algebraically closed field, or more generally every element of F is a square, then $\langle a\rangle \cong \langle 1\rangle$ and $W(F) = \mathbb{Z}/2$.

If $F = \mathbb{R}$, every bilinear form is classified by its rank and signature. For example, $\langle 1\rangle$ has signature 1 but H has signature 0, with $H \otimes H \cong H \oplus H$. Thus $K_0\mathbf{SBil}(\mathbb{R}) \cong \mathbb{Z}[H]/(H^2 - 2H)$ and the signature induces a ring isomorphism $W(\mathbb{R}) \cong \mathbb{Z}$.

If $F = \mathbb{F}_q$ is a finite field with q odd, then $I/I^2 \cong \mathbb{Z}/2$, and an elementary argument due to Steinberg shows that the ideal I^2 is zero. The structure of the ring $W(F)$ now follows from 5.6.6: if $q \equiv 3 \pmod 4$ then $W(F) = \mathbb{Z}/4$; if $q \equiv 1 \pmod 4$, $W(\mathbb{F}_q) = \mathbb{Z}/2[\eta]/(\eta^2)$, where $\eta = \langle a\rangle - 1$ for some $a \in F$.

If F is a finite field extension of the p-adic rationals, then $I^3 = 0$ and I^2 is cyclic of order 2. If p is odd and the residue field is $\mathbb{F}_q$, then $W(F)$ contains $\mathbb{Z}/2$ as a subring if $q \equiv 1 \pmod 4$ and contains $\mathbb{Z}/4$ if $q \equiv 3 \pmod 4$. If $p = 2$ then

$W(F)$ contains $\mathbb{Z}/2$ as a subring iff $\sqrt{-1} \in F$. Otherwise $W(F)$ contains $\mathbb{Z}/4$ or $\mathbb{Z}/8$, according to whether or not -1 is a sum of two squares, an issue which is somewhat subtle.

If $F = \mathbb{Q}$, the ring map $W(\mathbb{Q}) \to W(\mathbb{R}) = \mathbb{Z}$ is onto, with kernel N satisfying $N^3 = 0$. Since $I/I^2 = \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, the kernel is infinite but under control.

Quadratic Forms

The theory of symmetric bilinear forms is closely related to the theory of quadratic forms, which we now sketch.

Definition 5.7. Let V be a vector space over a field F. A quadratic form on V is a function $q\colon V \to F$ such that $q(av) = a^2\,q(v)$ for every $a \in F$ and $v \in V$, and such that the formula $B_q(v,w) = q(v+w) - q(v) - q(w)$ defines a symmetric bilinear form B_q on V. We call (V,q) a *quadratic space* if B_q is nondegenerate, and call (V,B_q) the underlying symmetric inner product space. We write $\mathbf{Quad}(F)$ for the category of quadratic spaces and form-preserving maps.

The orthogonal sum $(V,q)\oplus(V',q')$ of two quadratic spaces is defined to be $V\oplus V'$ equipped with the quadratic form $v\oplus v' \mapsto q(v)+q'(v')$. This is a quadratic space, whose underlying symmetric inner product space is the orthogonal sum $(V,B_q)\oplus(V',B_{q'})$. Thus $\mathbf{Quad}(F)$ is a symmetric monoidal category, and the underlying space functor $\mathbf{Quad}(F) \to \mathbf{SBil}(F)$ sending (V,q) to (V,B_q) is monoidal.

Here is one source of quadratic spaces. Suppose that β is a (possibly non-symmetric) bilinear form on V. The function $q(v) = \beta(v,v)$ is visibly quadratic, with associated symmetric bilinear form $B_q(v,w) = \beta(v,w) + \beta(w,v)$. By choosing an ordered basis of V, it is easy to see that every quadratic form arises in this way. Note that when β is symmetric we have $B_q = 2\beta$; if $\mathrm{char}(F) \neq 2$ this shows that $\beta \mapsto \frac{1}{2}q$ defines a monoidal functor $\mathbf{SBil}(F) \to \mathbf{Quad}(F)$ inverse to the underlying functor, and proves the following result.

Lemma 5.7.1. *If* $\mathrm{char}(F) \neq 2$ *then the underlying space functor* $\mathbf{Quad}(F) \to \mathbf{SBil}(F)$ *is an equivalence of monoidal categories.*

A quadratic space (V,q) is said to be *split* if it contains a subspace N so that $q(N) = 0$ and $\dim(V) = 2\dim(N)$. For example, the quadratic forms $q(x,y) = xy + cy^2$ on $V = F^2$ are split.

Definition 5.7.2. The group $WQ(F)$ is defined to be the quotient of the group $K_0\mathbf{Quad}(F)$ by the subgroup of all split quadratic spaces.

It follows from Ex. 5.10 that the underlying space functor defines a homomorphism $WQ(F) \to W(F)$. By Lemma 5.7.1, this is an isomorphism when $\mathrm{char}(F) \neq 2$.

When $\mathrm{char}(F) = 2$, the underlying symmetric inner product space of a quadratic space (V,q) is always hyperbolic, and V is always even-dimensional; see Ex. 5.12. In particular, $WQ(F) \to W(F)$ is the zero map when $\mathrm{char}(F) = 2$. By Ex. 5.12, $WQ(F)$ is a $W(F)$-module with $WQ(F)/I\cdot WQ(F)$ given by the Arf invariant. We will describe the rest of the filtration $I^n \cdot\ WQ(F)$ in III.7.10.4.

EXERCISES

5.1. Let R be a ring and let $\mathbf{P}^\infty(R)$ denote the category of all countably generated projective R-modules. Show that $K_0^\oplus \mathbf{P}^\infty(R) = 0$.

5.2. Suppose that the Krull-Schmidt Theorem holds in an additive category $\mathcal{C}$, *i.e.*, that every object of $\mathcal{C}$ can be written as a finite direct sum of indecomposable objects, in a way that is unique up to permutation. Show that $K_0^\oplus(\mathcal{C})$ is the free abelian group on the set of isomorphism classes of indecomposable objects.

5.3. Use Ex. 5.2 to prove Corollary 5.5.1.

5.4. Let R be a commutative ring, and let $H^0(\operatorname{Spec} R, \mathbb{Q}\times_{>0})$ denote the free abelian group of all continuous maps from $\operatorname{Spec}(R)$ to $\mathbb{Q}\times_{>0}$. Show that $[P] \mapsto \operatorname{rank}(P)$ induces a split surjection from $K_0\mathbf{FP}(R)$ onto $H^0(\operatorname{Spec} R, \mathbb{Q}\times_{>0})$. In the next two exercises, we shall show that the kernel of this map is isomorphic to $\widetilde{K}_0(R) \otimes \mathbb{Q}$.

5.5. Let R be a commutative ring, and let U_+ denote the subset of the ring $K_0(R) \otimes \mathbb{Q}$ consisting of all x such that $\operatorname{rank}(x)$ takes only positive values.

(a) Use the fact that the ideal $\widetilde{K}_0(R)$ is nilpotent to show that U_+ is an abelian group under multiplication, and that there is a split exact sequence

$$0 \to \widetilde{K}_0(R) \otimes \mathbb{Q} \xrightarrow{\exp} U_+ \xrightarrow{\text{rank}} H^0(\operatorname{Spec} R, \mathbb{Q}\times_{>0}) \to 0.$$

(b) Show that $P \mapsto [P] \otimes 1$ is an additive function from $\mathbf{FP}(R)$ to the multiplicative group U_+, and that it induces a map $K_0\mathbf{FP}(R) \to U_+$.

5.6. (Bass) Let R be a commutative ring. Show that the map $K_0\mathbf{FP}(R) \to U_+$ of the previous exercise is an isomorphism. *Hint:* The map is onto by Ex. 2.10. Conversely, if $[P] \otimes 1 = [Q] \otimes 1$ in U_+, show that $P \otimes R^n \cong Q \otimes R^n$ for some n.

5.7. Suppose that a finite group G acts freely on X, and let X/G denote the orbit space. Show that $\mathbf{VB}_G(X)$ is equivalent to the category $\mathbf{VB}(X/G)$, and conclude that $K_G^0(X) \cong KU(X/G)$.

5.8. Let R be a commutative ring. Show that the determinant of a projective module induces a monoidal functor $\det\colon \mathbf{P}(R) \to \mathbf{Pic}(R)$, and that the resulting map $K_0(\det)\colon K_0\mathbf{P}(R) \to K_0\mathbf{Pic}(R)$ is the determinant map $K_0(R) \to \operatorname{Pic}(R)$ of Proposition 2.6.

5.9. If $X = (V, B)$ and $X' = (V', B')$ are two inner product spaces, show that there is a nondegenerate bilinear form β on $V \otimes V'$ satisfying $\beta(v \otimes v', w \otimes w') = B(v, w)B'(v', w')$ for all $v, w \in V$ and $v', w' \in V'$. Writing $X \otimes X'$ for this inner product space, show that $X \otimes X' \cong X' \otimes X$ and $(X_1 \oplus X_2) \otimes X' \cong (X_1 \otimes X') \oplus (X_2 \otimes X')$. Then show that $X \otimes H \cong H \oplus \cdots \oplus H$.

5.10. A symmetric inner product space $S = (V, B)$ is called *split* if it has a basis so that B is represented by a matrix $\left(\begin{smallmatrix} 0 & I \\ I & A \end{smallmatrix}\right)$. Note that the sum of split spaces is also split, and that the hyperbolic plane is split. We define $W(F)$ to be the quotient of $K_0\mathbf{SBil}(F)$ by the subgroup of classes $[S]$ of split spaces.

(a) If $\operatorname{char}(F) \neq 2$, show that every split space S is hyperbolic. Conclude that this definition of $W(F)$ agrees with the definition given in 5.6.2.

(b) For any $a \in F^\times$, show that $\langle a \rangle \oplus \langle -a \rangle$ is split.

(c) If S is split, show that each $(V,B)\otimes S$ is split. In particular, $(V,B)\oplus(V,-B) = (V,B)\otimes\big(\langle 1\rangle\oplus\langle -1\rangle\big)$ is split. Conclude that $W(F)$ is also a ring when $\mathrm{char}(F)=2$.

(d) If $\mathrm{char}(F)=2$, show that the split space $S=\langle 1\rangle\oplus\langle 1\rangle$ is not hyperbolic, yet $\langle 1\rangle\oplus S\cong\langle 1\rangle\oplus H$. This shows that Witt Cancellation fails if $\mathrm{char}(F)=2$. *Hint:* consider the associated quadratic forms. Then consider the basis $(1,1,1)$, $(1,0,1)$, $(1,1,0)$ of $\langle 1\rangle\oplus S$.

5.11. If $a+b=1$ in F, show that $\langle a\rangle\oplus\langle b\rangle\cong\langle ab\rangle\oplus\langle 1\rangle$. Conclude that in both $K_0\mathbf{SBil}(F)$ and $W(F)$ we have the Steinberg identity $\big(\langle a\rangle-1\big)\big(\langle b\rangle-1\big)=0$.

5.12. Suppose that $\mathrm{char}(F)=2$ and that (V,q) is a quadratic form.

(a) Show that $B_q(v,v)=0$ for every $v\in V$.

(b) Show that the underlying inner product space (V,B_q) is hyperbolic, hence split in the sense of Ex. 5.10. This shows that $\dim(V)$ is even, and that the map $WQ(F)\to W(F)$ is zero. *Hint:* Find two elements x,y in V so that $B_q(x,y)=1$, and show that they span an orthogonal summand of V.

(c) If (W,β) is a symmetric inner product space, show that there is a unique quadratic form q' on $V'=V\otimes W$ satisfying $q'(v\oplus w)=q(v)\beta(w,w)$, such that the underlying bilinear form satisfies $B_{q'}(v\otimes w,v'\otimes w')=B_q(v,v')\beta(w,w')$. Show that this product makes $WQ(F)$ into a module over $W(F)$.

(d) (Arf invariant) Let $\wp\colon F\to F$ denote the additive map $\wp(a)=a^2+a$. By (b), we may choose a basis $x_1,\dots,x_n$, $y_1\dots,y_n$ of V so that each x_i,y_i span a hyperbolic plane. Show that the element $\Delta(V,q)=\sum q(x_i)q(y_i)$ of $F/\wp(F)$ is independent of the choice of basis, called the *Arf invariant* of the quadratic space (after C. Arf, who discovered it in 1941). Then show that Δ is an additive surjection. H. Sah showed that the Arf invariant and the module structure in (c) induces an isomorphism $WQ(F)/I\cdot WQ(F)\cong F/\wp(F)$.

(e) Consider the quadratic forms $q(a,b)=a^2+ab+b^2$ and $q'(a,b)=ab$ on $V=F^2$. Show that they are isometric if and only if F contains the field $\mathbb{F}_4$.

5.13. (Kato) If $\mathrm{char}(F)=2$, show that there is a ring homomorphism $W(F)\to F\otimes_{F^p}F$ sending $\langle a\rangle$ to $a^{-1}\otimes a$.

§6. K_0 of an Abelian Category

Another important situation in which we can define Grothendieck groups is when we have a (skeletally) small abelian category. This is due to the natural notion of exact sequence in an abelian category. We begin by quickly reminding the reader what an abelian category is, defining K_0 and then making a set-theoretic remark.

It helps to read the definitions below with some examples in mind. The reader should remember that the prototype abelian category is the category **mod**-R of right modules over a ring R, the morphisms being R-module homomorphisms. The full subcategory with objects the free R-modules $\{0, R, R^2, \dots\}$ is additive, and so is the slightly larger full subcategory $\mathbf{P}(R)$ of f.g. projective R-modules (this observation was already made in chapter I). For more information on abelian categories, see textbooks like [MacCW] or [WHomo].

Definitions 6.1. (1) An *additive category* is a category containing a zero object '0' (an object which is both initial and terminal), having all products $A \times B$, and such that every set $\mathrm{Hom}(A, B)$ is given the structure of an abelian group in such a way that composition is bilinear. In an additive category the product $A \times B$ is also the coproduct $A \amalg B$ of A and B; we call it the *direct sum* and write it as $A \oplus B$ to remind ourselves of this fact.

(2) An *abelian category* $\mathcal{A}$ is an additive category in which (i) every morphism $f\colon B \to C$ has a kernel and a cokernel, and (ii) every monic arrow is a kernel, and every epi is a cokernel. (Recall that $f\colon B \to C$ is called *monic* if $fe_1 \neq fe_2$ for every $e_1 \neq e_2\colon A \to B$; it is called *epi* if $g_1 f \neq g_2 f$ for every $g_1 \neq g_2\colon C \to D$.)

(3) In an abelian category, we call a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ *exact* if $\ker(g)$ equals $\mathrm{im}(f) \equiv \ker\{B \to \mathrm{coker}(f)\}$. A longer sequence is *exact* if it is exact at all places. By the phrase *short exact sequence* in an abelian category $\mathcal{A}$ we mean an exact sequence of the form:

$$0 \to A' \to A \to A'' \to 0. \qquad (*)$$

Definition 6.1.1 ($K_0\mathcal{A}$). Let $\mathcal{A}$ be an abelian category. Its *Grothendieck group* $K_0(\mathcal{A})$ is the abelian group presented as having one generator $[A]$ for each object A of $\mathcal{A}$, with one relation $[A] = [A'] + [A'']$ for every short exact sequence $(*)$ in $\mathcal{A}$.

Here are some useful identities which hold in $K_0(\mathcal{A})$.

a) $[0] = 0$ (take $A = A'$).

b) if $A \cong A'$ then $[A] = [A']$ (take $A'' = 0$).

c) $[A' \oplus A''] = [A'] + [A'']$ (take $A = A' \oplus A''$).

If two abelian categories are equivalent, their Grothendieck groups are naturally isomorphic, as *b*) implies they have the same presentation. By *c*), the group $K_0(\mathcal{A})$ is a quotient of the group $K_0^{\oplus}(\mathcal{A})$ defined in §5 by considering $\mathcal{A}$ as a symmetric monoidal category.

Universal Property 6.1.2. An *additive function* from $\mathcal{A}$ to an abelian group Γ is a function f from the objects of $\mathcal{A}$ to Γ such that $f(A) = f(A') + f(A'')$ for every short exact sequence $(*)$ in $\mathcal{A}$. By construction, the function $A \mapsto [A]$ defines an additive function from $\mathcal{A}$ to $K_0(\mathcal{A})$. This has the following universal property: any additive function f from $\mathcal{A}$ to Γ induces a unique group homomorphism $f\colon K_0(\mathcal{A}) \to \Gamma$, with $f([A]) = f(A)$ for every A.

For example, the direct sum $\mathcal{A}_1 \oplus \mathcal{A}_2$ of two abelian categories is also abelian. Using the universal property of K_0 it is clear that $K_0(\mathcal{A}_1 \oplus \mathcal{A}_2) \cong K_0(\mathcal{A}_1) \oplus K_0(\mathcal{A}_2)$. More generally, an arbitrary direct sum $\bigoplus \mathcal{A}_i$ of abelian categories is abelian, and we have $K_0(\bigoplus \mathcal{A}_i) \cong \bigoplus K_0(\mathcal{A}_i)$.

Set-theoretic Considerations 6.1.3. There is an obvious set-theoretic difficulty in defining $K_0\mathcal{A}$ when $\mathcal{A}$ is not small; recall that a category $\mathcal{A}$ is called *small* if the class of objects of $\mathcal{A}$ forms a set.

We will always implicitly assume that our abelian category $\mathcal{A}$ is *skeletally small*, *i.e.*, it is equivalent to a small abelian category $\mathcal{A}'$. In this case we define $K_0(\mathcal{A})$ to be $K_0(\mathcal{A}')$. Since any other small abelian category equivalent to $\mathcal{A}$ will also be equivalent to $\mathcal{A}'$, the definition of $K_0(\mathcal{A})$ is independent of this choice.

Example 6.1.4 (All R-modules). We cannot take the Grothendieck group of the abelian category $\mathbf{mod}$-R because it is not skeletally small. To finesse this difficulty, fix an infinite cardinal number κ and let $\mathbf{mod}_\kappa$-R denote the full subcategory of $\mathbf{mod}$-R consisting of all R-modules of cardinality $< \kappa$. As long as $\kappa \geq |R|$, $\mathbf{mod}_\kappa$-R is an abelian subcategory of $\mathbf{mod}$-R having a set of isomorphism classes of objects. The Eilenberg Swindle I.2.8 applies to give $K_0(\mathbf{mod}_\kappa\text{-}R) = 0$. In effect, the formula $M \oplus M^\infty \cong M^\infty$ implies that $[M] = 0$ for every module M.

The natural type of functor $F\colon \mathcal{A} \to \mathcal{B}$ between two abelian categories is an *additive* functor; this is a functor for which all the maps $\mathrm{Hom}(A, A') \to \mathrm{Hom}(FA, FA')$ are group homomorphisms. However, not all additive functors induce homomorphisms $K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

We say that an additive functor F is *exact* if it preserves exact sequences—that is, for every exact sequence $(*)$ in $\mathcal{A}$, the sequence $0 \to F(A') \to F(A) \to F(A'') \to 0$ is exact in $\mathcal{B}$. The presentation of K_0 implies that any exact functor F defines a group homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{B})$ by the formula $[A] \mapsto [F(A)]$.

Suppose given an inclusion $\mathcal{A} \subset \mathcal{B}$ of abelian categories. If the inclusion is an exact functor, we say that $\mathcal{A}$ is an *exact abelian subcategory* of $\mathcal{B}$. As with all exact functors, the inclusion induces a natural map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

Definition 6.2 (G_0R). If R is a (right) noetherian ring, let $\mathbf{M}(R)$ denote the subcategory of $\mathbf{mod}$-R consisting of all finitely generated R-modules. The noetherian hypothesis implies that $\mathbf{M}(R)$ is an abelian category, and we write $G_0(R)$ for $K_0\mathbf{M}(R)$. (We will give a definition of $\mathbf{M}(R)$ and $G_0(R)$ for non-noetherian rings in Example 7.1.4 below.)

The presentation of $K_0(R)$ in §2 shows that there is a natural map $K_0(R) \to G_0(R)$, which is called the *Cartan homomorphism* (send $[P]$ to $[P]$).

Associated to a ring homomorphism $f\colon R \to S$ are two possible maps on G_0: the contravariant transfer map and the covariant basechange map.

When S is finitely generated as an R-module (*e.g.*, $S = R/I$), there is a "transfer" homomorphism $f_*\colon G_0(S) \to G_0(R)$. It is induced from the forgetful functor $f_*\colon \mathbf{M}(S) \to \mathbf{M}(R)$, which is exact.

Whenever S is flat as an R-module, there is a "basechange" homomorphism $f^*\colon G_0(R) \to G_0(S)$. Indeed, the basechange functor $f^*\colon \mathbf{M}(R) \to \mathbf{M}(S)$, $f^*(M) = M \otimes_R S$, is exact iff S is flat over R. We will extend the definition of f^* in §7 to the case in which S has a finite resolution by flat R-modules using Serre's Formula (7.8.3): $f^*([M]) = \sum(-1)^i[\mathrm{Tor}_i^R(M,S)]$.

If F is a field then every exact sequence in $\mathbf{M}(F)$ splits, and it is easy to see that $G_0(F) \cong K_0(F) \cong \mathbb{Z}$. In particular, if R is an integral domain with field of fractions F, then there is a natural map $G_0(R) \to G_0(F) = \mathbb{Z}$, sending $[M]$ to the integer $\dim_F(M \otimes_R F)$.

Example 6.2.1 (Abelian groups). When $R = \mathbb{Z}$ the Cartan homomorphism is an isomorphism: $K_0(\mathbb{Z}) \cong G_0(\mathbb{Z}) \cong \mathbb{Z}$. To see this, first observe that the sequences

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

imply that $[\mathbb{Z}/n\mathbb{Z}] = [\mathbb{Z}] - [\mathbb{Z}] = 0$ in $G_0(\mathbb{Z})$ for every n. By the Fundamental Theorem of f.g. Abelian Groups, every f.g. abelian group M is a finite sum of copies of the groups $\mathbb{Z}$ and $\mathbb{Z}/n$. Hence $G_0(\mathbb{Z})$ is generated by $[\mathbb{Z}]$. To see that $G_0(\mathbb{Z}) \cong \mathbb{Z}$, observe that since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module there is a homomorphism from $G_0(\mathbb{Z})$ to $G_0(\mathbb{Q}) \cong \mathbb{Z}$ sending $[M]$ to $r(M) = \dim_{\mathbb{Q}}(M \otimes \mathbb{Q})$. In effect, $r(M)$ is an additive function; as such it induces a homomorphism $r\colon G_0(\mathbb{Z}) \to \mathbb{Z}$. As $r(\mathbb{Z}) = 1$, r is an isomorphism.

More generally, the Cartan homomorphism is an isomorphism whenever R is a principal ideal domain, and $K_0(R) \cong G_0(R) \cong \mathbb{Z}$. The proof is identical.

Example 6.2.2 (p-groups). Let $\mathbf{Ab}_p$ denote the abelian category of all finite p-groups for some prime p. Then $K_0(\mathbf{Ab}_p) \cong \mathbb{Z}$ on generator $[\mathbb{Z}/p]$. To see this, we observe that the length $\ell(M)$ of a composition series for a finite p-group M is well-defined by the Jordan-Hölder Theorem. Moreover ℓ is an additive function, and defines a homomorphism $K_0(\mathbf{Ab}_p) \to \mathbb{Z}$ with $\ell(\mathbb{Z}/p) = 1$. To finish we need only observe that $\mathbb{Z}/p$ generates $K_0(\mathbf{Ab}_p)$; this follows by induction on the length of a p-group, once we observe that any $L \subset M$ yields $[M] = [L] + [M/L]$ in $K_0(\mathbf{Ab}_p)$.

Example 6.2.3. The category $\mathbf{Ab}_{fin}$ of all finite abelian groups is the direct sum of the categories $\mathbf{Ab}_p$ of Example 6.2.2. Therefore $K_0(\mathbf{Ab}_{fin}) = \bigoplus K_0(\mathbf{Ab}_p)$ is the free abelian group on the set $\{[\mathbb{Z}/p], p \text{ prime}\}$.

Example 6.2.4. The category $\mathbf{M}(\mathbb{Z}/p^n)$ of all finite $\mathbb{Z}/p^n$-modules is an exact abelian subcategory of $\mathbf{Ab}_p$, and the argument above applies verbatim to prove that the simple module $[\mathbb{Z}/p]$ generates the group $G_0(\mathbb{Z}/p^n) \cong \mathbb{Z}$. In particular, the canonical maps from $G_0(\mathbb{Z}/p^n) = K_0\mathbf{M}(\mathbb{Z}/p^n)$ to $K_0(\mathbf{Ab}_p)$ are all isomorphisms.

Recall from Lemma 2.2 that $K_0(\mathbb{Z}/p^n) \cong \mathbb{Z}$ on $[\mathbb{Z}/p^n]$. The Cartan homomorphism from $K_0 \cong \mathbb{Z}$ to $G_0 \cong \mathbb{Z}$ is not an isomorphism; it sends $[\mathbb{Z}/p^n]$ to $n[\mathbb{Z}/p]$.

Definition 6.2.5 (G_0X). Let X be a noetherian scheme. The category $\mathbf{M}(X)$ of all coherent $\mathcal{O}_X$-modules is an abelian category. (See [Hart, II.5.7].) We write $G_0(X)$ for $K_0\mathbf{M}(X)$. When $X = \mathrm{Spec}(R)$ this agrees with Definition 6.2: $G_0(X) \cong G_0(R)$, because of the equivalence of $\mathbf{M}(X)$ and $\mathbf{M}(R)$.

If $f\colon X \to Y$ is a morphism of schemes, there is a *basechange functor* $f^*\colon \mathbf{M}(Y) \to \mathbf{M}(X)$ sending $\mathcal{F}$ to $f^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_X$; see I.5.2. When f is flat, the basechange f^* is exact and therefore the formula $f^*([\mathcal{F}]) = [\mathcal{F}]$ defines a homomorphism $f^*\colon G_0(Y) \to G_0(X)$. Thus G_0 is contravariant for flat maps.

If $f\colon X \to Y$ is a finite morphism, the direct image $f_*\mathcal{F}$ of a coherent sheaf $\mathcal{F}$ is coherent, and $f_*\colon \mathbf{M}(X) \to \mathbf{M}(Y)$ is an exact functor [EGA, I(1.7.8)]. In this case the formula $f_*([\mathcal{F}]) = [f_*\mathcal{F}]$ defines a "transfer" map $f_*\colon G_0(X) \to G_0(Y)$.

If $f\colon X \to Y$ is a proper morphism, the direct image $f_*\mathcal{F}$ of a coherent sheaf $\mathcal{F}$ is coherent, and so are its higher direct images $R^if_*\mathcal{F}$. (This is Serre's "Theorem B"; see I.5.2 or [EGA, III(3.2.1)].) The functor $f_*\colon \mathbf{M}(X) \to \mathbf{M}(Y)$ is not usually exact (unless f is finite). Instead we have:

Lemma 6.2.6. *If $f\colon X \to Y$ is a proper morphism of noetherian schemes, there is a "transfer" homomorphism $f_*\colon G_0(X) \to G_0(Y)$. It is defined by the formula $f_*([\mathcal{F}]) = \sum(-1)^i[R^if_*\mathcal{F}]$. The transfer homomorphism makes G_0 functorial for proper maps.*

Proof. For each coherent $\mathcal{F}$ the $R^if_*\mathcal{F}$ vanish for large i, so the sum is finite. By 6.2.1 it suffices to show that the formula gives an additive function. But if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence in $\mathbf{M}(X)$ there is a finite long exact sequence in $\mathbf{M}(Y)$:

$$0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to R^1f_*\mathcal{F}' \to R^1f_*\mathcal{F} \to R^1f_*\mathcal{F}'' \to R^2f_*\mathcal{F}' \to \cdots$$

and the alternating sum of the terms is $f_*[\mathcal{F}'] - f_*[\mathcal{F}] + f_*[\mathcal{F}'']$. This alternating sum must be zero by Proposition 6.6 below, so f_* is additive as desired. (Functoriality is relegated to Ex. 6.15.)

The next Lemma follows by inspection of the definition of $\mathcal{A}$.

Lemma 6.2.7 (Filtered colimits). *Suppose that $\{\mathcal{A}_i\}_{i\in I}$ is a filtered family of abelian categories and exact functors. Then the colimit (="direct limit") $\mathcal{A} = \varinjlim \mathcal{A}_i$ is also an abelian category, and*

$$K_0(\mathcal{A}) = \varinjlim K_0(\mathcal{A}_i).$$

Example 6.2.8 (S-torsion modules). Suppose that S is a multiplicatively closed set of elements in a noetherian ring R. Let $\mathbf{M}_S(R)$ be the subcategory of $\mathbf{M}(R)$ consisting of all f.g. R-modules M such that $Ms = 0$ for some $s \in S$. For example, if $S = \{p^n\}$ then $\mathbf{M}_S(\mathbb{Z}) = \mathbf{Ab}_p$ was discussed in Example 6.2.2. In general $\mathbf{M}_S(R)$ is not only the union of the $\mathbf{M}(R/RsR)$, but is also the union of the $\mathbf{M}(R/I)$ as I ranges over the ideals of R with $I \cap S \neq \phi$. By 6.2.7,

$$K_0\mathbf{M}_S(R) = \varinjlim_{I\cap S\neq\phi} G_0(R/I) = \varinjlim_{s\in S} G_0(R/RsR).$$

Devissage

The method behind the computation in Example 6.2.4 that $G_0(\mathbb{Z}/p^n) \cong K_0\mathbf{Ab}_p$ is called Devissage, a French word referring to the "unscrewing" of the composition series. Here is a formal statement of the process, due to Alex Heller.

Devissage Theorem 6.3. *Let $\mathcal{B} \subset \mathcal{A}$ be small abelian categories. Suppose that*
(a) $\mathcal{B}$ is closed in $\mathcal{A}$ under subobjects and quotient objects, and
(b) Every object A of $\mathcal{A}$ has a finite filtration $A = A_0 \supset A_1 \cdots \supset A_n = 0$ with all quotients A_i/A_{i+1} in $\mathcal{B}$.
Then the inclusion functor $\mathcal{B} \subset \mathcal{A}$ is exact and induces an isomorphism

$$K_0(\mathcal{B}) \cong K_0(\mathcal{A}).$$

Proof. It follows immediately from (a) that $\mathcal{B}$ is an exact abelian subcategory of $\mathcal{A}$; let $i_*: K_0(\mathcal{B}) \to K_0(\mathcal{A})$ denote the canonical homomorphism. To see that i_* is onto, observe that every filtration $A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ yields $[A] = \sum [A_i/A_{i+1}]$ in $K_0(\mathcal{A})$. This follows by induction on n, using the observation that $[A_i] = [A_{i+1}] + [A_i/A_{i+1}]$. Since by (b) such a filtration exists with the A_i/A_{i+1} in $\mathcal{B}$, this shows that the canonical i_* is onto.

For each A in $\mathcal{A}$, fix a filtration $A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ with each A_i/A_{i+1} in $\mathcal{B}$, and define $f(A)$ to be the element $\sum [A_i/A_{i+1}]$ of $K_0(\mathcal{B})$. We claim that $f(A)$ is independent of the choice of filtration. Because any two filtrations have equivalent refinements (Ex. 6.2), we only need check refinements of our given filtration. By induction we need only check for one insertion, say changing $A_i \supset A_{i+1}$ to $A_i \supset A' \supset A_{i+1}$. Appealing to the exact sequence

$$0 \to A'/A_{i+1} \to A_i/A_{i+1} \to A_i/A' \to 0,$$

we see that $[A_i/A_{i+1}] = [A_i/A'] + [A'/A_{i+1}]$ in $K_0(\mathcal{B})$, as claimed.

Given a short exact sequence $0 \to A' \to A \to A'' \to 0$, we may construct a filtration $\{A_i\}$ on A by combining our chosen filtration for A' with the inverse image in A of our chosen filtration for A''. For this filtration we have $\sum [A_i/A_{i+1}] = f(A') + f(A'')$. Therefore f is an additive function, and defines a map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$. By inspection, f is the inverse of the canonical map i_*.

Corollary 6.3.1. *Let I be a nilpotent ring of a noetherian ring R. Then the inclusion* **mod**-$(R/I) \subset$ **mod**-R *induces an isomorphism*

$$G_0(R/I) \cong G_0(R).$$

Proof. To apply Devissage, we need to observe that if M is a f.g. R-module, the filtration $M \supseteq IM \supseteq I^2M \supseteq \cdots \supseteq I^nM = 0$ is finite, and all the quotients $I^nM/I^{n+1}M$ are f.g. R/I-modules. Notice that this also proves the scheme version:

Corollary 6.3.2. *Let X be a noetherian scheme, and X_{red} the associated reduced scheme. Then $G_0(X) \cong G_0(X_{red})$.*

Application 6.3.3 (R-modules with support). Example 6.2.2 can be generalized as follows. Given a central element s in a ring R, let $\mathbf{M}_s(R)$ denote the abelian subcategory of $\mathbf{M}(R)$ consisting of all f.g. R-modules M such that $Ms^n = 0$ for some n. That is, modules such that $M \supset Ms \supset Ms^2 \supset \cdots$ is a finite filtration. By Devissage,

$$K_0\mathbf{M}_s(R) \cong G_0(R/sR).$$

More generally, suppose we are given an ideal I of R. Let $\mathbf{M}_I(R)$ be the (exact) abelian subcategory of $\mathbf{M}(R)$ consisting of all f.g. R-modules M such that the filtration $M \supset MI \supset MI^2 \supset \cdots$ is finite, *i.e.*, such that $MI^n = 0$ for some n. By Devissage,

$$K_0\mathbf{M}_I(R) \cong K_0\mathbf{M}(R/I) = G_0(R/I).$$

EXAMPLE 6.3.4. Let X be a noetherian scheme, and $i: Z \subset X$ the inclusion of a closed subscheme. Let $\mathbf{M}_Z(X)$ denote the abelian category of coherent $\mathcal{O}_X$-modules Z supported on Z, and $\mathcal{I}$ the ideal sheaf in $\mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I} \cong \mathcal{O}_Z$. Via the direct image $i_*: \mathbf{M}(Z) \subset \mathbf{M}(X)$, we can consider $\mathbf{M}(Z)$ as the subcategory of all modules M in $\mathbf{M}_Z(X)$ such that $\mathcal{I}M = 0$. Every M in $\mathbf{M}_Z(X)$ has a finite filtration $M \supset M\mathcal{I} \supset M\mathcal{I}^2 \supset \cdots$ with quotients in $\mathbf{M}(Z)$, so by Devissage:

$$K_0\mathbf{M}_Z(X) \cong K_0\mathbf{M}(Z) = G_0(Z).$$

The Localization Theorem

Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* of $\mathcal{A}$ is an abelian subcategory $\mathcal{B}$ which is closed under subobjects, quotients and extensions. That is, if $0 \to B \to C \to D \to 0$ is exact in $\mathcal{A}$ then

$$C \in \mathcal{B} \Leftrightarrow B, D \in \mathcal{B}.$$

Now assume for simplicity that $\mathcal{A}$ is small. If $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$, we can form a quotient abelian category $\mathcal{A}/\mathcal{B}$ as follows. Call a morphism f in $\mathcal{A}$ a $\mathcal{B}$-*iso* if $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathcal{B}$. The objects of $\mathcal{A}/\mathcal{B}$ are the objects of $\mathcal{A}$, and morphisms $A_1 \to A_2$ are equivalence classes of diagrams in $\mathcal{A}$:

$$A_1 \xleftarrow{f} A' \xrightarrow{g} A_2, \quad f \text{ a } \mathcal{B}\text{-iso}.$$

Such a morphism is equivalent to $A_1 \leftarrow A'' \to A_2$ if and only if there is a commutative diagram:

$$\begin{tikzcd} & A' \arrow[dl] \arrow[dr] & \\ A_1 & A \arrow[l] \arrow[u] \arrow[d] \arrow[r] & A_2 \\ & A'' \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \text{where } A' \leftarrow A \to A'' \text{ are } \mathcal{B}\text{-isos.}$$

The composition with $A_2 \xleftarrow{f'} A'' \xrightarrow{h} A_3$ is $A_1 \xleftarrow{f} A' \leftarrow A \to A'' \xrightarrow{h} A_3$, where A is the pullback of A' and A'' over A_2. The proof that $\mathcal{A}/\mathcal{B}$ is abelian, and that the quotient functor $\mathrm{loc}: \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is exact, may be found in [Swan, p.44ff] or [Gabriel]. (See the appendix to this chapter.)

It is immediate from the construction of $\mathcal{A}/\mathcal{B}$ that $\mathrm{loc}(A) \cong 0$ if and only if A is an object of $\mathcal{B}$, and that for a morphism $f: A \to A'$ in $\mathcal{A}$, $\mathrm{loc}(f)$ is an isomorphism iff f is a $\mathcal{B}$-iso. In fact $\mathcal{A}/\mathcal{B}$ solves a universal problem (see *op. cit.*): if $T: \mathcal{A} \to \mathcal{C}$ is an exact functor such that $T(B) \cong 0$ for all B in $\mathcal{B}$, then there is a unique exact functor $T': \mathcal{A}/\mathcal{B} \to \mathcal{C}$ so that $T = T' \circ \mathrm{loc}$.

Localization Theorem 6.4. *(Heller) Let $\mathcal{A}$ be a small abelian category, and $\mathcal{B}$ a Serre subcategory of $\mathcal{A}$. Then the following sequence is exact:*

$$K_0(\mathcal{B}) \to K_0(\mathcal{A}) \xrightarrow{loc} K_0(\mathcal{A}/\mathcal{B}) \to 0.$$

Proof. By the construction of $\mathcal{A}/\mathcal{B}$, $K_0(\mathcal{A})$ maps onto $K_0(\mathcal{A}/\mathcal{B})$ and the composition $K_0(\mathcal{B}) \to K_0(\mathcal{A}/\mathcal{B})$ is zero. Hence if Γ denotes the cokernel of the map $K_0(\mathcal{B}) \to K_0(\mathcal{A})$ there is a natural surjection $\Gamma \to K_0(\mathcal{A}/\mathcal{B})$; to prove the theorem it suffices to give an inverse. For this it suffices to show that $\gamma(\mathrm{loc}(A)) = [A]$ defines an additive function from $\mathcal{A}/\mathcal{B}$ to Γ, because the induced map $\gamma\colon K_0(\mathcal{A}/\mathcal{B}) \to \Gamma$ will be inverse to the natural surjection $\Gamma \to K_0(\mathcal{A}/\mathcal{B})$.

Since $\mathrm{loc}\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is a bijection on objects, γ is well-defined. We claim that if $\mathrm{loc}(A_1) \cong \mathrm{loc}(A_2)$ in $\mathcal{A}/\mathcal{B}$ then $[A_1] = [A_2]$ in Γ. To do this, represent the isomorphism by a diagram $A_1 \xleftarrow{f} A \xrightarrow{g} A_2$ with f a $\mathcal{B}$-iso. As $\mathrm{loc}(g)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$, g is also a $\mathcal{B}$-iso. In $K_0(\mathcal{A})$ we have

$$[A] = [A_1] + [\ker(f)] - [\mathrm{coker}(f)] = [A_2] + [\ker(g)] - [\mathrm{coker}(g)].$$

Hence $[A] = [A_1] = [A_2]$ in Γ, as claimed.

To see that γ is additive, suppose given an exact sequence in $\mathcal{A}/\mathcal{B}$ of the form:

$$0 \to \mathrm{loc}(A_0) \xrightarrow{i} \mathrm{loc}(A_1) \xrightarrow{j} \mathrm{loc}(A_2) \to 0;$$

we have to show that $[A_1] = [A_0] + [A_2]$ in Γ. Represent j by a diagram $A_1 \xleftarrow{f} A \xrightarrow{g} A_2$ with f a $\mathcal{B}$-iso. Since $[A] = [A_1] + [\ker(f)] - [\mathrm{coker}(f)]$ in $K_0(\mathcal{A})$, $[A] = [A_1]$ in Γ. Applying the exact functor loc to

$$0 \to \ker(g) \to A \xrightarrow{g} A_2 \to \mathrm{coker}(g) \to 0,$$

we see that $\mathrm{coker}(g)$ is in $\mathcal{B}$ and that $\mathrm{loc}(\ker(g)) \cong \mathrm{loc}(A_0)$ in $\mathcal{A}/\mathcal{B}$. Hence $[\ker(g)] \equiv [A_0]$ in Γ, and in Γ we have

$$[A_1] = [A] = [A_2] + [\ker(g)] - [\mathrm{coker}(g)] \equiv [A_0] + [A_2]$$

proving that γ is additive, and finishing the proof of the Localization Theorem.

Application 6.4.1. Let S be a central multiplicative set in a ring R, and let $\mathbf{mod}_S(R)$ denote the Serre subcategory of $\mathbf{mod}$-R consisting of S-torsion modules, *i.e.*, those R-modules M such that every $m \in M$ has $ms = 0$ for some $s \in S$. Then there is a natural equivalence between $\mathbf{mod}$-$(S^{-1}R)$ and the quotient category $\mathbf{mod}$-$R/\mathbf{mod}_S(R)$. If R is noetherian and $\mathbf{M}_S(R)$ denotes the Serre subcategory of $\mathbf{M}(R)$ consisting of f.g. S-torsion modules, then $\mathbf{M}(S^{-1}R)$ is equivalent to $\mathbf{M}(R)/\mathbf{M}_S(R)$. The Localization exact sequence becomes:

$$K_0\mathbf{M}_S(R) \to G_0(R) \to G_0(S^{-1}R) \to 0.$$

In particular, if $S = \{s^n\}$ for some s then by Application 6.3.3 we have an exact sequence

$$G_0(R/sR) \to G_0(R) \to G_0(R[\frac{1}{s}]) \to 0.$$

More generally, if I is an ideal of a noetherian ring R, we can consider the Serre subcategory $\mathbf{M}_I(R)$ of modules with some $MI^n = 0$ discussed in Application 6.3.3. The quotient category $\mathbf{M}(R)/\mathbf{M}_I(R)$ is known to be isomorphic to the category $\mathbf{M}(U)$ of coherent $\mathcal{O}_U$-modules, where U is the open subset of $\mathrm{Spec}(R)$ defined by I. The composition of the isomorphism $K_0\mathbf{M}(R/I) \cong K_0\mathbf{M}_I(R)$ of 6.3.3 with $K_0\mathbf{M}_I(R) \to K_0\mathbf{M}(R)$ is evidently the transfer map $i_*: G_0(R/I) \to G_0(R)$. Hence the Localization Sequence becomes the exact sequence

$$G_0(R/I) \xrightarrow{i_*} G_0(R) \to G_0(U) \to 0$$

Application 6.4.2. Let X be a scheme, and $i: Z \subset X$ a closed subscheme with complement $j: U \subset X$. Let $\mathbf{mod}_Z(X)$ denote the Serre subcategory of $\mathcal{O}_X$-$\mathbf{mod}$ consisting of all $\mathcal{O}_X$-modules $\mathcal{F}$ with support in Z, *i.e.*, such that $\mathcal{F}|_U = 0$. Gabriel proved in *Des catégories abeliennes*, Bull. Soc. Math. France 90 (1962), 323-448 that j^* induces an equivalence: $\mathcal{O}_U$-$\mathbf{mod} \cong \mathcal{O}_X$-$\mathbf{mod}/\mathbf{mod}_Z(X)$.

Morover, if X is noetherian and $\mathbf{M}_Z(X)$ denotes the category of coherent sheaves supported in Z, then $\mathbf{M}(X)/\mathbf{M}_Z(X) \cong \mathbf{M}(U)$. The inclusion $i: Z \subset X$ induces an exact functor $i_*: \mathbf{M}(Z) \subset \mathbf{M}(X)$, and $G_0(Z) \cong K_0\mathbf{M}_Z(X)$ by Example 6.3.4. Therefore the Localization sequence becomes:

$$G_0(Z) \xrightarrow{i_*} G_0(X) \xrightarrow{j^*} G_0(U) \to 0.$$

For example, if $X = \mathrm{Spec}(R)$ and $Z = \mathrm{Spec}(R/I)$, we recover the exact sequence in the previous application.

Application 6.4.3 (Higher Divisor Class Groups). Given a commutative noetherian ring R, let $D_i(R)$ denote the free abelian group on the set of prime ideals of height exactly i; this is generalizes the group of Weil divisors in Ch.I, §3. Let $\mathbf{M}_i(R)$ denote the category of f.g. R-modules M whose associated prime ideals all have height $\geq i$. Each $\mathbf{M}_i(R)$ is a Serre subcategory of $\mathbf{M}(R)$; see Ex. 6.9. Let $F^iG_0(R)$ denote the image of $K_0\mathbf{M}_i(R)$ in $G_0(R) = K_0\mathbf{M}(R)$. These subgroups form a filtration $\cdots \subset F^2 \subset F^1 \subset F^0 = G_0(R)$, called the *coniveau filtration* of $G_0(R)$.

It turns out that there is an equivalence $\mathbf{M}_i/\mathbf{M}_{i+1}(R) \cong \bigoplus \mathbf{M}_\mathfrak{p}(R_\mathfrak{p})$, $ht(\mathfrak{p}) = i$. By Application 6.3.3 of Devissage, $K_0\mathbf{M}_\mathfrak{p}(R_\mathfrak{p}) \cong G_0(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \cong \mathbb{Z}$, so there is an isomorphism $D_i(R) \xrightarrow{\cong} K_0\mathbf{M}_i/\mathbf{M}_{i+1}(R)$, $[\mathfrak{p}] \mapsto [R/\mathfrak{p}]$. By the Localization Theorem, we have an exact sequence

$$K_0\mathbf{M}_{i+1}(R) \to K_0\mathbf{M}_i(R) \to D_i(R) \to 0.$$

Thus $G_0(R)/F^1 \cong D_0(R)$, and each subquotient F^i/F^{i+1} is a quotient of $D_i(R)$.

For $i \geq 1$, the *generalized Weil divisor class group* $CH^i(R)$ is defined to be the subgroup of $K_0\mathbf{M}_{i-1}/\mathbf{M}_{i+1}(R)$ generated by the classes $[R/\mathfrak{p}]$, $ht(\mathfrak{p}) \geq i$. This definition is due to L. Claborn and R. Fossum; the notation reflects a theorem (in Chapter 5 below) that the kernel of $D_i(R) \to CH^i(R)$ is generated by rational equivalence. For example, we will see in Ex. 6.9 that if R is a Krull domain then $CH^1(R)$ is the usual divisor class group $Cl(R)$.

Similarly, if X is a noetherian scheme, there is a coniveau filtration on $G_0(X)$. Let $\mathbf{M}^i(X)$ denote the subcategory of $\mathbf{M}(X)$ consisting of coherent modules whose support has codimension $\geq i$, and let $D^i(X)$ denote the free abelian group on the set of points of X having codimension i. Then each $\mathbf{M}^i(X)$ is a Serre subcategory and $\mathbf{M}^i/\mathbf{M}^{i+1}(X) \cong \bigoplus \mathbf{M}_x(\mathcal{O}_{X,x})$, where x runs over all points of codimension i in X. Again by Devissage, there is an isomorphism $K_0\mathbf{M}^i/\mathbf{M}^{i+1}(X) \cong D^i(X)$ and hence $G_0(X)/F^1 \cong D^0(X)$. For $i \geq 1$, the *generalized Weil divisor class group* $CH^i(X)$ is defined to be the subgroup of $K_0\mathbf{M}_{i-1}/\mathbf{M}_{i+1}(X)$ generated by the classes $[\mathcal{O}_Z]$, $\operatorname{codim}_X(Z) = i$. We will see later on (in chapter 5) that $CH^i(X)$ is the usual Chow group of codimension i cycles on X modulo rational equivalence, as defined in [Fulton]. The verification that $CH^1(X) = Cl(X)$ is left to Ex. 6.10.

We now turn to a clasical application of the Localization Theorem: the Fundamental Theorem for G_0 of a noetherian ring R. Via the ring map $\pi: R[t] \to R$ sending t to zero, we have an inclusion $\mathbf{M}(R) \subset \mathbf{M}(R[t])$ and hence a transfer map $\pi_*: G_0(R) \to G_0(R[t])$. By 6.4.1 there is an exact localization sequence

$$G_0(R) \xrightarrow{\pi_*} G_0(R[t]) \xrightarrow{j^*} G_0(R[t,t^{-1}]) \to 0. \tag{6.4.4}$$

Given an R-module M, the exact sequence of $R[t]$-modules

$$0 \to M[t] \xrightarrow{t} M[t] \to M \to 0$$

shows that in $G_0(R[t])$ we have

$$\pi_*[M] = [M] = [M[t]] - [M[t]] = 0.$$

Thus $\pi_* = 0$, because every generator $[M]$ of $G_0(R)$ becomes zero in $G_0(R[t])$. From the Localization sequence (6.4.4) it follows that j^* is an isomorphism. This proves the easy part of the following result.

Fundamental Theorem for G_0-theory of Rings 6.5. *For every noetherian ring R, the inclusions $R \overset{i}{\hookrightarrow} R[t] \overset{j}{\hookrightarrow} R[t,t^{-1}]$ induce isomorphisms*

$$G_0(R) \cong G_0(R[t]) \cong G_0(R[t,t^{-1}]).$$

Proof. The ring inclusions are flat, so they induce maps $i^*: G_0(R) \to G_0(R[t])$ and $j^*: G_0(R[t]) \to G_0(R[t,t^{-1}])$. We have already seen that j^* is an isomorphism; it remains to show that i^* is an isomorphism.

Because $R = R[t]/tR[t]$, Serre's formula defines a map $\pi^*: G_0(R[t]) \to G_0(R)$ by the formula: $\pi^*[M] = [M/Mt] - [ann_M(t)]$, where $ann_M(t) = \{x \in M : xt = 0\}$. (See Ex. 6.6 or 7.8.3 below.) Since $\pi^* i^*[M] = \pi^*[M[t]] = [M]$, i^* is an injection split by π^*.

We shall present Grothendieck's proof that $i^*: G_0(R) \to G_0(R[t])$ is onto, which assumes that R is a commutative ring. A proof in the non-commutative case (due to Serre) will be sketched in Ex. 6.13.

If $G_0(R) \neq G_0(R[t])$, we proceed by noetherian induction to a contradiction. Among all ideals J for which $G_0(R/J) \neq G_0(R/J[t])$, there is a maximal one.

Replacing R by R/J, we may assume that $G_0(R/I) = G_0(R/I[t])$ for each $I \neq 0$ in R. Such a ring R must be reduced by Corollary 6.3.1. Let S be the set of non-zero divisors in R; by elementary ring theory $S^{-1}R$ is a finite product $\prod F_i$ of fields F_i, so $G_0(S^{-1}R) \cong \oplus G_0(F_i)$. Similarly $S^{-1}R[t] = \prod F_i[t]$ and $G_0(S^{-1}R[t]) \cong \oplus G_0(F_i[t])$. By Application 6.4.1 and Example 6.2.8 we have a diagram with exact rows:

$$\begin{array}{ccccccc}
\varinjlim G_0(R/sR) & \longrightarrow & G_0(R) & \longrightarrow & \oplus G_0(F_i) & \longrightarrow & 0 \\
\cong \big\downarrow i^* & & \big\downarrow i^* & & \big\downarrow & & \\
\varinjlim G_0(R/sR[t]) & \longrightarrow & G_0(R[t]) & \longrightarrow & \oplus G_0(F_i[t]) & \longrightarrow & 0.
\end{array}$$

Since the direct limits are taken over all $s \in S$, the left vertical arrow is an isomorphism by induction. Because each $F_i[t]$ is a principal ideal domain, (2.6.3) and Example 6.2.1 imply that the right vertical arrow is the sum of the isomorphisms

$$G_0(F_i) \cong K_0(F_i) \cong \mathbb{Z} \cong K_0(F_i[t]) \cong G_0(F_i[t]).$$

By the 5-lemma, the middle vertical arrow is onto, hence an isomorphism.

We can generalize the Fundamental Theorem from rings to schemes by a slight modification of the proof. For every scheme X, let $X[t]$ and $X[t,t^{-1}]$ denote the schemes $X \times \mathrm{Spec}(\mathbb{Z}[t])$ and $X \times \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$ respectively. Thus if $X = \mathrm{Spec}(R)$ we have $X[t] = \mathrm{Spec}(R[t])$ and $X[t,t^{-1}] = \mathrm{Spec}(R[t,t^{-1}])$. Now suppose that X is noetherian. Via the map $\pi\colon X \to X[t]$ defined by $t = 0$, we have an inclusion $\mathbf{M}(X) \subset \mathbf{M}(X[t])$ and hence a transfer map $\pi_*\colon G_0(X) \to G_0(X[t])$ as before. The argument we gave after (6.4.4) above goes through to show that $\pi_* = 0$ here too, because any generator $[\mathcal{F}]$ of $G_0(X)$ becomes zero in $G_0(X[t])$. By 6.4.2 we have an exact sequence

$$G_0(X) \xrightarrow{\pi_*} G_0(X[t]) \to G_0(X[t,t^{-1}]) \to 0$$

and therefore $G_0(X[t]) \cong G_0(X[t,t^{-1}])$.

Fundamental Theorem for G_0-theory of Schemes 6.5.1. *If X is a noetherian scheme then the flat maps $X[t,t^{-1}] \overset{j}{\hookrightarrow} X[t] \overset{i}{\hookrightarrow} X$ induce isomorphisms:*

$$G_0(X) \cong G_0(X[t]) \cong G_0(X[t,t^{-1}]).$$

Proof. We have already seen that j^* is an isomorphism. By Ex. 6.7 there is a map $\pi^*\colon G_0(X[t]) \to G_0(X)$ sending $[\mathcal{F}]$ to $[\mathcal{F}/t\mathcal{F}] - [ann_{\mathcal{F}}(t)]$. Since $\pi^* i^* [\mathcal{F}] = (i\pi)^*[\mathcal{F}] = [\mathcal{F}]$, we again see that i^* is an injection, split by π^*.

It suffices to show that i^* is a surjection for all X. By noetherian induction, we may suppose that the result is true for all proper closed subschemes Z of X. In particular, if Z is the complement of an affine open subscheme $U = \mathrm{Spec}(R)$ of X, we have a commutative diagram whose rows are exact by Application 6.4.2.

$$\begin{array}{ccccccc}
G_0(Z) & \longrightarrow & G_0(X) & \longrightarrow & G_0(R) & \longrightarrow & 0 \\
\cong \big\downarrow i^* & & \big\downarrow i^* & & \cong \big\downarrow i^* & & \\
G_0(Z[t]) & \longrightarrow & G_0(X[t]) & \longrightarrow & G_0(R[t]) & \longrightarrow & 0
\end{array}$$

The outside vertical arrows are isomorphisms, by induction and Theorem 6.5. By the 5-lemma, $G_0(X) \xrightarrow{i^*} G_0(X[t])$ is onto, and hence an isomorphism.

Euler Characteristics

Suppose that $C.\colon\ 0 \to C_m \to \cdots \to C_n \to 0$ is a bounded chain complex of objects in an abelian category $\mathcal{A}$. We define the *Euler characteristic* $\chi(C.)$ of $C.$ to be the following element of $K_0(\mathcal{A})$:

$$\chi(C.) = \sum(-1)^i[C_i].$$

PROPOSITION 6.6. *If $C.$ is a bounded complex of objects in $\mathcal{A}$, the element $\chi(C.)$ depends only upon the homology of $C.$:*

$$\chi(C.) = \sum(-1)^i[H_i(C.)].$$

In particular, if $C.$ is acyclic (exact as a sequence) then $\chi(C.) = 0$.

PROOF. Write Z_i and B_{i-1} for the kernel and image of the map $C_i \to C_{i-1}$, respectively. Since $B_{i-1} = C_i/Z_i$ and $H_i(C.) = Z_i/B_i$, we compute in $K_0(\mathcal{A})$:

$$\begin{aligned}\sum(-1)^i[H_i(C.)] &= \sum(-1)^i[Z_i] - \sum(-1)^i[B_i]\\ &= \sum(-1)^i[Z_i] + \sum(-1)^i[B_{i-1}]\\ &= \sum(-1)^i[C_i] = \chi(C.).\end{aligned}$$

Let $\mathbf{Ch}^{hb}(\mathcal{A})$ denote the abelian category of (possibly unbounded) chain complexes of objects in $\mathcal{A}$ having only finitely many nonzero homology groups. We call such complexes *homologically bounded.*

COROLLARY 6.6.1. *There is a natural surjection $\chi_H\colon K_0(\mathbf{Ch}^{hb}) \to K_0(\mathcal{A})$ sending $C.$ to $\sum(-1)^i[H_i(C.)]$. In particular, if $0 \to A. \to B. \to C. \to 0$ is a exact sequence of homologically bounded complexes then:*

$$\chi_H(B.) = \chi_H(A.) + \chi_H(C.).$$

EXERCISES

6.1 Let R be a ring and $\mathbf{mod}_{fl}(R)$ the abelian category of R-modules with finite length. Show that $K_0\mathbf{mod}_{fl}(R)$ is the free abelian group $\bigoplus_{\mathfrak{m}}\mathbb{Z}$, a basis being $\{[R/\mathfrak{m}], \mathfrak{m}$ a maximal right ideal of$R\}$. *Hint:* Use the Jordan-Hölder Theorem for modules of finite length.

6.2 *Schreier Refinement Theorem.* Let $A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_r = 0$ and $A = A'_0 \supseteq A'_1 \supseteq \cdots \supseteq A'_s = 0$ be two filtrations of an object A in an abelian category $\mathcal{A}$. Show that the subobjects $A_{i,j} = (A_i \cap A'_j) + A_{i+1}$, ordered lexicographically, form a filtration of A which refines the filtration $\{A_i\}$. By symmetry, there is also a filtration by the $A'_{j,i} = (A_i \cap A'_j) + A'_{j+1}$ which refines the filtration $\{A'_j\}$.

Prove *Zassenhaus' Lemma*, that $A_{i,j}/A_{i,j+1} \cong A'_{j,i}/A'_{j,i+1}$. This shows that the factors in the two refined filtrations are isomorphic up to a permutation; the slogan is that "any two filtrations have equivalent refinements."

6.3 *Jordan-Hölder Theorem in $\mathcal{A}$.* An object A in an abelian category $\mathcal{A}$ is called *simple* if it has no proper subobjects. We say that an object A has *finite length* if it has a composition series $A = A_0 \supset \cdots \supset A_s = 0$ in which all the quotients A_i/A_{i+1} are simple. By Ex. 6.2, the Jordan-Hölder Theorem holds in $\mathcal{A}_{fl}$: the simple factors in any composition series of A are unique up to permutation and isomorphism. Let $\mathcal{A}_{fl}$ denote the subcategory of objects in $\mathcal{A}$ finite length. Show that $\mathcal{A}_{fl}$ is a Serre subcategory of $\mathcal{A}$, and that $K_0(\mathcal{A}_{fl})$ is the free abelian group on the set of isomorphism classes of simple objects.

6.4 Let $\mathcal{A}$ be a small abelian category. If $[A_1] = [A_2]$ in $K_0(\mathcal{A})$, show that there are short exact sequences in $\mathcal{A}$

$$0 \to C' \to C_1 \to C'' \to 0, \quad 0 \to C' \to C_2 \to C'' \to 0$$

such that $A_1 \oplus C_1 \cong A_2 \oplus C_2$. *Hint:* First find sequences $0 \to D_i' \to D_i \to D_i'' \to 0$ such that $A_1 \oplus D_1' \oplus D_1'' \oplus D_2 \cong A_2 \oplus D_2' \oplus D_2'' \oplus D_1$, and set $C_i = D_i' \oplus D_i'' \oplus D_j$.

6.5 *Resolution.* Suppose that R is a regular noetherian ring, *i.e.*, that every R-module has a finite projective resolution. Show that the Cartan homomorphism $K_0(R) \to G_0(R)$ is onto. (We will see in Theorem 7.7 that it is an isomorphism.)

6.6 *Serre's Formula.* (Cf. 7.8.3) If s is a central element of a ring R, show that there is a map $\pi^*\colon G_0(R) \to G_0(R/sR)$ sending $[M]$ to $[M/Ms] - [ann_M(s)]$, where $ann_M(s) = \{x \in M : xs = 0\}$. Theorem 6.5 gives an example where π^* is onto, and if s is nilpotent the map is zero by Devissage 6.3.1. *Hint:* Use the map $M \xrightarrow{s} M$.

6.7 Let Y be a noetherian scheme over the ring $\mathbb{Z}[t]$, and let $X \overset{\pi}{\hookrightarrow} Y$ be the closed subscheme defined by $t = 0$. If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, let $ann_{\mathcal{F}}(t)$ denote the submodule of $\mathcal{F}$ annihilated by t. Show that there is a map $\pi^*\colon G_0(Y) \to G_0(X)$ sending $[\mathcal{F}]$ to $[\mathcal{F}/t\mathcal{F}] - [ann_{\mathcal{F}}(t)]$.

6.8 (Heller-Reiner) Let R be a commutative domain with field of fractions F. If $S = R - \{0\}$, show that there is a well-defined map $\Delta : F^\times \to K_0\mathbf{M}_S(R)$ sending the fraction $r/s \in F^\times$ to $[R/Rr] - [R/Rs]$. Then use Ex. 6.4 to show that the localization sequence extends to the exact sequence

$$1 \to R^\times \to F^\times \xrightarrow{\Delta} K_0\mathbf{M}_S(R) \to G_0(R) \to \mathbb{Z} \to 0.$$

6.9 *Weil Divisor Class groups.* Let R be a commutative noetherian ring.

(a) Show that each $\mathbf{M}_i(R)$ is a Serre subcategory of $\mathbf{M}(R)$.

(b) Show that $K_0\mathbf{M}_{i-1}/\mathbf{M}_{i+1}(R) \cong CH^i(R) \oplus D_{i-1}(R)$. In particular, if R is a 1-dimensional domain then $G_0(R) = \mathbb{Z} \oplus CH^1(R)$.

(c) Show that each $F^iG_0(R)/F^{i+1}G_0(R)$ is a quotient of the group $CH^i(R)$.

(d) Suppose that R is a domain with field of fractions F. As in Ex. 6.8, show that there is an exact sequence generalizing Proposition I.3.6:

$$0 \to R^\times \to F^\times \xrightarrow{\Delta} D_1(R) \to CH^1(R) \to 0.$$

In particular, if R is a Krull domain, conclude that $CH^1(R) \cong Cl(R)$.

6.10 Generalize the preceding exercise to a noetherian scheme X, as indicated in Application 6.4.3. *Hint:* F becomes the function field of X, and (d) becomes I.5.12.

6.11 If S is a multiplicatively closed set of central elements in a noetherian ring R, show that

$$K_0\mathbf{M}_S(R) \cong K_0\mathbf{M}_S(R)[t]) \cong K_0\mathbf{M}_S(R[t,t^{-1}]).$$

6.12 *Graded modules.* When $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ is a noetherian graded ring, let $\mathbf{M}_{gr}(S)$ denote the abelian category of f.g. graded S-modules. Write σ for the shift automorphism $M \mapsto M[-1]$ of the category $\mathbf{M}_{gr}(S)$. Show that:

(a) $K_0\mathbf{M}_{gr}(S)$ is a module over the ring $\mathbb{Z}[\sigma, \sigma^{-1}]$

(b) If S is flat over R, there is a map from the direct sum $G_0(R)[\sigma,\sigma^{-1}] = \bigoplus_{n\in\mathbb{Z}} G_0(R)\sigma^n$ to $K_0\mathbf{M}_{gr}(S)$ sending $[M]\sigma^n$ to $[\sigma^n(M \otimes S)]$.

(c) If $S = R$, the map in (b) is an isomorphism: $K_0\mathbf{M}_{gr}(R) \cong G_0(R)[\sigma,\sigma^{-1}]$.

(d) If $S = R[x_1,\cdots,x_m]$ with $x_1,\cdots,x_m$ in S_1, the map is surjective, *i.e.*, $K_0\mathbf{M}_{gr}(S)$ is generated by the classes $[\sigma^n M[x_1,...,x_m]]$. We will see in Ex. 7.14 that the map in (b) is an isomorphism for $S = R[x_1,\cdots,x_m]$.

(e) Let $\mathcal{B}$ be the subcategory of $\mathbf{M}_{gr}(R[x,y])$ of modules on which y is nilpotent. Show that $\mathcal{B}$ is a Serre subcategory, and that

$$K_0\mathcal{B} \cong K_0\mathbf{M}_{gr}(R) \cong G_0(R)[\sigma,\sigma^{-1}]$$

6.13 In this exercise we sketch Serre's proof of the Fundamental Theorem 6.5 when R is a non-commutative ring. We assume the results of the previous exercise. Show that the formula $j(M) = M/(y-1)M$ defines an exact functor $j\colon \mathbf{M}_{gr}(R[x,y]) \to \mathbf{M}(R[x])$, sending $\mathcal{B}$ to zero. In fact, j induces an equivalence

$$\mathbf{M}_{gr}(R[x,y])/\mathcal{B} \cong \mathbf{M}(R[x]).$$

Then use this equivalence to show that the map $i^*\colon G_0(R) \to G_0(R[x])$ is onto.

6.14 *G_0 of projective space.* Let k be a field and set $S = k[x_0,\cdots,x_m]$, with $X = \mathbb{P}^m_k$. Using the notation of Exercises 6.3 and 6.12, let $\mathbf{M}_{gr}(S)_{fl}$ denote the Serre subcategory of $\mathbf{M}_{gr}(S)$ consisting of graded modules of finite length. It is well-known (see [Hart, II.5.15]) that every coherent $\mathcal{O}_X$-module is of the form $\tilde{M}$ for some M in $\mathbf{M}_{gr}(S)$, *i.e.*, that the associated sheaf functor $\mathbf{M}_{gr}(S) \to \mathbf{M}(X)$ is onto, and that if M has finite length then $\tilde{M} = 0$. In fact, there is an equivalence

$$\mathbf{M}_{gr}(S)/\mathbf{M}_{gr}(S)_{fl} \cong \mathbf{M}(\mathbb{P}^m_k).$$

(See [Hart, Ex. II.5.9(c)].) Under this equivalence $\sigma^i(S)$ represents $\mathcal{O}_X(-i)$.

(a) Let F denote the graded S-module S^{m+1}, whose basis lies in degree 0. Use the Koszul sequence exact sequence of (I.5.3):

$$0 \to \sigma^n(\bigwedge^n F) \to \cdots \to \sigma^2(\bigwedge^2 F) \to \sigma\, F \xrightarrow{x_0,\ldots} S \to k \to 0$$

to show that in $K_0\mathbf{M}_{gr}(S)$ every f.g. k-module M satisfies

$$[M] = \sum(-1)^i \binom{m+1}{i}\sigma^i\,[M \otimes_k S] = (1-\sigma)^{m+1}[M \otimes_k S].$$

(b) Show that in $G_0(\mathbb{P}^m_k)$ every $[\mathcal{O}_X(n)]$ is a linear combination of the classes $[\mathcal{O}_X], [\mathcal{O}_X(-1)], \cdots, [\mathcal{O}_X(-m)]$.

(c) We will see in Ex. 7.14 that the map in Ex. 6.12(b) is an isomorphism:

$$K_0\mathbf{M}_{gr}(S) \cong G_0(R)[\sigma, \sigma^{-1}].$$

Assume this calculation, and show that

$$G_0(\mathbb{P}^m_k) \cong \mathbb{Z}^m \quad \text{on generators } [\mathcal{O}_X], [\mathcal{O}_X(-1)], \cdots, [\mathcal{O}_X(-m)].$$

6.15 *Naturality of f_*.* Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper morphisms between noetherian schemes. Show that $(gf)_* = g_* f_*$ as maps $G_0(X) \to G_0(Z)$.

§7. K_0 of an Exact Category

If $\mathcal{C}$ is an additive subcategory of an abelian category $\mathcal{A}$, we may still talk about exact sequences: an *exact sequence* in $\mathcal{C}$ is a sequence of objects (and maps) in $\mathcal{C}$ which is exact as a sequence in $\mathcal{A}$. With hindsight, we know that it helps to require $\mathcal{C}$ to be closed under extensions. Thus we formulate the following definitions.

DEFINITION 7.0 (EXACT CATEGORIES). An *exact category* is a pair $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is an additive category and $\mathcal{E}$ is a family of sequences in $\mathcal{C}$ of the form

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0, \tag{$\dagger$}$$

satisfying the following condition: there is an embedding of $\mathcal{C}$ as a full subcategory of an abelian category $\mathcal{A}$ so that

(1) $\mathcal{E}$ is the class of all sequences ($\dagger$) in $\mathcal{C}$ which are exact in $\mathcal{A}$;
(2) $\mathcal{C}$ is *closed under extensions* in $\mathcal{A}$ in the sense that if ($\dagger$) is an exact sequence in $\mathcal{A}$ with $B, D \in \mathcal{C}$ then $C \in \mathcal{C}$.

The sequences in $\mathcal{E}$ are called the *short exact sequences* of $\mathcal{C}$. We will often abuse notation and just say that $\mathcal{C}$ is an exact category when the class $\mathcal{E}$ is clear. We call a map in $\mathcal{C}$ an *admissible monomorphism* (resp. an *admissible epimorphism*) if it occurs as the monomorphism i (resp. as the epi j) in some sequence ($\dagger$) in $\mathcal{E}$.

The following hypothesis is commonly satisfied in applications, and is needed for Euler characteristics and the Resolution Theorem 7.5 below.
(7.0.1) We say that $\mathcal{C}$ is *closed under kernels of surjections* in $\mathcal{A}$ provided that whenever a map $f: B \to C$ in $\mathcal{C}$ is a surjection in $\mathcal{A}$ then $\ker(f) \in \mathcal{C}$. The well-read reader will observe that the definition of exact category in [Bass] is what we call an exact category closed under kernels of surjections.

An *exact functor* $F: \mathcal{B} \to \mathcal{C}$ between exact categories is an additive functor F carrying short exact sequences in $\mathcal{B}$ to exact sequences in $\mathcal{C}$. If $\mathcal{B}$ is a full subcategory of $\mathcal{C}$, and the exact sequences in $\mathcal{B}$ are precisely the sequences ($\dagger$) in $\mathcal{B}$ which are exact in $\mathcal{C}$, we call $\mathcal{B}$ an *exact subcategory* of $\mathcal{C}$. This is consistent with the notion of an exact abelian subcategory in §6.

DEFINITION 7.1 (K_0). Let $\mathcal{C}$ be a small exact category. $K_0(\mathcal{C})$ is the abelian group having generators $[C]$, one for each object C of $\mathcal{C}$, and relations $[C] = [B] + [D]$ for every short exact sequence $0 \to B \to C \to D \to 0$ in $\mathcal{C}$.

As in 6.1.1, we have $[0] = 0$, $[B \oplus D] = [B] + [D]$ and $[B] = [C]$ if B and C are isomorphic. As before, we could actually define $K_0(\mathcal{C})$ when $\mathcal{C}$ is only skeletally small, but we shall not dwell on these set-theoretic intricacies. Clearly, $K_0(\mathcal{C})$ satisfies the universal property 6.1.2 for additive functions from $\mathcal{C}$ to abelian groups.

EXAMPLE 7.1.1. The category $\mathbf{P}(R)$ of f.g. projective R-modules is exact by virtue of its embedding in $\mathbf{mod}$-R. As every exact sequence of projective modules splits, we have $K_0\mathbf{P}(R) = K_0(R)$.

Any additive category is a symmetric monoidal category under $\oplus$, and the above remarks show that $K_0(\mathcal{C})$ is a quotient of the group $K_0^{\oplus}(\mathcal{C})$ of §5. Since abelian categories are exact, Examples 6.2.1–4 show that these groups are not identical.

EXAMPLE 7.1.2 (SPLIT EXACT CATEGORIES). A *split exact* category $\mathcal{C}$ is an exact category in which every short exact sequence in $\mathcal{E}$ is split (*i.e.*, isomorphic to $0 \to B \to B \oplus D \to D \to 0$). In this case we have $K_0(\mathcal{C}) = K_0^{\oplus}(\mathcal{C})$ by definition. For example, the category $\mathbf{P}(R)$ is split exact.

If X is a topological space, the embedding of $\mathbf{VB}(X)$ in the abelian category of families of vector spaces over X makes $\mathbf{VB}(X)$ into an exact category. By the Subbundle Theorem I.4.1, $\mathbf{VB}(X)$ is a split exact category, so that $K^0(X) = K_0(\mathbf{VB}(X))$.

We will see in Exercise 7.7 that any additive category $\mathcal{C}$ may be made into a split exact category by equipping it with the class $\mathcal{E}_{split}$ of sequences isomorphic to $0 \to B \to B \oplus D \to D \to 0$

WARNING. Every abelian category $\mathcal{A}$ has a natural exact category structure, but it also has the split exact structure. These will yield different K_0 groups in general, unless something like a Krull-Schmidt Theorem holds in $\mathcal{A}$. We will always use the natural exact structure unless otherwise indicated.

EXAMPLE 7.1.3 (K_0 OF A SCHEME). Let X be a scheme (or more generally a ringed space). The category $\mathbf{VB}(X)$ of algebraic vector bundles on X, introduced in (I.5), is an exact category by virtue of its being an additive subcategory of the abelian category $\mathcal{O}_X$-$\mathbf{mod}$ of all $\mathcal{O}_X$-modules. We write $K_0(X)$ for $K_0\mathbf{VB}(X)$. If X is noetherian, the inclusion $\mathbf{VB}(X) \subset \mathbf{M}(X)$ yields a *Cartan homomorphism* $K_0(X) \to G_0(X)$. We saw in (I.5.3) that exact sequences in $\mathbf{VB}(X)$ do not always split, so $\mathbf{VB}(X)$ is not always a split exact category.

EXAMPLE 7.1.4 (G_0 OF NON-NOETHERIAN RINGS). If R is a non-noetherian ring, the category $\mathbf{mod}_{fg}(R)$ of all finitely generated R-modules will not be abelian, because $R \to R/I$ has no kernel inside this category. However, it is still an exact subcategory of $\mathbf{mod}$-R, so once again we might try to consider the group $K_0\mathbf{mod}_{fg}(R)$. However, it turns out that this definition does not have good properties (see Ex. 7.3 and 7.4).

Here is a more suitable definition, based upon [SGA6, I.2.9]. An R-module M is called *pseudo-coherent* if it has an infinite resolution $\cdots \to P_1 \to P_0 \to M \to 0$ by f.g. projective R-modules. Pseudo-coherent modules are clearly finitely presented, and if R is right noetherian then every finitely generated module is pseudo-coherent. Let $\mathbf{M}(R)$ denote the category of all pseudo-coherent R-modules. The "Horseshoe Lemma" [WHomo, 2.2.8] shows that $\mathbf{M}(R)$ is closed under extensions in $\mathbf{mod}$-R, so it is an exact category. (It is also closed under kernels of surjections, and cokernels of injections in $\mathbf{mod}$-R, as can be seen using the mapping cone.)

Now we define $G_0(R) = K_0\mathbf{M}(R)$. Note that if R is right noetherian then $\mathbf{M}(R)$ is the usual category of §6, and we have recovered the definition of $G_0(R)$ in 6.2.

EXAMPLE 7.1.5. The opposite category $\mathcal{C}^{op}$ has an obvious notion of exact sequence: turn the arrows around in the exact sequences of $\mathcal{C}$. Formally, this arises from the inclusion of $\mathcal{C}^{op}$ in $\mathcal{A}^{op}$. Clearly $K_0(\mathcal{C}) \cong K_0(\mathcal{C}^{op})$.

EXAMPLE 7.1.6. The direct sum $\mathcal{C}_1 \oplus \mathcal{C}_2$ of two exact categories is also exact, the ambient abelian category being $\mathcal{A}_1 \oplus \mathcal{A}_2$. Clearly $K_0(\mathcal{C}_1 \oplus \mathcal{C}_2) \cong K_0(\mathcal{C}_1) \oplus K_0(\mathcal{C}_2)$. More generally, the direct sum $\bigoplus \mathcal{C}_i$ of exact categories is an exact category (inside the abelian category $\oplus\mathcal{A}_i$), and as in 6.1.2 this yields $K_0(\oplus\mathcal{C}_i) \cong \oplus K_0(\mathcal{C}_i)$.

EXAMPLE 7.1.7 (FILTERED COLIMITS). Suppose that $\{\mathcal{C}_i\}$ is a filtered family of exact subcategories of a fixed abelian category $\mathcal{A}$. Then $\mathcal{C} = \cup\mathcal{C}_i$ is also an exact subcategory of $\mathcal{A}$, and by inspection of the definition we see that

$$K_0(\bigcup \mathcal{C}_i) = \varinjlim K_0(\mathcal{C}_i).$$

As a case in point, if a ring R is the union of subrings R_α then $\mathbf{P}(R)$ is the direct limit of the $\mathbf{P}(R_\alpha)$, and we have $K_0(R) = \varinjlim K_0(R_\alpha)$, as in §2.

COFINALITY LEMMA 7.2. *Let $\mathcal{B}$ be an exact subcategory of $\mathcal{C}$ which is closed under extensions in $\mathcal{C}$, and which is cofinal in the sense that for every C in $\mathcal{C}$ there is a C' in $\mathcal{C}$ so that $C \oplus C'$ is in $\mathcal{B}$. Then $K_0\mathcal{B}$ is a subgroup of $K_0\mathcal{C}$.*

PROOF. By (1.3) we know that $K_0^{\oplus}\mathcal{B}$ is a subgroup of $K_0^{\oplus}\mathcal{C}$. Given a short exact sequence $0 \to C_0 \to C_1 \to C_2 \to 0$ in $\mathcal{C}$, choose C_0' and C_2' in $\mathcal{C}$ so that $B_0 = C_0 \oplus C_0'$ and $B_2 = C_2 \oplus C_2'$ are in $\mathcal{B}$. Setting $B_1 = C_1 \oplus C_0' \oplus C_2'$, we have the short exact sequence $0 \to B_0 \to B_1 \to B_2 \to 0$ in $\mathcal{C}$. As $\mathcal{B}$ is closed under extensions in $\mathcal{C}$, $B_1 \in \mathcal{B}$. Therefore in $K_0^{\oplus}\mathcal{C}$:

$$[C_1] - [C_0] - [C_2] = [B_1] - [B_0] - [B_2].$$

Thus the kernel of $K_0^{\oplus}\mathcal{C} \to K_0\mathcal{C}$ equals the kernel of $K_0^{\oplus}\mathcal{B} \to K_0\mathcal{B}$, which implies that $K_0\mathcal{B} \to K_0\mathcal{C}$ is an injection.

Idempotent completion.

7.2.1. A category $\mathcal{C}$ is called *idempotent complete* if every idempotent endomorphism e of an object C factors as $C \to B \to C$ with the composite $B \to C \to B$ being the identity. Given $\mathcal{C}$, we can form a new category $\widehat{\mathcal{C}}$ whose objects are pairs (C, e) with e an idempotent endomorphism of an object C of $\mathcal{C}$; a morphism from (C, e) to (C', e') is a map $f: C \to C'$ in $\mathcal{C}$ such that $f = e'fe$. The category $\widehat{\mathcal{C}}$ is idempotent complete, since an idempotent endomorphism f of (C, e) factors through the object (C, efe).

$\widehat{\mathcal{C}}$ is called the *idempotent completion* of $\mathcal{C}$. To see why, consider the natural embedding of $\mathcal{C}$ into $\widehat{\mathcal{C}}$ sending C to (C, id). It is easy to see that any functor from $\mathcal{C}$ to an idempotent complete category $\mathcal{D}$ must factor through a functor $\widehat{\mathcal{C}} \to \mathcal{D}$ that is unique up to natural equivalence. In particular, if $\mathcal{C}$ is idempotent then $\mathcal{C} \cong \widehat{\mathcal{C}}$.

If $\mathcal{C}$ is an additive subcategory of an abelian category $\mathcal{A}$, then $\widehat{\mathcal{C}}$ is equivalent to a larger additive subcategory $\mathcal{C}'$ of $\mathcal{A}$ (see Ex. 7.6). Moreover, $\mathcal{C}$ is cofinal in $\widehat{\mathcal{C}}$, because (C, e) is a summand of C in $\mathcal{A}$. By the Cofinality Lemma 7.2, we see that $K_0(\mathcal{C})$ is a subgroup of $K_0(\widehat{\mathcal{C}})$.

EXAMPLE 7.2.2. Consider the subcategory $\mathbf{F}(R)$ of $\mathbf{M}(R)$ consisting of f.g. free R-modules. The idempotent completion of $\mathbf{F}(R)$ is the category $\mathbf{P}(R)$ of f.g. projective modules. Thus the cyclic group $K_0\mathbf{F}(R)$ is a subgroup of $K_0(R)$. If R satisfies the Invariant Basis Property (IBP), then $K_0\mathbf{F}(R) \cong \mathbb{Z}$ and we have recovered the conclusion of Lemma 2.1.

EXAMPLE 7.2.3. Let $R \to S$ be a ring homomorphism, and let $\mathcal{B}$ denote the full subcategory of $\mathbf{P}(S)$ on the modules of the form $P \otimes_R S$ for P in $\mathbf{P}(R)$. Since it contains all the free modules S^n, $\mathcal{B}$ is cofinal in $\mathbf{P}(S)$, so $K_0\mathcal{B}$ is a subgroup of $K_0(S)$. Indeed, $K_0\mathcal{B}$ is the image of the natural map $K_0(R) \to K_0(S)$.

Products

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be exact categories. A functor $F : \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ is called *biexact* if $F(A,-)$ and $F(-,B)$ are exact functors for every A in $\mathcal{A}$ and B in $\mathcal{B}$, and $F(0,-)=F(-,0)=0$. (The last condition, not needed in this chapter, can always be arranged by replacing $\mathcal{C}$ by an equivalent category.) The following result is completely elementary.

LEMMA 7.3. *A biexact functor $F\colon\mathcal{A}\times\mathcal{B}\to\mathcal{C}$ induces a bilinear map*

$$\begin{aligned} K_0\mathcal{A}\otimes K_0\mathcal{B} &\to K_0\mathcal{C}. \\ [A]\otimes[B] &\mapsto [F(A,B)] \end{aligned}$$

APPLICATION 7.3.1. Let R be a commutative ring. The tensor product $\otimes_A$ defines a biexact functor $\mathbf{P}(R)\times\mathbf{P}(R)\to\mathbf{P}(R)$, as well as a biexact functor $\mathbf{P}(R)\times\mathbf{M}(R)\to\mathbf{M}(R)$. The former defines the product $[P][Q]=[P\otimes Q]$ in the commutative ring $K_0(R)$, as we saw in §2. The latter defines an action of $K_0(R)$ on $G_0(R)$, making $G_0(R)$ into a $K_0(R)$-module.

APPLICATION 7.3.2. Let X be a scheme (or more generally a locally ringed space) The tensor product of vector bundles defines a biexact functor $\mathbf{VB}(X)\times\mathbf{VB}(X)\to\mathbf{VB}(X)$ (see I.5.3). This defines a product on $K_0(X)$ satisfying $[\mathcal{E}][\mathcal{F}]=[\mathcal{E}\otimes\mathcal{F}]$. This product is clearly commutative and associative, so it makes $K_0(X)$ into a commutative ring. We will discuss this ring further in the next section.

If X is noetherian, recall from 6.2.5 that $G_0(X)$ denotes $K_0\mathbf{M}(X)$. Since the tensor product of a vector bundle and a coherent module is coherent, we have a functor $\mathbf{VB}(X)\times\mathbf{M}(X)\to\mathbf{M}(X)$. It is biexact (why?), so it defines an action of $K_0(X)$ on $G_0(X)$, making $G_0(X)$ into a $K_0(X)$-module.

APPLICATION 7.3.3 (ALMKVIST). If R is a ring, let $\mathbf{End}(R)$ denote the exact category whose objects (P,α) are pairs, where P is a fin. gen. projective R-module and α is an endomorphism of P. A morphism $(P,\alpha)\to(Q,\beta)$ in $\mathbf{End}(R)$ is a morphism $f\colon P\to Q$ in $\mathbf{P}(R)$ such that $f\alpha=\beta f$, and exactness in $\mathbf{End}(R)$ is determined by exactness in $\mathbf{P}(R)$.

If R is commutative, the tensor product of modules gives a biexact functor

$$\begin{aligned} \otimes_R:\ \mathbf{End}(R)\times\mathbf{End}(R) &\to \mathbf{End}(R), \\ ((P,\alpha),(Q,\beta)) &\mapsto (P\otimes_R Q,\alpha\otimes_R\beta). \end{aligned}$$

As $\otimes_R$ is associative and symmetric up to isomorphism, the induced product makes $K_0\mathbf{End}(R)$ into a commutative ring with unit $[(R,1)]$. The inclusion of $\mathbf{P}(R)$ in $\mathbf{End}(R)$ by $\alpha=0$ is split by the forgetful functor, and the kernel $End_0(R)$ of $K_0\mathbf{End}(R)\to K_0(R)$ is not only an ideal but a commutative ring with unit $1=[(R,1)]-[(R,0)]$. Almkvist proved that $(P,\alpha)\mapsto\det(1-\alpha t)$ defines an isomorphism of $End_0(R)$ with the subgroup of the multiplicative group $W(R)=1+tR[[t]]$ consisting of all quotients $f(t)/g(t)$ of polynomials in $1+tR[t]$ (see Ex. 7.18). Almkvist also proved that $End_0(R)$ is a subring of $W(R)$ under the ring structure of 4.3.

If A is an R-algebra, $\otimes_R$ is also a pairing $\mathbf{End}(R)\times\mathbf{End}(A)\to\mathbf{End}(A)$, making $End_0(A)$ into an $End_0(R)$-module. We leave the routine details to the reader.

Example 7.3.4. If R is a ring, let $\mathbf{Nil}(R)$ denote the category whose objects (P,ν) are pairs, where P is a f.g. projective R-module and ν is a nilpotent endomorphism of P. This is an exact subcategory of $\mathbf{End}(R)$. The forgetful functor $\mathbf{Nil}(R) \to \mathbf{P}(R)$ sending (P,ν) to P is exact, and is split by the exact functor $\mathbf{P}(R) \to \mathbf{Nil}(R)$ sending P to $(P,0)$. Therefore $K_0(R) = K_0\mathbf{P}(R)$ is a direct summand of $K_0\mathbf{Nil}(R)$. We write $Nil_0(R)$ for the kernel of $K_0\mathbf{Nil}(R) \to \mathbf{P}(R)$, so that there is a direct sum decomposition $K_0\mathbf{Nil}(R) = K_0(R) \oplus Nil_0(R)$. Since $[P,\nu] = [P\oplus Q, \nu\oplus 0] - [Q,0]$ in $K_0\mathbf{Nil}(R)$, we see that $Nil_0(R)$ is generated by elements of the form $\big[(R^n,\nu)\big] - n\big[(R,0)\big]$ for some n and some nilpotent matrix ν.

If A is an R-algebra, then the tensor product pairing on $\mathbf{End}$ restricts to a biexact functor $F\colon \mathbf{End}(R)\times\mathbf{Nil}(A) \to \mathbf{Nil}(A)$. The resulting bilinear map $K_0\mathbf{End}(R) \times K_0\mathbf{Nil}(A) \to K_0\mathbf{Nil}(A)$ is associative, and makes $Nil_0(A)$ into a module over the ring $End_0(R)$, and makes $Nil_0(A) \to End_0(A)$ an $End_0(R)$-module map.

Any additive functor $T : \mathbf{P}(A) \to \mathbf{P}(B)$ induces an exact functor $\mathbf{Nil}(A) \to \mathbf{Nil}(B)$ and a homomorphism $Nil_0(A) \to Nil_0(B)$. If A and B are R-algebras and T is R-linear, $Nil_0(A) \to Nil_0(B)$ is an $End_0(R)$-module homomorphism. (Exercise!)

Example 7.3.5. If R is a commutative regular ring, and $A = R[x]/(x^N)$, we will see in III.3.8.1 that $Nil_0(A) \to End_0(A)$ is an injection, identifying $Nil_0(A)$ with the ideal $(1+xtA[t])^\times$ of $End_0(A)$, and identifying $[(A,x)]$ with $1-xt$.

This isomorphism $End_0(A) \cong (1+xtA[t])^\times$ is universal in the following sense. If B is an R-algebra and (P,ν) is in $\mathbf{Nil}(B)$, with $\nu^N=0$, we may regard P as an A–B bimodule. By 2.8, this yields an R-linear functor $\mathbf{Nil}_0(A) \to \mathbf{Nil}_0(B)$ sending (A,x) to (P,ν). By 7.3.4, there is an $End_0(R)$-module homomorphism $(1+xtA[t])^\times \to Nil_0(B)$ sending $1-xt$ to $[(P,\nu)]$.

The following result shows that Euler characteristics can be useful in exact categories as well as in abelian categories, and is the analogue of Proposition 6.6.

Proposition 7.4. *Suppose that $\mathcal{C}$ is closed under kernels of surjections in an abelian category $\mathcal{A}$. If $C.$ is a bounded chain complex in $\mathcal{C}$ whose homology $H_i(C.)$ is also in $\mathcal{C}$ then in $K_0(\mathcal{C})$:*

$$\chi(C.) = \sum(-1)^i[C_i] \quad \textit{equals} \quad \sum(-1)^i[H_i(C.)].$$

In particular, if $C.$ is any exact sequence in $\mathcal{C}$ then $\chi(C.)=0$.

Proof. The proof we gave in 6.6 for abelian categories will go through, provided that the Z_i and B_i are objects of $\mathcal{C}$. Consider the exact sequences:

$$0 \to Z_i \to C_i \to B_i \to 0$$

$$0 \to B_i \to Z_i \to H_i(C.) \to 0.$$

Since $B_i = 0$ for $i << 0$, the following inductive argument shows that all the B_i and Z_i belong to $\mathcal{C}$. If $B_{i-1}\in\mathcal{C}$ then the first sequence shows that $Z_i\in\mathcal{C}$; since $H_i(C.)$ is in $\mathcal{C}$, the second sequence shows that $B_i\in\mathcal{C}$.

Corollary 7.4.1. *Suppose $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$. If $f\colon C'_. \to C_.$ is a morphism of bounded complexes in $\mathcal{C}$, inducing an isomorphism on homology, then*

$$\chi(C'_.) = \chi(C_.).$$

Proof. Form the mapping cone cone(f), which has $C_n \oplus C'_{n-1}$ in degree n. By inspection, $\chi(\text{cone}(f)) = \chi(C_.) - \chi(C'_.)$. But cone$(f)$ is an exact complex because f is a homology isomorphism, so $\chi(\text{cone}(f)) = 0$.

The Resolution Theorem

We need a definition in order to state our next result. Suppose that $\mathcal{P}$ is an additive subcategory of an abelian category $\mathcal{A}$. A *$\mathcal{P}$-resolution* $P_. \to C$ of an object C of $\mathcal{A}$ is an exact sequence in $\mathcal{A}$

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to C \to 0$$

in which all the P_i are in $\mathcal{P}$. The *$\mathcal{P}$-dimension* of C is the minimum n (if it exists) such that there is a resolution $P_. \to C$ with $P_i = 0$ for $i > n$.

Resolution Theorem 7.5. *Let $\mathcal{P} \subset \mathcal{C} \subset \mathcal{A}$ be an inclusion of additive categories with $\mathcal{A}$ abelian ($\mathcal{A}$ gives the notion of exact sequence to $\mathcal{P}$ and $\mathcal{C}$). Assume that:*
(a) Every object C has finite $\mathcal{P}$-dimension; and
(b) $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$.
Then the inclusion $\mathcal{P} \subset \mathcal{C}$ induces an isomorphism $K_0(\mathcal{P}) \cong K_0(\mathcal{C})$.

Proof. To see that $K_0(\mathcal{P})$ maps onto $K_0(\mathcal{C})$, observe that if $P_. \to C$ is a finite $\mathcal{P}$-resolution, then the exact sequence

$$0 \to P_n \to \cdots \to P_0 \to C \to 0$$

has $\chi = 0$ by 7.4, so $[C] = \sum(-1)^i[P] = \chi(P_.)$ in $K_0(\mathcal{C})$. To see that $K_0(\mathcal{P}) \cong K_0(\mathcal{C})$, we will show that the formula $\chi(C) = \chi(P_.)$ defines an additive function from $\mathcal{C}$ to $K_0(\mathcal{P})$. For this, we need the following lemma, due to Grothendieck.

Lemma 7.5.1. *Given a map $f\colon C' \to C$ in $\mathcal{C}$ and a finite $\mathcal{P}$-resolution $P_. \to C$, there is a finite $\mathcal{P}$-resolution $P'_. \to C'$ and a commutative diagram*

$$\begin{array}{ccccccccccccccc}
0 & \to & P'_m & \to & \cdots & \to & P'_n & \to & \cdots & \to & P'_1 & \to & P'_0 & \to & C' & \to & 0 \\
 & & & & & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
 & & & & 0 & \to & P_n & \to & \cdots & \to & P_1 & \to & P_0 & \to & C & \to & 0
\end{array}$$

We will prove this lemma in a moment. First we shall use it to finish the proof of Theorem 7.5. Suppose given two finite $\mathcal{P}$-resolutions $P_. \to C$ and $P'_. \to C$ of an object C. Applying the lemma to the diagonal map $C \to C \oplus C$ and $P_. \oplus P'_. \to C \oplus C$, we get a $\mathcal{P}$-resolution $P''_. \to C$ and a map $P''_. \to P_. \oplus P'_.$ of complexes. Since the maps $P_. \leftarrow P''_. \to P'_.$ are quasi-isomorphisms, Corollary 7.4.1 implies that

$\chi(P.) = \chi(P''.) = \chi(P'.)$. Hence $\chi(C) = \chi(P.)$ is independent of the choice of $\mathcal{P}$-resolution.

Given a short exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$ and a $\mathcal{P}$-resolution $P. \to C$, the lemma provides a $\mathcal{P}$-resolution $P'. \to C'$ and a map $f\colon P'. \to P.$. Form the mapping cone complex cone(f), which has $P_n \oplus P'_n[-1]$ in degree n, and observe that $\chi(\text{cone}(f)) = \chi(P.) - \chi(P'.)$. The homology exact sequence

$$H_i(P') \to H_i(P) \to H_i(\text{cone}(f)) \to H_{i-1}(P') \to H_{i-1}(P)$$

shows that $H_i\text{cone}(f) = 0$ for $i \neq 0$, and $H_0(\text{cone}(f)) = C''$. Thus $\text{cone}(f) \to C''$ is a finite $\mathcal{P}$-resolution, and so

$$\chi(C'') = \chi(\text{cone}(f)) = \chi(P.) - \chi(P'.) = \chi(C) - \chi(C').$$

This proves that χ is an additive function, so it induces a map $\chi\colon K_0\mathcal{C} \to K_0(\mathcal{P})$. If P is in $\mathcal{P}$ then evidently $\chi(P) = [P]$, so χ is the inverse isomorphism to the map $K_0(\mathcal{P}) \to K_0(\mathcal{C})$. This finishes the proof of the Resolution Theorem 7.5.

Proof of Lemma 7.5.1. We proceed by induction on the length n of $P.$. If $n = 0$, we may choose any $\mathcal{P}$-resolution of C'; the only nonzero map $P'_n \to P_n$ is $P'_0 \to C' \to C \cong P_0$. If $n \geq 1$, let Z denote the kernel (in $\mathcal{A}$) of $\varepsilon\colon P_0 \to C$ and let B denote the kernel (in $\mathcal{A}$) of $(\varepsilon, -f)\colon P_0 \oplus C' \to C$. As $\mathcal{C}$ is closed under kernels, both Z and B are in $\mathcal{C}$. Moreover, the sequence

$$0 \to Z \to B \to C' \to 0$$

is exact in $\mathcal{C}$ (because it is exact in $\mathcal{A}$). Choose a surjection $P'_0 \to B$ with P'_0 in $\mathcal{P}$ and let Y denote the kernel of the surjection $P'_0 \to B \to C'$. By induction applied to $Y \to Z$, we can find a $\mathcal{P}$-resolution $P'.[+1]$ of Y and maps $f_i\colon P'_i \to P_i$ making the following diagram commute (the rows are not exact at Y and Z):

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P'_2 & \longrightarrow & P'_1 & \longrightarrow & Y & \longrightarrow & P'_0 & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & Z & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Splicing the rows by deleting Y and Z yields the desired $\mathcal{P}$-resolution of C'.

Definition 7.6 ($\mathbf{H}(R)$). Given a ring R, let $\mathbf{H}(R)$ denote the category of all R-modules M having a finite resolution by f.g. projective modules, and let $\mathbf{H}_n(R)$ denote the subcategory in which the resolutions have length $\leq n$.

By the Horseshoe Lemma [WHomo, 2.2.8], both $\mathbf{H}(R)$ and $\mathbf{H}_n(R)$ are exact subcategories of **mod**-R. The following Lemma shows that they are also closed under kernels of surjections in **mod**-R.

Lemma 7.6.1. *If* $0 \to L \to M \xrightarrow{f} N \to 0$ *is a short exact sequence of modules, with* M *in* $\mathbf{H}_m(R)$ *and* N *in* $\mathbf{H}_n(R)$*, then* L *is in* $\mathbf{H}_\ell(R)$*, where* $\ell = \min\{m, n-1\}$.

Proof. If $P. \to M$ and $Q. \to N$ are projective resolutions, and $P. \to Q.$ lifts f, then the kernel P'_0 of the surjection $P_0 \oplus Q_1 \to Q_0$ is f.g. projective, and the truncated mapping cone $\cdots \to P_1 \oplus Q_2 \to P'_0$ is a resolution of L.

Corollary 7.6.2. *$K_0(R) \cong K_0\mathbf{H}(R) \cong K_0\mathbf{H}_n(R)$ for all $n \geq 1$.*

Proof. Apply the Resolution Theorem to $\mathbf{P}(R) \subset \mathbf{H}(R)$.

Here is a useful variant of the above construction. Let S be a multiplicatively closed set of central nonzerodivisors in a ring R. We say module M is S-torsion if $Ms = 0$ for some $s \in S$ (*cf*. Example 6.2.8), and write $\mathbf{H}_S(R)$ for the exact subcategory $\mathbf{H}(R) \cap \mathbf{M}_S(R)$ of S-torsion modules M in $\mathbf{H}(R)$. Similarly, we write $\mathbf{H}_{n,S}(R)$ for the S-torsion modules in $\mathbf{H}_n(R)$. Note that $\mathbf{H}_{0,S}(R) = 0$, and that the modules R/sR belong to $\mathbf{H}_{1,S}(R)$.

Corollary 7.6.3. *$K_0\mathbf{H}_S(R) \cong K_0\mathbf{H}_{n,S}(R) \cong K_0\mathbf{H}_{1,S}(R)$ for all $n \geq 1$.*

Proof. We apply the Resolution Theorem with $\mathcal{P} = \mathbf{H}_{1,S}(R)$. By Lemma 7.6.1, each $\mathbf{H}_{n,S}(R)$ is closed under kernels of surjections. Every N in $\mathbf{H}_{n,S}(R)$ is finitely generated, so if $Ns = 0$ there is an exact sequence $0 \to L \to (R/sR)^m \to N \to 0$. If $n \geq 2$ then L is in $\mathbf{H}_{n-1,S}(R)$ by Lemma 7.6.1. By induction, L and hence N has a $\mathcal{P}$-resolution.

Corollary 7.6.4. *If S is a multiplicatively closed set of central nonzerodivisors in a ring R, the sequence $K_0\mathbf{H}_S(R) \to K_0(R) \to K_0(S^{-1}R)$ is exact.*

Proof. If $[P] - [R^n] \in K_0(R)$ vanishes in $K_0(S^{-1}R)$, $S^{-1}P$ is stably free (Cor. 1.3). Hence there is an isomorphism $(S^{-1}R)^{m+n} \to S^{-1}P \oplus (S^{-1}R)^m$. Clearing denominators yields a map $f\colon R^{m+n} \to P \oplus R^m$ whose kernel and cokernel are S-torsion. But $\ker(f) = 0$ because S consists of nonzerodivisors, and therefore $M = \operatorname{coker}(f)$ is in $\mathbf{H}_{1,S}(R)$. But the map $K_0\mathbf{H}_S(R) \to K_0\mathbf{H}(R) = K_0(R)$ sends $[M]$ to $[M] = [P] - [R^n]$.

Let R be a regular noetherian ring. Since every module has finite projective dimension, $\mathbf{H}(R)$ is the abelian category $\mathbf{M}(R)$ discussed in §6. Combining Corollary 7.6.2 with the Fundamental Theorem for G_0 6.5, we have:

Fundamental Theorem for K_0 of regular rings 7.7. *If R is a regular noetherian ring, then $K_0(R) \cong G_0(R)$. Moreover,*

$$K_0(R) \cong K_0(R[t]) \cong K_0(R[t, t^{-1}]).$$

If R is not regular, we can still use the localization sequence 7.6.4 to get a partial result, which will be considerably strengthened by the Fundamental Theorem for K_0 in chapter III.

Proposition 7.7.1. *The map $K_0(R[t]) \to K_0(R[t, t^{-1}])$ is injective for every ring R.*

To prove this, we need the following lemma. Recall from Example 7.3.4 that $\mathbf{Nil}(R)$ is the category of pairs (P, ν) with ν a nilpotent endomorphism of $P \in \mathbf{P}(R)$.

Lemma 7.7.2. *Let S be the multiplicative set $\{t^n\}$ in the polynomial ring $R[t]$. Then $\mathbf{Nil}(R)$ is equivalent to the category $\mathbf{H}_{1,S}(R[t])$ of t-torsion $R[t]$-modules M in $\mathbf{H}_1(R[t])$.*

Proof. If (P, ν) is in $\mathbf{Nil}(R)$, let P_ν denote the $R[t]$-module P on which t acts as ν. It is a t-torsion module because $t^n P_\nu = \nu^n P = 0$ for large n. A projective resolution of P_ν is given by the "characteristic sequence" of ν:

$$0 \to P[t] \xrightarrow{t-\nu} P[t] \to P_\nu \to 0, \tag{7.7.3}$$

Thus P_ν is an object of $\mathbf{H}_{1,S}(R[t])$. Conversely, each M in $\mathbf{H}_{1,S}(R[t])$ has a projective resolution $0 \to P \to Q \to M \to 0$ by f.g. projective $R[t]$-modules, and M is killed by some power t^n of t. From the exact sequence

$$0 \to \mathrm{Tor}_1^{R[t]}\big(M, R[t]/(t^n)\big) \to P/t^nP \to Q/t^nQ \to M \to 0$$

and the identification of the first term with M we obtain the exact sequence $0 \to M \xrightarrow{t^n} P/t^nP \to P/t^nQ \to 0$. Since P/t^nP is a projective R-module and $pd_R(P/t^nQ) \le 1$, we see that M must be a projective R-module. Thus (M, t) is an object of $\mathbf{Nil}(R)$.

Combining Lemma 7.7.2 with Corollary 7.6.3 yields:

Corollary 7.7.4. $K_0\mathbf{Nil}(R) \cong K_0\mathbf{H}_S(R[t])$.

Proof of Proposition 7.7.1. By Corollaries 7.6.4 and 7.7.4, we have an exact sequence

$$K_0\mathbf{Nil}(R) \to K_0(R[t]) \to K_0(R[t, t^{-1}]).$$

The result will follow once we calculate that the left map is zero. This map is induced by the forgetful functor $\mathbf{Nil}(R) \to \mathbf{H}(R[t])$ sending (P, ν) to P. Since the characteristic sequence (7.7.3) of ν shows that $[P] = 0$ in $K_0(R[t])$, we are done.

Basechange and Transfer Maps for Rings

7.8. Let $f: R \to S$ be a ring homomorphism. We have already seen that the basechange $\otimes_R S: \mathbf{P}(R) \to \mathbf{P}(S)$ is an exact functor, inducing $f^*: K_0(R) \to K_0(S)$. If $S \in \mathbf{P}(R)$, we observed in (2.8.1) that the forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$ is exact, inducing the transfer map $f_*: K_0(S) \to K_0(R)$.

Using the Resolution Theorem, we can also define a transfer map f_* if $S \in \mathbf{H}(R)$. In this case every f.g. projective S-module is in $\mathbf{H}(R)$, because if $P \oplus Q = S^n$ then $pd(P) \le pd(S^n) = pd(S) < \infty$. Hence there is an (exact) forgetful functor $\mathbf{P}(S) \to \mathbf{H}(R)$, and we define the transfer map to be the induced map

$$f_*: K_0(S) = K_0\mathbf{P}(S) \to K_0\mathbf{H}(R) \cong K_0(R). \tag{7.8.1}$$

A similar trick works to construct basechange maps for the groups G_0. We saw in 6.2 that if S is flat as an R-module then $\otimes_R S$ is an exact functor $\mathbf{M}(R) \to \mathbf{M}(S)$ and we obtained a map $f^*: G_0(R) \to G_0(S)$. More generally, suppose that S has finite flat dimension $fd_R(S) = n$ as a left R-module, *i.e.*, that there is an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to S \to 0$$

of R-modules, with the F_i flat. Let $\mathcal{F}$ denote the full subcategory of $\mathbf{M}(R)$ consisting of all f.g. R-modules M with $\mathrm{Tor}_i^R(M, S) = 0$ for $i \neq 0$; $\mathcal{F}$ is an exact category concocted so that $\otimes_R S$ defines an exact functor from $\mathcal{F}$ to $\mathbf{M}(S)$. Not only does

$\mathcal{F}$ contain $\mathbf{P}(R)$, but from homological algebra one knows that (if R is noetherian) every f.g. R-module has a finite resolution by objects in $\mathcal{F}$: for any projective resolution $P. \to M$ the kernel of $P_n \to P_{n-1}$ (the n^{th} *syzygy*) of any projective resolution will be in $\mathcal{F}$. The long exact Tor sequence shows that $\mathcal{F}$ is closed under kernels, so the Resolution Theorem applies to yield $K_0(\mathcal{F}) \cong K_0(\mathbf{M}(R)) = G_0(R)$. Therefore if R is noetherian and $fd_R(S) < \infty$ we can define the basechange map $f^*: G_0(R) \to G_0(S)$ as the composite

$$G_0(R) \cong K_0(\mathcal{F}) \xrightarrow{\otimes} K_0\mathbf{M}(S) = G_0(S). \tag{7.8.2}$$

The following formula for f^* was used in §6 to show that $G_0(R) \cong G_0(R[x])$.

SERRE'S FORMULA 7.8.3. *Let $f: R \to S$ be a map between noetherian rings with $fd_R(S) < \infty$. Then the basechange map $f^*: G_0(R) \to G_0(S)$ of (7.8.2) satisfies:*

$$f^*([M]) = \sum (-1)^i \left[\mathrm{Tor}_i^R(M, S)\right].$$

PROOF. Choose an $\mathcal{F}$-resolution $L. \to M$ (by R-modules L_i in $\mathcal{F}$):

$$0 \to L_n \to \cdots \to L_1 \to L_0 \to M \to 0.$$

From homological algebra, we know that $\mathrm{Tor}_i^R(M, S)$ is the i^{th} homology of the chain complex $L. \otimes_R S$. By Prop. 7.4, the right-hand side of (7.8.3) equals

$$\chi(L. \otimes_R S) = \sum (-1)^i [L_i \otimes_R S] = f^*(\sum (-1)^i [L_i]) = f^*([M]).$$

EXERCISES

7.1 Suppose that $\mathbf{P}$ is an exact subcategory of an abelian category $\mathcal{A}$, closed under kernels of surjections in $\mathcal{A}$. Suppose further that every object of $\mathcal{A}$ is a quotient of an object of $\mathbf{P}$ (as in Corollary 7.6.2). Let $\mathbf{P}_n \subset \mathcal{A}$ be the full subcategory of objects having $\mathbf{P}$-dimension $\leq n$. Show that each $\mathbf{P}_n$ is an exact category closed under kernels of surjections, so that by the Resolution Theorem $K_0(\mathbf{P}) \cong K_0(\mathbf{P}_n)$. *Hint.* If $0 \to L \to P \to M \to 0$ is exact with $P \in \mathbf{P}$ and $M \in \mathbf{P}_1$, show that $L \in \mathbf{P}$.

7.2 Let $\mathcal{A}$ be a small exact category. If $[A_1] = [A_2]$ in $K_0(\mathcal{A})$, show that there are short exact sequences in $\mathcal{A}$

$$0 \to C' \to C_1 \to C'' \to 0, \quad 0 \to C' \to C_2 \to C'' \to 0$$

such that $A_1 \oplus C_1 \cong A_2 \oplus C_2$. (Cf. Ex. 6.4.)

7.3 This exercise shows why the noetherian hypothesis was needed for G_0 in Corollary 6.3.1, and motivates the definition of $G_0(R)$ in 7.1.4. Let R be the ring $k \oplus I$, where I is an infinite-dimensional vector space over a field k, with multiplication given by $I^2 = 0$.

(a) (Swan) Show that $K_0\mathbf{mod}_{fg}(R) = 0$ but $K_0\mathbf{mod}_{fg}(R/I) = G_0(R/I) = \mathbb{Z}$.

(b) Show that every pseudo-coherent R-module is isomorphic to R^n for some n. Conclude that $G_0(R) = \mathbb{Z}$.

7.4 The groups $G_0(\mathbb{Z}[G])$ and $K_0\mathbf{mod}_{fg}(\mathbb{Z}[G])$ are very different for the free group G on two generators x and y. Let I be the two-sided ideal of $\mathbb{Z}[G]$ generated by y, so that $\mathbb{Z}[G]/I = \mathbb{Z}[x, x^{-1}]$. As a right module, $\mathbb{Z}[G]/I$ is not finitely presented.

(a) (Lück) Construct resolutions $0 \to \mathbb{Z}[G]^2 \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z}[G]/I \to \mathbb{Z}[G]/I \to \mathbb{Z} \to 0$, and conclude that $K_0\mathbf{mod}_{fg}(\mathbb{Z}[G]) = 0$

(b) Gersten proved in [Ger74] that $K_0(\mathbb{Z}[G]) = \mathbb{Z}$ by showing that every finitely presented $\mathbb{Z}[G]$-module is in $\mathbf{H}(\mathbb{Z}[G])$, i.e., has a finite resolution by f. g. projective modules. Show that $G_0(\mathbb{Z}[G]) \cong K_0(\mathbb{Z}[G]) \cong \mathbb{Z}$.

7.5 *Naturality of basechange.* Let $R \xrightarrow{f} S \xrightarrow{g} T$ be maps between noetherian rings, with $\mathrm{fd}_R(S)$ and $\mathrm{fd}_S(T)$ finite. Show that $g^* f^* = (gf)^*$ as maps $G_0(R) \to G_0(T)$.

7.6 *Idempotent completion.* Suppose that $(\mathcal{C}, \mathcal{E})$ is an exact category. Show that there is a natural way to make the idempotent completion $\widehat{\mathcal{C}}$ of $\mathcal{C}$ into an exact category, with $\mathcal{C}$ an exact subcategory. As noted in 7.2.1, this proves that $K_0(\mathcal{C})$ is a subgroup of $K_0(\widehat{\mathcal{C}})$.

7.7 Let $\mathcal{C}$ be a small additive category, and $\mathcal{A} = \mathbf{Ab}^{\mathcal{C}}$ the (abelian) category of all additive functors from $\mathcal{C}$ to $\mathbf{Ab}$. The Yoneda embedding $h: \mathcal{C} \to \mathcal{A}$, defined by $h(C) = \mathrm{Hom}_{\mathcal{C}}(-, C)$, embeds $\mathcal{C}$ as a full subcategory of $\mathcal{A}$. Show that every object of $\mathcal{C}$ is a projective object in $\mathcal{A}$. Then conclude that this embedding makes $\mathcal{C}$ into a split exact category (see 7.1.2).

7.8 (Quillen). Let $\mathcal{C}$ be an exact category, with the family $\mathcal{E}$ of short exact sequences (and admissible monics i and admissible epis j)

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0 \tag{\dagger}$$

as in Definition 7.0. Show that the following three conditions hold:

(1) Any sequence in $\mathcal{C}$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$. If (†) is a sequence in $\mathcal{E}$ then i is a kernel for j (resp. j is a cokernel for i) in $\mathcal{C}$. The class $\mathcal{E}$ contains all of the sequences

$$0 \to B \xrightarrow{(1,0)} B \oplus D \xrightarrow{(0,1)} D \to 0.$$

(2) The class of admissible epimorphisms (resp. monomorphisms) is closed under composition. If (†) is in $\mathcal{E}$ and $B \to B''$, $D' \to D$ are maps in $\mathcal{C}$ then the base-change sequence $0 \to B \to (C \times_D D') \to D' \to 0$ and the cobase-change sequence $0 \to B'' \to (B'' \amalg_B C) \to D \to 0$ are in $\mathcal{E}$.

(3) If $C \to D$ is a map in $\mathcal{C}$ possessing a kernel, and there is a map $C' \to C$ in $\mathcal{C}$ so that $C' \to D$ is an admissible epimorphism, then $C \to D$ is an admissible epimorphism. Dually, if $B \to C$ has a cokernel and some $B \to C \to C''$ is admissible monomorphism, then so is $B \to C$.

Keller [Ke90, App. A] has proven that (1) and (2) imply (3).

Quillen observed that a converse is true: let $\mathcal{C}$ be an additive category, equipped with a family $\mathcal{E}$ of sequences of the form (†). If conditions (1) and (2) hold, then $\mathcal{C}$ is an exact category in the sense of definition 7.0. The ambient abelian category $\mathcal{A}(\mathcal{E})$ used in 7.0 is the category of contravariant additive functors $F: \mathcal{C} \to \mathbf{Ab}$ which carry each (†) to a "left" exact sequence

$$0 \to F(D) \to F(C) \to F(B),$$

and the embedding $\mathcal{C} \subset \mathcal{A}(\mathcal{E})$ is the Yoneda embedding.

We refer the reader to Appendix A of [TT] for a detailed proof that $\mathcal{E}$ is the class of sequences in $\mathcal{C}$ which are exact in $\mathcal{A}(\mathcal{E})$, as well as the following useful result: If $\mathcal{C}$ is idempotent complete then it is closed under kernels of surjections in $\mathcal{A}(\mathcal{E})$.

7.9 Let $\{\mathcal{C}_i\}$ be a filtered system of exact categories and exact functors. Use Ex. 7.8 to generalize Example 7.1.7, showing that $\mathcal{C} = \varinjlim \mathcal{C}_i$ is an exact category and that $K_0(\mathcal{C}) = \varinjlim K_0(\mathcal{C}_i)$.

7.10 *Projection Formula for rings.* Suppose that R is a commutative ring, and A is an R-algebra which as an R-module is in $\mathbf{H}(R)$. By Ex. 2.1, $\otimes_R$ makes $K_0(A)$ into a $K_0(R)$-module. Generalize Ex. 2.2 to show that the transfer map $f_*: K_0(A) \to K_0(R)$ is a $K_0(R)$-module map, *i.e.*, that the *projection formula* holds:

$$f_*(x \cdot f^*y) = f_*(x) \cdot y \quad \text{for every} \quad x \in K_0(A),\ y \in K_0(R).$$

7.11 For a localization $f : R \to S^{-1}R$ at a central set of nonzerodivisors, every $\alpha : S^{-1}P \to S^{-1}Q$ has the form $\alpha = \gamma/s$ for some $\gamma \in \mathrm{Hom}_R(P, Q)$ and $s \in S$. Show that $[(P, \gamma/s, Q)] \mapsto [Q/\gamma(P)] - [Q/sQ]$ defines an isomorphism $K_0(f) \to K_0\mathbf{H}_S(R)$ identifying the sequences (2.10.1) and 7.6.4.

7.12 This exercise generalizes the Localization Theorem 6.4. Let $\mathcal{C}$ be an exact subcategory of an abelian category $\mathcal{A}$, closed under extensions and kernels of surjections, and suppose that $\mathcal{C}$ contains a Serre subcategory $\mathcal{B}$ of $\mathcal{A}$. Let $\mathcal{C}/\mathcal{B}$ denote the full subcategory of $\mathcal{A}/\mathcal{B}$ on the objects of $\mathcal{C}$. Considering $\mathcal{B}$-isos $A \to C$ with C in $\mathcal{C}$, show that the following sequence is exact:

$$K_0(\mathcal{B}) \to K_0(\mathcal{C}) \xrightarrow{\text{loc}} K_0(\mathcal{C}/\mathcal{B}) \to 0.$$

7.13 *δ-functors.* Let $T = \{T_i: \mathcal{C} \to \mathcal{A}, i \geq 0\}$ be a homological δ-functor from an exact category $\mathcal{C}$ to an abelian category $\mathcal{A}$, *i.e.*, for every exact sequence (†) in $\mathcal{C}$ we have a long exact sequence in $\mathcal{A}$:

$$\cdots \to T_1(D) \xrightarrow{\delta} T_0(B) \to T_0(C) \to T_0(D) \to 0.$$

Let $\mathcal{F}$ denote the category of all C in $\mathcal{C}$ such that $T_i(C) = 0$ for all $i > 0$, and assume that every C in $\mathcal{C}$ is a quotient of some object of $\mathcal{F}$.

(a) Show that $K_0(\mathcal{F}) \cong K_0(\mathcal{C})$, and that T defines a map $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ sending $[C]$ to $\sum(-1)^i[T_iC]$. (Cf. Ex. 6.6.)

(b) Suppose that $f: X \to Y$ is a map of noetherian schemes, and that $\mathcal{O}_X$ has finite flat dimension over $f^{-1}\mathcal{O}_Y$. Show that there is a basechange map $f^*: G_0(Y) \to G_0(X)$ satisfying $f^*g^* = (gf)^*$, generalizing (7.8.2) and Ex. 7.5.

7.14 This exercise is a refined version of Ex. 6.12. Consider $S = R[x_0, \cdots, x_m]$ as a graded ring with $x_1, \cdots, x_n$ in S_1, and let $\mathbf{M}_{gr}(S)$ denote the exact category of f.g. graded S-modules.

(a) Use Ex. 7.13 with $T_i = \mathrm{Tor}_i^S(-, R)$ to show that $K_0\mathbf{M}_{gr}(S) \cong G_0(R)[\sigma, \sigma^{-1}]$.

(b) Use (a) and Ex. 6.12(e) to obtain an exact sequence

$$G_0(R)[\sigma, \sigma^{-1}] \xrightarrow{i} G_0(R)[\sigma, \sigma^{-1}] \to G_0(R[x]) \to 0.$$

Then show that the map i sends α to $\alpha - \sigma\alpha$.

(c) Conclude that $G_0(R) \cong G_0(R[x])$.

7.15 Let R be a noetherian ring. Show that the groups $K_0\mathbf{M}_i(R)$ of Application 6.4.3 are all $K_0(R)$-modules, and that the subgroups F^i in the coniveau filtration of $G_0(R)$ are $K_0(R)$-submodules. Conclude that if R is regular then the F^i are ideals in the ring $K_0(R)$.

7.16 (Grayson) Show that the operations $\lambda^n(P,\alpha) = (\wedge^n P, \wedge^n \alpha)$ make $K_0\mathbf{End}(R)$ and $End_0(R)$ into λ-rings. Then show that the ring map $End_0(R) \to W(R)$ (of 7.3.3) is a λ-ring injection, where $W(R)$ is the ring of big Witt vectors of R (see Example 4.3). Conclude that $End_0(R)$ is a special λ-ring (4.3.1).

7.17 This exercise is a refinement of 7.3.4. Let $F_n\mathbf{Nil}(R)$ denote the full subcategory of $\mathbf{Nil}(R)$ on the (P,ν) with $\nu^n = 0$. Show that $F_n\mathbf{Nil}(R)$ is an exact subcategory of $\mathbf{Nil}(R)$. If R is an algebra over a commutative ring k, show that the kernel $F_nNil_0(R)$ of $K_0F_n\mathbf{Nil}(R) \to K_0\mathbf{P}(R)$ is an $End_0(k)$-module, and $F_nNil_0(R) \to Nil_0(R)$ is a module map.

7.18 Let $\alpha_n = \alpha_n(a_1,...,a_n)$ denote the $n \times n$ matrix over a commutative ring R:

$$\alpha_n(a_1,...,a_n) = \begin{pmatrix} 0 & & & -a_n \\ 1 & 0 & & -a_{n-1} \\ & \ddots & \ddots & \vdots \\ & & 1 & -a_1 \end{pmatrix}.$$

(a) Show that $[(R^n,\alpha_n)] = 1 + a_1t + \cdots + a_nt^n$ in $W(R)$. Conclude that the image of the map $End_0(R) \to W(R)$ in 7.3.3 is indeed the subgroup of all quotients $f(t)/g(t)$ of polynomials in $1 + tR[t]$.

(b) Let A be an R-algebra. Recall that $(R^n,\alpha_n) * [(P,\nu)] = [(P^n,\alpha_n\nu)]$ in the $End_0(R)$-module $Nil_0(A)$ (see 7.3.4). Show that $(R^{n+1},\alpha_{n+1}(a_1,\dots,a_n,0)) * [(P,\nu)] = (R^n,\alpha_n) * [(P,\nu)]$.

(c) Use 7.3.5 with $R = \mathbb{Z}[a_1,...,a_n]$ to show that $(R^n,\alpha_n) * [(P,\nu)] = [(P^n,\beta)]$, $\beta = \alpha_n(a_1\nu,...,a_n\nu^n)$. If $\nu^N = 0$, this is clearly independent of the a_i for $i \geq N$.

(d) Conclude that the $End_0(R)$=module structure on $Nil_0(A)$ extends to a $W(R)$-module structure by the formula

$$\left(1 + \sum a_it^i\right) * [(P,\nu)] = (R^n,\alpha_n(a_1,...,a_n)) * [(P,\nu)], \quad n \gg 0.$$

7.19 (Lam) If R is a commutative ring, and Λ is an R-algebra, we write $G_0^R(\Lambda)$ for $K_0\mathbf{Rep}_R(\Lambda)$, where $\mathbf{Rep}_R(\Lambda)$ denotes the full subcategory of **mod**-Λ consisting of modules M which are finitely generated and projective as R-modules. If $\Lambda = R[G]$ is the group ring of a group G, the tensor product $M \otimes_R N$ of two $R[G]$-modules is again an $R[G]$-module where $g \in G$ acts by $(m \otimes n)g = mg \otimes ng$. Show that:

(a) $\otimes_R$ makes $G_0^R(R[G])$ an associative, commutative ring with identity $[R]$.

(b) $G_0^R(R[G])$ is an algebra over the ring $K_0(R)$, and $K_0(R[G])$ is a $G_0^R(R[G])$-module.

(c) If R is a regular ring and Λ is f.g. projective as an R-module, $G_0^R(\Lambda) \cong G_0(\Lambda)$.

(d) If R is regular and G is finite, then $G_0(R[G])$ is a commutative $K_0(R)$-algebra, and that $K_0(R[G])$ is a module over $G_0(R[G])$.

7.20 A *filtered object* in an abelian category $\mathcal{A}$ is an object A together with a finite filtration $\cdots \subseteq W_nA \subseteq W_{n+1}A \subseteq \cdots$. The category $\mathcal{A}_{\text{filt}}$ of filtered objects in $\mathcal{A}$ is additive but not abelian (because images and coimages can differ). Let $\mathcal{E}$ denote the collection of all sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}_{\text{filt}}$ such that each subsequence $0 \to W_nA \to W_nB \to W_nC \to 0$ is exact in $\mathcal{A}$.
(a) Show that $(\mathcal{A}_{\text{filt}}, \mathcal{E})$ is an exact category.
(b) Show that $K_0(\mathcal{A}_{\text{filt}}) \cong \mathbb{Z} \times K_0(\mathcal{A})$.

7.21 *Replete exact categories.* A sequence $0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0$ in an additive category $\mathcal{C}$ is called *replete* if i is the categorical kernel of j, and j is the categorical cokernel of i. Let $\mathcal{E}_{rep}$ denote the class of all replete sequences, and show that $(\mathcal{C}, \mathcal{E}_{rep})$ is an exact category.

7.22 Consider the full subcategory $\mathcal{C}$ of the category $\mathbf{Ab}_p$ of all finite abelian p-groups arising as direct sums of the group $\mathbb{Z}/p^{2i}$ where p is some prime. Show that $\mathcal{C}$ is an additive category, but not an exact subcategory of $\mathbf{Ab}_p$. Let $\mathcal{E}$ be the sequences in $\mathcal{C}$ which are exact in $\mathbf{Ab}_p$; is $(\mathcal{C}, \mathcal{E})$ an exact category?

7.23 Give an example of a cofinal exact subcategory $\mathcal{B}$ of an exact category $\mathcal{C}$, such that the map $K_0\mathcal{B} \to K_0\mathcal{C}$ is not an injection (see 7.2).

7.24 Suppose that $\mathcal{C}_i$ are exact categories. Show that the product category $\prod \mathcal{C}_i$ is an exact category. Need $K_0(\prod \mathcal{C}_i) \to \prod K_0(\mathcal{C}_i)$ be an isomorphism?

7.25 (Claborn-Fossum). Set $R_n = \mathbb{C}[x_0, \cdots, x_n]/(\sum x_i^2 = 1)$. This is the complex coordinate ring of the n-sphere; it is a regular ring for every n, and $R_1 \cong \mathbb{C}[z, z^{-1}]$. In this exercise, we show that

$$\tilde{K}_0(R_n) \cong \widetilde{KU}(S^n) \cong \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbb{Z} & \text{if } n \text{ is even, } (n \neq 0) \end{cases}.$$

(a) Set $z = x_0 + ix_1$ and $\bar{z} = x_0 - ix_1$, so that $z\bar{z} = x_0^2 + x_1^2$. Show that

$$R_n[z^{-1}] \cong \mathbb{C}[z, z^{-1}, x_2, \ldots, x_n]$$

$$R_n/zR_n \cong R_{n-2}[\bar{z}], \quad n \geq 2.$$

(b) Use (a) to show that $\tilde{K}_0(R_n) = 0$ for n odd, and that if n is even there is a surjection $\beta\colon K_0(R_{n-2}) \to \tilde{K}_0(R_n)$.
(c) If n is even, show that β sends $[R_{n-2}]$ to zero, and conclude that there is a surjection $\mathbb{Z} \to \tilde{K}_0(R_n)$.

Fossum produced a f.g. projective R_{2n}-module P_n such that the map $\tilde{K}_0(R_{2n}) \to \widetilde{KU}(S^{2n}) \cong \mathbb{Z}$ sends $[P_n]$ to the generator. (See [Foss].)
(d) Use the existence of P_n to finish the calculation of $K_0(R_n)$.

§8. K_0 of Schemes and Varieties

We have already introduced the Grothendieck group $K_0(X)$ of a scheme X in Example 7.1.3. By definition, it is $K_0\mathbf{VB}(X)$, where $\mathbf{VB}(X)$ denotes the (exact) category of vector bundles on X. The tensor product of vector bundles makes $K_0(X)$ into a commutative ring, as we saw in 7.3.2. This ring structure is natural in X: K_0 is a contravariant functor from schemes to commutative rings. Indeed, we saw in I.5.2 that a morphism of schemes $f\colon X \to Y$ induces an exact basechange functor $f^*\colon \mathbf{VB}(Y) \to \mathbf{VB}(X)$, preserving tensor products, and such an exact functor induces a (ring) homomorphism $f^*\colon K_0(Y) \to K_0(X)$.

In this section we shall study $K_0(X)$ in more depth. Such a study requires that the reader has somewhat more familiarity with algebraic geometry than we assumed in the previous section, which is why this study has been isolated in its own section. We begin with two general invariants: the rank and determinant of a vector bundle.

Let $H^0(X;\mathbb{Z})$ denote the ring of continuous functions $X \to \mathbb{Z}$. We saw in I.5.1 that the rank of a vector bundle $\mathcal{F}$ is a continuous function, so $\operatorname{rank}(\mathcal{F}) \in H^0(X;\mathbb{Z})$. Similarly, we saw in I.5.3 that the determinant of $\mathcal{F}$ is a line bundle on X, *i.e.*, $\det(\mathcal{F}) \in \operatorname{Pic}(X)$.

THEOREM 8.1. *Let X be a scheme. Then $H^0(X;\mathbb{Z})$ is a subring of $K_0(X)$, and the rank of a vector bundle induces a split surjection of rings*

$$\operatorname{rank}\colon K_0(X) \to H^0(X;\mathbb{Z}).$$

Similarly, the determinant of a vector bundle induces a surjection of abelian groups

$$\det\colon K_0(X) \to \operatorname{Pic}(X).$$

The sum $\operatorname{rank} \oplus \det\colon K_0(X) \to H^0(X;\mathbb{Z}) \oplus \operatorname{Pic}(X)$ *is a surjective ring map.*

PROOF. Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of vector bundles on X. At any point x of X we have an isomorphism of free $\mathcal{O}_x$-modules $\mathcal{F}_x \cong \mathcal{E}_x \oplus \mathcal{G}_x$, so $\operatorname{rank}_x(\mathcal{F}) = \operatorname{rank}_x(\mathcal{E}) + \operatorname{rank}_x(\mathcal{G})$. Hence each rank_x is an additive function on $\mathbf{VB}(X)$. As x varies rank becomes an additive function with values in $H^0(X;\mathbb{Z})$, so by 6.1.2 it induces a map $\operatorname{rank}\colon K_0(X) \to H^0(X;\mathbb{Z})$. This is a ring map, since the formula $\operatorname{rank}(\mathcal{E} \otimes \mathcal{F}) = \operatorname{rank}(\mathcal{E}) \cdot \operatorname{rank}(\mathcal{F})$ may be checked at each point x. If $f\colon X \to \mathbb{N}$ is continuous, the componentwise free module $\mathcal{O}_X^f$ has rank f. It follows that rank is onto. Since componentwise free $\mathcal{O}_X$-modules are closed under $\oplus$ and $\otimes$, the elements $[\mathcal{O}_X^f] - [\mathcal{O}_X^g]$ in $K_0(X)$ form a subring isomorphic to $H^0(X;\mathbb{Z})$.

Similarly, det is an additive function, because we have $\det(\mathcal{F}) \cong \det(\mathcal{E}) \otimes \det(\mathcal{G})$ by Ex. I.5.4. Hence det induces a map $K_0(X) \to \operatorname{Pic}(X)$ by 6.1.2. If $\mathcal{L}$ is a line bundle on X, then the element $[\mathcal{L}] - [\mathcal{O}_X]$ of $K_0(X)$ has rank zero and determinant $\mathcal{L}$. Hence $\operatorname{rank} \oplus \det$ is onto; the proof that it is a ring map is given in Ex. 8.5.

DEFINITION 8.1.1. As in 2.3 and 2.6.1, the ideal $\widetilde{K}_0(X)$ of $K_0(X)$ is defined to be the kernel of the rank map, so that $K_0(X) = H^0(X;\mathbb{Z}) \oplus \widetilde{K}_0(X)$ as an abelian group. In addition, we let $SK_0(X)$ denote the kernel of $\operatorname{rank} \oplus \det$. By Theorem 8.1, these are both ideals of the ring $K_0(X)$. In fact, they form the beginning of the γ-filtration.

Regular Noetherian Schemes and the Cartan Map

Historically, the group $K_0(X)$ first arose in [RR], when X is a smooth projective variety. The following theorem was central to Grothendieck's proof of the Riemann-Roch Theorem.

Recall from §6 that $G_0(X)$ is the Grothendieck group of the category $\mathbf{M}(X)$ of coherent $\mathcal{O}_X$-modules. The inclusion $\mathbf{VB}(X) \subset \mathbf{M}(X)$ induces a natural map $K_0(X) \to G_0(X)$, called the *Cartan homomorphism* (see 7.1.3).

THEOREM 8.2. *If X is a separated regular noetherian scheme, then the Cartan homomorphism is an isomorphism:*

$$K_0(X) \xrightarrow{\cong} G_0(X).$$

PROOF. By [SGA6, II, 2.2.3 and 2.2.7.1], we know that every coherent $\mathcal{O}_X$-module $\mathcal{F}$ has a finite resolution by vector bundles. Hence the Resolution Theorem 7.5 applies to the inclusion $\mathbf{VB}(X) \subset \mathbf{M}(X)$.

PROPOSITION 8.2.1 (NONSINGULAR CURVES). *Let X be a 1-dimensional separated regular noetherian scheme. Then $SK_0(X) = 0$, and $K_0(X) = H^0(X;\mathbb{Z}) \oplus \mathrm{Pic}(X)$.*

PROOF. Given Theorem 8.2, this does follow from Ex. 6.10 (see Example 8.2.2 below). However, we shall give a slightly different proof here.

Without loss of generality, we may assume that X is irreducible. If X is affine, this is just Corollary 2.6.3. Otherwise, choose any closed point P on X. By [Hart, Ex. IV.1.3] the complement $U = X - P$ is affine, say $U = \mathrm{Spec}(R)$. Under the isomorphism $\mathrm{Pic}(X) \cong Cl(X)$ of I.5.14, the line bundles $\mathcal{L}(P)$ correspond to the class of the Weil divisor $[P]$. Hence the right-hand square commutes in the following diagram

$$\begin{array}{ccccccc}
 & G_0(P) & \xrightarrow{i_*} & \widetilde{K}_0(X) & \to & \widetilde{K}_0(R) & \to 0 \\
 & \vdots\downarrow & & \downarrow \det & & \cong\downarrow \det & \\
0 \to & \mathbb{Z} & \xrightarrow{\{\mathcal{L}(P)\}} & \mathrm{Pic}(X) & \to & \mathrm{Pic}(R) & \to 0.
\end{array}$$

The top row is exact by 6.4.2 (and 8.2), and the bottom row is exact by I.5.14 and Ex. I.5.11. The right vertical map is an isomorphism by 2.6.2.

Now $G_0(P) \cong \mathbb{Z}$ on the class $[\mathcal{O}_P]$. From the exact sequence $0 \to \mathcal{L}(-P) \to \mathcal{O}_X \to \mathcal{O}_P \to 0$ we see that $i_*[\mathcal{O}_P] = [\mathcal{O}_X] - [\mathcal{L}(-P)]$ in $K_0(X)$, and $\det(i_*[\mathcal{O}_P]) = \det \mathcal{L}(-P)^{-1}$ in $\mathrm{Pic}(X)$. Hence the isomorphism $G_0(P) \cong \mathbb{Z}$ is compatible with the above diagram. A diagram chase yields $\widetilde{K}_0(X) \cong \mathrm{Pic}(X)$.

EXAMPLE 8.2.2 (CLASSES OF SUBSCHEMES). Let X be a separated noetherian regular scheme. Given a subscheme Z of X, it is convenient to write $[Z]$ for the element $[\mathcal{O}_Z] \in K_0\mathbf{M}(X) = K_0(X)$. By Ex. 6.10(d) we see that $SK_0(X)$ is the subgroup of $K_0(X)$ generated by the classes $[Z]$ as Z runs through the irreducible subschemes of codimension ≥ 2. In particular, if $\dim(X) = 2$ then $SK_0(X)$ is generated by the classes $[P]$ of closed points (of codimension 2).

Transfer for finite and proper maps to regular schemes 8.2.3. Let $f: X \to Y$ be a finite morphism of separated noetherian schemes with Y regular. As pointed out in 6.2.5, the direct image f_* is an exact functor $\mathbf{M}(X) \to \mathbf{M}(Y)$. In this case we have a transfer map f_* on K_0 sending $[\mathcal{F}]$ to $[f_*\mathcal{F}]$: $K_0(X) \to G_0(X) \to G_0(Y) \cong K_0(X)$.

If $f: X \to Y$ is a proper morphism of separated noetherian schemes with Y regular, we can use the transfer $G_0(X) \to G_0(Y)$ of Lemma 6.2.6 to get a functorial transfer map $f_*: K_0(X) \to K_0(Y)$, this time sending $[\mathcal{F}]$ to $\sum(-1)^i[R^i f_*\mathcal{F}]$.

A Non-separated Example 8.2.4. Here is an example of a regular but non-separated scheme X with $K_0(X) \neq G_0(X)$. Let X be "affine n-space with a double origin" over a field F, where $n \geq 2$. This scheme is the union of two copies of $\mathbb{A}^n = \mathrm{Spec}(F[x_1, ..., x_n])$ along $\mathbb{A}^n - \{0\}$. Using the localization sequence for either origin and the Fundamental Theorem 6.5, one can show that $G_0(X) = \mathbb{Z} \oplus \mathbb{Z}$. However the inclusion $\mathbb{A}^n \subset X$ is known to induce an equivalence $\mathbf{VB}(X) \cong \mathbf{VB}(\mathbb{A}^n)$ (see [EGA, IV(5.9)]), so by Theorem 7.7 we have $K_0(X) \cong K_0(F[x_1, ..., x_n]) \cong \mathbb{Z}$.

Definition 8.3. Let $\mathbf{H}(X)$ denote the category consisting of all quasicoherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that: $\mathcal{F}|_U$ has a finite resolution by vector bundles for each affine open subscheme $U = \mathrm{Spec}(R)$ of X. Since $\mathcal{F}|_U$ is defined by the finitely generated R-module $M = \mathcal{F}(U)$ this just means that M is in the category $\mathbf{H}(R)$ of Definition 7.6.

If X is regular, then we saw in the proof of Theorem 8.2 that $\mathbf{H}(X) = \mathbf{M}(X)$. If $X = \mathrm{Spec}(R)$, it is easy to see that $\mathbf{H}(X)$ is equivalent to the category $\mathbf{H}(R)$.

$\mathbf{H}(X)$ is an exact subcategory of $\mathcal{O}_X$-**mod**, closed under kernels of surjections, because each $\mathbf{H}(R)$ is closed under extensions and kernels of surjections in R-**mod**.

To say much more about the relation between $\mathbf{H}(X)$ and $K_0(X)$, we need to restrict our attention to quasi-compact schemes such that every $\mathcal{F}$ in $\mathbf{H}(X)$ is a quotient of a vector bundle $\mathcal{E}_0$. This implies that every module $\mathcal{F} \in \mathbf{H}(X)$ has a finite resolution $0 \to \mathcal{E}_d \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ by vector bundles. Indeed, the kernel $\mathcal{F}'$ of a quotient map $\mathcal{E}_0 \to \mathcal{F}$ is always locally of lower projective dimension than $\mathcal{F}$, and X has a finite affine cover by $U_i = \mathrm{Spec}(R_i)$, it follows that the d^{th} syzygy is a vector bundle, where $d = \max\{pd_{R_i} M_i\}$, $M_i = \mathcal{F}(U_i)$.

For this condition to hold, it is easiest to assume that X is *quasi-projective* (over a commutative ring k), *i.e.*, a locally closed subscheme of some projective space $\mathbb{P}^n_k$. By [EGA II, 4.5.5 and 4.5.10], this implies that every quasicoherent $\mathcal{O}_X$-module of finite type $\mathcal{F}$ is a quotient of some vector bundle $\mathcal{E}_0$ of the form $\mathcal{E}_0 = \bigoplus \mathcal{O}_X(n_i)$.

Proposition 8.3.1. *If X is quasi-projective (over a commutative ring), then $K_0(X) \cong K_0\mathbf{H}(X)$.*

Proof. Because $\mathbf{H}(X)$ is closed under kernels of surjections in $\mathcal{O}_X$-**mod**, and every object in $\mathbf{H}(X)$ has a finite resolution by vector bundles, the Resolution Theorem 7.5 applies to $\mathbf{VB}(X) \subset \mathbf{H}(X)$.

Technical remark 8.3.2. Another assumption that guarantees that every $\mathcal{F}$ in $\mathbf{H}(X)$ is a quotient of a vector bundle is that X be quasi-separated and quasi-compact with an ample family of line bundles. Such schemes are called *divisorial* in [SGA6, II.2.2.4]. For such schemes, the proof of 8.3.1 goes through to show that we again have $K_0(X) \cong K_0\mathbf{H}(X)$.

RESTRICTING BUNDLES 8.3.3. Given an open subscheme U of a quasi-projective scheme X, let $\mathcal{B}$ denote the full subcategory of $\mathbf{VB}(U)$ consisting of vector bundles $\mathcal{F}$ whose class in $K_0(U)$ is in the image of $j^*\colon K_0(X) \to K_0(U)$. We claim that the category $\mathcal{B}$ is cofinal in $\mathbf{VB}(U)$, so that $K_0\mathcal{B}$ is a subgroup of $K_0(U)$ by the Cofinality Lemma 7.2. To see this, note that each vector bundle $\mathcal{F}$ on U fits into an exact sequence $0 \to \mathcal{F}' \to \mathcal{E}_0 \to \mathcal{F} \to 0$, where $\mathcal{E}_0 = \bigoplus \mathcal{O}_U(n_i)$. But then $\mathcal{F} \oplus \mathcal{F}'$ is in $\mathcal{B}$, because in $K_0(U)$

$$[\mathcal{F} \oplus \mathcal{F}'] = [\mathcal{F}] + [\mathcal{F}'] = [\mathcal{E}_0] = \sum j^*[\mathcal{O}_X(n_i)].$$

Transfer Maps for Schemes

8.4 We can define a transfer map $f_*\colon K_0(X) \to K_0(Y)$ with $(gf)_* = g_*f_*$ associated to various morphisms $f\colon X \to Y$. If Y is regular, we have already done this in 8.2.3.

Suppose first that f is a finite map. In this case, the inverse image of any affine open $U = \operatorname{Spec}(R)$ of Y is an affine open $f^{-1}U = \operatorname{Spec}(S)$ of X, S is finitely generated as an R-module, and the direct image sheaf $f_*\mathcal{O}_X$ satisfies $f_*\mathcal{O}(U) = S$. Thus the direct image functor f_* is an exact functor from $\mathbf{VB}(X)$ to $\mathcal{O}_Y$-modules (as pointed out in 6.2.5).

If f is finite and $f_*\mathcal{O}_X$ is a vector bundle then f_* is an exact functor from $\mathbf{VB}(X)$ to $\mathbf{VB}(Y)$. Indeed, locally it sends each f.g. projective S-module to a f.g. projective R-module, as described in Example 2.8.1. Thus there is a canonical transfer map $f_*\colon K_0(X) \to K_0(Y)$ sending $[\mathcal{F}]$ to $[f_*\mathcal{F}]$.

If f is finite and $f_*\mathcal{O}_X$ is in $\mathbf{H}(X)$ then f_* sends $\mathbf{VB}(X)$ into $\mathbf{H}(X)$, because locally it is the forgetful functor $\mathbf{P}(S) \to \mathbf{H}(R)$ of (7.8.1). Therefore f_* defines a homomorphism $K_0(X) \to K_0\mathbf{H}(Y)$. If Y is quasi-projective then composition with $K_0\mathbf{H}(Y) \cong K_0(Y)$ yields a “finite” transfer map $K_0(X) \to K_0(Y)$.

Now suppose that $f\colon X \to Y$ is a proper map between quasi-projective noetherian schemes. The transfer homomorphism $f_*\colon G_0(X) \to G_0(Y)$ was constructed in Lemma 6.2.6, with $f_*[\mathcal{F}] = \sum(-1)^i[R^if_*\mathcal{F}]$.

If in addition f has finite Tor-dimension, then we can also define a transfer map $f_*\colon K_0(X) \to K_0(Y)$, following [SGA 6, IV.2.12.3]. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is called *f_*-acyclic* if $R^qf_*\mathcal{F} = 0$ for all $q > 0$. Let $\mathbf{P}(f)$ denote the category of all vector bundles $\mathcal{F}$ on X such that $\mathcal{F}(n)$ is f_*-acyclic for all $n \geq 0$. By the usual yoga of homological algebra, $\mathbf{P}(f)$ is an exact category, closed under cokernels of injections, and f_* is an exact functor from $\mathbf{P}(f)$ to $\mathbf{H}(Y)$. Hence the following lemma allows us to define the transfer map as

$$K_0(X) \xleftarrow{\cong} K_0\mathbf{P}(f) \xrightarrow{f_*} K_0\mathbf{H}(Y) \xleftarrow{\cong} K_0(Y) \tag{8.4.1}$$

LEMMA 8.4.2. *Every vector bundle $\mathcal{F}$ on X has a finite resolution*

$$0 \to \mathcal{F} \to \mathcal{P}_0 \to \cdots \to \mathcal{P}_m \to 0$$

by vector bundles in $\mathbf{P}(f)$. Hence by the Resolution Theorem $K_0\mathbf{P}(f) \cong K_0(X)$.

PROOF. For $n \geq 0$ the vector bundle $\mathcal{O}_X(n)$ is generated by global sections. Dualizing the resulting surjection $\mathcal{O}_X^r \to \mathcal{O}_X(n)$ and twisting n times yields a short exact sequence of vector bundles $0 \to \mathcal{O}_X \to \mathcal{O}_X(n)^r \to \mathcal{E} \to 0$. Hence for every vector bundle $\mathcal{F}$ on X we have a short exact sequence of vector bundles $0 \to \mathcal{F} \to \mathcal{F}(n)^r \to \mathcal{E} \otimes \mathcal{F} \to 0$. For all large n, the sheaf $\mathcal{F}(n)$ is f_*-acyclic (see [EGA, III.3.2.1] or [Hart, III.8.8]), and $\mathcal{F}(n)$ is in $\mathbf{P}(f)$. Repeating this process with $\mathcal{E} \otimes \mathcal{F}$ in place of $\mathcal{F}$, we obtain the desired resolution of $\mathcal{F}$.

Like the transfer map for rings, the transfer map f_* is a $K_0(Y)$-module homomorphism. (This is the *projection formula*; see Ex. 7.10 and Ex. 8.3.)

Projective Bundles

Let $\mathcal{E}$ be a vector bundle of rank $r+1$ over a quasi-compact scheme X, and let $\mathbb{P} = \mathbb{P}(\mathcal{E})$ denote the projective space bundle of Example I.5.8. (If $\mathcal{E}|_U$ is free over $U \subseteq X$ then $\mathbb{P}|_U$ is the usual projective space $\mathbb{P}^r_U$.) Via the structural map $\pi\colon \mathbb{P} \to X$, the basechange map is a ring homomorphism $\pi^*\colon K_0(X) \to K_0(\mathbb{P})$, sending $[\mathcal{M}]$ to $[f^*\mathcal{M}]$, where $f^*\mathcal{M} = \mathcal{O}_{\mathbb{P}} \otimes_X \mathcal{M}$. In this section we give Quillen's proof [Q341, §8] of the following result, originally due to Berthelot [SGA6, VI.1.1].

Projective Bundle Theorem 8.6. *Let $\mathbb{P}$ be the projective bundle over a quasi-compact scheme X. Then $K_0(\mathbb{P})$ is a free $K_0(X)$-module with basis the twisting line bundles $\{1 = [\mathcal{O}_{\mathbb{P}}], [\mathcal{O}_{\mathbb{P}}(-1)], ..., [\mathcal{O}_{\mathbb{P}}(-r)]\}$.*

To prove this result, we would like to apply the direct image functor π_* to a vector bundle $\mathcal{F}$ and get a vector bundle. This requires a vanishing condition. The proof of this result rests upon the following notion, which is originally due to Castelnuovo. It is named after David Mumford, who exploited it in [Mum].

Definition 8.6.1. A quasicoherent $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{F}$ is called *Mumford-regular* if for all $q > 0$ the higher derived sheaves $R^q\pi_*(\mathcal{F}(-q))$ vanish. Here $\mathcal{F}(n)$ is $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(n)$, as in Example I.5.3.1. We write **MR** for the additive category of all Mumford-regular vector bundles, and abbreviate $\otimes_X$ for $\otimes_{\mathcal{O}_X}$.

Examples 8.6.2. If $\mathcal{N}$ is a quasicoherent $\mathcal{O}_X$-module then the standard cohomology calculations on projective spaces show that $\pi^*\mathcal{N} = \mathcal{O}_{\mathbb{P}} \otimes_X \mathcal{N}$ is Mumford-regular, with $\pi_*\pi^*\mathcal{N} = \mathcal{N}$. More generally, if $n \geq 0$ then $\pi^*\mathcal{N}(n)$ is Mumford-regular, with $\pi_*\pi^*\mathcal{N}(n) = Sym_n\mathcal{E} \otimes_X \mathcal{N}$. For $n < 0$ we have $\pi_*\pi^*\mathcal{N}(n) = 0$. In particular, $\mathcal{O}_{\mathbb{P}}(n) = \pi^*\mathcal{O}_X(n)$ is Mumford-regular for all $n \geq 0$.

If X is noetherian and $\mathcal{F}$ is coherent, then for $n \gg 0$ the twists $\mathcal{F}(n)$ are Mumford-regular, because the higher derived functors $R^q\pi_*\mathcal{F}(n)$ vanish for large n and also for $q > r$ (see [Hart, III.8.8]).

The following facts were discovered by Castelnuovo when $X = \mathrm{Spec}(\mathbb{C})$, and proven in [Mum, Lecture 14] as well as [Q341, §8]:

Proposition 8.6.3. *If $\mathcal{F}$ is Mumford-regular, then*

(1) *The twists $\mathcal{F}(n)$ are Mumford-regular for all $n \geq 0$;*
(2) *Mumford-regular modules are π_*-acyclic, and in fact $R^q\pi_*\mathcal{F}(n) = 0$ for all $q > 0$ and $n \geq -q$;*
(3) *The canonical map $\varepsilon\colon \pi^*\pi_*(\mathcal{F}) \to \mathcal{F}$ is onto.*

Remark. Suppose that X is affine. Since $\pi^*\pi_*(\mathcal{F}) = \mathcal{O}_{\mathbb{P}} \otimes_X \pi_*\mathcal{F}$, and $\pi_*\mathcal{F}$ is quasicoherent, item (3) states that Mumford-regular sheaves are generated by their global sections.

LEMMA 8.6.4. *Mumford-regular modules form an exact subcategory of $\mathcal{O}_{\mathbb{P}}$-**mod**, and π_* is an exact functor from Mumford-regular modules to $\mathcal{O}_X$-modules.*

PROOF. Suppose that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of $\mathcal{O}_{\mathbb{P}}$-modules with both $\mathcal{F}'$ and $\mathcal{F}''$ Mumford-regular. From the long exact sequence

$$R^q\pi_*\mathcal{F}'(-q) \to R^q\pi_*\mathcal{F}(-q) \to R^q\pi_*\mathcal{F}''(-q)$$

we see that $\mathcal{F}$ is also Mumford-regular. Thus Mumford-regular modules are closed under extensions, *i.e.*, they form is an exact subcategory of $\mathcal{O}_{\mathbb{P}}$-**mod**. Since $\mathcal{F}'(1)$ is Mumford-regular, $R^1\pi_*\mathcal{F}' = 0$, and so we have an exact sequence

$$0 \to \pi_*\mathcal{F}' \to \pi_*\mathcal{F} \to \pi_*\mathcal{F}'' \to 0.$$

This proves that π_* is an exact functor.

The following results were proven by Quillen in [Q341, §8].

LEMMA 8.6.5. *Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}$.*

(1) *$\mathcal{F}(n)$ is a Mumford-regular vector bundle on $\mathbb{P}$ for all large enough n;*
(2) *If $\mathcal{F}(n)$ is π_*-acyclic for all $n \geq 0$ then $\pi_*\mathcal{F}$ is a vector bundle on X.*
(3) *Hence by 8.6.3, if $\mathcal{F}$ is Mumford-regular then $\pi_*\mathcal{F}$ is a vector bundle on X.*
(4) *$\pi^*\mathcal{N} \otimes_{\mathbb{P}} \mathcal{F}(n)$ is Mumford-regular for all large enough n, and all quasicoherent $\mathcal{O}_X$-modules $\mathcal{N}$.*

DEFINITION 8.6.6 (T_n). Given a Mumford-regular $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{F}$, we define a natural sequence of $\mathcal{O}_X$-modules $T_n = T_n\mathcal{F}$ and $\mathcal{O}_{\mathbb{P}}$-modules $Z_n = Z_n\mathcal{F}$, starting with $T_0\mathcal{F} = \pi_*\mathcal{F}$ and $Z_{-1} = \mathcal{F}$. Let Z_0 be the kernel of the natural map $\varepsilon\colon \pi^*\pi_*\mathcal{F} \to \mathcal{F}$ of Proposition 8.6.3. Inductively, we define $T_n\mathcal{F} = \pi_*Z_{n-1}(n)$ and define Z_n to be $\ker(\varepsilon)(-n)$, where ε is the canonical map from $\pi^*T_n = \pi^*\pi_*Z_{n-1}(n)$ to $Z_{n-1}(n)$.

Thus we have sequences (exact except possibly at $Z_{n-1}(n)$)

$$0 \to Z_n(n) \to \pi^*(T_n\mathcal{F}) \xrightarrow{\varepsilon} Z_{n-1}(n) \to 0 \qquad (8.6.7)$$

whose twists fit together into the sequence of the following theorem.

QUILLEN RESOLUTION THEOREM 8.6.8. *If $\mathcal{F}$ is Mumford-regular then $Z_r = 0$, and the sequences (8.6.7) are exact for $n \geq 0$, so there is an exact sequence*

$$0 \to (\pi^*T_r\mathcal{F})(-r) \xrightarrow{\varepsilon(-r)} \cdots \to (\pi^*T_i\mathcal{F})(-i) \xrightarrow{\varepsilon(-i)} \cdots \xrightarrow{\varepsilon(-1)} (\pi^*T_0\mathcal{F}) \xrightarrow{\varepsilon} \mathcal{F} \to 0.$$

Moreover, each $\mathcal{F} \mapsto T_i\mathcal{F}$ is an exact functor from Mumford-regular modules to $\mathcal{O}_X$-modules.

PROOF. We first prove by induction on $n \geq 0$ that (a) the module $Z_{n-1}(n)$ is Mumford-regular, (b) $\pi_*Z_n(n) = 0$ and (c) the canonical map $\varepsilon\colon \pi^*T_n \to Z_{n-1}(n)$ is onto, *i.e.*, that (8.6.7) is exact for n.

We are given that (a) holds for $n = 0$, so we suppose that (a) holds for n. This implies part (c) for n by Proposition 8.6.3. Inductively then, we are given that (8.6.7) is exact, so $\pi_*Z_n(n) = 0$ and the module $Z_n(n+1)$ is Mumford-regular by

Ex. 8.6. That is, (b) holds for n and (a) holds for $n+1$. This finishes the first proof by induction.

Using (8.6.7), another induction on n shows that (d) each $\mathcal{F} \mapsto Z_{n-1}\mathcal{F}(n)$ is an exact functor from Mumford-regular modules to itself, and (e) each $\mathcal{F} \mapsto T_n\mathcal{F}$ is an exact functor from Mumford-regular modules to $\mathcal{O}_X$-modules. Note that (d) implies (e) by Lemma 8.6.4, since $T_n = \pi_* Z_{n-1}(n)$.

Since the canonical resolution is obtained by splicing the exact sequences (8.6.7) together for $n = 0, ..., r$, all that remains is to prove that $Z_r = 0$, or equivalently, that $Z_r(r) = 0$. From (8.6.7) we get the exact sequence

$$R^{q-1}\pi_* Z_{n+q-1}(n) \to R^q\pi_* Z_{n+q}(n) \to R^q\pi_*(\pi^* T_n(-q))$$

which allows us to conclude, starting from (b) and 8.6.2, that $R^q\pi_*(Z_{n+q}) = 0$ for all $n, q \geq 0$. Since $R^q\pi_* = 0$ for all $q > r$, this shows that $Z_r(r)$ is Mumford-regular. Since $\pi^*\pi_* Z_r(r) = 0$ by (b), we see from Proposition 8.6.3(3) that $Z_r(r) = 0$ as well.

Corollary 8.6.9. *If $\mathcal{F}$ is Mumford-regular, each $T_i\mathcal{F}$ is a vector bundle on X.*

Proof. For every $n \geq 0$, the n^{th} twist of the Quillen resolution 8.6.8 yields exact sequences of π_*-acyclic modules. Thus applying π_* yields an exact sequence of $\mathcal{O}_X$-modules, which by 8.6.2 is

$$0 \to T_n \to \mathcal{E} \otimes T_{n-1} \to \cdots \to Sym_{n-i}\mathcal{E} \otimes T_i \to \cdots \to \pi_*\mathcal{F}(n) \to 0.$$

The result follows from this sequence and induction on i, since $\pi_*\mathcal{F}(n)$ is a vector bundle by Lemma 8.6.5(3).

Let $\mathbf{MR}(n)$ denote the n^{th} twist of $\mathbf{MR}$; it is the full subcategory of $\mathbf{VB}(\mathbb{P})$ consisting of vector bundles $\mathcal{F}$ such that $\mathcal{F}(-n)$ is Mumford-regular. Since twisting is an exact functor, each $\mathbf{MR}(n)$ is an exact category.

Proposition 8.6.10. *The inclusions $\mathbf{MR}(n) \subset \mathbf{VB}(\mathbb{P})$ induce isomorphisms $K_0\mathbf{MR} \cong K_0\mathbf{MR}(n) \cong K_0(\mathbb{P})$.*

Proof. By Lemma 8.6.3 we have $\mathbf{MR}(n) \subset \mathbf{MR}(n-1)$, and the union of the $\mathbf{MR}(n)$ is $\mathbf{VB}(\mathbb{P})$ by Lemma 8.6.5(1). By Example 7.1.7 we have $K_0\mathbf{VB}(\mathbb{P}) = \varinjlim K_0\mathbf{MR}(n)$, so it suffices to show that each inclusion $\mathbf{MR}(n) \subset \mathbf{MR}(n-1)$ induces an isomorphism on K_0. Let $u_i\colon \mathbf{MR}(n-1) \to \mathbf{MR}(n)$ be the exact functor $\mathcal{F} \mapsto \mathcal{F} \otimes_X \wedge^i\mathcal{E}$. It induces a homomorphism $u_i\colon K_0\mathbf{MR}(n-1) \to K_0\mathbf{MR}(n)$. By Proposition 7.4 (Additivity), we see that the map $\sum_{i>0}(-1)^{i-1}u_i$ is an inverse to the map $\iota_n\colon K_0\mathbf{MR}(n) \to K_0\mathbf{MR}(n-1)$ induced by the inclusion. Hence ι_n is an isomorphism, as desired.

Proof of Projective Bundle Theorem 8.6. Each T_n is an exact functor from $\mathbf{MR}$ to $\mathbf{VB}(X)$ by Theorem 8.6.8 and 8.6.9. Hence we have a homomorphism

$$t\colon K_0\mathbf{MR} \to K_0(X)^{r+1}, \qquad [\mathcal{F}] \mapsto ([T_0\mathcal{F}], -[T_1\mathcal{F}], \ldots, (-1)^r[T_r\mathcal{F}]).$$

This fits into the diagram

$$K_0(\mathbb{P}) \xleftarrow{\cong} K_0\mathbf{MR} \xrightarrow{t} K_0(X)^{r+1} \xrightarrow{u} K_0(\mathbb{P}) \xleftarrow{\cong} K_0\mathbf{MR} \xrightarrow{v} K_0(X)^{r+1}$$

where $u(a_0, ..., a_r) = \pi^* a_0 + \pi^* a_1 \cdot [\mathcal{O}_{\mathbb{P}}(-1)] + \cdots + \pi^* a_r \cdot [\mathcal{O}_{\mathbb{P}}(-r)]$ and $v[\mathcal{F}] = ([\pi_* \mathcal{F}], [\pi_* \mathcal{F}(1)], \dots, [\pi_* \mathcal{F}(r)])$. The composition ut sends $[\mathcal{F}]$ to the alternating sum of the $[(\pi^* T_i \mathcal{F})(-i)]$, which equals $[\mathcal{F}]$ by Quillen's Resolution Theorem. Hence u is a surjection.

Since the (i,j) component of vu sends $\mathcal{N}_j$ to $\pi_*(\pi^* \mathcal{N}_j(i-j)) = Sym_{i-j}\mathcal{E} \otimes_X \mathcal{N}_j$ by Example 8.6.2, it follows that the composition vu is given by a lower triangular matrix with ones on the diagonal. Therefore vu is an isomorphism, so u is injective.

λ-operations in $K_0(X)$

The following result was promised in Example 4.1.5.

PROPOSITION 8.7. *The operations $\lambda^k[\mathcal{F}] = [\wedge^k \mathcal{F}]$ are well-defined on $K_0(X)$, and make $K_0(X)$ into a λ-ring.*

PROOF. It suffices to show that the formula $\lambda_t(\mathcal{F}) = \sum [\wedge^k \mathcal{F}] t^k$ defines an additive homomorphism from $\mathbf{VB}(X)$ to the multiplicative group $1 + tK_0(X)[[t]]$. Note that the constant term in $\lambda_t(\mathcal{F})$ is 1 because $\wedge^0 \mathcal{F} = \mathcal{O}_X$. Suppose given an exact sequence of vector bundles $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. By Ex. I.5.4, each $\wedge^k \mathcal{F}$ has a finite filtration whose associated quotient modules are the $\wedge^i \mathcal{F}' \otimes \wedge^{k-i} \mathcal{F}''$, so in $K_0(X)$ we have

$$[\wedge^k \mathcal{F}] = \sum [\wedge^i \mathcal{F}' \otimes \wedge^{k-i} \mathcal{F}''] = \sum [\wedge^i \mathcal{F}'] \cdot [\wedge^{k-i} \mathcal{F}''].$$

Assembling these equations yields the formula $\lambda_t(\mathcal{F}) = \lambda_t(\mathcal{F}')\lambda_t(\mathcal{F}'')$ in the group $1 + tK_0(X)[[t]]$, proving that λ_t is additive. Hence λ_t (and each coefficient λ^k) is well-defined on $K_0(X)$.

SPLITTING PRINCIPLE 8.7.1 (SEE 4.2.2). *Let $f\colon \mathbb{F}(\mathcal{E}) \to X$ be the flag bundle of a vector bundle $\mathcal{E}$ over a quasi-compact scheme X. Then $K_0(\mathbb{F}(\mathcal{E}))$ is a free module over the ring $K_0(X)$, and $f^*[\mathcal{E}]$ is a sum of line bundles $\sum [\mathcal{L}_i]$.*

PROOF. Let $f\colon \mathbb{F}(\mathcal{E}) \to X$ be the flag bundle of $\mathcal{E}$; by Theorem I.5.9 the bundle $f^*\mathcal{E}$ has a filtration by sub-vector bundles whose successive quotients $\mathcal{L}_i$ are line bundles. Hence $f^*[\mathcal{E}] = \sum [\mathcal{L}_i]$ in $K_0(\mathbb{F}(\mathcal{E}))$. Moreover, we saw in I.5.8 that the flag bundle is obtained from X by a sequence of projective bundle extensions, beginning with $\mathbb{P}(\mathcal{E})$. By the Projective Bundle Theorem 8.6, $K_0(\mathbb{F}(\mathcal{E}))$ is obtained from $K_0(X)$ by a sequence of finite free extensions.

The λ-ring $K_0(X)$ has a positive structure in the sense of Definition 4.2.1. The "positive elements" are the classes $[\mathcal{F}]$ of vector bundles, and the augmentation $\varepsilon\colon K_0(X) \to H^0(X;\mathbb{Z})$ is given by Theorem 8.1. In this vocabulary, the "line elements" are the classes $[\mathcal{L}]$ of line bundles on X, and the subgroup L of units in $K_0(X)$ is just $\mathrm{Pic}(X)$. The following corollary now follows from Theorems 4.2.3 and 4.7.

COROLLARY 8.7.2. *$K_0(X)$ is a special λ-ring. Consequently, the first two ideals in the γ-filtration of $K_0(X)$ are $F^1_\gamma = \widetilde{K}_0(R)$ and $F^2_\gamma = SK_0(R)$. In particular,*

$$F^0_\gamma / F^1_\gamma \cong H^0(X;\mathbb{Z}) \text{ and } \quad F^1_\gamma / F^2_\gamma \cong \mathrm{Pic}(X).$$

Corollary 8.7.3. *For every commutative ring R, $K_0(R)$ is a special λ-ring.*

Proposition 8.7.4. *If X is quasi-projective, or more generally if X has an ample line bundle $\mathcal{L}$ then every element of $\widetilde{K}_0(X)$ is nilpotent. Hence $\widetilde{K}_0(X)$ is a nil ideal of $K_0(X)$.*

Proof. By Ex. 4.5, it suffices to show that $\ell = [\mathcal{L}]$ is an ample line element. Given $x = [\mathcal{E}] - [\mathcal{F}]$ in $\widetilde{K}_0(X)$, the fact that $\mathcal{L}$ is ample implies that $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for all large n. Hence there are short exact sequences

$$0 \to \mathcal{G}_n \to \mathcal{O}_X^{r_n} \to \mathcal{F}(n) \to 0$$

and therefore in $K_0(X)$ we have the required equation:

$$\ell^n x = [\mathcal{E}(n)] - [\mathcal{O}_X^{r_n}] + [\mathcal{G}_n] = [\mathcal{E}(n) \oplus \mathcal{G}_n] - r_n.$$

Remark 8.7.5 (Nilpotence). If X is noetherian and quasiprojective of dimension d, then $\widetilde{K}_0(X)^{d+1} = 0$, because it lies inside F_γ^{d+1}, which vanishes by [SGA6, VI.6.6] or [FL, V.3.10]. (See Example 4.8.2.)

Limits of Schemes 8.8. The following construction is the analogue for schemes of the fact that every commutative ring is the filtered union of its finitely generated (noetherian) subrings. By [EGA, IV.8.2.3], every quasi-compact separated scheme X is the inverse limit of a filtered inverse system $i \mapsto X_i$ of noetherian schemes, each finitely presented over $\mathbb{Z}$, with affine transition maps.

Let $i \mapsto X_i$ be any filtered inverse system of schemes such that the transition morphisms $X_i \to X_j$ are affine, and let X be the inverse limit scheme $\varprojlim X_i$. This scheme exists by [EGA, IV.8.2]. In fact, over an affine open subset $\mathrm{Spec}(R_j)$ of any X_j we have affine open subsets $\mathrm{Spec}(R_i)$ of each X_i, and the corresponding affine open of X is $\mathrm{Spec}(\varinjlim R_i)$. By [EGA, IV.8.5] every vector bundle on X comes from a bundle on some X_j, and two bundles on X_j are isomorphic over X just in case they are isomorphic over some X_i. Thus the filtered system of groups $K_0(X_i)$ has the property that

$$K_0(X) = \varinjlim K_0(X_i).$$

EXERCISES

8.1 Suppose that Z is a closed subscheme of a quasi-projective scheme X, with complement U. Let $\mathbf{H}_Z(X)$ denote the subcategory of $\mathbf{H}(X)$ consisting of modules supported on Z.

(a) (Deligne) Let $(R, \mathfrak{m})$ be a 2-dimensional local noetherian domain which is not Cohen-Macauley, meaning that for every x, y in $\mathfrak{m}$ there is a $z \notin xR$ with $yz \in xR$. Setting $X = \mathrm{Spec}(R)$ and $Z = \{\mathfrak{m}\}$, show that $\mathbf{H}_Z(X) = 0$.

(b) Suppose that $U = \mathrm{Spec}(R)$ for some ring R, and that Z is locally defined by a nonzerodivisor. (The ideal $\mathcal{I}_Z$ is invertible; see §I.5.12.) As in Cor. 7.6.4, show that there is an exact sequence: $K_0\mathbf{H}_Z(R) \to K_0(X) \to K_0(U)$.

(c) Suppose that Z is contained in an open subset V of X which is regular. Show that $\mathbf{H}_Z(X)$ is the abelian category $\mathbf{M}_Z(X)$ of 6.4.2, so that $K_0\mathbf{H}_Z(X) \cong G_0(Z)$. Then apply Ex. 7.12 to show that there is an exact sequence

$$G_0(Z) \to K_0(X) \to K_0(U) \to 0.$$

8.2 Let X be a curve over an algebraically closed field. By Ex. I.5.7, $K_0(X)$ is generated by classes of line bundles. Show that $K_0(X) = H^0(X;\mathbb{Z}) \oplus \mathrm{Pic}(X)$.

8.3 *Projection Formula for schemes.* Suppose that $f\colon X \to Y$ is a proper map between quasi-projective schemes, both of which have finite Tor-dimension.

(a) Given $\mathcal{E}$ in $\mathbf{VB}(X)$, consider the subcategory $\mathbf{L}(f)$ of $\mathbf{M}(Y)$ consisting of coherent $\mathcal{O}_Y$-modules which are Tor-independent of both $f_*\mathcal{E}$ and $f_*\mathcal{O}_X$. Show that $K_0(Y) \cong K_0\mathbf{L}(f)$.

(b) Using 7.3.2 and the ring map $f^*\colon K_0(Y) \to K_0(X)$, both $K_0(X)$ and $G_0(X)$ are $K_0(Y)$-modules. Show that the transfer maps $f_*\colon G_0(X) \to G_0(Y)$ of Lemma 6.2.6 and $f_*\colon K_0(X) \to K_0(Y)$ of (8.4.1) are $K_0(Y)$-module homomorphisms, *i.e.*, that the *projection formula* holds for every $y \in K_0(Y)$:

$$f_*(x \cdot f^*y) = f_*(x) \cdot y \quad \text{for every } x \in K_0(X) \text{ or } x \in G_0(X).$$

8.4 Suppose given a commutative square of quasi-projective schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with $X' = X \times_Y Y'$ and f proper. Assume that g has finite flat dimension, and that X and Y' are Tor-independent over Y, *i.e.*, for $q > 0$ and all $x \in X$, $y' \in Y'$ and $y \in Y$ with $y = f(x) = g(y')$ we have

$$Tor_q^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0.$$

Show that $g^*f_* = f'_*g'^*$ as maps $G_0(X) \to G_0(Y')$.

8.5 Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be vector bundles of ranks r_1 and r_2, respectively. Modify Ex. I.2.7 to show that $\det(\mathcal{F}_1 \otimes \mathcal{F}_2) \cong (\det \mathcal{F}_1)^{r_2} \otimes (\det \mathcal{F}_2)^{r_1}$. Conclude that $K_0(X) \to H^0(X;\mathbb{Z}) \oplus \mathrm{Pic}(X)$ is a ring map.

8.6 Let $\pi\colon \mathbb{P} \to X$ be a projective bundle as in 8.6, and let $\mathcal{F}$ be a Mumford-regular $\mathcal{O}_{\mathbb{P}}$-module. Let $\mathcal{N}$ denote the kernel of the canonical map $\varepsilon\colon \pi^*\pi_*\mathcal{F} \to \mathcal{F}$. Show that $\mathcal{N}(1)$ is Mumford-regular, and that $\pi_*\mathcal{N} = 0$.

§9. K_0 of a Waldhausen category

It is useful to be able to define the Grothendieck group $K_0(\mathcal{C})$ of a more general type of category than exact categories, by adding a notion of weak equivalence. A structure that generalizes well to higher K-theory is that of a category of cofibrations and weak equivalences, which we shall call a "Waldhausen category" for brevity. The definitions we shall use are due to Friedhelm Waldhausen, although the ideas for K_0 are due to Grothendieck and were used in [SGA6].

We need to consider two families of distinguished morphisms in a category $\mathcal{C}$, the cofibrations and the weak equivalences. For this we use the following device. Suppose that we are given a family $\mathcal{F}$ of distinguished morphisms in a category $\mathcal{C}$. We assume that these distinguished morphisms are closed under composition, and contain every identity. It is convenient to regard these distinguished morphisms as the morphisms of a subcategory of $\mathcal{C}$, which by abuse of notation we also call $\mathcal{F}$.

Definition 9.1. Let $\mathcal{C}$ be a category equipped with a subcategory $co = co(\mathcal{C})$ of morphisms in a category $\mathcal{C}$, called "cofibrations" (and indicated with feathered arrows $\rightarrowtail$). The pair $(\mathcal{C}, co)$ is called a *category with cofibrations* if the following axioms are satisfied:

(W0) Every isomorphism in $\mathcal{C}$ is a cofibration;

(W1) There is a zero object '0' in $\mathcal{C}$, and the unique map $0 \rightarrowtail A$ in $\mathcal{C}$ is a cofibration for every A in $\mathcal{C}$;

(W2) If $A \rightarrowtail B$ is a cofibration, and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \cup_A C$ of these two maps exists in $\mathcal{C}$, and moreover the map $C \rightarrowtail B \cup_A C$ is a cofibration.

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ C \arrow[r, tail] & B \cup_A C \end{tikzcd}$$

These axioms imply that two constructions make sense in $\mathcal{C}$: (1) the coproduct $B \amalg C$ of any two objects exists in $\mathcal{C}$ (it is the pushout $B \cup_0 C$), and (2) every cofibration $i\colon A \rightarrowtail B$ in $\mathcal{C}$ has a cokernel B/A (this is the pushout $B \cup_A 0$ of i along $A \to 0$). We refer to $A \rightarrowtail B \twoheadrightarrow B/A$ as a *cofibration sequence* in $\mathcal{C}$.

For example, any abelian category is naturally a category with cofibrations: the cofibrations are the monomorphisms. More generally, we can regard any exact category as a category with cofibrations by letting the cofibrations be the admissible monics; axiom (W2) follows from Ex. 7.8(2). In an exact category, the cofibration sequences are exactly the admissible exact sequences.

Definition 9.1.1. A *Waldhausen category* $\mathcal{C}$ is a category with cofibrations, together with a family $w(\mathcal{C})$ of morphisms in $\mathcal{C}$ called "weak equivalences" (abbreviated '$w.e.$' and indicated with decorated arrows $\xrightarrow{\sim}$). Every isomorphism in $\mathcal{C}$ is to be a weak equivalence, and weak equivalences are to be closed under composition (so we may regard $w(\mathcal{C})$ as a subcategory of $\mathcal{C}$). In addition, the following "Glueing axiom" must be satisfied:

(W3) *Glueing for weak equivalences.* For every commutative diagram of the form

$$\begin{array}{ccccc} C & \leftarrow & A & \rightarrowtail & B \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ C' & \leftarrow & A' & \rightarrowtail & B' \end{array}$$

(in which the vertical maps are weak equivalences and the two right horizontal maps are cofibrations), the induced map

$$B \cup_A C \to B' \cup_{A'} C'$$

is also a weak equivalence.

Although a Waldhausen category is really a triple $(\mathcal{C}, co, w)$, we will usually drop the (co, w) from the notation and just write $\mathcal{C}$.

Definition 9.1.2 ($K_0\mathcal{C}$). Let $\mathcal{C}$ be a Waldhausen category. $K_0(\mathcal{C})$ is the abelian group presented as having one generator $[C]$ for each object C of $\mathcal{C}$, subject to the relations

(1) $[C] = [C']$ if there is a weak equivalence $C \xrightarrow{\sim} C'$
(2) $[C] = [B] + [C/B]$ for every cofibration sequence $B \rightarrowtail C \twoheadrightarrow C/B$.

Of course, in order for this to be set-theoretically meaningful, we must assume that the weak equivalence classes of objects form a set. We shall occasionally use the notation $K_0(w\mathcal{C})$ for $K_0(\mathcal{C})$ to emphasize the choice of $w\mathcal{C}$ as weak equivalences.

These relations imply that $[0] = 0$ and $[B \amalg C] = [B] + [C]$, as they did in §6 for abelian categories. Because pushouts preserve cokernels, we also have $[B \cup_A C] = [B] + [C] - [A]$. However, weak equivalences add a new feature: $[C] = 0$ whenever $0 \simeq C$.

Example 9.1.3. Any exact category $\mathcal{A}$ becomes a Waldhausen category, with cofibrations being admissible monics and weak equivalences being isomorphisms. By construction, the Waldhausen definition of $K_0(\mathcal{A})$ agrees with the exact category definition of $K_0(\mathcal{A})$ given in §7.

More generally, any category with cofibrations $(\mathcal{C}, co)$ may be considered as a Waldhausen category in which the category of weak equivalences is the category $\mathrm{iso}\,\mathcal{C}$ of all isomorphisms. In this case $K_0(\mathcal{C}) = K_0(\mathrm{iso}\,\mathcal{C})$ has only the relation (2). We could of course have developed this theory in §7 as an easy generalization of the preceding paragraph.

Topological Example 9.1.4. To show that we need not have additive categories, we give a topological example due to Waldhausen. Let $\mathcal{R} = \mathcal{R}(*)$ be the category of based CW complexes with countably many cells (we need a bound on the cardinality of the cells for set-theoretic reasons). Morphisms are cellular maps, and $\mathcal{R}_f = \mathcal{R}_f(*)$ is the subcategory of finite based CW complexes. Both are Waldhausen categories: "cofibration" is a cellular inclusion, and "weak equivalence" means weak homotopy equivalence (isomorphism on homotopy groups). The coproduct $B \vee C$ is obtained from the disjoint union of B and C by identifying their basepoints.

The Eilenberg Swindle shows that $K_0\mathcal{R} = 0$. In effect, the infinite coproduct C^∞ of copies of a fixed complex C exists in $\mathcal{R}$, and equals $C \vee C^\infty$. In contrast, the finite complexes have interesting K-theory:

Proposition 9.1.5. *$K_0\mathcal{R}_f \cong \mathbb{Z}$.*

Proof. The inclusion of S^{n-1} in the n-disk D^n has $D^n/S^{n-1} \cong S^n$, so $[S^{n-1}]+[S^n] = [D^n] = 0$. Hence $[S^n] = (-1)^n[S^0]$. If C is obtained from B by attaching an n-cell, $C/B \cong S^n$ and $[C] = [B] + [S^n]$. Hence $K_0\mathcal{R}_f$ is generated by $[S^0]$. Finally, the reduced Euler characteristic $\chi(C) = \sum(-1)^i \dim \tilde{H}^i(X;\mathbb{Q})$ defines a surjection from $K_0\mathcal{R}_f$ onto $\mathbb{Z}$, which must therefore be an isomorphism.

BiWaldhausen Categories 9.1.6. In general, the opposite $\mathcal{C}^{op}$ need not be a Waldhausen category, because the quotients $B \twoheadrightarrow B/A$ need not be closed under composition: the family $\mathrm{quot}(\mathcal{C})$ of these quotient maps need not be a subcategory of $\mathcal{C}^{op}$. We call $\mathcal{C}$ a *category with bifibrations* if $\mathcal{C}$ is a category with cofibrations, $\mathcal{C}^{op}$ is a category with cofibrations $co(\mathcal{C}^{op}) = \mathrm{quot}(\mathcal{C})$, the canonical map $A \amalg B \to A \times B$ is always an isomorphism, and A is the kernel of each quotient map $B \twoheadrightarrow B/A$. We call $\mathcal{C}$ a *biWaldhausen category* if $\mathcal{C}$ is a category with bifibrations, having a subcategory $w(\mathcal{C})$ so that both $(\mathcal{C}, co, w)$ and $(\mathcal{C}^{op}, \mathrm{quot}, w^{op})$ are Waldhausen categories. The notions of bifibrations and biWaldhausen category are self-dual, so we have:

Lemma 9.1.6.1. *$K_0(\mathcal{C}) \cong K_0(\mathcal{C}^{op})$ for every biWaldhausen category.*

Example 9.1.3 shows that exact categories are biWaldhausen categories. We will see in 9.2 below that chain complexes form another important family of biWaldhausen categories.

Exact Functors 9.1.7. A functor $F: \mathcal{C} \to \mathcal{D}$ between Waldhausen categories is called an *exact functor* if it preserves all the relevant structure: zero, cofibrations, weak equivalences and pushouts along a cofibration. The last condition means that the canonical map $FB \cup_{FA} FC \to F(B \cup_A C)$ is an isomorphism for every cofibration $A \rightarrowtail B$. Clearly, an exact functor induces a group homomorphism $K_0(F): K_0\mathcal{C} \to K_0\mathcal{D}$.

A *Waldhausen subcategory* $\mathcal{A}$ of a Waldhausen category $\mathcal{C}$ is a subcategory which is also a Waldhausen category in such a way that: (i) the inclusion $\mathcal{A} \subseteq \mathcal{C}$ is an exact functor, (ii) the cofibrations in $\mathcal{A}$ are the maps in $\mathcal{A}$ which are cofibrations in $\mathcal{C}$ and whose cokernel lies in $\mathcal{A}$, and (iii) the weak equivalences in $\mathcal{A}$ are the weak equivalences of $\mathcal{C}$ which lie in $\mathcal{A}$.

For example, suppose that $\mathcal{C}$ and $\mathcal{D}$ are exact categories (in the sense of §7), considered as Waldhausen categories. A functor $F: \mathcal{C} \to \mathcal{D}$ is exact in the above sense if and only if F is additive and preserves short exact sequences, *i.e.*, F is an exact functor between exact categories in the sense of §7. The routine verification of this assertion is left to the reader.

Here is an elementary consequence of the definition of exact functor. Let $\mathcal{A}$ and $\mathcal{C}$ be Waldhausen categories and F, F', F'' three exact functors from $\mathcal{A}$ to $\mathcal{C}$. Suppose moreover that there are natural transformations $F' \Rightarrow F \Rightarrow F''$ so that for all A in $\mathcal{A}$

$$F'A \rightarrowtail FA \twoheadrightarrow F''A \tag{9.1.8}$$

is a cofibration sequence in $\mathcal{C}$. Then $[FA] = [F'A] + [F''A]$ in $K_0\mathcal{C}$, so as maps from $K_0\mathcal{A}$ to $K_0\mathcal{C}$ we have $K_0(F) = K_0(F') + K_0(F'')$.

Chain complexes

9.2 Historically, one of the most important families of Waldhausen categories are those arising from chain complexes. The definition of K_0 for a category of (co)chain complexes dates to the 1960's, being used in [SGA6] to study the Riemann-Roch Theorem. We will work with chain complexes here, although by reindexing we could equally well work with cochain complexes.

Given a small abelian category $\mathcal{A}$, let $\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$ denote the category of all chain complexes in $\mathcal{A}$, and let $\mathbf{Ch}^b$ denote the full subcategory of all bounded complexes. The following structure makes $\mathbf{Ch}$ into a Waldhausen category, with $\mathbf{Ch}^b(\mathcal{A})$ as a Waldhausen subcategory. We will show below that $K_0\mathbf{Ch} = 0$ but that $K_0\mathbf{Ch}^b \cong K_0\mathcal{A}$.

A cofibration $C \to D$ is a chain map such that every map $C_n \rightarrowtail D_n$ is monic in $\mathcal{A}$. Thus a cofibration sequence is just a short exact sequence of chain complexes. A weak equivalence $C \xrightarrow{\sim} D$ is a quasi-isomorphism, *i.e.*, a chain map inducing isomorphisms on homology.

Here is a slightly more general construction, taken from [SGA6, IV(1.5.2)]. Suppose that $\mathcal{C}$ is an exact category, embedded in an abelian category $\mathcal{A}$. Let $\mathbf{Ch}(\mathcal{C})$, resp. $\mathbf{Ch}^b(\mathcal{C})$, denote the category of all (resp. all bounded) chain complexes in $\mathcal{C}$. A cofibration $A. \rightarrowtail B.$ in $\mathbf{Ch}(\mathcal{C})$ (resp. $\mathbf{Ch}^b\mathcal{C}$) is a map which is a degreewise admissible monomorphism, *i.e.*, such that each $C_n = B_n/A_n$ is in $\mathcal{C}$, yielding short exact sequences $A_n \rightarrowtail B_n \twoheadrightarrow C_n$ in $\mathcal{C}$. To define the weak equivalences, we use the notion of homology in the ambient abelian category $\mathcal{A}$: let $w\mathbf{Ch}(\mathcal{C})$ denote the family of all chain maps in $\mathbf{Ch}(\mathcal{C})$ which are quasi-isomorphisms of complexes in $\mathbf{Ch}(\mathcal{A})$. With this structure, both $\mathbf{Ch}(\mathcal{C})$ and $\mathbf{Ch}^b(\mathcal{C})$ become Waldhausen subcategories of $\mathbf{Ch}(\mathcal{A})$.

Subtraction in $K_0\mathbf{Ch}$ and $K_0\mathbf{Ch}^b$ is given by shifting indices on complexes. To see this, recall from [WHomo, 1.2.8] that the n^{th} *translate* of C is defined to be the chain complex $C[n]$ which has C_{i+n} in degree i. (If we work with cochain complexes then C^{i-n} is in degree i.) Moreover, the *mapping cone complex* $\mathrm{cone}(f)$ of a chain complex map $f\colon B \to C$ fits into a short exact sequence of complexes:

$$0 \to C \to \mathrm{cone}(f) \to B[-1] \to 0.$$

Therefore in K_0 we have $[C] + [B[-1]] = [\mathrm{cone}(f)]$. In particular, if f is the identity map on C, the cone complex is exact and hence *w.e.* to 0. Thus we have $[C] + [C[-1]] = [\mathrm{cone}(\mathrm{id})] = 0$. We record this observation as follows.

LEMMA 9.2.1. *Let* $\mathbf{C}$ *be any Waldhausen subcategory of* $\mathbf{Ch}(\mathcal{A})$ *closed under translates and the formation of mapping cones. Then* $[C[n]] = (-1)^n[C]$ *in* $K_0(\mathbf{C})$. *In particular, this is true in* $K_0\mathbf{Ch}(\mathcal{C})$ *and* $K_0\mathbf{Ch}^b(\mathcal{C})$ *for every exact subcategory* $\mathcal{C}$ *of* $\mathcal{A}$.

A chain complex C is called *bounded below* (resp. *bounded above*) if $C_n = 0$ for all $n \ll 0$ (resp. all $n \gg 0$). If C is bounded above, then each infinite direct sum $C_n \oplus C_{n+2} \oplus \cdots$ is finite, so the infinite direct sum of shifts

$$B = C \oplus C[2] \oplus C[4] \oplus \cdots \oplus C[2n] \oplus \cdots$$

is defined in **Ch**. From the exact sequence $0 \to B[2] \to B \to C \to 0$, we see that in $K_0\mathbf{Ch}$ we have the Eilenberg swindle: $[C] = [B] - [B[2]] = [B] - [B] = 0$. A similar argument shows that $[C] = 0$ if C is bounded below. But every chain complex C fits into a short exact sequence

$$0 \to B \to C \to D \to 0$$

in which B is bounded above and D is bounded below. (For example, take $B_n = 0$ for $n > 0$ and $B_n = C_n$ otherwise.) Hence $[C] = [B] + [D] = 0$ in $K_0\mathbf{Ch}$. This shows that $K_0\mathbf{Ch} = 0$, as asserted.

If $\mathcal{C}$ is any exact category, the natural inclusion of $\mathcal{C}$ into $\mathbf{Ch}^b(\mathcal{C})$ as the chain complexes concentrated in degree zero is an exact functor. Hence it induces a homomorphism $K_0(\mathcal{C}) \to K_0\mathbf{Ch}^b(\mathcal{C})$.

Theorem 9.2.2 ([SGA6], I.6.4). *Let $\mathcal{A}$ be an abelian category. Then*

$$K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A}),$$

and the class $[C]$ of a chain complex C in $K_0\mathcal{A}$ is the same as its Euler characteristic, namely $\chi(C) = \sum(-1)^i[C_i]$.

Similarly, if $\mathcal{C}$ is an exact category closed under kernels of surjections in an abelian category (in the sense of 7.0.1), then $K_0(\mathcal{C}) \cong K_0\mathbf{Ch}^b(\mathcal{C})$, and again we have $\chi(C) = \sum(-1)^i[C_i]$ in $K_0(\mathcal{C})$.

Proof. We give the proof for $\mathcal{A}$; the proof for $\mathcal{C}$ is the same, except one cites 7.4 in place of 6.6. As in Proposition 6.6 (or 7.4), the Euler characteristic $\chi(C)$ of a bounded complex is the element $\sum(-1)^i[C_i]$ of $K_0(\mathcal{A})$. We saw in 6.6 (and 7.4.1) that $\chi(B) = \chi(C)$ if $B \to C$ is a weak equivalence (quasi-isomorphism). If $B \rightarrowtail C \twoheadrightarrow D$ is a cofibration sequence in $\mathbf{Ch}^b$, then from the short exact sequences $0 \to B_n \to C_n \to D_n \to 0$ in $\mathcal{A}$ we obtain $\chi(C) = \chi(B) + \chi(C/B)$ by inspection (as in 7.4.1). Hence χ satisfies the relations needed to define a homomorphism χ from $K_0(\mathbf{Ch}^b)$ to $K_0(\mathcal{A})$. If C is concentrated in degree 0 then $\chi(C) = [C_0]$, so the composite map $K_0(\mathcal{A}) \to K_0(\mathbf{Ch}^b) \to K_0(\mathcal{A})$ is the identity.

It remains to show that $[C] = \chi(C)$ in $K_0\mathbf{Ch}^b$ for every complex

$$C\colon 0 \to C_m \to \cdots \to C_n \to 0.$$

If $m = n$, then $C = C_n[-n]$ is the object C_n of $\mathcal{A}$ concentrated in degree n; we have already observed that $[C] = (-1)^n[C_n[0]] = (-1)^n[C_n]$ in this case. If $m > n$, let B denote the subcomplex consisting of C_n in degree n, and zero elsewhere. Then $B \rightarrowtail C$ is a cofibration whose cokernel C/B has shorter length than C. By induction, we have the desired relation in $K_0\mathbf{Ch}^b$, finishing the proof:

$$[C] = [B] + [C/B] = \chi(B) + \chi(C/B) = \chi(C).$$

Remark 9.2.3 (K_0 and derived categories). Let $\mathcal{C}$ be an exact category. Theorem 9.2.2 states that the group $K_0\mathbf{Ch}^b(\mathcal{C})$ is independent of the choice of ambient abelian category $\mathcal{A}$, as long as $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$. This is the group $k(\mathcal{C})$ introduced in [SGA6], Expose IV(1.5.2). (The context of [SGA6] was triangulated categories, and the main observation in *loc. cit.* is that this definition only depends upon the derived category $D^b_{\mathcal{C}}(\mathcal{A})$. See Ex. 9.5 below.)

We warn the reader that if $\mathcal{C}$ is not closed under kernels of surjections in $\mathcal{A}$, then $K_0\mathbf{Ch}^b(\mathcal{C})$ can differ from $K_0(\mathcal{C})$. (See Ex. 9.11).

If $\mathcal{A}$ is an abelian category, or even an exact category, the category $\mathbf{Ch}^b = \mathbf{Ch}^b(\mathcal{A})$ has another Waldhausen structure with the same weak equivalences: we redefine cofibration so that $B \to C$ is a cofibration iff each $B_i \to C_i$ is a *split* injection in $\mathcal{A}$. If split$\mathbf{Ch}^b$ denotes $\mathbf{Ch}^b$ with this new Waldhausen structure, then the inclusion split$\mathbf{Ch}^b \to \mathbf{Ch}^b$ is an exact functor, so it induces a surjection $K_0(\text{split}\mathbf{Ch}^b) \to K_0(\mathbf{Ch}^b)$.

LEMMA 9.2.4. *If $\mathcal{A}$ is an exact category then*

$$K_0(\textit{split}\mathbf{Ch}^b) \cong K_0(\mathbf{Ch}^b) \cong K_0(\mathcal{A}).$$

PROOF. Lemma 9.2.1 and enough of the proof of 9.2.2 go through to prove that $[C[n]] = (-1)^n[C]$ and $[C] = \sum(-1)^n[C_n]$ in $K_0(\text{split}\mathbf{Ch}^b)$. Hence it suffices to show that $A \mapsto [A]$ defines an additive function from $\mathcal{A}$ to $K_0(\text{split}\mathbf{Ch}^b)$. If A is an object of $\mathcal{A}$, let $[A]$ denote the class in $K_0(\text{split}\mathbf{Ch}^b)$ of the complex which is A concentrated in degree zero. Any short exact sequence $E\colon 0 \to A \to B \to C \to 0$ in $\mathcal{A}$ may be regarded as an (exact) chain complex concentrated in degrees 0, 1 and 2 so:

$$[E] = [A] - [B] + [C]$$

in $K_0(\text{split}\mathbf{Ch}^b)$. But E is weakly equivalent to zero, so $[E] = 0$. Hence $A \mapsto [A]$ is an additive function, defining a map $K_0(\mathcal{A}) \to K_0(\text{split}\mathbf{Ch}^b)$.

EXTENSION CATEGORIES 9.3. If $\mathcal{C}$ is a Waldhausen category, the cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{C}$ form the objects of a category $\mathcal{E}$. A morphism $E \to E'$ in $\mathcal{E}$ is a commutative diagram:

$$\begin{array}{ccccccc} E: & & A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E': & & A' & \rightarrowtail & B' & \twoheadrightarrow & C' \end{array}$$

We can make $\mathcal{E}$ in to a Waldhausen category as follows. A morphism $E \to E'$ in $\mathcal{E}$ is a cofibration if $A \to A'$, $C \to C'$ and $A' \cup_A B \to B'$ are cofibrations in $\mathcal{C}$. This is required by axiom (W2), and implies that the composite $B \rightarrowtail A' \cup_A B \rightarrowtail B'$ is a cofibration too. A morphism in $\mathcal{E}$ is a weak equivalence if its component maps $A \to A'$, $B \to B'$, $C \to C'$ are weak equivalences in $\mathcal{C}$.

There is an exact functor $\amalg\colon \mathcal{C} \times \mathcal{C} \to \mathcal{E}$, sending (A, C) to $A \rightarrowtail A \amalg C \twoheadrightarrow C$. Conversely, there are three exact functors (s, t and q) from $\mathcal{E}$ to $\mathcal{C}$, which send $A \rightarrowtail B \twoheadrightarrow C$ to A, B and C, respectively. By the above remarks, $t_* = s_* + q_*$ as maps $K_0(\mathcal{E}) \to K_0(\mathcal{C})$.

PROPOSITION 9.3.1. *$K_0(\mathcal{E}) \cong K_0(\mathcal{C}) \times K_0(\mathcal{C})$.*

PROOF. Since (s, q) is a left inverse to $\amalg$, $\amalg_*$ is a split injection from $K_0(\mathcal{C}) \times K_0(\mathcal{C})$ to $K_0(\mathcal{E})$. Thus it suffices to show that for every $E\colon A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}$ we have $[E] = [\amalg(A, 0)] + [\amalg(0, C)]$ in $K_0(\mathcal{E})$. This relation follows from the fundamental relation (2) of K_0, given that

$$\begin{array}{ccccccc} \amalg(A,0): & & A & \overset{=}{\rightarrowtail} & A & \twoheadrightarrow & 0 \\ \downarrow & & \| & & \downarrow & & \downarrow \\ E: & & A & \rightarrowtail & B & \twoheadrightarrow & C \end{array}$$

is a cofibration in $\mathcal{E}$ with cokernel $\amalg(0, C) : 0 \rightarrowtail C \twoheadrightarrow C$.

EXAMPLE 9.3.2 (HIGHER EXTENSION CATEGORIES). Here is a generalization of the extension category $\mathcal{E} = \mathcal{E}_2$ constructed above. Let $\mathcal{E}_n$ be the category whose objects are sequences of n cofibrations in a Waldhausen category $\mathcal{C}$:

$$A: \quad 0 = A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n.$$

A morphism $A \to B$ in $\mathcal{E}_n$ is a natural transformation of sequences, and is a weak equivalence if each component $A_i \to B_i$ is a $w.e.$ in $\mathcal{C}$. It is a cofibration when for each $0 \le i < j < k \le n$ the map of cofibration sequences

$$\begin{array}{ccccc} A_j/A_i & \rightarrowtail & A_k/A_i & \twoheadrightarrow & A_k/A_j \\ \downarrow & & \downarrow & & \downarrow \\ B_j/B_i & \rightarrowtail & B_k/B_i & \twoheadrightarrow & Bk/B_j \end{array}$$

is a cofibration in $\mathcal{E}$. The reader is encouraged in Ex. 9.4 to check that $\mathcal{E}_n$ is a Waldhausen category, and to compute $K_0(\mathcal{E}_n)$.

COFINALITY THEOREM 9.4. *Let $\mathcal{B}$ be a Waldhausen subcategory of $\mathcal{C}$ closed under extensions. If $\mathcal{B}$ is cofinal in $\mathcal{C}$ (in the sense that for all C in $\mathcal{C}$ there is a C' in $\mathcal{C}$ so that $C \amalg C'$ is in $\mathcal{B}$), then $K_0(\mathcal{B})$ is a subgroup of $K_0(\mathcal{C})$.*

PROOF. Considering $\mathcal{B}$ and $\mathcal{C}$ as symmetric monoidal categories with product $\amalg$, we have $K_0^{\amalg}(\mathcal{B}) \subset K_0^{\amalg}(\mathcal{C})$ by (1.3). The proof of cofinality for exact categories (Lemma 7.2) goes through verbatim to prove that $K_0(\mathcal{B}) \subset K_0(\mathcal{C})$.

Products

9.5 Our discussion in 7.3 about products in exact categories carries over to the Waldhausen setting. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Waldhausen categories, and suppose given a functor $F: \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. The following result is completely elementary:

LEMMA 9.5.1. *If each $F(A, -): \mathcal{B} \to \mathcal{C}$ and $F(-, B): \mathcal{A} \to \mathcal{C}$ is an exact functor, then $F: \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces a bilinear map*

$$\begin{aligned} K_0\mathcal{A} \otimes K_0\mathcal{B} &\to K_0\mathcal{C} \\ [A] \otimes [B] &\mapsto [F(A, B)]. \end{aligned}$$

Note that the 3×3 diagram in $\mathcal{C}$ determined by $F(A \rightarrowtail A', B \rightarrowtail B')$ yields the following relation in $K_0(\mathcal{C})$.

$$[F(A', B')] = [F(A, B)] + [F(A'/A, B)] + [F(A, B'/B)] + [F(A'/A, B'/B)]$$

Higher K-theory will need this relation to follow from more symmetric considerations, viz. that $F(A \rightarrowtail A', B \rightarrowtail B')$ should represent a cofibration in the category $\mathcal{E}$ of all cofibration sequences in $\mathcal{C}$. With this in mind, we introduce the following definition.

DEFINITION 9.5.2. A functor $F: \mathcal{A} \times \mathcal{B} \rightarrowtail \mathcal{C}$ between Waldhausen categories is called *biexact* if each $F(A, -)$ and $F(-, B)$ is exact, and the following condition is satisfied:

For every pair of cofibrations ($A \rightarrowtail A'$ in $\mathcal{A}$, $B \rightarrowtail B'$ in $\mathcal{B}$) the map

$$F(A', B) \cup_{F(A,B)} F(A, B') \rightarrowtail F(A', B')$$

must be a cofibration in $\mathcal{C}$.

Our next result requires some notation. Suppose that a category with cofibrations $\mathcal{C}$ has two notions of weak equivalence, a weak one v and a stronger one w. (Every map in v belongs to w.) We write $v\mathcal{C}$ and $w\mathcal{C}$ for the two Waldhausen categories $(\mathcal{C}, co, v)$ and $(\mathcal{C}, co, w)$. The identity on $\mathcal{C}$ is an exact functor $v\mathcal{C} \to w\mathcal{C}$.

Let $\mathcal{C}^w$ denote the full subcategory of all w-acyclic objects in $\mathcal{C}$, *i.e.*, those C for which $0 \rightarrowtail C$ is in $w(\mathcal{C})$; $\mathcal{C}^w$ is a Waldhausen subcategory (9.1.7) of $v\mathcal{C}$, *i.e.*, of the category $\mathcal{C}$ with the v-notion of weak equivalence.

We say that a Waldhausen category $\mathcal{B}$ is *saturated* if: whenever f, g are composable maps and fg is a weak equivalence, f is a weak equivalence iff g is.

Localization Theorem 9.6. *Suppose that $\mathcal{C}$ is a category with cofibrations, endowed with two notions ($v \subset w$) of weak equivalence, with w saturated, and that $\mathcal{C}^w$ is defined as above.*

Assume in addition that every map $f: C_1 \to C_2$ in $\mathcal{C}$ factors as the composition of a cofibration $C_1 \rightarrowtail C$ and an equivalence $C \xrightarrow{\sim} C_2$ in $v(\mathcal{C})$.

Then the exact inclusions $\mathcal{C}^w \to v\mathcal{C} \to w\mathcal{C}$ induce an exact sequence

$$K_0(\mathcal{C}^w) \to K_0(v\mathcal{C}) \to K_0(w\mathcal{C}) \to 0.$$

Proof. Our proof of this is similar to the proof of the Localization Theorem 6.4 for abelian categories. Clearly $K_0(v\mathcal{C})$ maps onto $K_0(w\mathcal{C})$ and $K_0(\mathcal{C}^w)$ maps to zero. Let L denote the cokernel of $K_0(\mathcal{C}^w) \to K_0(v\mathcal{C})$; we will prove the theorem by showing that $\lambda(C) = [C]$ induces a map $K_0(w\mathcal{C}) \to L$ inverse to the natural surjection $L \to K_0(w\mathcal{C})$. As $v\mathcal{C}$ and $w\mathcal{C}$ have the same notion of cofibration, it suffices to show that $[C_1] = [C_2]$ in L for every equivalence $f: C_1 \to C_2$ in $w\mathcal{C}$. Our hypothesis that f factors as $C_1 \rightarrowtail C \xrightarrow{\sim} C_2$ implies that in $K_0(v\mathcal{C})$ we have $[C_2] = [C] = [C_1] + [C/C_1]$. Since $w\mathcal{C}$ is saturated, it contains $C_1 \rightarrowtail C$. The following lemma implies that C/C_1 is in $\mathcal{C}^w$, so that $[C_2] = [C_1]$ in L. This is the relation we needed to have λ define a map $K_0(w\mathcal{C}) \to L$, proving the theorem.

Lemma 9.6.1. *If $B \overset{\sim}{\rightarrowtail} C$ is both a cofibration and a weak equivalence in a Waldhausen category, then $0 \rightarrowtail C/B$ is also a weak equivalence.*

Proof. Apply the Glueing Axiom (W3) to the diagram:

$$\begin{array}{ccccc} 0 & \leftarrow & B & = & B \\ \| & & \| \sim & & \downarrow\sim \\ 0 & \leftarrow & B & \rightarrowtail & C. \end{array}$$

Here is a simple application of the Localization Theorem. Let $(\mathcal{C}, co, v)$ be a Waldhausen category, and G an abelian group. Given a surjective homomorphism $\pi: K_0(\mathcal{C}) \to G$, we let $\mathcal{C}^\pi$ denote the Waldhausen subcategory of $\mathcal{C}$ consisting of all objects C such that $\pi([C]) = 0$.

Proposition 9.6.2. *Assume that every morphism in a Waldhausen category $\mathcal{C}$ factors as the composition of a cofibration and a weak equivalence. There is a short exact sequence*

$$0 \to K_0(\mathcal{C}^\pi) \to K_0(\mathcal{C}) \xrightarrow{\pi} G \to 0.$$

Proof. Define $w\mathcal{C}$ to be the family of all morphisms $A \to B$ in $\mathcal{C}$ with $\pi([A]) = \pi([B])$. This satisfies axiom (W3) because $[C \cup_A B] = [B] + [C] - [A]$, and the factorization hypothesis ensures that the Localization Theorem 9.6 applies to $v \subseteq w$. Since $\mathcal{C}^\pi$ is the category $\mathcal{C}^w$ of w-acyclic objects, this yields exactness at $K_0(\mathcal{C})$. Exactness at $K_0(\mathcal{C}^\pi)$ will follow from the Cofinality Theorem 9.4, provided we show that $\mathcal{C}^\pi$ is cofinal. Given an object C, factor the map $C \to 0$ as a cofibration $C \rightarrowtail C''$ followed by a weak equivalence $C'' \xrightarrow{\sim} 0$. If C' denotes C''/C, we compute in G that

$$\pi([C \amalg C']) = \pi([C]) + \pi([C']) = \pi([C] + [C']) = \pi([C'']) = 0.$$

Hence $C \amalg C'$ is in $\mathcal{C}^\pi$, and $\mathcal{C}^\pi$ is cofinal in $\mathcal{C}$.

Approximation Theorem 9.7. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between two Waldhausen categories. Suppose also that F satisfies the following conditions:*

(a) A morphism f in $\mathcal{A}$ is a weak equivalence if and only if $F(f)$ is a w.e. in $\mathcal{B}$.

(b) Given any map $b\colon F(A) \to B$ in $\mathcal{B}$, there is a cofibration $a\colon A \rightarrowtail A'$ in $\mathcal{A}$ and a weak equivalence $b'\colon F(A') \xrightarrow{\sim} B$ in $\mathcal{B}$ so that $b = b' \circ F(a)$.

(c) If b is a weak equivalence, we may choose a to be a weak equivalence in $\mathcal{A}$.

Then F induces an isomorphism $K_0\mathcal{A} \cong K_0\mathcal{B}$.

Proof. Applying (*b*) to $0 \rightarrowtail B$, we see that for every B in $\mathcal{B}$ there is a weak equivalence $F(A') \xrightarrow{\sim} B$. If $F(A) \xrightarrow{\sim} B$ is a weak equivalence, so is $A \xrightarrow{\sim} A'$ by (c). Therefore not only is $K_0\mathcal{A} \to K_0\mathcal{B}$ onto, but the set W of weak equivalence classes of objects of $\mathcal{A}$ is isomorphic to the set of *w.e.* classes of objects in $\mathcal{B}$.

Now $K_0\mathcal{B}$ is obtained from the free abelian group $\mathbb{Z}[W]$ on the set W by modding out by the relations $[C] = [B] + [C/B]$ corresponding to the cofibrations $B \rightarrowtail C$ in $\mathcal{B}$. Given $F(A) \xrightarrow{\sim} B$, hypothesis (*b*) yields $A \rightarrowtail A'$ in $\mathcal{A}$ and a weak equivalence $F(A') \xrightarrow{\sim} C$ in $\mathcal{B}$. Finally, the Glueing Axiom (W3) applied to

$$\begin{array}{ccccc} 0 & \leftarrow & F(A) & \rightarrowtail & F(A') \\ \| & & \downarrow\sim & & \downarrow\sim \\ 0 & \leftarrow & B & \rightarrowtail & C \end{array}$$

implies that the map $F(A'/A) \to C/B$ is a weak equivalence. Therefore the relation $[C] = [B] + [C/B]$ is equivalent to the relation $[A'] = [A] + [A'/A]$ in the free abelian group $\mathbb{Z}[W]$, and already holds in $K_0\mathcal{A}$. This yields $K_0\mathcal{A} \cong K_0\mathcal{B}$, as asserted.

Saturated categories 9.7.1. We say that a Waldhausen category $\mathcal{B}$ is *saturated* if: whenever f, g are composable maps and fg is a weak equivalence, f is a weak equivalence iff g is. If $\mathcal{B}$ is saturated, then condition (c) is redundant in the Approximation Theorem, because $F(a)$ is a weak equivalence by (b) and hence by (a) the map a is a *w.e.* in $\mathcal{A}$.

All the categories mentioned earlier in this section are saturated.

Example 9.7.2. Recall from Example 9.1.4 that the category $\mathcal{R}(*)$ of based CW complexes is a Waldhausen category. Let $\mathcal{R}_{hf}(*)$ denote the Waldhausen subcategory of all based CW-complexes weakly homotopic to a finite CW complex. The Approximation Theorem applies to the inclusion of $\mathcal{R}_f(*)$ into $\mathcal{R}_{hf}(*)$; this

may be seen by using the Whitehead Theorem and elementary obstruction theory. Hence
$$K_0\mathcal{R}_{hf}(*) \cong K_0\mathcal{R}_f(*) \cong \mathbb{Z}.$$

EXAMPLE 9.7.3. If $\mathcal{A}$ is an exact category, the Approximation Theorem applies to the inclusion split$\mathbf{Ch}^b \subset \mathbf{Ch}^b = \mathbf{Ch}^b(\mathcal{A})$ of Lemma 9.2.4, yielding a more elegant proof that $K_0(\text{split}\mathbf{Ch}^b) = K_0(\mathbf{Ch}^b)$. To see this, observe that any chain complex map $f: A \to B$ factors through the mapping cylinder complex $\text{cyl}(f)$ as the composite $A \rightarrowtail \text{cyl}(f) \xrightarrow{\sim} B$, and that split$\mathbf{Ch}^b$ is saturated (see 9.7.1).

EXAMPLE 9.7.4 (HOMOLOGICALLY BOUNDED COMPLEXES). Fix an abelian category $\mathcal{A}$, and consider the Waldhausen category $\mathbf{Ch}(\mathcal{A})$ of all chain complexes over $\mathcal{A}$, as in (9.2). We call a complex $C.$ *homologically bounded* if it is exact almost everywhere, *i.e.*, if only finitely many of the $H_i(C)$ are nonzero. Let $\mathbf{Ch}^{hb}(\mathcal{A})$ denote the Waldhausen subcategory of $\mathbf{Ch}(\mathcal{A})$ consisting of the homologically bounded complexes, and let $\mathbf{Ch}_-^{hb}(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$ denote the Waldhausen subcategory of all bounded above, homologically bounded chain complexes $0 \to C_n \to C_{n-1} \to \cdots$. These are all saturated biWaldhausen categories (see 9.1.6 and 9.7.1). We will prove that
$$K_0\mathbf{Ch}^{hb}(\mathcal{A}) \cong K_0\mathbf{Ch}_-^{hb}(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A}) \cong K_0(\mathcal{A}),$$
the first isomorphism being Theorem 9.2.2. From this and Proposition 6.6 it follows that if C is homologically bounded then
$$[C] = \sum(-1)^i[H_i(\mathcal{A})] \text{ in } K_0\mathcal{A}.$$

We first claim that the Approximation Theorem 9.7 applies to $\mathbf{Ch}^b \subset \mathbf{Ch}_-^{hb}$, yielding $K_0\mathbf{Ch}^b \cong K_0\mathbf{Ch}_-^{hb}$. If $C.$ is bounded above then each good truncation $\tau_{\ge n}C = (\cdots C_{n+1} \to Z_n \to 0)$ of C is a bounded subcomplex of C such that $H_i(\tau_{\ge n}C)$ is $H_i(C)$ for $i \ge n$, and 0 for $i < n$. (See [WHomo, 1.2.7].) Therefore $\tau_{\ge n}C \xrightarrow{\sim} C$ is a quasi-isomorphism for small n ($n \ll 0$). If B is a bounded complex, any map $f: B \to C$ factors through $\tau_{\ge n}C$ for small n; let A denote the mapping cylinder of $B \to \tau_{\ge n}C$ (see [WHomo, 1.5.8]). Then A is bounded and f factors as the cofibration $B \rightarrowtail A$ composed with the weak equivalence $A \xrightarrow{\sim} \tau_{\ge n}C \xrightarrow{\sim} C$. Thus we may apply the Approximation Theorem, as claimed.

The Approximation Theorem does not apply to $\mathbf{Ch}_-^{hb} \subset \mathbf{Ch}^{hb}$, but rather to $\mathbf{Ch}_+^{hb} \subset \mathbf{Ch}^{hb}$, where the "+" indicates bounded below chain complexes. The argument for this is the same as for $\mathbf{Ch}^b \subset \mathbf{Ch}_-^{hb}$. Since these are biWaldhausen categories, we can apply 9.1.6.1 to $\mathbf{Ch}_-^{hb}(\mathcal{A})^{op} = \mathbf{Ch}_+^{hb}(\mathcal{A}^{op})$ and $\mathbf{Ch}^{hb}(\mathcal{A})^{op} = \mathbf{Ch}^{hb}(\mathcal{A}^{op})$ to get
$$K_0\mathbf{Ch}_-^{hb}(\mathcal{A}) = K_0\mathbf{Ch}_+^{hb}(\mathcal{A}^{op}) \cong K_0\mathbf{Ch}^{hb}(\mathcal{A}^{op}) = K_0\mathbf{Ch}^{hb}(\mathcal{A}).$$

This completes our calculation that $K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^{hb}(\mathcal{A})$.

EXAMPLE 9.7.5 (K_0 AND PERFECT COMPLEXES). Let R be a ring. A chain complex $M.$ of R-modules is called *perfect* if there is a quasi-isomorphism $P. \xrightarrow{\sim} M.$, where $P.$ is a bounded complex of f.g. projective R-modules, *i.e.*, $P.$ is a complex in $\mathbf{Ch}^b(\mathbf{P}(R))$. The perfect complexes form a Waldhausen subcategory $\mathbf{Ch}_{\mathrm{perf}}(R)$ of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$. We claim that the Approximation Theorem applies to $\mathbf{Ch}^b(\mathbf{P}(R)) \subset \mathbf{Ch}_{\mathrm{perf}}(R)$, so that

$$K_0\mathbf{Ch}_{\mathrm{perf}}(R) \cong K_0\mathbf{Ch}^b\mathbf{P}(R) \cong K_0(R).$$

To see this, consider the intermediate Waldhausen category $\mathbf{Ch}^b_{\mathrm{perf}}$ of bounded perfect complexes. The argument of Example 9.7.4 applies to show that $K_0\mathbf{Ch}^b_{\mathrm{perf}} \cong K_0\mathbf{Ch}_{\mathrm{perf}}(R)$, so it suffices to show that the Approximation Theorem applies to $\mathbf{Ch}^b\mathbf{P}(R) \subset \mathbf{Ch}^b_{\mathrm{perf}}$. This is an elementary application of the projective lifting property, which we relegate to Exercise 9.2.

EXAMPLE 9.7.6 (G_0 AND PSEUDO-COHERENT COMPLEXES). Let R be a ring. A complex $M.$ of R-modules is called *pseudo-coherent* if there exists a quasi-isomorphism $P. \xrightarrow{\sim} M.$, where $P.$ is a bounded below complex $\cdots \to P_{n+1} \to P_n \to 0$ of f.g. projective R-modules, *i.e.*, $P.$ is a complex in $\mathbf{Ch}_+(\mathbf{P}(R))$. For example, if R is noetherian we can consider any finitely generated module M as a pseudo-coherent complex concentrated in degree zero. Even if R is not noetherian, it follows from Example 7.1.4 that M is pseudo-coherent as an R-module iff it is pseudo-coherent as a chain complex. (See [SGA6], I.2.9.)

The pseudo-coherent complexes form a Waldhausen subcategory $\mathbf{Ch}_{pcoh}(R)$ of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$, and the category $\mathbf{Ch}^{hb}_{pcoh}$ of homologically bounded pseudo-coherent complexes is also Waldhausen. Moreover, the above remarks show that $\mathbf{M}(R)$ is a Waldhausen subcategory of both of them. We will see in Ex. 9.7 that the Approximation Theorem applies to the inclusions $\mathbf{M}(R) \subset \mathbf{Ch}^{hb}_+\mathbf{P}(R) \subset \mathbf{Ch}^{hb}_{pcoh}$, so that in particular we have

$$K_0\mathbf{Ch}^{hb}_{pcoh} \cong G_0(R).$$

Chain complexes with support

Suppose that S is a multiplicatively closed set of central elements in a ring R. Let $\mathbf{Ch}^b_S\mathbf{P}(R)$ denote the Waldhausen subcategory of $\mathcal{C} = \mathbf{Ch}^b\mathbf{P}(R)$ consisting of complexes E such that $S^{-1}E$ is exact, and write $K_0(R \text{ on } S)$ for $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$.

The category $\mathbf{Ch}^b_S\mathbf{P}(R)$ is the category $\mathcal{C}^w$ of the Localization Theorem 9.6, where w is the family of all morphisms $P \to Q$ in $\mathcal{C}$ such that $S^{-1}P \to S^{-1}Q$ is a quasi-isomorphism. By Theorem 9.2.2 we have $K_0(\mathcal{C}) = K_0(R)$. Hence there is an exact sequence

$$K_0(R \text{ on } S) \to K_0(R) \to K_0(w\mathcal{C}) \to 0.$$

THEOREM 9.8. *The localization $w\mathcal{C} \to \mathbf{Ch}^b\mathbf{P}(S^{-1}R)$ induces an injection on K_0, so there is an exact sequence*

$$K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R).$$

PROOF. Let $\mathcal{B}$ denote the category of $S^{-1}R$-modules of the form $S^{-1}P$ for P in $\mathbf{P}(R)$. By Example 7.2.3 and Theorem 9.2.2, $K_0\mathbf{Ch}^b(\mathcal{B}) = K_0(\mathcal{B})$ is a subgroup of $K_0(S^{-1}R)$. Therefore the result follows from the following Proposition.

PROPOSITION 9.8.1. *The Approximation Theorem 9.7 applies to* $w\mathcal{C} \to \mathbf{Ch}^b(\mathcal{B})$.

PROOF. Let P be a complex in $\mathbf{Ch}^b\mathbf{P}(R)$ and $b\colon S^{-1}P \to B$ a map in $\mathcal{B}$. Because each B_n has the form $S^{-1}Q_n$ and each $B_n \to B_{n-1}$ is $s_n^{-1}d_n$ for some $s_n \in S$ and $d_n\colon Q_n \to Q_{n-1}$ such that $d_n d_{n-1} = 0$, B is isomorphic to the localization $S^{-1}Q$ of a bounded complex Q in $\mathbf{P}(R)$, and some sb is the localization of a map $f\colon P \to Q$ in $\mathbf{Ch}^b\mathbf{P}(R)$. Hence f factors as $P \rightarrowtail \mathrm{cyl}(f) \xrightarrow{\sim} Q$. Since b is the localization of f, followed by an isomorphism $S^{-1}Q \cong B$ in $\mathcal{B}$, it factors as desired.

EXERCISES

9.1 *Retracts of a space.* Fix a CW complex X and let $\mathcal{R}(X)$ be the category of CW complexes Y obtained from X by attaching cells, and having a retraction $Y \to X$. Let $\mathcal{R}_f(X)$ be the subcategory of those Y obtained by attaching only finitely many cells. Let $\mathcal{R}_{fd}(X)$ be the subcategory of those Y which are finitely dominated, *i.e.*, are retracts up to homotopy of spaces in $\mathcal{R}_f(X)$. Show that $K_0\mathcal{R}_f(X) \cong \mathbb{Z}$ and $K_0\mathcal{R}_{fd}(X) \cong K_0(\mathbb{Z}[\pi_1 X])$. *Hint*: The cellular chain complex of the universal covering space $\tilde{Y}$ is a chain complex of free $\mathbb{Z}[\pi_1 X]$-modules.

9.2 Let R be a ring. Use the projective lifting property to show that the Approximation Theorem applies to the inclusion $\mathbf{Ch}^b\mathbf{P}(R) \subset \mathbf{Ch}^b_{\mathrm{perf}}$ of Example 9.7.5. Conclude that $K_0(R) = K_0\mathbf{Ch}_{\mathrm{perf}}(R)$.

If S is a multiplicatively closed set of central elements of R, show that the Approximation Theorem also applies to the inclusion of $\mathbf{Ch}^b_S\mathbf{P}(R)$ in $\mathbf{Ch}_{\mathrm{perf},S}(R)$, and conclude that $K_0(R \text{ on } S) \cong K_0\mathbf{Ch}_{\mathrm{perf},S}(R)$.

9.3 Consider the category $\mathbf{Ch}^b = \mathbf{Ch}^b(\mathcal{A})$ of Theorem 9.2.2 as a Waldhausen category in which the weak equivalences are the isomorphisms, $\mathrm{iso}\,\mathbf{Ch}^b$, as in Example 9.1.3. Let $\mathbf{Ch}^b_{acyc}$ denote the subcategory of complexes whose differentials are all zero. Show that $\mathbf{Ch}^b_{acyc}$ is equivalent to the category $\oplus_{n\in\mathbb{Z}}\mathcal{A}$, and that the inclusion in $\mathbf{Ch}^b$ induces an isomorphism

$$K_0(\mathrm{iso}\,\mathbf{Ch}^b) \cong \bigoplus_{n\in\mathbb{Z}} K_0(\mathcal{A}).$$

9.4 *Higher Extension categories.* Consider the category $\mathcal{E}_n$ constructed in Example 9.3.2, whose objects are sequences of n cofibrations in a Waldhausen category $\mathcal{C}$. Show that $\mathcal{E}_n$ is a Waldhausen category, and that

$$K_0(\mathcal{E}_n) \cong \bigoplus_{i=1}^{n} K_0(\mathcal{C}).$$

9.5 ([SGA6, IV(1.6)]) Let $\mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{A}$, or more generally any exact subcategory of $\mathcal{A}$ closed under extensions and kernels of surjections. Let $\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A})$ denote the Waldhausen subcategory of $\mathbf{Ch}^b(\mathcal{A})$ of bounded complexes C with $H_i(C)$ in $\mathcal{B}$ for all i. Show that

$$K_0\mathcal{B} \cong K_0\mathbf{Ch}^b(\mathcal{B}) \cong K_0\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A}).$$

9.6 *Perfect injective complexes.* Let R be a ring and let $\mathbf{Ch}^+_{inj}(R)$ denote the Waldhausen subcategory of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$ consisting of perfect bounded below cochain complexes of injective R-modules $0 \to I^m \to I^{m+1} \cdots$. (Recall from Example 9.7.5 that $I^\cdot$ is called *perfect* if it is quasi-isomorphic to a bounded complex $P^\cdot$ of f.g. projective modules.) Show that

$$K_0\mathbf{Ch}^+_{inj}(R) \cong K_0(R).$$

9.7 *Pseudo-coherent complexes and $G_0(R)$.* Let R be a ring. Recall from Example 9.7.6 that $\mathbf{Ch}^{hb}_{pcoh}$ denotes the Waldhausen category of all homologically bounded pseudo-coherent chain complexes of R-modules. Show that:

(a) The category $\mathbf{M}(R)$ is a Waldhausen subcategory of $\mathbf{Ch}^{bh}_{pcoh}$.

(b) $K_0(\mathbf{Ch}_{pcoh}) = K_0\mathbf{Ch}_+\mathbf{P}(R) = 0$

(c) The Approximation Theorem applies to $\mathbf{M}(R) \subset \mathbf{Ch}^{hb}_+\mathbf{P}(R) \subset \mathbf{Ch}^{hb}_{pcoh}$, and therefore $G_0(R) \cong K_0\mathbf{Ch}^{hb}_+\mathbf{P}(R) \cong K_0(\mathbf{Ch}^{hb}_{pcoh})$.

9.8 *Pseudo-coherent complexes and G_0^{der}.* Let X be a scheme. A cochain complex $E^\cdot$ of $\mathcal{O}_X$-modules is called *strictly pseudo-coherent* if it is bounded above complex of vector bundles, and *pseudo-coherent* if it is locally quasi-isomorphic to a strictly pseudo-coherent complex, *i.e.*, if every point $x \in X$ has a neighborhood U, a strictly pseudo-coherent complex $P^\cdot$ on U and a quasi-isomorphism $P^\cdot \to E^\cdot|_U$. Let $\mathbf{Ch}^{hb}_{pcoh}(X)$ denote the Waldhausen category of all pseudo-coherent complexes $E^\cdot$ which are homologically bounded, and set $G_0^{der}(X) = K_0\mathbf{Ch}^{hb}_{pcoh}$; this is the definition used in [SGA6], Expose IV(2.2).

(a) If X is a noetherian scheme, show that every coherent $\mathcal{O}_X$-module is a pseudo-coherent complex concentrated in degree zero, so that we may consider $\mathbf{M}(X)$ as a Waldhausen subcategory of $\mathbf{Ch}^{hb}_{pcoh}(X)$. Then show that a complex $E^\cdot$ is pseudo-coherent iff is it homologically bounded and all the homology sheaves of $E^\cdot$ are coherent $\mathcal{O}_X$-modules.

(b) If X is a noetherian scheme, show that $G_0(X) \cong G_0^{der}(X)$.

(c) If $X = \mathrm{Spec}(R)$ for a ring R, show that $G_0^{der}(X)$ is isomorphic to the group $K_0\mathbf{Ch}^{hb}_{pcoh}(R)$ of the previous exercise.

9.10 *Perfect complexes and K_0^{der}.* Let X be a scheme. A complex $E^\cdot$ of $\mathcal{O}_X$-modules is called *strictly perfect* if it is a bounded complex of vector bundles, *i.e.*, a complex in $\mathbf{Ch}^b\mathbf{VB}(X)$. A complex is called *perfect* if it is locally quasi-isomorphic to a strictly perfect complex, *i.e.*, if every point $x \in X$ has a neighborhood U, a strictly perfect complex $P^\cdot$ on U and a quasi-isomorphic $P^\cdot \to E^\cdot|_U$. Write $\mathbf{Ch}_{\mathrm{perf}}(X)$ for the Waldhausen category of all perfect complexes, and $K_0^{der}(X)$ for $K_0\mathbf{Ch}_{\mathrm{perf}}(X)$; this is the definition used in [SGA6], Expose IV(2.2).

(a) If $X = \mathrm{Spec}(R)$, show that $K_0(R) \cong K_0^{der}(X)$. *Hint:* show that the Approximation Theorem 9.7 applies to $\mathbf{Ch}_{\mathrm{perf}}(R) \subset \mathbf{Ch}_{\mathrm{perf}}(X)$.

(b) If X is noetherian, show that the category $\mathcal{C} = \mathbf{Ch}^{qc}_{\mathrm{perf}}$ of perfect complexes of quasi-coherent $\mathcal{O}_X$-modules also has $K_0(\mathcal{C}) = K_0^{der}(X)$.

(c) If X is a regular noetherian scheme, show that a homologically bounded complex is perfect iff it is pseudo-coherent, and conclude that $K_0^{der}(X) \cong G_0(X)$.

(d) Let X be the affine plane with a double origin over a field k, obtained by glueing two copies of $\mathbb{A}^2 = Spec(k[x, y])$ together. X is a regular noetherian scheme. Show that $K_0(X) = \mathbb{Z}$ but $K_0^{der}(X) = \mathbb{Z} \oplus \mathbb{Z}$. *Hint.* Use the fact that $\mathbb{A}^2 \to X$ induces an isomorphism $\mathbf{VB}(X) \cong \mathbf{VB}(\mathbb{A}^2)$ and the identification of $K_0^{der}(X)$ with $G_0(X)$ from part (c).

9.11 Give an example of an exact subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ in which $K_0(\mathcal{C}) \neq K_0\mathbf{Ch}^b(\mathcal{C})$. Here $\mathbf{Ch}^b(\mathcal{C})$ is the Waldhausen category described before Definition 9.2. Note that $\mathcal{C}$ cannot be closed under kernels of surjections, by Theorem 9.2.2.

9.12 *Finitely dominated complexes.* Let $\mathcal{C}$ be a small exact category, closed under extensions and kernels of surjections in an ambient abelian category $\mathcal{A}$ (Definition 7.0.1). A bounded below complex $C.$ of objects in $\mathcal{C}$ is called *finitely dominated* if there is a bounded complex $B.$ and two maps $C. \to B. \to C.$ whose composite $C. \to C.$ is chain homotopic to the identity. Let $\mathbf{Ch}_+^{fd}(\mathcal{C})$ denote the category of finitely dominated chain complexes of objects in $\mathcal{C}$. (If $\mathcal{C}$ is abelian, this is the category $\mathbf{Ch}_+^{hb}(\mathcal{C})$ of Example 9.7.4.)

(a) Let e be an idempotent endomorphism of an object C, and let $\mathrm{tel}(e)$ denote the nonnegative complex

$$\cdots \xrightarrow{e} C \xrightarrow{1-e} C \xrightarrow{e} C \to 0.$$

Show that $\mathrm{tel}(e)$ is finitely dominated.

(b) Let $\hat{\mathcal{C}}$ denote the idempotent completion 7.2.1 of $\mathcal{C}$. Show that there is a map from $K_0(\hat{\mathcal{C}})$ to $K_0\mathbf{Ch}_+^{fd}(\mathcal{C})$ sending $[(C,e)]$ to $[\mathrm{tel}(e)]$.

(c) Show that the map in (b) induces an isomorphism $K_0(\hat{\mathcal{C}}) \cong K_0\mathbf{Ch}_+^{fd}(\mathcal{C})$.

9.13 Let S be a multiplicatively closed set of central nonzerodivisors in a ring R. Show that $K_0\mathbf{H}_S(R) \cong K_0(R \text{ on } S)$, and compare Cor. 7.6.4 to Theorem 9.8.

Appendix. Localizing by categories of fractions

If $\mathcal{C}$ is a category and S is a collection of morphisms in $\mathcal{C}$, then the *localization of $\mathcal{C}$ with respect to S* is a category $\mathcal{C}_S$, together with a functor loc: $\mathcal{C} \to \mathcal{C}_S$ such that

(1) For every $s \in S$, loc(s) is an isomorphism
(2) If $F: \mathcal{C} \to \mathcal{D}$ is any functor sending S to isomorphisms in D, then F factors uniquely through loc: $\mathcal{C} \to \mathcal{C}_S$.

Example. We may consider any ring R as an additive category $\mathcal{R}$ with one object. If S is a central multiplicative subset of R, there is a ring $S^{-1}R$ obtained by localizing R at S, and the corresponding category is $\mathcal{R}_S$. The useful fact that every element of the ring $S^{-1}R$ may be written in standard form $s^{-1}r = rs^{-1}$ generalizes to morphisms in a localization $\mathcal{C}_S$, provided that S is a "locally small multiplicative system" in the following sense.

Definition A.1. A collection S of morphisms in $\mathcal{C}$ is called a *multiplicative system* if it satisfies the following three self-dual axioms:

(FR1) S is closed under composition and contains the identity morphisms 1_X of all objects X of $\mathcal{C}$. That is, S forms a subcategory of $\mathcal{C}$ with the same objects.

(FR2) (Ore condition) If $t : Z \to Y$ is in S, then for every $g : X \to Y$ in $\mathcal{C}$ there is a commutative diagram in $\mathcal{C}$ with $s \in S$:

$$\begin{array}{ccc} W & \xrightarrow{f} & Z \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{g} & Y. \end{array}$$

(The slogan is "$t^{-1}g = fs^{-1}$ for some f and s.") Moreover, the symmetric statement (whose slogan is "$fs^{-1} = t^{-1}g$ for some t and g") is also valid.

(FR3) (Cancellation) If $f, g: X \to Y$ are parallel morphisms in $\mathcal{C}$, then the following two conditions are equivalent:
(a) $sf = sg$ for some $s: Y \to Z$ in S
(b) $ft = gt$ for some $t: W \to X$ in S.

Example A.1.1. If S is a multiplicatively closed subset of a ring R, then S forms a multiplicative system if and only if S is a "2–sided denominator set."

Example A.1.2 (Gabriel). Let $\mathcal{B}$ be a Serre subcategory (see §6) of an abelian category $\mathcal{A}$, and let S be the collection of all $\mathcal{B}$-*isos*, *i.e.*, those maps f such that $\ker(f)$ and $\operatorname{coker}(f)$ is in $\mathcal{B}$. Then S is a multiplicative system in $\mathcal{A}$; the verification of axioms (FR2), (FR3) is a pleasant exercise in diagram chasing. In this case, $\mathcal{A}_S$ is the quotient abelian category $\mathcal{A}/\mathcal{B}$ discussed in the Localization Theorem 6.4.

We would like to say that every morphism $X \to Z$ in $\mathcal{C}_S$ is of the form fs^{-1}. However, the issue of whether or this construction makes sense (in our universe) involves delicate set-theoretic questions. The following notion is designed to avoid these set-theoretic issues.

We say that S is *locally small* (on the left) if for each X in $\mathcal{C}$ there is a *set* S_X of morphisms $X' \xrightarrow{s} X$ in S such that every map $Y \to X$ in S factors as $Y \to X' \xrightarrow{s} X$ for some $s \in S_X$.

Definition A.2 (Fractions). A (left) *fraction* between X and Y is a chain in $\mathcal{C}$ of the form:

$$fs^{-1}: \quad X \xleftarrow{s} X_1 \xrightarrow{f} Y, \qquad s \in S.$$

Call fs^{-1} *equivalent* to $X \leftarrow X_2 \rightarrow Y$ just in case there is a chain $X \leftarrow X_3 \rightarrow Y$ fitting into a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} & X_1 \arrow[dl] \arrow[dr] & \\ X & X_3 \arrow[l] \arrow[u] \arrow[d] \arrow[r] & Y \\ & X_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

It is easy to see that this is an equivalence relation. Write $\mathrm{Hom}_S(X,Y)$ for the equivalence classes of such fractions between X and Y. ($\mathrm{Hom}_S(X,Y)$ is a set when S is locally small.)

We cite the following theorem without proof from [WHomo, 10.3.7], relegating its routine proof to Exercises A.1 and A.2.

Gabriel-Zisman Theorem A.3. *Let S be a locally small multiplicative system of morphisms in a category $\mathcal{C}$. Then the localization $\mathcal{C}_S$ of $\mathcal{C}$ exists, and may be constructed as follows.*

$\mathcal{C}_S$ has the same objects as $\mathcal{C}$, but $\mathrm{Hom}_{\mathcal{C}_S}(X,Y)$ *is the set of equivalence classes of chains $X \leftarrow X' \rightarrow Y$ with $X' \rightarrow X$ in S, and composition is given by the Ore condition. The functor loc: $\mathcal{C} \rightarrow \mathcal{C}_S$ sends $X \rightarrow Y$ to the chain $X \xleftarrow{=} X \rightarrow Y$, and if $s: X \rightarrow Y$ is in S its inverse is represented by $Y \leftarrow X \xrightarrow{=} X$.*

Corollary A.3.1. *Two parallel arrows $f, g: X \rightarrow Y$ become identified in $\mathcal{C}_S$ iff the conditions of (FR3) hold.*

Corollary A.3.2. *Suppose that $\mathcal{C}$ has a zero object, and that S is a multiplicative system in $\mathcal{C}$. Assume that S is saturated in the sense that if s and st are in S then so is t. Then for every X in $\mathcal{C}$:*

$$loc(X) \cong 0 \Leftrightarrow \textit{The zero map } \; X \xrightarrow{0} X \textit{ is in } \; S.$$

Proof. Since $\mathrm{loc}(0)$ is a zero object in $\mathcal{C}_S, \mathrm{loc}(X) \cong 0$ iff the parallel maps $0, 1: X \rightarrow X$ become identified in $\mathcal{C}_S$.

Now let $\mathcal{A}$ be an abelian category, and $\mathbf{C}$ a full subcategory of the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$, closed under translation and the formation of mapping cones. Let $\mathbf{K}$ be the quotient category of $\mathbf{C}$, obtained by identifying chain homotopic maps in $\mathbf{C}$. Let Q denote the family of (chain homotopy equivalence classes of) quasi-isomorphisms in $\mathbf{C}$. The following result states that Q forms a multiplicative system in $\mathbf{K}$, so that we can form the localization $\mathbf{K}_Q$ of $\mathbf{K}$ with respect to Q by the calculus of fractions.

LEMMA A.4. *The family Q of quasi-isomorphisms in the chain homotopy category* $\mathbf{K}$ *forms a multiplicative system.*

PROOF. (FR1) is trivial. To prove (FR2), consider a diagram $X \xrightarrow{u} Y \xleftarrow{s} Z$ with $s \in Q$. Set $C = \mathrm{cone}(s)$, and observe that C is acyclic. If $f: Y \to C$ is the natural map, set $W = \mathrm{cone}(fu)$, so that the natural map $t: W \to X[-1]$ is a quasi-isomorphism. Now the natural projections from each $W_n = Z_{n-1} \oplus Y_n \oplus X_{n-1}$ to Z_{n-1} form a morphism $v: W \to Z$ of chain complexes making the following diagram commute:

$$\begin{tikzcd} & X \arrow[r, "fu"] \arrow[d, "u"'] & C \arrow[r] \arrow[d, equal] & W \arrow[r, "t"] \arrow[d, "v"] & X[-1] \arrow[d] \\ Z \arrow[r, "s"] & Y \arrow[r, "f"] & C \arrow[r] & Z[-1] \arrow[r, "{s[-1]}"] & Y[-1]. \end{tikzcd}$$

Applying $X \mapsto X[1]$ to the right square gives the first part of (FR2); the second part is dual and is proven similarly.

To prove (FR3), we suppose given a quasi-isomorphism $s: Y \to Y'$ and set $C = \mathrm{cone}(s)$; from the long exact sequence in homology we see that C is acyclic. Moreover, if v denotes the map $C[1] \to Y$ then there is an exact sequence:

$$\mathrm{Hom}_{\mathbf{K}}(X, C[1]) \xrightarrow{v} \mathrm{Hom}_{\mathbf{K}}(X, Y) \xrightarrow{s} \mathrm{Hom}_{\mathbf{K}}(X, Y')$$

(see [WHomo, 10.2.8]). Given f and g, set $h = f - g$. If $sh = 0$ in $\mathbf{K}$, there is a map $w: X \to C[1]$ such that $h = vw$. Setting $X' = \mathrm{cone}(w)[1]$, the natural map $X' \xrightarrow{t} X$ must be a quasi-isomorphism because C is acyclic. Moreover, $wt = 0$, so we have $ht = vwt = 0$, *i.e.*, $ft = gt$.

DEFINITION A.5. Let $\mathbf{C} \subset \mathbf{Ch}(\mathcal{A})$ be a full subcategory closed under translation and the formation of mapping cones. The *derived category* of $\mathbf{C}$, $\mathbf{D}(\mathbf{C})$, is defined to be the localization $\mathbf{K}_Q$ of the chain homotopy category $\mathbf{K}$ at the multiplicative system Q of quasi-isomorphisms. The *derived category* of $\mathcal{A}$ is $\mathbf{D}(\mathcal{A}) = \mathbf{D}(\mathbf{Ch}(\mathcal{A}))$.

Another application of calculus of fractions is Verdier's formation of quotient triangulated categories by thick subcategories. We will use Rickard's definition of thickness, which is equivalent to Verdier's.

DEFINITION A.6. Let $\mathbf{K}$ be any triangulated category (see [WHomo, 10.2.1]). A full additive subcategory $\mathcal{E}$ of $\mathbf{K}$ is called *thick* if:

(1) In any distinguished triangle $A \to B \to C \to A[1]$, if two out of A, B, C are in $\mathcal{E}$ then so is the third.
(2) if $A \oplus B$ is in $\mathcal{E}$ then both A and B are in $\mathcal{E}$.

If $\mathcal{E}$ is a thick subcategory of $\mathbf{K}$, we can form a quotient triangulated category $\mathbf{K}/\mathcal{E}$, parallel to Gabriel's construction of a quotient abelian category in A.1.2. That is, $\mathbf{K}/\mathcal{E}$ is defined to be $S^{-1}\mathbf{K}$, where S is the family of maps whose cone is in $\mathcal{E}$. By Ex. A.6, S is a saturated multiplicative system of morphisms, so $S^{-1}\mathbf{K}$ can be constructed by the calculus of fractions (theorem A.3).

To justify this definition, note that because S is saturated it follows from A.3.2 and A.6(2) that: (a) $X \cong 0$ in $\mathbf{K}/\mathcal{E}$ if and only if X is in $\mathcal{E}$, and (b) a morphism $f : X \to Y$ in $\mathbf{K}$ becomes an isomorphism in $\mathbf{K}/\mathcal{E}$ if and only if f is in S.

EXERCISES

A.1 Show that the construction of the Gabriel-Zisman Theorem A.3 makes $\mathcal{C}_S$ into a category by showing that composition is well-defined and associative.

A.2 If $F: \mathcal{C} \to \mathcal{D}$ is a functor sending S to isomorphisms, show that F factors uniquely through the Gabriel-Zisman category $\mathcal{C}_S$ of the previous exercise as $\mathcal{C} \to \mathcal{C}_S \to \mathcal{D}$. This proves the Gabriel-Zisman Theorem A.3, that $\mathcal{C}_S$ is indeed the localization of $\mathcal{C}$ with respect to S.

A.3 Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$, and let S be a multiplicative system in $\mathcal{C}$ such that $S \cap \mathcal{B}$ is a multiplicative system in $\mathcal{B}$. Assume furthermore that one of the following two conditions holds:

(a) Whenever $s: C \to B$ is in S with B in $\mathcal{B}$, there is a morphism $f: B' \to C$ with B' in $\mathcal{B}$ such that $sf \in S$

(b) Condition (a) with the arrows reversed, for $s: B \to C$.

Show that the natural functor $\mathcal{B}_S \to \mathcal{C}_S$ is fully faithful, so that $\mathcal{B}_S$ can be identified with a full subcategory of $\mathcal{C}_S$.

A.4 Let $F: \mathcal{A} \to \mathcal{A}'$ be an exact functor between two abelian categories, and let S be the family of morphisms s in $\mathbf{Ch}(\mathcal{A})$ such that $F(s)$ is a quasi-isomorphism. Show that S is a multiplicative system in $\mathbf{Ch}\mathcal{A}$.

A.5 Suppose that $\mathbf{C}$ is a subcategory of $\mathbf{Ch}(\mathcal{A})$ closed under translation and the formation of mapping cones, and let Σ be the family of all chain homotopy equivalences in $\mathbf{C}$. Show that the localization $\mathbf{C}_\Sigma$ is the quotient category $\mathbf{K}$ of $\mathbf{C}$ described before Lemma A.4. Conclude that the derived category $\mathbf{D}(\mathbf{C})$ is the localization of $\mathbf{C}$ at the family of all quasi-isomorphisms. *Hint:* If two maps $f_1, f_2: X \to Y$ are chain homotopic then they factor through a common map $f: \mathrm{cyl}(X) \to Y$ out of the mapping cylinder of X.

A.6 Let $\mathcal{E}$ be a thick subcategory of a triangulated category $\mathbf{K}$, and S the morphisms whose cone is in $\mathcal{E}$, as in A.6. Show that S is a multiplicative system of morphisms. Then show that S is saturated in the sense of A.3.2.

CHAPTER III

K_1 AND K_2 OF A RING

Let R be an associative ring with unit. In this chapter, we introduce the classical definitions of the groups $K_1(R)$ and $K_2(R)$. These definitions use only linear algebra and elementary group theory, as applied to the groups $GL(R)$ and $E(R)$. We also define relative groups for K_1 and K_2, as well as the negative K-groups $K_{-n}(R)$ and the Milnor K-groups $K_n^M(R)$.

In the next chapter we will give another definition: $K_n(R) = \pi_n K(R)$ for all $n \geq 0$, where $K(R)$ is a certain topological space built using the category $\mathbf{P}(R)$ of f.g. projective R-modules. We will then have to prove that these topologically defined groups agree with the definition of $K_0(R)$ in chapter II, as well as with the classical constructions of $K_1(R)$ and $K_2(R)$ in this chapter.

§1. The Whitehead Group K_1 of a ring

Let R be an associative ring with unit. Identifying each $n \times n$ matrix g with the larger matrix $\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ gives an embedding of $GL_n(R)$ into $GL_{n+1}(R)$. The union of the resulting sequence

$$GL_1(R) \subset GL_2(R) \subset \cdots \subset GL_n(R) \subset GL_{n+1}(R) \subset \cdots$$

is called the *infinite general linear group* $GL(R)$.

Recall that the commutator subgroup $[G, G]$ of a group G is the subgroup generated by its commutators $[g, h] = ghg^{-1}h^{-1}$. It is always a normal subgroup of G, and has a universal property: the quotient $G/[G, G]$ is an abelian group, and every homomorphism from G to an abelian group factors through $G/[G, G]$.

DEFINITION 1.1. $K_1(R)$ is the abelian group $GL(R)/[GL(R), GL(R)]$.

The universal property of $K_1(R)$ is this: every homomorphism from $GL(R)$ to an abelian group must factor through the natural quotient $GL(R) \to K_1(R)$. Depending upon our situation, we will sometimes think of $K_1(R)$ as an additive group, and sometimes as a multiplicative group.

A ring map $R \to S$ induces a natural map from $GL(R)$ to $GL(S)$, and hence from $K_1(R)$ to $K_1(S)$. That is, K_1 is a functor from rings to abelian groups.

EXAMPLE 1.1.1 (SK_1). If R happens to be commutative, the determinant of a matrix provides a group homomorphism from $GL(R)$ onto the group $R^\times$ of units of R. It is traditional to write $SK_1(R)$ for the kernel of the induced surjection $\det\colon K_1(R) \to R^\times$. The *special linear group* $SL_n(R)$ is the subgroup of $GL_n(R)$ consisting of matrices with determinant 1, and $SL(R)$ is their union. Since the natural inclusion of the units $R^\times$ in $GL(R)$ as $GL_1(R)$ is split by the homomorphism det: $GL(R) \to R^\times$, we see that $GL(R)$ is the semidirect product $SL(R) \rtimes R^\times$, and there is a direct sum decomposition: $K_1(R) = R^\times \oplus SK_1(R)$.

EXAMPLE 1.1.2. If F is a field, then $K_1(F) = F^\times$. We will see this below (see Lemma 1.2.2 and 1.3.1 below), but it is more fun to deduce this from an 1899 theorem of L.E.J. Dickson, that $SL_n(F)$ is the commutator subgroup of both $GL_n(F)$ and $SL_n(F)$, with only two exceptions: $GL_2(\mathbb{F}_2) = SL_2(\mathbb{F}_2) \cong S_3$, which has order 6, and $GL_2(\mathbb{F}_3) = \mathbb{F}_3^\times \times SL_2(\mathbb{F}_3)$, in which $SL_2(\mathbb{F}_3)$ has order 24.

EXAMPLE 1.1.3. If R is the product $R' \times R''$ of two rings, then $K_1(R) = K_1(R') \oplus K_1(R'')$. Indeed, $GL(R)$ is the product $GL(R') \times GL(R'')$, and the commutator subgroup decomposes accordingly.

We will show that the commutator subgroup of $GL(R)$ is the subgroup $E(R)$ generated by "elementary" matrices. These are defined as follows.

DEFINITION 1.2. If $i \neq j$ are distinct positive integers and $r \in R$ then the *elementary matrix* $e_{ij}(r)$ is the matrix in $GL(R)$ which has 1 in every diagonal spot, has r in the (i,j)-spot, and is zero elsewhere.

$E_n(R)$ denotes the subgroup of $GL_n(R)$ generated by all elementary matrices $e_{ij}(r)$ with $1 \leq i, j \leq n$, and the union $E(R)$ of the $E_n(R)$ is the subgroup of $GL(R)$ generated by all elementary matrices.

EXAMPLE 1.2.1. A *signed permutation matrix* is one which permutes the standard basis $\{e_i\}$ up to sign, *i.e.*, it permutes the set $\{\pm e_1, \dots, \pm e_n\}$. The following signed permutation matrix belongs to $E_2(R)$:

$$\bar{w}_{12} = e_{12}(1)e_{21}(-1)e_{12}(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By changing the subscripts, we see that the signed permutation matrices $\bar{w}_{ij}$ belong to $E_n(R)$ for $n \geq i, j$. Since the products $\bar{w}_{jk}\bar{w}_{ij}$ correspond to cyclic permutations of 3 basis elements, every matrix corresponding to an even permutation of basis elements belongs to $E_n(R)$. Moreover, if $g \in GL_n(R)$ then we see by Ex. I.1.11 that $E_{2n}(R)$ contains the matrix $\left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right)$.

1.2.2. If we interpret matrices as linear operators on column vectors, then $e_{ij}(r)$ is the elementary row operation of adding r times row j to row i, and $E_n(R)$ is the subset of all matrices in $GL_n(R)$ which may be reduced to the identity matrix using only these row operations. The quotient set $GL_n(R)/E_n(R)$ measures the obstruction to such a reduction.

If F is a field this obstruction is $F^\times$, and is measured the determinant. That is, $E_n(F) = SL_n(F)$ for all $n \geq 1$. Indeed, standard linear algebra shows that every matrix of determinant 1 is a product of elementary matrices.

REMARK 1.2.3 (SURJECTIONS). If I is an ideal of R, each homomorphism $E_n(R) \to E_n(R/I)$ is onto, because the generators $e_{ij}(r)$ of $E_n(R)$ map onto the generators $e_{ij}(\bar{r})$ of $E_n(R/I)$. In contrast, the maps $GL_n(R) \to GL_n(R/I)$ are usually not onto unless I is a radical ideal (Ex. I.12(iv)). Indeed, the obstruction is measured by the group $K_0(I) = K_0(R, I)$; see Proposition 2.3 below.

Division rings 1.2.4. The same linear algebra that we invoked for fields shows that if D is a division ring (a "skew field") then every invertible matrix may be reduced to a diagonal matrix $\operatorname{diag}(r, 1, ..., 1)$, and that $E_n(D)$ is a normal subgroup of $GL_n(D)$. Thus each $GL_n(D)/E_n(D)$ is a quotient group of the nonabelian group $D^\times$. Dieudonné proved in 1943 that in fact $GL_n(D)/E_n(D) = D^\times/[D^\times, D^\times]$ for all n. A proof of this result is sketched in Exercise 1.1 below.

If D is a finite-dimensional algebra over its center F (which must be a field), then the action of D upon its underlying vector space induces an inclusion $D \subset M_n(F)$, where $n = \dim_F(D)$. The composite map $D^\times \subset GL_n(F) \xrightarrow{\det} F^\times$ is called the *reduced norm* N_{red}. We define $SK_1(D)$ to be the kernel of the induced map

$$N_{red}: K_1(D) \to K_1 M_n(F) \cong K_1(F) = F^\times.$$

In 1950 S. Wang showed that $SK_1(D) = 1$ if F is a number field, or if the *Schur index* $i = \sqrt{n}$ of D is squarefree. In 1976 V. Platanov produced the first examples of a D with $SK_1(D) \neq 1$, by constructing a map from $SK_1(D)$ to a subquotient of the Brauer group $Br(F)$.

Remark 1.2.5. There is no a priori reason to believe that the subgroups $E_n(R)$ are normal, except in special cases. For example, we shall show in Ex. 1.2 that if R has stable range $d+1$ then $E_n(R)$ is a normal subgroup of $GL_n(R)$ for all $n \geq d+2$. Vaserstein proved [V69] that $K_1(R) = GL_n(R)/E_n(R)$ for all $n \geq d+2$.

If R is commutative, we can do better: $E_n(R)$ is a normal subgroup of $GL_n(R)$ for all $n \geq 3$. This theorem was proven by A. Suslin in [S77]; we give Suslin's proof in Ex. 1.9. Suslin also gave examples of Dedekind domains for which $E_2(R)$ is not normal in $GL_2(R)$ in [S81]. For noncommutative rings, the $E_n(R)$ are only known to be normal for large n, and only then when the ring R has finite stable range in the sense of Ex. I.1.5; see Ex. 1.2 below.

Commutators 1.3. Here are some easy-to-check formulas for multiplying elementary matrices. Fixing the indices, we have $e_{ij}(r)e_{ij}(s) = e_{ij}(r+s)$, and $e_{ij}(-r)$ is the inverse of $e_{ij}(r)$. The commutator of two elementary matrices is easy to compute and simple to describe (unless $j = k$ and $i = \ell$):

$$(1.3.1) \qquad [e_{ij}(r), e_{k\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell \\ e_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell \\ e_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases}$$

Recall that a group is called *perfect* if $G = [G, G]$. If a subgroup H of G is perfect, then $H \subseteq [G, G]$. The group $E(R)$ is perfect, as are most of its finite versions:

Lemma 1.3.2. *If $n \geq 3$ then $E_n(R)$ is a perfect group.*

Proof. If i, j, k are distinct then $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$.

We know from Example 1.1.2 that $E_2(R)$ is not always perfect; in fact $E_2(\mathbb{F}_2)$ and $E_2(\mathbb{F}_3)$ are solvable groups.

Rather than become enmeshed in technical issues, it is useful to "stabilize" by increasing the size of the matrices we consider. One technical benefit of stability is given in Ex. 1.3. The following stability result was proven by J.H.C. Whitehead in the 1950 paper [Wh], and in some sense is the origin of K-theory.

WHITEHEAD'S LEMMA 1.3.3. *$E(R)$ is the commutator subgroup of $GL(R)$. Hence $K_1(R) = GL(R)/E(R)$.*

PROOF. The commutator subgroup contains $E(R)$ by Lemma 1.3.2. Conversely, every commutator in $GL_n(R)$ can be expressed as a product in $GL_{2n}(R)$:

$$[g,h] = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{pmatrix}. \tag{1.3.4}$$

But we saw in Example 1.2.1 that each of these terms is in $E_{2n}(R)$.

EXAMPLE 1.3.5. If F is a field then $K_1(F) = F^\times$, because we have already seen that $E(R) = SL(R)$. Similarly, if R is a Euclidean domain such as $\mathbb{Z}$ or $F[t]$ then it is easy to show that $SK_1(R) = 0$ and hence $K_1(R) = R^\times$; see Ex. 1.4. In particular, $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{\pm 1\}$ and $K_1(F[t]) = F^\times$.

To get a feeling for the non-commutative situation, suppose that D is a division ring. Diedonné's calculation of $GL_n(D)/E_n(D)$ (described in 1.2.4 and Ex. 1.1) gives an isomorphism $K_1(D) \cong D^\times/[D^\times, D^\times]$.

EXAMPLE 1.3.6. If F is a finite field extension of $\mathbb{Q}$ (a number field) and R is an integrally closed subring of F, then Bass, Milnor and Serre proved in [BMS, 4.3] that $SK_1(R) = 0$, so that $K_1(R) \cong R^\times$. We mention that, by the Dirichlet Unit Theorem, $K_1(R) = R^\times$ is a finitely generated abelian group isomorphic to $\mu(F) \oplus \mathbb{Z}^{s-1}$, where $\mu(F)$ denotes the cyclic group of all roots of unity in F and s is the number of "places at infinity" for R.

LEMMA 1.4. *If R is a semilocal ring then the natural inclusion of $R^\times = GL_1(R)$ into $GL(R)$ induces an isomorphism $K_1(R) \cong R^\times/[R^\times, R^\times]$.*

If R is a commutative semilocal ring, then

$$SK_1(R) = 0 \quad \textit{and} \quad K_1(R) = R^\times.$$

PROOF. By Example 1.1.1 (and Ex. 1.1 in the noncommutative case), it suffices to prove that $R^\times$ maps onto $K_1(R)$. This will follow by induction on n once we show that $GL_n(R) = E_n(R)GL_{n-1}(R)$. Let J denote the Jacobson radical of R, so that R/J is a finite product of division rings. By examples 1.1.3 and 1.2.4, $(R/J)^\times$ maps onto $K_1(R/J)$. That is, every $\bar{g} \in GL_n(R/J)$ is a product $\bar{e}\bar{g}_1$, where $\bar{e} \in E_n(R/J)$ and $\bar{g}_1 \in GL_1(R/J)$.

Given $g \in GL_n(R)$, its reduction $\bar{g}$ in $GL_n(R/J)$ may be decomposed as above: $\bar{g} = \bar{e}\bar{g}_1$. By Remark 1.2.3, we can lift $\bar{e}$ to an element $e \in E_n(R)$. The matrix $e^{-1}g$ is congruent to the diagonal matrix $\bar{g}_1$ modulo J, so its diagonal entries are all units and its off-diagonal entries lie in J. Using elementary row operations $e_{ij}(r)$ with $r \in J$, it is an easy matter to reduce $e^{-1}g$ to a diagonal matrix, say to $D = \mathrm{diag}(r_1, ..., r_n)$. By Ex. I.1.11, the matrix $\mathrm{diag}(1, ..., 1, r_n, r_n^{-1})$ is in $E_n(R)$. Multiplying D by this matrix yields a matrix in $GL_{n-1}(R)$, finishing the induction and the proof.

Commutative Banach Algebras

Let R be a commutative Banach algebra over the real or complex numbers. For example, R could be the ring $\mathbb{R}^X$ of continuous real-valued functions of a compact space X. As subspaces of the metric space of $n \times n$ matrices over R, the groups $SL_n(R)$ and $GL_n(R)$ are topological groups.

PROPOSITION 1.5. *$E_n(R)$ is the path component of the identity matrix in the special linear group $SL_n(R)$, $n \geq 2$. Hence we may identify the group $SK_1(R)$ with the group $\pi_0 SL(R)$ of path components of the topological space $SL(R)$.*

PROOF. To see that $E_n(R)$ is path-connected, fix an element $g = \prod e_{i_\alpha j_\alpha}(r_\alpha)$. The formula $t \mapsto \prod e_{i_\alpha j_\alpha}(r_\alpha t)$, $0 \leq t \leq 1$ defines a path in $E_n(R)$ from the identity to g. To prove that $E_n(R)$ is open subset of $SL_n(R)$ (and hence a path-component), it suffices to prove that $E_n(R)$ contains U_{n-1}, the set of matrices $1+(r_{ij})$ in $SL_n(R)$ with $||r_{ij}|| < \frac{1}{n-1}$ for all i, j. We will actually show that each matrix in U_{n-1} can be expressed naturally as a product of $n^2 + 5n - 6$ elementary matrices, each of which depends continuously upon the entries $r_{ij} \in R$.

Set $u = 1 + r_{11}$. Since $\frac{n-2}{n-1} < ||u||$, u has an inverse v with $||v|| < \frac{n-1}{n-2}$. Subtracting vr_{1j} times the first column from the j^{th} we obtain a matrix $1 + r'_{ij}$ whose first row is $(u, 0, ..., 0)$ and

$$||r'_{ij}|| < \frac{1}{n-1} + \frac{n-1}{n-2}\left(\frac{1}{n-1}\right)^2 = \frac{1}{n-2}.$$

We can continue to clear out entries in this way so that after $n(n-1)$ elementary operations we have reduced the matrix to diagonal form.

By Ex. I.1.10, any diagonal matrix $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$ is the product of 6 elementary matrices. By induction, it follows that any diagonal $n \times n$ matrix of determinant 1 can be written naturally as a product of $6(n-1)$ elementary matrices.

Let V denote the path component of 1 in the topological group $R^\times$, *i.e.*, the kernel of $R^\times \to \pi_0 R^\times$. By Ex. 1.8, V is a quotient of the additive group R.

COROLLARY 1.5.1. *If R is a commutative Banach algebra, there is a natural surjection from $K_1(R)$ onto $\pi_0 GL(R) = \pi_0(R^\times) \times \pi_0 SL(R)$. The kernel of this map is the divisible subgroup V of $R^\times$.*

EXAMPLE 1.5.2. If $R = \mathbb{R}$ then $K_1(\mathbb{R}) = \mathbb{R}^\times$ maps onto $\pi_0 GL(\mathbb{R}) = \{\pm 1\}$, and the kernel is the uniquely divisible multiplicative group $V = (0, \infty)$. If $R = \mathbb{C}$ then $V = \mathbb{C}^\times$, because $K_1(\mathbb{C}) = \mathbb{C}^\times$ but $\pi_0 GL(\mathbb{C}) = 0$.

EXAMPLE 1.5.3. Let X be a compact space with a nondegenerate basepoint. Then $SK_1(\mathbb{R}^X)$ is the group $\pi_0 SL(\mathbb{R}^X) = [X, SL(\mathbb{R})] = [X, SO]$ of homotopy classes of maps from X to the infinite special orthogonal group SO. By Ex. II.3.11 we have $\pi_0 GL(\mathbb{R}^X) = [X, O] = KO^{-1}(X)$, and there is a short exact sequence

$$0 \to \mathbb{R}^X \xrightarrow{\exp} K_1(\mathbb{R}^X) \to KO^{-1}(X) \to 0.$$

Similarly, $SK_1(\mathbb{C}^X)$ is the group $\pi_0 SL(\mathbb{C}^X) = [X, SL(\mathbb{C})] = [X, SU]$ of homotopy classes of maps from X to the infinite special unitary group SU. Since $\pi_0 GL(\mathbb{C}^X) = [X, U] = KU^{-1}(X)$ by II.3.5.1 and Ex. II.3.11, there is a natural surjection from $K_1(\mathbb{C}^X)$ onto $KU^{-1}(X)$, and the kernel V is the divisible group of all contractible maps $X \to \mathbb{C}^\times$.

EXAMPLE 1.5.4. When X is the circle S^1 we have $SK_1(\mathbb{R}^{S^1}) = [S^1, SO] = \pi_1 SO = \mathbb{Z}/2$. On the other hand, we have $\pi_0 SL_2(\mathbb{R}^{S^1}) = \pi_1 SL_2(\mathbb{R}) = \pi_1 SO_2 = \mathbb{Z}$, generated by the matrix $A = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$. Since $\pi_1 SO_2(\mathbb{R}) \to \pi_1 SO$ is onto, the matrix A represents the nonzero element of $SK_1(\mathbb{R}^{S^1})$.

The ring $R = \mathbb{R}[x,y]/(x^2+y^2-1)$ may be embedded in the ring $\mathbb{R}^{S^1}$ by $x \mapsto \cos(\theta), y \mapsto \sin(\theta)$. Since the matrix $\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$ maps to A, it represents a nontrivial element of $SK_1(R)$. In fact it is not difficult to show that $SK_1(R) \cong \mathbb{Z}/2$ using Mennicke symbols (Ex. 1.10).

K_1 and projective modules

Now let P be a f.g. projective R-module. Choosing an isomorphism $P \oplus Q \cong R^n$ gives a group homomorphism from $\mathrm{Aut}(P)$ to $GL_n(R)$. (Send α to $\alpha \oplus 1_Q$.)

LEMMA 1.6. *The homomorphism from* $\mathrm{Aut}(P)$ *to* $GL(R) = \bigcup GL_n(R)$ *is well-defined up to inner automorphism of* $GL(R)$. *Hence there is a well-defined homomorphism* $\mathrm{Aut}(P) \to K_1(R)$.

PROOF. First suppose that Q and n are fixed. Two different isomorphisms between $P \oplus Q$ and R^n must differ by an automorphism of R^n, *i.e.*, by an element $g \in GL_n(R)$. Thus if $\alpha \in \mathrm{Aut}(P)$ maps to the matrices A and B, respectively, we must have $A = gBg^{-1}$. Next we observe that there is no harm in stabilizing, *i.e.*, replacing Q by $Q \oplus R^m$ and $P \oplus Q \cong R^n$ by $P \oplus (Q \oplus R^m) \cong R^{n+m}$. This is because $GL_n(R) \to GL(R)$ factors through $GL_{n+m}(R)$. Finally, suppose given a second isomorphism $P \oplus Q' \cong R^m$. Since $Q \oplus R^m \cong R^n \oplus Q'$, we may stabilize both Q and Q' to make them isomorphic, and invoke the above argument.

COROLLARY 1.6.1. *If R and S are rings, there is a natural external product operation* $K_0(R) \otimes K_1(S) \to K_1(R \otimes S)$.

If R is commutative and S is an R-algebra, there is a natural product operation $K_0(R) \otimes K_1(S) \to K_1(S)$, *making* $K_1(S)$ *into a module over the ring* $K_0(R)$.

PROOF. For each f.g. projective R-module P and each m, Lemma 1.6 provides a homomorphism $\mathrm{Aut}(P \otimes S^m) \to K_1(R \otimes S)$. For each $\beta \in GL_m(S)$, let $[P] \cdot \beta$ denote the image of the automorphism $1_P \otimes \beta$ of $P \otimes S^m$ under this map. Fixing β and m, the isomorphism $(P \oplus P') \otimes S^m \cong (P \otimes S^m) \oplus (P' \otimes S^m)$ yields the identity $[P \oplus P'] \cdot \beta = [P] \cdot \beta + [P'] \cdot \beta$ in $K_1(R \otimes S)$. Hence $P \mapsto [P] \cdot \beta$ is an additive function of $P \in \mathbf{P}(R)$, which means that it factors through $K_0(R)$. Now fix P and vary β; the resulting map from $GL_m(S)$ to $K_1(R \otimes S)$ is compatible with stabilization in m. Thus the map $\beta \mapsto [P] \cdot \beta$ factors through a map $GL(S) \to K_1(R \otimes S)$, and through a map $K_1(S) \to K_1(R \otimes S)$. This shows that the product is well-defined and bilinear.

When R is commutative, $K_0(R)$ is a ring by II, §2. If S is an R-algebra, there is a ring map $R \otimes S \to S$. Composing the external product with $K_1(R \otimes S) \to K_1(S)$ yields a natural product operation $K_0(R) \otimes K_1(S) \to K_1(S)$. The verification that $[P \otimes Q] \cdot \beta = [P] \cdot ([Q] \cdot \beta$ is routine.

Here is a homological interpretation of $K_1(R)$. Recall that the first homology $H_1(G; \mathbb{Z})$ of any group G is naturally isomorphic to $G/[G,G]$. (See [WHomo, 6.1.11] for a proof.) By Whitehead's Lemma, we have

$$K_1(R) = H_1(GL(R);\mathbb{Z}) = \lim_{n\to\infty} H_1(GL_n(R);\mathbb{Z}). \tag{1.6.2}$$

By Lemma 1.6, we also have well-defined compositions

$$H_1(\mathrm{Aut}(P);\mathbb{Z}) \to H_1(GL_n(R);\mathbb{Z}) \to K_1(R),$$

which are independent of the choice of isomorphism $P \oplus Q \cong R^n$.

Here is another description of $K_1(R)$ in terms of the category $\mathbf{P}(R)$ of f.g. projective R-modules. Consider the translation category $t\mathbf{P}$ of $\mathbf{P}(R)$: its objects are isomorphism classes of f.g. projective modules, and the morphisms between P and P' are the isomorphism classes of Q such that $P \oplus Q \cong P'$. This is a filtering category [WHomo, 2.6.13], and $P \mapsto H_1(\mathrm{Aut}(P);\mathbb{Z})$ is a well-defined functor from $t\mathbf{P}$ to abelian groups. Hence we can take the filtered direct limit of this functor. Since the free modules are cofinal in $t\mathbf{P}$, we see from (1.6.2) that we have

Corollary 1.6.3 (Bass). $K_1(R) \cong \varinjlim_{P\in t\mathbf{P}} H_1(\mathrm{Aut}(P);\mathbb{Z})$.

Recall from II.2.7 that if two rings R and S are *Morita equivalent* then the categories $\mathbf{P}(R)$ and $\mathbf{P}(S)$ are equivalent. By Corollary 1.6.3 we have the following:

Proposition 1.6.4 (Morita invariance of K_1). *The group $K_1(R)$ depends only upon the category $\mathbf{P}(R)$. That is, if R and S are Morita equivalent rings then $K_1(R) \cong K_1(S)$. In particular, the maps $R \to M_n(R)$ induce isomorphisms on K_1:*

$$K_1(R) \cong K_1(M_n(R)).$$

Transfer maps

Let $f\colon R \to S$ be a ring homomorphism. We will see later on that a *transfer homomorphism* $f_*\colon K_1(S) \to K_1(R)$ is defined whenever S has a finite R-module resolution by f.g. projective R-modules. This construction requires a definition of K_1 for an exact category such as $\mathbf{H}(R)$, and is analogous to the transfer map in II(7.8.1) for K_0. Without this machinery, we can still construct the transfer map when S is f.g. projective as an R-module, using the analogue of the method used for the K_0 transfer map in example II.2.8.1.

Lemma 1.7. *Suppose that S is f.g. projective as an R-module. Then there is a natural transfer homomorphism $f_*\colon K_1(S) \to K_1(R)$.*

If R is commutative, the composite

$$K_1(R) \xrightarrow{f^*} K_1(S) \xrightarrow{f_*} K_1(R)$$

is multiplication by $[S] \in K_0(R)$.

Proof. If P is a f.g. projective S-module, then we can also consider P to be a f.g. projective R-module, and $\mathrm{Aut}_S(P) \subset \mathrm{Aut}_R(P)$. Applying H_1 gives a natural transformation of functors defined on the translation category $t\mathbf{P}(S)$ of all f.g. projective S-modules. By Corollary 1.6.3, $K_1(S) = \varinjlim_{P\in\mathbf{P}(S)} H_1(\mathrm{Aut}_S(P);\mathbb{Z})$.

The map $t\mathbf{P}(S) \to t\mathbf{P}(R)$ induces a map

$$K_1(S) \to \varinjlim_{P\in\mathbf{P}(S)} H_1(\mathrm{Aut}_R(P);\mathbb{Z}) \to \varinjlim_{Q\in\mathbf{P}(R)} H_1(\mathrm{Aut}_R(Q);\mathbb{Z}) = K_1(R).$$

The composite $K_1(S) \to K_1(R)$ is the transfer map f_*.

To compute the composite f_*f^*, we compute its effect upon an element $\alpha \in GL_n(R)$. Taking $P = S^n$, we can identify the R-module S^n with $S \otimes_R R^n$. Thus the matrix $f^*(\alpha) = 1\otimes\alpha$ lies in $GL_n(S) = \mathrm{Aut}_S(S\otimes_R R^n)$. To apply f_* we consider $1 \otimes \alpha$ as an element of the group $\mathrm{Aut}_R(S^n) = \mathrm{Aut}_R(S \otimes_R R^n)$, which we then map into $GL(R)$. But this is just the product $[S] \cdot \alpha$ of 1.6.1.

EXAMPLE 1.8 (WHITEHEAD GROUP Wh_1). If R is the group ring $\mathbb{Z}G$ of a group G, the (first) Whitehead group $Wh_1(G)$ is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup generated by ± 1 and the elements of G, considered as elements of GL_1. If G is abelian, then $\mathbb{Z}G$ is a commutative ring and $\pm G$ is a subgroup of $K_1(\mathbb{Z}G)$, so by 1.3.4 we have $Wh_1(G) = (\mathbb{Z}G)^\times/\pm G \oplus SK_1(\mathbb{Z}G)$. If G is finite then $Wh_1(G)$ is a finitely generated group whose rank is $r - q$, where r and q are the number of simple factors in $\mathbb{R}G$ and $\mathbb{Q}G$, respectively. This and other calculations related to $Wh_1(G)$ may be found in R. Oliver's excellent sourcebook [Oliver].

The group $Wh_1(G)$ arose in Whitehead's 1950 study [Wh] of simple homotopy types. Two finite CW complexes have the same simple homotopy type if they are connected by a finite sequence of "elementary expansions and collapses." Given a homotopy equivalence $f\colon K \to L$ of complexes with fundamental group G, the *torsion* of f is an element $\tau(f) \in Wh_1(G)$. Whitehead proved that $\tau(f) = 0$ iff f is a simple homotopy equivalence, and that every element of $Wh_1(G)$ is the torsion of some f. An excellent source for the geometry behind this is [Cohen].

Here is another area of geometric topology in which Whitehead torsion has played a crucial role, piecewise-linear ("PL") topology. We say that a triple (W, M, M') of compact PL manifolds is an *h-cobordism* if the boundary of W is the disjoint union of M and M', and both inclusions $M \subset W$, $M' \subset W$ are simple homotopy equivalences. In this case we can define the torsion τ of $M \subset W$, as an element of $Wh_1(G)$, $G = \pi_1 M$. The *s-cobordism theorem* states that if M is fixed with $\dim(M) \geq 5$ then $(W, M, M') \cong (M \times [0, 1], M \times 0, M \times 1)$ iff $\tau = 0$. Moreover, every element of $Wh_1(G)$ arises as the torsion of some h-cobordism (W, M, M').

To apply the s-cobordism, suppose given an h-cobordism (W, M, M'), and let N be the union of W, the cone on M and the cone on M'. Then N is PL homeomorphic to the suspension SM of M iff $(W, M, M') \cong (M \times [0, 1], M \times 0, M \times 1)$ iff $\tau = 0$.

This gives a counterexample to the "Hauptvermutung" that two homeomorphic complexes would be PL homeomorphic. Indeed, if (W, M, M') is an h-cobordism with nonzero torsion, then N and SM cannot be PL homeomorphic, yet the theory of "engulfing" implies that they must be homeomorphic manifolds.

Another application, due to Smale, is the Generalized Poincaré Conjecture. Let N be an n-dimensional PL manifold of the homotopy type of the sphere S^n, $n \geq 5$. Then N is PL homeomorphic to S^n. To see this, let W be obtained by removing two small disjoint n-discs D_1, D_2 from N. The boundary of these discs is the boundary of W, and (W, S^{n-1}, S^{n-1}) is an h-cobordism. Its torsion must be zero since $\pi_1(S^{n-1}) = 0$ and $Wh_1(0) = 0$. Hence W is $S^{n-1} \times [0,1]$, and this implies that $N = W \cup D_1 \cup D_2$ is S^n.

EXERCISES

1.1 *Semilocal rings.* Let R be a noncommutative semilocal ring (Ex. II.2.6). Show that there exists a unique "determinant" map from $GL_n(R)$ onto the abelian group $R^\times/[R^\times, R^\times]$ with the following properties: (i) $\det(e) = 1$ for every elementary matrix e, and (ii) If $\rho = \mathrm{diag}(r, 1, ..., 1)$ and $g \in GL_n(R)$ then $\det(\rho \cdot g) = r \cdot \det(g)$. Then show that det is a group homomorphism: $\det(gh) = \det(g)\det(h)$. Conclude that $K_1(R) \cong R^\times/[R^\times, R^\times]$.

1.2 Suppose that a ring R has stable range $sr(R) = d+1$ in the sense of Ex. I.1.5. (For example, R could be a d-dimensional commutative noetherian ring.) This condition describes the action of $E_{d+2}(R)$ on unimodular rows in R^{d+2}.

(a) Show that $GL_n(R) = GL_{d+1}(R)E_n(R)$ for all $n > d+1$, and deduce that $GL_{d+1}(R)$ maps onto $K_1(R)$.
(b) Show that $E_n(R)$ is a normal subgroup of $GL_n(R)$ for all $n \geq d+2$. *Hint:* Conjugate $e_{nj}(r)$ by $g \in GL_{d+2}(R)$.

1.3 Let R be the polynomial ring $F[x, y]$ over a field F. Show that the matrix $g = \begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix}$ is in $E_3(R) \cap GL_2(R)$. P.M. Cohn proved in 1966 that g is not in $E_2(R)$.

1.4 Let R be a Euclidean domain, such as $\mathbb{Z}$ or the polynomial ring $F[t]$ over a field. Show that $E_n(R) = SL_n(R)$ for all n, and hence that $SK_1(R) = 0$.

1.5 Here is another interpretation of the group law for K_1. For each m, n, let $\oplus_{mn}$ denote the group homomorphism $GL_m(R) \times GL_n(R) \to GL_{m+n}(R)$ sending (α, β) to the block diagonal matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$. Show that in $K_1(R)$ we have $[\alpha \oplus_{mn} \beta] = [\alpha][\beta]$.

1.6 Let $E = \mathrm{End}_R(R^\infty)$ be the ring of infinite row-finite matrices over R of Ex. I.1.7. Show that $K_1(E) = 0$. *Hint:* If $\alpha \in GL_n(E)$, form the block diagonal matrix $\alpha^\infty = \mathrm{diag}(\alpha, \alpha, \dots)$ in $GL(E)$, and show that $\alpha \oplus \alpha^\infty$ is conjugate to α^∞.

1.7 In this exercise we show that the center of $E(R)$ is trivial. First show that any matrix in $GL_n(R)$ commuting with $E_n(R)$ must be a diagonal matrix $\mathrm{diag}(r, ..., r)$ with r in the center of R. Conclude that no element in $E_{n-1}(R)$ is in the center of $E_n(R)$, and pass to the limit as $n \to \infty$.

1.8 If R is a commutative Banach algebra, let $\exp(R)$ denote the image of the exponential map $R \to R^\times$. Show that $\exp(R)$ is the path component of 1 in $R^\times$.

1.9 In this exercise we suppose that R is a commutative ring, and give Suslin's proof that $E_n(R)$ is a normal subgroup of $GL_n(R)$ when $n \geq 3$. Let $v = \sum_{i=1}^n v_i e_i$ be a column vector, and let u, w be row vectors such that $u \cdot v = 1$ and $w \cdot v = 0$.

(a) Show that $w = \sum_{i<j} r_{ij}(v_j e_i - v_i e_j)$, where $r_{ij} = w_i u_j - w_j u_i$.
(b) Conclude that the matrix $I_n + (v \cdot w)$ is in $E_n(R)$ if $n \geq 3$.
(c) If $g \in GL_n(R)$ and $i < j$, let v be the i^{th} column of g and w the j^{th} row of g^{-1}, so that $w \cdot v = 0$. Show that $g e_{ij}(r) g^{-1} = I_n + (v \cdot rw)$ for all $r \in R$. By (b), this proves that $E_n(R)$ is normal.

1.10 *Mennicke symbols.* Let (r,s) be a unimodular row over a commutative ring R. Choosing $t,u\in R$ such that $ru-st=1$, we define the *Mennicke symbol* $\left[\begin{smallmatrix}s\\r\end{smallmatrix}\right]$ to be the class in $SK_1(R)$ of the matrix $\left(\begin{smallmatrix}r&s\\t&u\end{smallmatrix}\right)$. Show that this Mennicke symbol is independent of the choice of t and u, that $\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right]=\left[\begin{smallmatrix}s\\r\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}s\\r\end{smallmatrix}\right]\left[\begin{smallmatrix}s'\\r\end{smallmatrix}\right]=\left[\begin{smallmatrix}ss'\\r\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}s\\r\end{smallmatrix}\right]=\left[\begin{smallmatrix}s+xr\\r\end{smallmatrix}\right]$.

If R is noetherian of dimension 1, or more generally has $sr(R)\le 2$, then we know by Ex. 1.2 that $GL_2(R)$ maps onto $K_1(R)$, and hence $SK_1(R)$ is generated by Mennicke symbols.

1.11 *Transfer.* Suppose that R is a Dedekind domain and $\mathfrak{p}$ is a prime ideal of R. Show that there is a map π_* from $K_1(R/\mathfrak{p})=(R/\mathfrak{p})^\times$ to $SK_1(R)$ sending $\bar{s}\in(R/\mathfrak{p})^\times$ to the Mennicke symbol $\left[\begin{smallmatrix}s\\r\end{smallmatrix}\right]$, where $s\in R$ maps to $\bar{s}$ and $r\in R$ is an element of $\mathfrak{p}-\mathfrak{p}^2$ relatively prime to s. Another construction of the transfer map π_* will be given in chapter V.

1.12 If H is a normal subgroup of a group G, then G acts upon H and hence its homology $H_*(H;\mathbb{Z})$ by conjugation. Since H always acts trivially upon its homology [WHomo, 6.7.8]), the group G/H acts upon $H_*(H;\mathbb{Z})$. Taking $H=E(R)$ and $G=GL(R)$, use Example 1.2.1 to show that $GL(R)$ and $K_1(R)$ act trivially upon the homology of $E(R)$.

§2. Relative K_1

Let I be an ideal in a ring R. We write $GL(I)$ for the kernel of the natural map $GL(R)\to GL(R/I)$; the notation reflects the fact that $GL(I)$ is independent of R (see Ex. I.1.10). In addition, we define $E(R,I)$ to be the smallest normal subgroup of $E(R)$ containing the elementary matrices $e_{ij}(x)$ with $x\in I$. More generally, for each n we define $E_n(R,I)$ to be the normal subgroup of $E_n(R)$ generated by the matrices $e_{ij}(x)$ with $x\in I$ and $1\le i\ne j\le n$. Clearly $E(R,I)$ is the union of the subgroups $E_n(R,I)$.

Relative Whitehead Lemma 2.1. *$E(R,I)$ is a normal subgroup of $GL(I)$, and contains the commutator subgroup of $GL(I)$.*

Proof. For any matrix $g=1+\alpha\in GL(I)$, the identity

$$\begin{pmatrix}g&0\\0&g^{-1}\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\\alpha&1\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&g^{-1}\alpha\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-g\alpha&1\end{pmatrix}.$$

shows that the matrix $\begin{pmatrix}g&0\\0&g^{-1}\end{pmatrix}$ is in $E_{2n}(R,I)$. Hence if $h\in E_n(R,I)$ then the conjugate

$$\begin{pmatrix}ghg^{-1}&0\\0&1\end{pmatrix}=\begin{pmatrix}g&0\\0&g^{-1}\end{pmatrix}\begin{pmatrix}h&0\\0&1\end{pmatrix}\begin{pmatrix}g^{-1}&0\\0&g\end{pmatrix}$$

is in $E(R,I)$. Finally, if $g,h\in GL_n(I)$ then $[g,h]$ is in $E_{2n}(R,I)$ by equation (1.3.4).

Definition 2.2. The relative group $K_1(R,I)$ is defined to be the quotient $GL(I)/E(R,I)$. By the Relative Whitehead Lemma, it is an abelian group.

The inclusion of $GL(I)$ in $GL(R)$ induces a map $K_1(R,I) \to K_1(R)$. More generally, if $R \to S$ is a ring map sending I into an ideal J of S, the natural maps $GL(I) \to GL(J)$ and $E(R) \to E(S)$ induce a map $K_1(R,I) \to K_1(S,J)$.

REMARK 2.2.1. Suppose that $R \to S$ is a ring map sending an ideal I of R isomorphically onto an ideal of S. The induced map from $K_1(R,I)$ to $K_1(S,I)$ must be a surjection, as both groups are quotients of $GL(I)$ (Ex. I.1.10ii). However, Swan discovered that they need not be isomorphic. One of his simple examples is given in Ex. 2.3 below.

PROPOSITION 2.3. *There is an exact sequence*

$$K_1(R,I) \to K_1(R) \to K_1(R/I) \xrightarrow{\partial} K_0(I) \to K_0(R) \to K_0(R/I).$$

PROOF. By Ex. II.2.3 there is an exact sequence

$$1 \to GL(I) \to GL(R) \to GL(R/I) \xrightarrow{\partial} K_0(I) \to K_0(R) \to K_0(R/I).$$

Since the K_1 groups are quotients of the GL groups, and $E(R)$ maps onto $E(R/I)$, this gives exactness except at $K_1(R)$. Suppose $g \in GL(R)$ maps to zero under $GL(R) \to K_1(R) \to K_1(R/I)$. Then the reduction $\bar{g}$ of g mod I is in $E(R/I)$. Since $E(R)$ maps onto $E(R/I)$, there is a matrix $e \in E(R)$ mapping to $\bar{g}$, *i.e.*, ge^{-1} is in the kernel $GL(I)$ of $GL(R) \to GL(R/I)$. Hence the class of ge^{-1} in $K_1(R,I)$ is defined, and maps to the class of g in $K_1(R)$. This proves exactness at the remaining spot.

The relative group $SK_1(R,I)$

If R happens to be commutative, the determinant map $K_1(R) \to R^\times$ of Example 1.1.1 induces a relative determinant map det: $K_1(R,I) \to GL_1(I)$, since the determinant of a matrix in $GL(I)$ is congruent to 1 modulo I. It is traditional to write $SK_1(R,I)$ for the kernel of det, so the canonical map $GL_1(I) \to K_1(R,I)$ induces a direct sum decomposition $K_1(R,I) = GL_1(I) \oplus SK_1(R,I)$ compatible with the decomposition $K_1(R) = R^\times \oplus SK_1(R)$ of Example 1.1.1. Here are two important cases in which $SK_1(R,I)$ vanishes:

LEMMA 2.4. *Let I be a radical ideal in R. Then:*
1. *$K_1(R,I)$ is a quotient of the multiplicative group $1+I = GL_1(I)$.*
2. *If R is a commutative ring, then $SK_1(R,I) = 0$ and $K_1(R,I) = 1+I$.*

PROOF. As in the proof of Lemma 1.4, it suffices to show that $GL_n(I) = E_n(R,I)GL_{n-1}(I)$ for $n \geq 2$. If (x_{ij}) is a matrix in $GL_n(I)$ then x_{nn} is a unit of R, and for $i < n$ the enties x_{in}, x_{ni} are in I. Multiplying by the diagonal matrix $\mathrm{diag}(1,\dots,1,x_{nn},x_{nn}^{-1})$, we may assume that $x_{nn} = 1$. Now multiplying on the left by the matrices $e_{in}(-x_{in})$ and on the right by $e_{ni}(-x_{ni})$ reduces the matrix to one in $GL_{n-1}(I)$.

The next theorem extends the calculation mentioned in Example 1.3.6 above. We cite it from [BMS, 4.3], mentioning only that its proof involves calculations with Mennicke symbols (see Ex. 1.10 and 2.5).

BASS-MILNOR-SERRE THEOREM 2.5. *Let R be an integrally closed subring of a number field F, and I an ideal of R. Then*

(1) *If F has any embedding into $\mathbb{R}$ then $SK_1(R,I)=0$.*
(2) *If F is "totally imaginary" (has no embedding into $\mathbb{R}$), then $SK_1(R,I)\cong C_n$ is a finite cyclic group whose order n divides the order m of the group of roots of unity in R. The exponent $\operatorname{ord}_p n$ of p in the integer n is the minimum over all prime ideals $\mathfrak{p}$ of R containing I of the integer*

$$\inf\left\{\operatorname{ord}_p m, \sup\{0, \left[\frac{\operatorname{ord}_{\mathfrak{p}}(I)}{\operatorname{ord}_{\mathfrak{p}}(p)} - \frac{1}{p-1}\right]\right\}$$

The Mayer-Vietoris Exact Sequence

Suppose we are given a ring map $f\colon R\to S$ and an ideal I of R mapped isomorphically into an ideal of S. Then we have a Milnor square of rings, as in I.2:

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar f} & S/I \end{array}$$

THEOREM 2.6 (MAYER-VIETORIS). *Given a Milnor square as above, there is an exact sequence*

$$K_1(R)\xrightarrow{\Delta} K_1(S)\oplus K_1(R/I)\to K_1(S/I)\xrightarrow{\partial} K_0(R)\xrightarrow{\Delta} K_0(S)\oplus K_0(R/I)\xrightarrow{\pm} K_0(S/I).$$

PROOF. By Theorem II.2.9 we have an exact sequence

$$GL(S/I)\xrightarrow{\partial} K_0(R)\xrightarrow{\Delta} K_0(S)\oplus K_0(R/I)\xrightarrow{\pm} K_0(S/I).$$

Since $K_0(R)$ is abelian, we may replace $GL(S/I)$ by $K_1(S/I)$ in this sequence. This gives the sequence of the Theorem, and exactness at all the K_0 places. Also by II.2.9, the image of $\partial\colon K_1(S/I)\to K_0(R)$ is the double coset space

$$GL(S)\backslash GL(S/I)/GL(R/I).$$

Therefore the kernel of ∂ is the subgroup of $K_1(S/I)$ generated by the images of $GL(S)$ and $GL(R/I)$, and the sequence is exact at $K_1(S/I)$. To prove exactness at the final spot, suppose given $\bar g\in GL_n(R/I)$, $h\in GL_n(S)$ and an elementary matrix $\bar e\in E(S/I)$ such that $\bar f(\bar g)\bar e\equiv h \pmod I$. Lifting $\bar e$ to an $e\in E_n(S)$ (by Remark 1.2.3) yields $\bar f(\bar g)\equiv he^{-1} \pmod I$. Since R is the pullback of S and R/I, there is a $g\in GL_n(R)$, equivalent to $\bar g$ modulo I, such that $f(g)=he^{-1}$. This establishes exactness at the final spot.

EXERCISES

2.1 Suppose we are given a Milnor square in which R and S are commutative rings. Using the Units-Pic sequence (I.3.10), conclude that there are exact sequences

$$SK_1(R,I) \to SK_1(R) \to SK_1(R/I) \xrightarrow{\partial} SK_0(I) \to SK_0(R) \to SK_0(R/I),$$

$$SK_1(R) \to SK_1(S) \oplus SK_1(R/I) \xrightarrow{\partial} SK_0(R) \to SK_0(S) \oplus SK_0(R/I) \to SK_0(S/I).$$

2.2 *Rim Squares.* Let C_p be a cyclic group of prime order p with generator t, and let $\zeta = e^{2\pi i/p}$. The ring $\mathbb{Z}[\zeta]$ is the integral closure of $\mathbb{Z}$ in the number field $\mathbb{Q}(\zeta)$. Let $f\colon \mathbb{Z}C_p \to \mathbb{Z}[\zeta]$ be the ring surjection sending t to ζ, and let I denote the kernel of the augmentation $\mathbb{Z}C_p \to \mathbb{Z}$.

(a) Show that I is isomorphic to the ideal of $\mathbb{Z}[\zeta]$ generated by $\zeta - 1$, so that we have a Milnor square with the rings $\mathbb{Z}C_p$, $\mathbb{Z}[\zeta]$, $\mathbb{Z}$ and $\mathbb{F}_p$.

(b) Show that for each $k = 1, ..., p-1$ the "cyclotomic" element $(\zeta^k - 1)/(\zeta - 1) = 1 + \cdots + \zeta^{k-1}$ is a unit of $\mathbb{Z}[\zeta]$, mapping onto $k \in \mathbb{F}_p^\times$. If $p \geq 3$, the Dirichlet Unit Theorem says that the units of $\mathbb{Z}[\zeta]$ split as the direct sum of the finite group $\{\pm\zeta^k\}$ of order $2p$ ($p \neq 2$) and a free abelian group of rank $(p-3)/2$.

(c) Conclude that if $p > 3$ then both $K_1(\mathbb{Z}C_p)$ and $Wh_1(C_p)$ are nonzero. In fact, $SK_1(\mathbb{Z}C_p) = 0$.

2.3 *Failure of Excision for K_1.* Here is Swan's simple example to show that $K_1(R,I)$ depends upon R. Let F be a field and let R be the algebra of all upper triangular matrices $r = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ in $M_2(F)$. Let I be the ideal of all such matrices with $x = z = 0$, and let R_0 be the commutative subalgebra $F \oplus I$. Show that $K_1(R_0, I) \cong F$ but that $K_1(R,I) = 0$. *Hint:* Calculate $e_{21}(r)e_{12}(y)e_{21}(-r)$.

2.4 (Vaserstein) If I is an ideal of R, and $x \in I$ and $r \in R$ are such that $(1 + rx)$ is a unit, show that $(1 - xr)$ is also a unit, and that $(1 + rx)(1 + xr)^{-1}$ is in $E(R,I)$. *Hint:* Start by calculating $e_{21}(-x)e_{12}(-r)(1 + rx)e_{21}(x)e_{12}(r)$, and then use the Relative Whitehead Lemma.

2.5 *Mennicke symbols.* If I is an ideal of a commutative ring R, $r \in (1 + I)$ and $s \in I$, we define the *Mennicke symbol* $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$ to be the class in $SK_1(R,I)$ of the matrix $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$, where $t \in I$ and $u \in (1 + I)$ satisfy $ru - st = 1$. Show that this Mennicke symbol is independent of the choice of t and u, with $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\left[\begin{smallmatrix} s \\ r' \end{smallmatrix}\right] = \left[\begin{smallmatrix} s \\ rr' \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\left[\begin{smallmatrix} s' \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} ss' \\ r \end{smallmatrix}\right]$. (*Hint:* Use Ex. 1.10.) Finally, show that if $t \in I$ then

$$\begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} s + rt \\ r \end{bmatrix} = \begin{bmatrix} s \\ r + st \end{bmatrix}.$$

2.6 *The obstruction to excision.* Let $R \to S$ be a map of commutative rings, sending an ideal I of R isomorphically onto an ideal of S. Given $x \in I$ and $s \in S$, let $\psi(x,s)$ denote the Mennicke symbol $\left[\begin{smallmatrix} x \\ 1-sx \end{smallmatrix}\right]$ in $SK_1(R,I)$.

(a) Verify that $\psi(x,s)$ vanishes in $SK_1(S,I)$.

(b) Prove that ψ is bilinear, and that $\psi(x,s) = 1$ if either $x \in I^2$ or $s \in R$. Thus ψ induces a map from $(I/I^2) \otimes (S/R)$ to $SK_1(R,I)$.

(c) Prove that the Leibniz rule holds: $\psi(x, ss') = \psi(sx, s')\psi(s'x, s)$.

For every map $R \to S$, the S-module $\Omega_{S/R}$ of *relative Kähler differentials* is presented with generators ds, $s \in S$, subject to the following relations: $d(s + s') = ds + ds'$, $d(ss') = s\,ds' + s'\,ds$, and if $r \in R$ then $dr = 0$. (See [WHomo].)

(d) (Vorst) Show that $\Omega_{S/R} \otimes_S I/I^2$ is the quotient of $(S/R) \otimes (I/I^2)$ by the subgroup generated by the elements $s \otimes s'x + s' \otimes sx - ss' \otimes x$. Then conclude that ψ induces a map $\Omega_{S/R} \otimes_S I/I^2 \to SK_1(R, I)$.

Swan proved in [Swan71] that the resulting sequence is exact:

$$\Omega_{S/R} \otimes_S I/I^2 \xrightarrow{\psi} SK_1(R, I) \to SK_1(S, I) \to 1.$$

2.7 Suppose that the ring map $R \to R/I$ is split by a map $R/I \to R$. Show that $K_1(R) \cong K_1(R/I) \oplus K_1(R, I)$. The corresponding decomposition of $K_0(R)$ follows from the ideal sequence 2.3, or from the definition of $K_0(I)$, since $R \cong R/I \oplus I$; see Ex. II.2.4.

§3. The Fundamental Theorems for K_1 and K_0

The Fundamental Theorem for K_1 is a calculation of $K_1(R[t, t^{-1}])$, and describes one of the many relationships between K_1 and K_0. The core of this calculation depends upon the construction of an exact sequence (see 3.2 below and II.7.7.1):

$$K_1(R[t]) \to K_1(R[t, t^{-1}]) \xrightarrow{\partial} K_0\mathbf{H}_{\{t^n\}}(R[t]) \to 0$$

We will construct a localization sequence connecting K_1 and K_0 in somewhat greater generality first. Recall from chapter II, Theorem 9.8 that for any multiplicatively closed set S of central elements in a ring R there is an exact sequence $K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R)$, where $K_0(R \text{ on } S)$ denotes K_0 of the Waldhausen category $\mathbf{Ch}^b_S\mathbf{P}(R)$. If S consists of nonzerodivisors, $K_0(R \text{ on } S)$ also equals $K_0\mathbf{H}_S(R)$ by Ex. II.9.13; see Corollary II.7.6.4.

Our first goal is to extend this sequence to the left using K_1, and we begin by constructing the boundary map ∂.

Let α be an endomorphism of R^n. We say that α is an *S-isomorphism* if $S^{-1}\ker(\alpha) = S^{-1}\mathrm{coker}(\alpha) = 0$, or equivalently, $\alpha/1 \in GL_n(S^{-1}R)$. Write $\mathrm{cone}(\alpha)$ for the *mapping cone* of α, which is the chain complex $R^n \xrightarrow{-\alpha} R^n$ concentrated in degrees 0 and 1; see [WHomo, 1.5.1]. It is clear that α is an S-isomorphism if and only if $\mathrm{cone}(\alpha) \in \mathbf{Ch}^b_S\mathbf{P}(R)$.

LEMMA 3.1. *Let S be a multiplicatively closed set of central elements in a ring R. Then there is a group homomorphism*

$$K_1(S^{-1}R) \xrightarrow{\partial} K_0(R \text{ on } S)$$

sending each S-isomorphism α to the class $[\mathrm{cone}(\alpha)]$ of the mapping cone of α. In particular, each $s \in S$ is an endomorphism of R so $\partial(s)$ is the class of the chain complex $\mathrm{cone}(s) : R \xrightarrow{-s} R$.

Before proving this lemma, we give one important special case. When S consists of nonzerodivisors, every S-isomorphism α must be an injection, and $\mathrm{coker}(\alpha)$ is a module of projective dimension one, *i.e.*, an object of $\mathbf{H}_S(R)$. Moreover, under the isomorphism $K_0\mathbf{Ch}^b_S\mathbf{P}(R) \cong K_0\mathbf{H}_S(R)$ of Ex. II.9.13, the class of cone(α) in $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$ corresponds to the element $[\mathrm{coker}(\alpha)] \in K_0\mathbf{H}_S(R)$. Thus we immediately have:

Corollary 3.1.1. *If S consists of nonzerodivisors then there is a homomorphism $K_1(S^{-1}R) \xrightarrow{\partial} K_0\mathbf{H}_S(R)$ sending each S-isomorphism α to $[coker(\alpha)]$, and sending $s \in S$ to $[R/sR]$.*

Proof of 3.1. If $\beta \in \mathrm{End}(R^m)$ is also an S-isomorphism, then the diagram

$$\begin{array}{ccccc}
0 & \rightarrowtail & R^n & \overset{=}{\twoheadrightarrow} & R^n \\
\downarrow & & \downarrow (1,\beta) & & \downarrow \beta \\
R^n & \overset{(10)}{\rightarrowtail} & R^n \oplus R^n & \overset{(01)}{\twoheadrightarrow} & R^n \\
\alpha\beta \downarrow & & \downarrow \binom{\alpha\beta}{-\alpha} & & \downarrow \\
R^n & \overset{=}{\rightarrowtail} & R^n & \twoheadrightarrow & 0
\end{array}$$

is a short exact sequence in $\mathbf{Ch}^b_S\mathbf{P}(R)$, where we regard the columns as chain complexes. Since the middle column of the diagram is quasi-isomorphic to its subcomplex $0 \to 0 \oplus R^n \xrightarrow{-\alpha} R^n$, we get the relation

$$\big[\mathrm{cone}(\alpha)\big] - \big[\mathrm{cone}(\alpha\beta)\big] = \big[\mathrm{cone}(\beta)[-1]\big] = -\big[\mathrm{cone}(\beta)\big],$$

or

$$\big[\mathrm{cone}(\alpha)\big] + \big[\mathrm{cone}(\beta)\big] = \big[\mathrm{cone}(\alpha\beta)\big] \tag{3.1.2}$$

in $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$. In particular, if β is the diagonal matrix diag$(t, ..., t)$ then cone(β) is the direct sum of n copies of cone(t), so we have

$$\big[\mathrm{cone}(\alpha t)\big] = \big[\mathrm{cone}(\alpha)\big] + n\big[\mathrm{cone}(t)\big]. \tag{3.1.3}$$

Every $g \in GL_n(S^{-1}R)$ can be represented as α/s for some S-isomorphism α and some $s \in S$, and we define $\partial(g) = \partial(\alpha/s)$ to be the element $\big[\mathrm{cone}(\alpha)\big] - n\big[\mathrm{cone}(s)\big]$ of $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$. By (3.1.3) we have $\partial(\alpha/s) = \partial(\alpha t/st)$, which implies that $\partial(g)$ is independent of the choice of α and s. By (3.1.2) this implies that ∂ is a well-defined homomorphism from each $GL_n(S^{-1}R)$ to $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$. Finally, the maps ∂ are compatible with the inclusions $GL_n \subset GL_{n+1}$, because

$$\begin{aligned}
\partial\begin{pmatrix} \alpha/s & 0 \\ 0 & 1 \end{pmatrix} = \partial\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}/s\right) &= \left[\mathrm{cone}\begin{pmatrix} \alpha & 0 \\ 0 & s \end{pmatrix}\right] - (n+1)\big[\mathrm{cone}(s)\big] \\
&= \big[\mathrm{cone}(\alpha)\big] + \big[\mathrm{cone}(s)\big] - (n+1)\big[\mathrm{cone}(s)\big] = \partial(\alpha/s).
\end{aligned}$$

Hence ∂ extends to $GL(S^{-1}R)$, and hence must factor through the universal map to $K_1(S^{-1}R)$.

KEY EXAMPLE 3.1.4. For the Fundamental Theorem, we shall need the following special case of this construction. Let T be the multiplicative set $\{t^n\}$ in the polynomial ring $R[t]$. Then the map ∂ goes from $K_1(R[t,t^{-1}])$ to $K_0\mathbf{H}_T(R[t])$. If ν is a nilpotent endomorphism of R^n then $t-\nu$ is a T-isomorphism, because its inverse is the polynomial $t^{-1}(1+\nu t^{-1}+\nu^2 t^{-2}+\dots)$. If (R^n,ν) denotes the $R[t]$-module R^n on which t acts as ν,

$$\partial(t-\nu) = \big[R[t]^n/(t-\nu)\big] = \big[(R^n,\nu)\big],$$

$$\partial(1-\nu t^{-1}) = \partial(t-\nu) - \partial(t\cdot \mathrm{id}_n) = \big[(R^n,\nu)\big] - n\big[(R,0)\big].$$

We can also compose ∂ with the product $K_0(R)\otimes K_1(\mathbb{Z}[t,t^{-1}]) \xrightarrow{\cdot} K_1(R[t,t^{-1}])$ of Corollary 1.6.1. Given a f.g. projective R-module P, the product $[P]\cdot t$ is the image of $t\cdot \mathrm{id}_{P[t,t^{-1}]}$ under the map $\mathrm{Aut}(P[t,t^{-1}]) \to K_1(R[t,t^{-1}])$ of Lemma 1.6. To compute $\partial([P]\cdot t)$, choose Q such that $P\oplus Q\cong R^n$. Since the cokernel of $t\cdot\mathrm{id}_{P[t]}\colon P[t]\to P[t]$ is the $R[t]$-module $(P,0)$, we have an exact sequence of $R[t]$-modules:

$$0\to R[t]^n \xrightarrow{t\cdot \mathrm{id}_{P[t]}\oplus 1\cdot \mathrm{id}_{Q[t]}} R[t]^n \to (P,0)\to 0.$$

Therefore we have the formula $\partial([P]\cdot t) = \big[(P,0)\big]$.

LEMMA 3.1.5. *$K_0\mathbf{Ch}^b_S\mathbf{P}(R)$ is generated by the classes $[Q.]$ of chain complexes concentrated in degrees 0 and 1, i.e., by complexes $Q.$ of the form $Q_1\to Q_0$.*

The kernel of $K_0\mathbf{Ch}^b_S\mathbf{P}(R)\to K_0(R)$ is generated by the complexes $R^n\xrightarrow{\alpha} R^n$ associated to S-isomorphisms, i.e., by the classes $\partial(\alpha)=\big[cone(\alpha)\big]$.

PROOF. By the Shifting Lemma II.9.2.1, K_0 is generated by bounded complexes of the form $0\to P_n\to\cdots\to P_0\to 0$. If $n\ge 2$, choose a free R-module $F=R^N$ mapping onto $H_0(P.)$. By assumption, we have $sH_0(P.)=0$ for some $s\in S$. By the projective lifting property, there are maps f_0, f_1 making the diagram

$$\begin{tikzcd} F \arrow[r, "s"] \arrow[d, "f_1"'] & F \arrow[r] \arrow[d, "f_0"'] & F/sF \arrow[r] \arrow[d] & 0 \\ P_1 \arrow[r] & P_0 \arrow[r] & H_0(P) \arrow[r] & 0 \end{tikzcd}$$

commute. Thus if $Q.$ denotes the complex $F\xrightarrow{s}F$ we have a chain map $Q.\xrightarrow{f}P.$ inducing a surjection on H_0. The mapping cone of f fits into a cofibration sequence $P.\rightarrowtail \mathrm{cone}(f)\twoheadrightarrow Q.[-1]$ in $\mathbf{Ch}^b_S\mathbf{P}(R)$, so we have $[P.]=[Q.]+[\mathrm{cone}(f)]$ in $K_0(R \text{ on } S)$. Moreover, $H_0(\mathrm{cone}(f))=0$, so there is a decomposition $P_1\oplus F\cong P_0\oplus P_1'$ so that the mapping cone is the direct sum of an exact complex $P_0\xrightarrow{\cong}P_0$ and a complex $P'_.$ of the form $0\to P_n\to\cdots\to P_3\to P_2\oplus F\to P_1'\to 0$. Since $P'_.$ has length $n-1$, induction on n implies that $[\mathrm{cone}(f)]=[P'_.]$ is a sum of terms of the form $[Q_1\to Q_0]$.

Hence every element of K_0 has the form $x=[P_1\xrightarrow{\alpha}P_0]-[Q_1\xrightarrow{\beta}Q_0]$. Choose $s\in S$ so that $s\beta^{-1}$ is represented by an S-isomorphism $Q_0\xrightarrow{\gamma}Q_1$; adding γ to both terms of x, as well as the appropriate zero term $Q'\xrightarrow{=}Q'$, we may assume that $Q_1 = Q_0 = R^n$, *i.e.*, that the second term of x is the mapping cone of some S-isomorphism $\beta \in \mathrm{End}(R^n)$. With this reduction, the map to $K_0(R)$ sends x to $[P_1] - [P_0]$. If this vanishes, then P_1 and P_0 are stably isomorphic. Adding the appropriate $P' \xrightarrow{=} P'$ makes $P_1 = P_0 = R^m$ for some m, and writes x in the form

$$x = \mathrm{cone}(\alpha) - \mathrm{cone}(\beta) = \partial(\alpha) - \partial(\beta).$$

THEOREM 3.2. *Let S be a multiplicatively closed set of central elements in a ring R. Then the map ∂ of Lemma 3.1 fits into an exact sequence*

$$K_1(R) \to K_1(S^{-1}R) \xrightarrow{\partial} K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R).$$

PROOF. We have proven exactness at $K_0(R)$ in Theorem 9.8, and the composition of any two consecutive maps is zero by inspection. Exactness at $K_0(R \text{ on } S)$ was proven in Lemma 3.1.5. Hence it suffices to establish exactness at $K_1(S^{-1}R)$.

For reasons of exposition, we shall give the proof when S consists of nonzerodivisors, relegating the general proof (which is similar but more technical) to Exercise 3.5. The point of this simplification is that we can work with the exact category $\mathbf{H}_S(R)$. In particular, for every S-isomorphism α the class of the module $\mathrm{coker}(\alpha)$ is simpler to manipulate than the class of the mapping cone.

Recall from the proof of Lemma 3.1 that every element of $GL_n(S^{-1}R)$ can be represented as α/s for some S-isomorphism $\alpha \in \mathrm{End}(R^n)$ and some $s \in S$, and that $\partial(\alpha/s)$ is defined to be $\big[\mathrm{coker}(\alpha)\big] - [R^n/sR^n]$. If $\partial(\alpha/s) = 0$, then from Ex. II.7.2 there are short exact sequences in $\mathbf{H}_S(R)$

$$0 \to C' \to C_1 \to C'' \to 0, \quad 0 \to C' \to C_2 \to C'' \to 0$$

such that $\mathrm{coker}(\alpha) \oplus C_1 \cong (R^n/sR^n) \oplus C_2$. By Ex. 3.4 we may add terms to C', C'' to assume that $C' = \mathrm{coker}(\alpha')$ and $C'' = \mathrm{coker}(\alpha'')$ for appropriate S-isomorphisms of some R^m. By the Horseshoe Lemma ([WHomo, 2.2.8]) we can construct two exact sequences of projective resolutions

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^m & \longrightarrow & R^{2m} & \longrightarrow & R^m & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha'}\downarrow & & {\scriptstyle\alpha_i}\downarrow & & {\scriptstyle\alpha''}\downarrow & & \\
0 & \longrightarrow & R^m & \longrightarrow & R^{2m} & \longrightarrow & R^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C_i & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Inverting S makes each α_i an isomorphism conjugate to $\left(\begin{smallmatrix}\alpha' & 0\\ 0 & \alpha''\end{smallmatrix}\right)$. Thus in $K_1(S^{-1}R)$ we have $[\alpha_1]=[\alpha']+[\alpha'']=[\alpha_2]$. On the other hand, the two endomorphisms $\alpha\oplus\alpha_1$ and $s\cdot\mathrm{id}_n\oplus\alpha_2$ of R^{2m+n} have isomorphic cokernels by construction. Lemma 3.2.1 below implies that in $K_1(S^{-1}R)$ we have

$$[\alpha/s]=[\alpha\oplus\alpha_1]-[s\cdot\mathrm{id}_n\oplus\alpha_2]=g \qquad \text{for some } g\in GL(R).$$

This completes the proof of Theorem 3.2.

LEMMA 3.2.1. *Suppose that S consists of nonzerodivisors. If $\alpha,\beta\in\mathrm{End}_R(R^n)$ are S-isomorphisms with $R^n/\alpha R^n$ isomorphic to $R^n/\beta R^n$, then there is a $g\in GL_{4n}(R)$ such that $[\alpha]=[g][\beta]$ in $K_1(S^{-1}R)$.*

PROOF. Put $M=\mathrm{coker}(\alpha)\oplus\mathrm{coker}(\beta)$, and let $\gamma\colon R^n/\alpha R^n\cong R^n/\beta R^n$ be an automorphism. By Ex. 3.3(b) with $Q=R^{2n}$ we can lift the automorphism $\left(\begin{smallmatrix}0 & \gamma^{-1}\\ \gamma & 0\end{smallmatrix}\right)$ of M to an automorphism γ_0 of R^{4n}. If π_1 and π_2 denote the projections $R^{4n}\xrightarrow{(pr,0,0,0)}\mathrm{coker}(\alpha)$, and $R^{4n}\xrightarrow{(0,pr,0,0)}\mathrm{coker}(\beta)$, respectively, then we have $\gamma\pi_1=\pi_2\gamma_0$. This yields a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^{4n} & \xrightarrow{\alpha,1,1,1} & R^{4n} & \xrightarrow{\pi_1} & R^n/\alpha R^n & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\gamma_1} & & \cong\downarrow{\scriptstyle\gamma_0} & & \cong\downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & R^{4n} & \xrightarrow{1,\beta,1,1} & R^{4n} & \xrightarrow{\pi_2} & R^n/\beta R^n & \longrightarrow & 0
\end{array}$$

in which γ_1 is the induced map. Since γ and γ_0 are isomorphisms, so is γ_1. Because $\gamma_0(\alpha,1,1,1)=(1,\beta,1,1)\gamma_1$ in $GL_{4n}(S^{-1}R)$, we have $[\gamma_0]+[\alpha]=[\beta]+[\gamma_1]$, or $[\alpha]=[\gamma_1\gamma_0^{-1}][\beta]$ in $K_1(S^{-1}R)$.

NK_1 and the group Nil_0

DEFINITION 3.3 (NF). If F is any functor from rings to abelian groups, we write $NF(R)$ for the cokernel of the natural map $F(R)\to F(R[t])$; NF is also a functor on rings. Moreover, the ring map $R[t]\xrightarrow{t=1}R$ provides a splitting $F(R[t])\to F(R)$ of the natural map, so we have a decomposition $F(R[t])\cong F(R)\oplus NF(R)$.

In particular, when F is K_i $(i=0,1)$ we have functors NK_i and a decomposition $K_i(R[t])\cong K_i(R)\oplus NK_i(R)$. Since the ring maps $R[t]\xrightarrow{t=r}R$ are split surjections for every $r\in R$, we see by Proposition 2.3 and Ex. 2.7 that for every r we also have

$$NK_0(R)\cong K_0(R[t],(t-r)) \quad\text{and}\quad NK_1(R)\cong K_1(R[t],(t-r)).$$

We will sometimes speak about NF for functors F defined on any category of rings closed under polynomial extensions and containing the map "$t=1$," such as k-algebras or commutative rings. For example, the functors NU and $N\,\mathrm{Pic}$ were discussed briefly for commutative rings in chapter I, Ex. 3.17 and 3.19.

DEFINITION 3.3.1 (F-REGULAR RINGS). We say that a ring R is *F-regular* if $F(R) = F(R[t_1,\dots,t_n])$ for all n. Since $NF(R[t]) = NF(R) \oplus N^2F(R)$, we see by induction on n that R is F-regular if and only if $N^nF(R) = 0$ for all $n \geq 1$.

For example, we saw in II.6.5 that any regular ring is K_0-regular. We will see in chapter V that regular rings are also K_1-regular, and more generally that they are K_m-regular for every m. Rosenberg has also shown that commutative C^*-algebras are K_m-regular for all m; see [Ro96].

LEMMA 3.3.2. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring. If $NF(R) = 0$ then $F(R) \cong F(R_0)$.*

PROOF. Write $F(t = r)$ for the map $F(R[t]) \to F(R)$ arising from evaluation at "$t = r$". By assumption, the maps $F(t = r)$ are all identical, equal to the inverse of the natural map $F(R) \to F(R[t])$. Let f denote the ring map $R \to R[t]$ defined by $f(r_n) = r_nt^n$ for every $r_n \in R_n$. Since the composition of f and "$t = 1$" is the identity on R, $F(f)$ is an isomorphism with inverse $F(t = 1) = F(t = 0)$. But the composition of f and "$t = 0$" is the projection $R \to R_0 \to R$, so the composition $F(R) \to F(R_0) \to F(R)$ must be the identity on $F(R)$. The result follows, since $F(R_0) \to F(R) \to F(R_0)$ is the identity on $F(R_0)$.

3.4 We are going to describe the group $NK_1(R)$ in terms of nilpotent matrices. For this, we need the following trick, which was published by Graham Higman in 1940. For clarity, if $I = fA$ is an ideal in A we write $GL(A, f)$ for $GL(I)$.

HIGMAN'S TRICK 3.4.1. *For every $g \in GL(R[t], t)$ there is a nilpotent matrix ν over R such that $[g] = [1 - \nu t]$ in $K_1(R[t])$.*

Similarly, for every $g \in GL(R[t, t^{-1}], t - 1)$ there is a nilpotent matrix ν over R such that $[g] = [1 - \nu(t - 1)]$ in $K_1(R[t, t^{-1}], t - 1)$.

PROOF. Every invertible $p \times p$ matrix over $R[t]$ can be written as a polynomial $g = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \cdots + \gamma_n t^n$ with the γ_i in $M_p(R)$. If g is congruent to the identity modulo t, then $\gamma_0 = 1$. If $n \geq 2$ and we write $g = 1 - ht + \gamma_n t^n$, then modulo $E_{2p}(R[t], t)$ we have

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} g & \gamma_n t^{n-1} \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 - ht & \gamma_n t^{n-1} \\ -t & 1 \end{pmatrix} = 1 - \begin{pmatrix} h & -\gamma_n t^{n-2} \\ 1 & 0 \end{pmatrix} t.$$

By induction on n, $[g]$ is represented by a matrix of the form $1 - \nu t$. The matrix ν is nilpotent by Ex. 3.1.

Over $R[t, t^{-1}]$ we can use a similar argument. After multiplying by a power of t, we may write g as a polynomial in t. Such a polynomial may be rewritten as a polynomial $\sum \gamma_i x^i$ in $x = (t - 1)$. If g is congruent to the identity modulo $(t - 1)$ then again we have $\gamma_0 = 1$. By Higman's trick (applied to x), we may reduce g to a matrix of the form $1 - \nu x$, and again ν must be nilpotent by Ex. 3.1.

We will also need the category $\mathbf{Nil}(R)$ of II.7.3.4. Recall that the objects of this category are pairs (P, ν), where P is a f.g. projective R-module and ν is a nilpotent endomorphism of P. Let T denote the multiplicative set $\{t^n\}$ in $R[t]$. From II.7.7.4 we have

$$K_0(R[t] \text{ on } T) \cong K_0\mathbf{Nil}(R) \cong K_0(R) \oplus Nil_0(R),$$

where $Nil_0(R)$ is the subgroup generated by elements of the form $\big[(R^n,\nu)\big] - n\big[(R,0)\big]$ for some n and some nilpotent matrix ν.

LEMMA 3.4.2. *For every ring R, the product with $t \in K_1(\mathbb{Z}[t,t^{-1}])$ induces a split injection $K_0(R) \xrightarrow{\cdot t} K_1(R[t,t^{-1}])$.*

PROOF. Since the forgetful map $K_0\mathbf{Nil}(R) \to K_0(R)$ sends $\big[(P,\nu)\big]$ to $[P]$, the calculation in Example 3.1.4 shows that the composition

$$K_0(R) \xrightarrow{\cdot t} K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0\mathbf{Nil}(R) \to K_0(R)$$

is the identity map. Hence the first map is a split injection.

Momentarily changing variables from t to s, we now define an additive function τ from $\mathbf{Nil}(R)$ to $K_1(R[s])$. Given an object (P,ν), let $\tau(P,\nu)$ be the image of the automorphism $1-\nu s$ of $P[s]$ under the natural map $\mathrm{Aut}(P[s]) \to K_1(R[s])$ of Lemma 1.6. Given a short exact sequence

$$0 \to (P',\nu') \to (P,\nu) \to (P'',\nu'') \to 0$$

in $\mathbf{Nil}(R)$, a choice of a splitting $P \cong P' \oplus P''$ allows us to write

$$(1-\nu s) = \begin{pmatrix} 1-\nu' s & \gamma s \\ 0 & 1-\nu'' s \end{pmatrix} = \begin{pmatrix} 1-\nu' s & 0 \\ 0 & 1-\nu'' s \end{pmatrix} \begin{pmatrix} 1 & \gamma' s \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{Aut}(P[s])$. Hence in $K_1(R[s])$ we have $[1-\nu s] = [1-\nu' s][1-\nu'' s]$. Therefore τ is an additive function, and induces a homomorphism $\tau\colon K_0\mathbf{Nil}(R) \to K_1(R[s])$. Since $\tau(P,0) = 1$ for all P and $1-\nu s$ is congruent to 1 modulo s, we see that τ is actually a map from $Nil_0(R)$ to $K_1(R[s],s)$.

PROPOSITION 3.4.3. *$Nil_0(R) \cong NK_1(R)$, and $K_0\mathbf{Nil}(R) \cong K_0(R) \oplus NK_1(R)$.*

PROOF. For convenience, we identify s with t^{-1}, so that $R[s,s^{-1}] = R[t,t^{-1}]$. Applying Lemma 3.1 to $R[t]$ and $T = \{1,t,t^2,\dots\}$, we form the composition

$$\begin{aligned} &K_1(R[s],s) \to K_1(R[s]) \to K_1(R[s,s^{-1}]) \\ &\qquad = K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0(R \text{ on } T) \to Nil_0(R). \end{aligned} \tag{3.4.4}$$

Let us call this composition δ. We claim that τ is the inverse of δ. By Higman's Trick, every element of $K_1(R[s],s)$ is represented by a matrix $1-\nu s$ with ν nilpotent. In Example 3.1.4 we saw that $\delta(1-\nu s) = \big[(R^n,\nu)\big] - n\big[(R,0)\big]$. By the construction of τ we have the desired equations: $\tau\delta(1-\nu s) = \tau\big[(R^n,\nu)\big] = (1-\nu s)$ and

$$\delta\tau\Big(\big[(R^n,\nu)\big] - n\big[(R,0)\big]\Big) = \delta(1-\nu s) = \big[(R^n,\nu)\big] - n\big[(R,0)\big].$$

COROLLARY 3.4.5. *$K_1(R[s]) \to K_1(R[s,s^{-1}])$ is an injection for every ring R.*

PROOF. By Ex. 2.7, we have $K_1(R[s]) \cong K_1(R) \oplus K_1(R[s],s)$. Since $K_1(R)$ is a summand of $K_1(R[s,s^{-1}])$, the isomorphism $\delta\colon K_1(R[s],s) \cong Nil_0(R)$ of (3.3.4) factors through $K_1(R[s],s) \to K_1(R[s,s^{-1}])/K_1(R)$. This quotient map must then be an injection. The result follows.

The Fundamental Theorems for K_1 and K_0

FUNDAMENTAL THEOREM FOR K_1 3.5. *For every ring R, there is a split surjection $K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0(R)$, with inverse $[P] \mapsto [P]\cdot t$. This map fits into a naturally split exact sequence:*

$$0 \to K_1(R) \xrightarrow{\Delta} K_1(R[t]) \oplus K_1(R[t^{-1}]) \xrightarrow{\pm} K_1(R[t,t^{-1}] \xrightarrow{\partial} K_0(R) \to 0.$$

Consequently, we have a natural direct sum decomposition:

$$K_1(R[t,t^{-1}] \cong K_1(R) \oplus K_0(R) \oplus NK_1(R) \oplus NK_1(R).$$

PROOF. We merely assemble the pieces of the proof from §3.4. The first assertion is just Lemma 3.4.2. The natural maps from $K_1(R)$ into $K_1(R[t])$, $K_1(R[t^{-1}])$ and $K_1(R[t,t^{-1}])$ are injections, split by "$t = 1$" (as in 3.4), so the obviously exact sequence

$$0 \to K_1(R) \xrightarrow{\Delta} K_1(R) \oplus K_1(R) \xrightarrow{\pm} K_1(R) \to 0 \tag{3.5.1}$$

is a summand of the sequence we want to prove exact. From Proposition II.7.7.1, Theorem 3.2 and Corollary 3.4.5, we have an exact sequence

$$0 \to K_1(R[t]) \to K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0\mathbf{Nil}(R) \to 0. \tag{3.5.2}$$

Since $K_0\mathbf{Nil}(R) \cong K_0(R) \oplus Nil_0(R)$, the map ∂ in (3.5.2) is split by the maps of 3.4.2 and 3.4.3. The sequence in the Fundamental Theorem for K_1 is obtained by rearranging the terms in sequences (3.5.1) and (3.5.2).

In order to formulate the corresponding Fundamental Theorem for K_0, we define $K_{-1}(R)$ to be the cokernel of the map $K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t,t^{-1}])$. We will reprove the following result more formally in the next section.

FUNDAMENTAL THEOREM FOR K_0 3.6. *For every ring R, there is a naturally split exact sequence:*

$$0 \to K_0(R) \xrightarrow{\Delta} K_0(R[t]) \oplus K_0(R[t^{-1}]) \xrightarrow{\pm} K_0(R[t,t^{-1}] \xrightarrow{\partial} K_{-1}(R) \to 0.$$

Consequently, we have a natural direct sum decomposition:

$$K_0(R[t,t^{-1}] \cong K_0(R) \oplus K_{-1}(R) \oplus NK_0(R) \oplus NK_0(R).$$

PROOF. Let s be a second indeterminate. The Fundamental Theorem for K_1, applied to the variable t, gives a natural decomposition

$$K_1(R[s,t,t^{-1}]) \cong K_1(R[s]) \oplus NK_1(R[s]) \oplus NK_1(R[s]) \oplus K_0(R[s]),$$

and similar decompositions for the other terms in the map

$$K_1(R[s,t,t^{-1}]) \oplus K_1(R[s^{-1},t,t^{-1}]) \to K_1(R[s,s^{-1},t,t^{-1}]).$$

Therefore the cokernel of this map also has a natural splitting. But the cokernel is $K_0(R[t,t^{-1}])$, as we see by applying the Fundamental Theorem for K_1 to the variable s.

EXERCISES

3.1 Let A be a ring and $a \in A$, show that the following are equivalent: (i) a is nilpotent; (ii) $1 - at$ is a unit of $A[t]$; (iii) $1 - a(t-1)$ is a unit of $A[t, t^{-1}]$.

3.2 Let $\alpha, \beta\colon P \to Q$ be two maps between fin. gen. projective R-modules. If S is a central multiplicatively closed set in R and $S^{-1}\alpha, S^{-1}\beta$ are isomorphisms, then $g = \beta^{-1}\alpha$ is an automorphism of $S^{-1}P$. Show that $\partial(g) = [\mathrm{cone}(\alpha)] - [\mathrm{cone}(\beta)]$. In particular, if S consists of nonzerodivisors, show that $\partial(g) = [\mathrm{coker}(\alpha)] - [\mathrm{coker}(\beta)]$.

3.3 (Bass) Prove that every module M in $\mathbf{H}(R)$ has a projective resolution $P. \to M$ such that every automorphism α of M lifts to an *automorphism* of the chain complex $P.$. To do so, proceed as follows.

(a) Fix a surjection $\pi\colon Q \to M$, and use Ex. I.1.11 to lift the automorphism $\alpha \oplus \alpha^{-1}$ of $M \oplus M$ to an automorphism β of $Q \oplus Q$.
(b) Defining $e\colon Q \oplus Q \to M$ to be $e(x, y) = \pi(x)$, show that every automorphism of M can be lifted to an automorphism of $Q \oplus Q$.
(c) Set $P_0 = Q \oplus Q$, and repeat the construction on $Z_0 = \ker(e)$ to get a finite resolution $P.$ of M with the desired property.

3.4 Suppose that S consists of nonzerodivisors, and that M is a module in $\mathbf{H}_S(R)$.

(a) Prove that there is a module M' and an S-isomorphism $\alpha \in \mathrm{End}(R^m)$ so that $\mathrm{coker}(\alpha) = M \oplus M'$. *Hint:* Modify the proof of Lemma 3.1.5, where M is the cokernel of a map $P_0 \xrightarrow{\beta} P_1$.
(b) Given S-isomorphisms $\alpha', \alpha'' \in \mathrm{End}(R^m)$ and a short exact sequence of S-torsion modules $0 \to \mathrm{coker}(\alpha') \to M \to \mathrm{coker}(\alpha'') \to 0$, show that there is an S-isomorphism $\alpha \in R^{2m}$ with $M \cong \mathrm{coker}(\alpha)$.

3.5 Modify the proofs of the previous two exercises to prove Theorem 3.2 when S contains zerodivisors.

3.6 *Noncommutative localization.* By definition, a multiplicatively closed set S in a ring R is called a *right denominator set* if it satisfies the following two conditions: (i) For any $s \in S$ and $r \in R$ there exists an $s' \in S$ and $r' \in R$ such that $sr' = rs'$; (ii) if $sr = 0$ for any $r \in R$, $s \in S$ then $rs' = 0$ for some $s' \in S$. This is the most general condition under which a (right) ring of fractions $S^{-1}R$ exists, in which every element of $S^{-1}R$ has the form $r/s = rs^{-1}$, and if $r/1 = 0$ then some $rs = 0$ in R.

Prove Theorem 3.2 when S is a right denominator set consisting of nonzerdivisors. To do this, proceed as follows.

(a) Show that for any finite set of elements x_i in $S^{-1}R$ there is an $s \in S$ and $r_i \in R$ so that $x_i = r_i/s$ for all i.
(b) Reprove II.7.6.3 and II.9.8 for denominator sets, using (a); this yields exactness at $K_0(R)$.
(c) Modify the proof of Lemma 3.1 and 3.1.5 to construct the map ∂ and prove exactness at $K_0\mathbf{H}_S(R)$.
(d) Modify the proof of Theorem 3.2 to prove exactness at $K_1(S^{-1}R)$.

§4. Negative K-theory

In the last section, we defined $K_{-1}(R)$ to be the cokernel of the map $K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t, t^{-1}])$. Of course we can keep going, and define all the negative K-groups by induction on n:

DEFINITION 4.1. For $n > 0$, we inductively define $K_{-n}(R)$ to be the cokernel of the map

$$K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \to K_{-n+1}(R[t, t^{-1}]).$$

Clearly, each K_{-n} is a functor from rings to abelian groups.

To describe the properties of these negative K-groups, it is convenient to cast the Fundamental Theorems above in terms of Bass' notion of *contracted functors*. With this in mind, we make the following definitions.

DEFINITION 4.1.1 (CONTRACTED FUNCTORS). Let F be a functor from rings to abelian groups. For each R, we define $LF(R)$ to be the cokernel of the map $F(R[t]) \oplus F(R[t^{-1}]) \to F(R[t, t^{-1}])$. We write $Seq(F, R)$ for the following sequence:

$$0 \to F(R) \xrightarrow{\Delta} F(R[t]) \oplus F(R[t^{-1}]) \xrightarrow{\pm} F(R[t, t^{-1}]) \to LF(R) \to 0.$$

We say that F is *acyclic* if $Seq(F, R)$ is exact for all R. We say that F is a *contracted functor* if F is acyclic and in addition there is a splitting $h = h_{t,R}$ of the defining map $F(R[t, t^{-1}]) \to LF(R)$, a splitting which natural in both t and R.

By iterating this definition, we can speak about the functors NLF, L^2F, etc. For example, Definition 4.1 states that $K_{-n} = L^n(K_0)$.

EXAMPLE 4.1.2 (FUNDAMENTAL THEOREM FOR K_{-n}). The Fundamental Theorems for K_1 and K_0 may be restated as the assertions that these are contracted functors. It follows from Proposition 4.2 below that each K_{-n} is a contracted functor; by Definition 4.1, this means that there is a naturally split exact sequence:

$$0 \to K_{-n}(R) \xrightarrow{\Delta} K_{-n}(R[t]) \oplus K_{-n}(R[t^{-1}]) \xrightarrow{\pm} K_{-n}(R[t, t^{-1}]) \xrightarrow{\partial} K_{-n-1}(R) \to 0.$$

As with the definition of NF, we can define LF for functors F which are defined on a category of rings closed under polynomial and Laurent polynomial extensions, such as the functors U and Pic, which are defined only for commutative rings.

EXAMPLE 4.1.3 (UNITS). Let $U(R) = R^\times$ denote the group of units in a commutative ring R. By Ex. I.3.17, U is a contracted functor with contraction $LU(R) = [\mathrm{Spec}(R), \mathbb{Z}]$; the splitting map $LU(R) \to U(R[t, t^{-1}]$ sends a function $f\colon \mathrm{Spec}(R) \to \mathbb{Z}$ to the unit t^f of $R[t, t^{-1}]$. From Ex. 4.2 below we see that the functors L^2U and NLU are zero. Thus we can write a simple formula for the units of any extension $R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. If R is reduced, so that $NU(R)$ vanishes (Ex. I.3.17), then we just have

$$U(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]) = U(R) \times \coprod_{i=1}^{n} [\mathrm{Spec}(R), \mathbb{Z}] \cdot t_i.$$

EXAMPLE 4.1.4 (Pic). Recall from chapter I, §3 that the Picard group $\mathrm{Pic}(R)$ of a commutative ring is a functor, and that $N\,\mathrm{Pic}(R) = 0$ exactly when R_{red} is seminormal. By Ex. I.3.18 the sequence $Seq(\mathrm{Pic}, R)$ is exact. In fact Pic is a contracted functor with $NL\,\mathrm{Pic} = L^2\,\mathrm{Pic} = 0$; see [Weib91]. The group $L\,\mathrm{Pic}(R)$ is the étale cohomology group $H^1_{et}(\mathrm{Spec}(R), \mathbb{Z})$.

A *morphism of contracted functors* is a natural transformation $\eta: F \Rightarrow F'$ between two contracted functors such that the following square commutes for all R.

$$\begin{array}{ccc} LF(R) & \xrightarrow{h} & F(R[t,t^{-1}]) \\ {\scriptstyle (L\eta)_R}\downarrow & & \downarrow{\scriptstyle \eta_{R[t,t^{-1}]}} \\ LF'(R) & \xrightarrow{h'} & F'(R[t,t^{-1}]) \end{array}$$

PROPOSITION 4.2. *Let $\eta: F \Rightarrow F'$ be a morphism of contracted functors. Then both* $\ker(\eta)$ *and coker(η) are also contracted functors.*

In particular, if F is contracted, then NF and LF are also contracted functors. Moreover, there is a natural isomorphism of contracted functors $NLF \cong LNF$.

PROOF. If $C \xrightarrow{\phi} D$ is a morphism between split exact sequences, which have compatible spittings, then the sequences $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are always split exact, with splittings induced from the splittings of C and D. Applying this remark to $Seq(F, R) \to Seq(F', R)$ shows that both $Seq(\ker(\eta), R)$ and $Seq(\mathrm{coker}(\eta), R)$ are split exact. That is, both $\ker(\eta)$ and $\mathrm{coker}(\eta)$ are contracted functors. It also shows that

$$0 \to \ker(\eta)(R) \to F(R) \xrightarrow{\eta_R} F'(R) \to \mathrm{coker}(\eta)(R) \to 0$$

is an exact sequence of contracted functors.

Since $NF(R)$ is the cokernel of the morphism $F(R) \to F'(R) = F(R[t])$ and $LF(R)$ is the cokernel of the morphism $\pm$ in $Seq(F, R)$, both NF and LF are contracted functors. Finally, the natural isomorphism $NLF(R) \cong LNF(R)$ arises from inspecting one corner of the large commutative diagram represented by

$$0 \to Seq\,F(R[s], s) \to Seq\,F(R[s]) \to Seq\,F(R) \to 0.$$

EXAMPLE 4.2.1 (SK_1). If R is a commutative ring, it follows from Examples 1.1.1 and 4.1.3 that $\det: K_1(R) \to U(R)$ is a morphism of contracted functors. Hence SK_1 is a contracted functor. The contracted map $L\det$ is the map $\mathrm{rank}: K_0(R) \to H_0(R) = [\mathrm{Spec}(R), \mathbb{Z}]$ of II.2.3; it follows that $L(SK_1)(R) = \widetilde{K}_0(R)$. From Ex. 4.2 we also have $L^2(SK_1)(R) = L\widetilde{K}_0(R) = K_{-1}(R)$.

We can give an elegant formula for $F(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$, using the following notation. If $p(N, L) = \sum m_{ij} N^i L^j$ is any formal polynomial in N and L with integer coefficients $m_{ij} > 0$, and F is a functor from rings to abelian groups, we set $p(N, L)F$ equal to the direct sum of m_{ij} copies of each group $N^i L^j F(R)$.

COROLLARY 4.2.2. *$F(R[t_1, \dots, t_n]) \cong (1 + N)^n F(R)$ for every F. If F is a contracted functor, then $F(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]) \cong (1 + 2N + L)^n F(R)$.*

PROOF. The case $n = 1$ follows from the definitions, and the general case follows from induction.

For example, if $L^2F = 0$ and R is F-regular, then $(1+2N+L)^nF(R)$ stands for $F(R) \oplus n\,LF(R)$. In particular, the formula for units in Example 4.1.3 is just the case $F = U$ of 4.2.2.

EXAMPLE 4.2.3. Since $L^jK_0 = K_{-j}$, $K_0(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$ is the direct sum of many pieces $N^iK_{-j}(R)$, including $K_{-n}(R)$ and $\binom{n}{j}$ copies of $K_{-j}(R)$.

CONJECTURE 4.2.4. Let R be a commutative noetherian ring of Krull dimension d. It is conjectured that $K_{-j}(R)$ vanishes for all $j > d$, and that R is K_{-d}-regular; see [Weib80]. This is so for $d = 0, 1$ by exercises 4.3 and 4.4, and Example 4.3.1 below shows that the bound is best possible.

The Mayer-Vietoris sequence

Suppose that $f\colon R \to S$ is a ring map, and I is an ideal of R mapped isomorphically into an ideal of S. By Theorem 2.6 there is an exact "Mayer-Vietoris" sequence:

$$K_1(R) \xrightarrow{\Delta} K_1(S) \oplus K_1(R/I) \to K_1(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I).$$

Applying the contraction operation L to this sequence gives a sequence relating K_0 to K_{-1}, whose first three terms are identical to the last three terms of the displayed sequence. Splicing these together yields a longer sequence. Repeatedly applying L and splicing sequences leads to the following result.

THEOREM 4.3 (MAYER-VIETORIS). *Suppose we are given a ring map $f\colon R \to S$ and an ideal I of R mapped isomorphically into an ideal of S. Then the Mayer-Vietoris sequence of Theorem 2.6 continues as a long exact Mayer-Vietoris sequence of negative K-groups.*

$$\cdots \xrightarrow{\Delta} \begin{matrix} K_0(S) \oplus \\ K_0(R/I) \end{matrix} \xrightarrow{\pm} K_0(S/I) \xrightarrow{\partial} K_{-1}(R) \xrightarrow{\Delta} \begin{matrix} K_{-1}(S) \oplus \\ K_{-1}(R/I) \end{matrix} \xrightarrow{\pm} K_{-1}(S/I) \xrightarrow{\partial} K_{-2}(R) \to$$
$$\cdots \to K_{-n+1}(S/I) \xrightarrow{\partial} K_{-n}(R) \xrightarrow{\Delta} \begin{matrix} K_{-n}(S) \oplus \\ K_{-n}(R/I) \end{matrix} \xrightarrow{\pm} K_{-n}(S/I) \xrightarrow{\partial} K_{-n-1}(R) \cdots$$

EXAMPLE 4.3.1 (B. DAYTON). Fix a regular ring R, and let $\Delta^n(R)$ denote the coordinate ring $R[t_0, \dots, t_n]/(f)$, $f = t_0 \cdots t_n(1 - \sum t_i)$ of the n-dimensional tetrahehron over R. Using $I = (1 - \sum t_i)\Delta^n(R)$ and $\Delta^n(R)/I \cong R[t_1, \dots, t_n]$ via $t_0 \mapsto 1 - (t_1 + \cdots + t_n)$, we have a Milnor square

$$\begin{array}{ccc} \Delta^n(R) & \longrightarrow & A_n \\ \downarrow & & \downarrow \\ R[t_1, \dots, t_n] & \longrightarrow & \Delta^{n-1}(R) \end{array}$$

where $A_n = R[t_0, \dots, t_n]/(t_0 \cdots t_n)$. By Ex. 4.7, the negative K-groups of A_n vanish and $K_i(A_n) = K_i(R)$ for $i = 0, 1$. Therefore we have $K_0(\Delta^n(R)) \cong K_1(\Delta^{n-1}(R))$, and $K_{-j}(\Delta^n(R)) \cong K_{1-j}(\Delta^{n-1}(R))$ for $j > 0$. These groups vanish for $j > n$, with

$K_{-n}(\Delta^n(R)) \cong K_0(R)$. In particular, if F is a field then $\Delta^n(F)$ is an n-dimensional noetherian ring with $K_{-n}(\Delta^n(F)) \cong \mathbb{Z}$; see Conjecture 4.2.4.

When we have introduced higher K-theory, we will see that in fact $K_0(\Delta^n(R)) \cong K_n(R)$ and $K_1(\Delta^n(R)) \cong K_{n+1}(R)$. This is just one way in which higher K-theory appears in classical K-theory.

EXERCISES

4.1 Suppose that $0 \to F' \to F \to F'' \to 0$ is an exact sequence of functors, with F' and F'' contracted. Show that F is acyclic, but need not be contracted.

4.2 For a commutative ring R, let $H_0(R)$ denote the group $[\mathrm{Spec}(R), \mathbb{Z}]$ of all continuous functions from $\mathrm{Spec}(R)$ to $\mathbb{Z}$. Show that $NH_0 = LH_0 = 0$, *i.e.*, that $H_0(R) = H_0(R[t]) = H_0(R[t, t^{-1}])$.

4.3 Let R be an Artinian ring. Show that R is K_0-regular, and that $K_{-n}(R) = 0$ for all $n > 0$.

4.4 (Bass-Murthy) Let R be a 1-dimensional commutative noetherian ring with finite normalization $\widetilde{R}$ and conductor ideal I. Show that R is K_{-1}-regular, and that $K_{-n}(R) = 0$ for all $n \geq 2$. If $h_0(R)$ denotes the rank of the free abelian group $H_0(R) = [\mathrm{Spec}(R), \mathbb{Z}]$, show that $K_{-1}(R) \cong L\,\mathrm{Pic}(R) \cong \mathbb{Z}^r$, where $r = h_0(R) - h_0(\widetilde{R}) + h_0(\widetilde{R}/I) - h_0(R/I)$.

4.5 (Carter) Let S be a multiplicative set in a commutative ring R. Regarding $F(A) = K_0(A \text{ on } S)$ as a functor on R-algebras, define $K_{-n}(A \text{ on } S)$ to be $L^n K_0(A \text{ on } S)$. Deduce from the Localization sequence that these are acyclic functors, and that the Localization Sequence of Theorem 3.2 continues into negative K-theory for every R-algebra A.

$$\cdots \to K_0(A) \to K_0(S^{-1}A) \xrightarrow{\partial} K_{-1}(A \text{ on } S) \to K_{-1}(A) \to$$
$$K_{-1}(S^{-1}A) \xrightarrow{\partial} K_{-2}(A \text{ on } S) \to K_{-2}(A) \to \cdots$$

With the help of higher K-theory to define $K_1(A \text{ on } S)$, and construct the product "$\cdot t$", it will follow that the $K_i(A \text{ on } S)$ are actually contracted functors.

4.6 Let G be a finite group of order n, and let $\widetilde{R}$ be a "maximal order" in $\mathbb{Q}[G]$. It is well known that $\widetilde{R}$ is a regular ring containing $\mathbb{Z}[G]$, and that $I = n\widetilde{R}$ is an ideal of $\mathbb{Z}[G]$; see [Bass, p. 560]. Show that $K_{-n}\mathbb{Z}[G] = 0$ for $n \geq 2$, and that K_{-1} has the following resolution by free abelian groups:

$$0 \to \mathbb{Z} \to H_0(\widetilde{R}) \oplus H_0(\mathbb{Z}/n[G]) \to H_0(\widetilde{R}/n\widetilde{R}) \to K_{-1}(\mathbb{Z}[G]) \to 0.$$

D. Carter has shown in [Carter] that $K_{-1}\mathbb{Z}[G] \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^s$, where s equals the number of simple components $M_{n_i}(D_i)$ of the semisimple ring $\mathbb{Q}[G]$ such that the Schur index of D is even (see 1.2.4), but the Schur index of D_p is odd at each prime p dividing n. In particular, if G is abelian then $K_{-1}\mathbb{Z}[G]$ is torsionfree (see [Bass, p. 695]).

4.7 *Coordinate hyperplanes.* Let R be a regular ring. By induction on n, show that the graded rings $A_n = R[t_0, \dots, t_n]/(t_0 \cdots t_n)$ are K_i-regular for all $i \leq 1$. Conclude that $K_1(A_n) = K_1(R)$, $K_0(A_n) = K_0(R)$ and $K_i(A_n) = 0$ for all $i < 0$.

§5. Milnor's K_2 of a ring

The group K_2 of a ring was defined by J. Milnor in 1967, following a 1962 paper by R. Steinberg on Universal Central Extensions of Chevalley groups. Milnor's 1971 book [Milnor] is still the best source for the fundamental theorems about it. In this section we will give an introduction to the subject, but we will not prove the harder theorems.

Following Steinberg, we define a group in terms of generators and relations designed to imitate the behavior of the elementary matrices, as described in (1.2.1). To avoid technical complications, we shall avoid any definition of $St_2(R)$.

DEFINITION 5.1. For $n \geq 3$ the *Steinberg group* $St_n(R)$ of a ring R is the group defined by generators $x_{ij}(r)$, with i, j a pair of distinct integers between 1 and n and $r \in R$, subject to the following "Steinberg relations"

(5.1.1) $x_{ij}(r)x_{ij}(s) = x_{ij}(r + s)$

(5.1.2) $[x_{ij}(r), x_{k\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell \\ x_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases}$

As observed in (1.3.1), the Steinberg relations are also satisfied by the elementary matrices $e_{ij}(r)$ which generate the subgroup $E_n(R)$ of $GL_n(R)$. Hence there is a canonical group surjection $\phi_n\colon St_n(R) \to E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$.

The Steinberg relations for $n+1$ include the Steinberg relations for n, so there is an obvious map $St_n(R) \to St_{n+1}(R)$. We write $St(R)$ for $\varinjlim St_n(R)$, and observe that by stabilizing the ϕ_n induce a surjection $\phi\colon St(R) \to E(R)$.

DEFINITION 5.2. The group $K_2(R)$ is the kernel of $\phi\colon St(R) \to E(R)$. Thus there is an exact sequence of groups

$$1 \to K_2(R) \to St(R) \xrightarrow{\phi} GL(R) \to K_1(R) \to 1.$$

It will follow from Theorem 5.3 below that $K_2(R)$ is an abelian group. Moreover, it is clear that St and K_2 are both covariant functors from rings to groups, just as GL and K_1 are.

THEOREM 5.2.1 (STEINBERG). *$K_2(R)$ is an abelian group. In fact it is precisely the center of $St(R)$.*

PROOF. If $x \in St(R)$ commutes with every element of $St(R)$, then $\phi(x)$ must commute with all of $E(R)$. But the center of $E(R)$ is trivial (by Ex. 1.7) so $\phi(x) = 1$, *i.e.*, $x \in K_2(R)$. Thus the center of $St(R)$ is contained in $K_2(R)$.

Conversely, suppose that $y \in St(R)$ satisfies $\phi(y) = 1$. Then in $E(R)$ we have

$$\phi([y, p]) = \phi(y)\phi(p)\phi(y)^{-1}\phi(p)^{-1} = \phi(p)\phi(p)^{-1} = 1$$

for every $p \in St(R)$. Choose an integer n large enough that y can be expressed as a word in the symbols $x_{ij}(r)$ with $i, j < n$. For each element $p = x_{kn}(s)$ with $k < n$ and $s \in R$, the Steinberg relations imply that the commutator $[y, p]$ is an element of the subgroup P_n of $St(R)$ generated by the symbols $x_{in}(r)$ with $i < n$.

On the other hand, we know by Ex. 5.2 that ϕ maps P_n injectively into $E(R)$. Since $\phi([y,p]) = 1$ this implies that $[y,p] = 1$. Hence y commutes with every generator $x_{kn}(s)$ with $k < n$.

By symmetry, this proves that y also commutes with every generator $x_{nk}(s)$ with $k < n$. Hence y commutes with all of $St_n(R)$, since it commutes with every $x_{kl}(s) = [x_{kn}(s), x_{nl}(1)]$ with $k, l < n$. Since n can be arbitrarily large, this proves that y is in the center of $St(R)$.

EXAMPLE 5.2.2. The group $K_2(\mathbb{Z})$ is cyclic of order 2. This calculation uses the Euclidean Algorithm to rewrite elements of $St(\mathbb{Z})$, and is given in §10 of [Milnor]. In fact, Milnor proves that the symbol $\{-1,-1\} = \{x_{12}(1)x_{21}(-1)x_{12}(1)\}^4$ is the only nontrivial element of $\ker(\phi_n)$ for all $n \geq 3$. It is easy to see that $\{-1,-1\}$ is in the kernel of each ϕ_n, because the 2×2 matrix $e_{12}(1)e_{21}(-1)e_{12}(1) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has order 4 in $GL_n(\mathbb{Z})$. We will see in Example 6.2.1 below that $\{-1,-1\}$ is still nonzero in $K_2(\mathbb{R})$.

Tate has used the same Euclidean Algorithm type techniques to show that $K_2(\mathbb{Z}[\sqrt{-7}])$ and $K_2(\mathbb{Z}[\sqrt{-15}])$ are also cyclic of order 2, generated by the symbol $\{-1,-1\}$, while $K_2(R) = 1$ for the imaginary quadratic rings $R = \mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-3}]$, $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-11}]$. See the appendix to [BT] for details.

EXAMPLE 5.2.3. For every field F we have $K_2(F[t]) = K_2(F)$. This was originally proven by K. Dennis and J. Sylvester using the same Euclidean Algorithm type techniques as in the provious example. We shall not describe the details, because we shall see in chapter V that $K_2(R[t]) = K_2(R)$ for every regular ring.

Universal Central Extensions

The Steinberg group $St(R)$ can be described in terms of universal central extensions, and the best exposition of this is [Milnor, §5]. Properly speaking, this is a subject in pure group theory; see [Suz, 2.9]. However, since extensions of a group G are classified by the cohomology group $H^2(G)$, the theory of universal central extensions is also a part of homological algebra; see [WHomo, §6.9]. Here are the relevant definitions.

Let G be a group and A an abelian group. A *central extension* of G by A is a short exact sequence of groups $1 \to A \to X \xrightarrow{\pi} G \to 1$ such that A is in the center of X. We say that a central extension is *split* if it is isomorphic to an extension of the form $1 \to A \to A \times G \xrightarrow{pr} G \to 1$, where $pr(a,g) = g$.

If A or π is clear from the context, we may omit it from the notation. For example, $1 \to K_2(R) \to St(R) \to E(R) \to 1$ is a central extension by Steinberg's Theorem 5.2.1, but we usually just say that $St(R)$ is a central extension of $E(R)$.

Two extensions X and Y of G by A are said to be *equivalent* if there is an isomorphism $f\colon X \to Y$ which is the identity on A and which induces the identity map on G. It is well-known that the equivalence classes of central extensions of G by a fixed group A are in 1–1 correspondence with the elements of the cohomology group $H^2(G; A)$; see [WHomo, §6.6].

More generally, by a *homomorphism over* G from $X \xrightarrow{\pi} G$ to another central extension $1 \to B \to Y \xrightarrow{\tau} G \to 1$ we mean a group map $f\colon X \to Y$ such that $\pi = \tau f$.

DEFINITION 5.3.1. A *universal central extension* of G is a central extension $X \xrightarrow{\pi} G$ such that for every other central extension $Y \xrightarrow{\tau} G$ there is a unique homomorphism f over G from X to Y. Clearly a universal central extension is unique up to isomorphism over G, provided it exists.

LEMMA 5.3.2. *If G has a universal central extension X, then both G and X must be perfect groups.*

PROOF. Otherwise $B = X/[X,X]$ is nontrivial, and there would be two homomorphisms over G from X to the central extension $1 \to B \to B \times G \to G \to 1$, namely the maps $(0, \pi)$ and (pr, π), where pr is the natural projection $X \to B$.

LEMMA 5.3.3. *If X and Y are central extensions of G, and X is a perfect group, there is at most one homomorphism over G from X to Y.*

PROOF. If f and f_1 are two such homomorphisms, then for any x and x' in X we can write $f_1(x) = f(x)c$, $f_1(x') = f(x')c'$ for elements c and c' in the center of Y. Therefore $f_1(xx'x^{-1}(x')^{-1}) = f(xx'x^{-1}(x')^{-1})$. Since the commutators $[x,x'] = xx'x^{-1}(x')^{-1}$ generate X we must have $f_1 = f$.

EXAMPLE 5.3.4. Every presentation of G gives rise to two natural central extensions as follows. A presentation corresponds to the choice of a free group F mapping onto G, and a description of the kernel $R \subset F$. Since $[R,F]$ is a normal subgroup of F, we may form the following central extensions:

$$1 \to R/[R,F] \to F/[R,F] \to G \to 1,$$

$$1 \to (R \cap [F,F])/[R,F] \to [F,F]/[R,F] \to [G,G] \to 1. \tag{5.3.5}$$

The group $(R \cap [F,F])/[R,F]$ in (5.3.5) is the homology group $H_2(G;\mathbb{Z})$; this identity was discovered in 1941 by Hopf [WHomo, 6.8.8]. If $G = [G,G]$, both are extensions of G, and (5.3.5) is the universal central extension by the following theorem.

RECOGNITION THEOREM 5.4. *Every perfect group G has a universal central extension, namely the extension (5.3.5):*

$$1 \to H_2(G;\mathbb{Z}) \to [F,F]/[R,F] \to G \to 1.$$

Let X be any central extension of G, the following are equivalent: (1) X is a universal central extension; (2) X is perfect, and every central extension of X splits; (3) $H_1(X;\mathbb{Z}) = H_2(X;\mathbb{Z}) = 0$.

PROOF. Given any central extension X of G, the map $F \to G$ lifts to a map $h\colon F \to X$ because F is free. Since $h(R)$ is in the center of X, $h([R,F]) = 1$. Thus h induces a map from $[F,F]/[R,F]$ to X over G. This map is unique by Lemma 5.3.3. This proves that (5.3.5) is a universal central extension, and proves the equivalence of (1) and (3). The implication (1) $\Rightarrow$ (2) is Lemma 5.3.2 and Ex. 5.7, and (2) $\Rightarrow$ (1) is immediate.

THEOREM 5.5 (KERVAIRE, STEINBERG). *The Steinberg group $St(R)$ is the universal central extension of $E(R)$. Hence*

$$K_2(R) \cong H_2(E(R);\mathbb{Z}).$$

This theorem follows immediately from the Recognition Theorem 5.4, and the following splitting result:

PROPOSITION 5.5.1. *If $n \geq 5$, every central extension $Y \xrightarrow{\pi} St_n(R)$ is split. Hence $St_n(R)$ is the universal central extension of $E_n(R)$.*

PROOF. We first show that if $j \neq k$ and $l \neq i$ then every two elements $y, z \in Y$ with $\pi(y) = x_{ij}(r)$ and $\pi(z) = x_{kl}(s)$ must commute in Y. Pick t distinct from i, j, k, l and choose $y', y'' \in Y$ with $\pi(y') = x_{it}(1)$ and $\pi(y'') = x_{tj}(r)$. The Steinberg relations imply that both $[y', z]$ and $[y'', z]$ are in the center of Y, and since $\pi(y) = \pi[y', y'']$ this implies that z commutes with $[y', y'']$ and y.

We now choose distinct indices i, j, k, l and elements $u, v, w \in Y$ with

$$\pi(u) = x_{ij}(1), \quad \pi(v) = x_{jk}(s) \quad \text{and } \pi(w) = x_{kl}(r).$$

If G denotes the subgroup of Y generated by u, v, w then its commutator subgroup $[G, G]$ is generated by elements mapping under π to $x_{ik}(s)$, $x_{jl}(sr)$ or $x_{il}(sr)$. From the first paragraph of this proof it follows that $[u, w] = 1$ and that $[G, G]$ is abelian. By Ex. 5.3 we have $[[u, v], w] = [u, [v, w]]$. Therefore if $\pi(y) = x_{ik}(s)$ and $\pi(z) = x_{jl}(sr)$ we have $[y, w] = [u, z]$. Taking $s = 1$, this identity proves that the element

$$y_{il}(r) = [u, z], \qquad \text{where } \pi(u) = x_{ij}(1),\ \pi(z) = x_{jl}(r)$$

doesn't depend upon the choice of j, nor upon the lifts u and z of $x_{ij}(1)$ and $x_{jl}(r)$.

We claim that the elements $y_{ij}(r)$ satisfy the Steinberg relations, so that there is a group homomorphism $St_n(R) \to Y$ sending $x_{ij}(r)$ to $y_{ij}(r)$. Such a homomorphism will provide the desired splitting of the extension π. The first paragraph of this proof implies that if $j \neq k$ and $l \neq i$ then $y_{ij}(r)$ and $y_{kl}(s)$ commute. The identity $[y, w] = [u, z]$ above may be rewritten as

$$[y_{ik}(r), y_{kl}(s)] = y_{il}(rs) \quad \text{for } i, k, l \text{ distinct.}$$

The final relation $y_{ij}(r)y_{ij}(s) = y_{ij}(r+s)$ is a routine calculation with commutators left to the reader.

REMARK 5.5.2 (STABILITY FOR K_2). The kernel of $St_n(R) \to E_n(R)$ is written as $K_2(n, R)$, and there are natural maps $K_2(n, R) \to K_2(R)$. If R is noetherian of dimension d, or more generally has $sr(R) = d+1$, then the following stability result holds: $K_2(n, R) \cong K_2(R)$ for all $n \geq d+3$. This result evolved in the mid-1970's as a sequence of results by Dennis, Vaserstein, van der Kallen and Suslin-Tulenbaev. We refer the reader to section 19C25 of Math Reviews for more details.

Transfer maps on K_2

Here is a description of $K_2(R)$ in terms of the translation category $t\mathbf{P}(R)$ of f.g. projective R-modules, analogous to the description given for K_1 in Corollary 1.6.3.

PROPOSITION 5.6 (BASS). $K_2(R) \cong \varinjlim_{P \in t\mathbf{P}} H_2([\mathrm{Aut}(P), \mathrm{Aut}(P)]; \mathbb{Z})$.

PROOF. If G is a group, then G acts by conjugation upon $[G, G]$ and hence upon the homology $H_2([G, G]; \mathbb{Z})$. Taking coinvariants, we obtain the functor H_2' from groups to abelian groups defined by $H_2'(G) = H_0(G; H_2([G, G]; \mathbb{Z}))$. By construction, G acts trivially upon $H_2'(G)$ and commutes with direct limits of groups.

Note that if G acts trivially upon $H_2([G, G]; \mathbb{Z})$ then $H_2'(G) = H_2([G, G]; \mathbb{Z})$. For example, $GL(R)$ acts trivially upon the homology of $E(R) = [GL(R), GL(R)]$ by Ex. 1.12. By Theorem 5.5 this implies that $H_2'(GL(R)) = H_2(E(R); \mathbb{Z}) = K_2(R)$.

Since morphisms in the translation category $t\mathbf{P}(R)$ are well-defined up to isomorphism, it follows that $P \mapsto H_2'(\mathrm{Aut}(P))$ is a well-defined functor from $t\mathbf{P}(R)$ to abelian groups. Hence we can take the filtered colimit of this functor, as we did in 1.6.3. Since the free modules are cofinal in $t\mathbf{P}(R)$, the result follows from the identification of the colimit as

$$\lim_{n \to \infty} H_2'(GL_n(R)) \cong H_2'(GL(R)) = K_2(R).$$

COROLLARY 5.6.1 (MORITA INVARIANCE OF K_2). *The group $K_2(R)$ depends only upon the category $\mathbf{P}(R)$. That is, if R and S are Morita equivalent rings (see II.2.7) then $K_2(R) \cong K_2(S)$. In particular, the maps $R \to M_n(R)$ induce isomorphisms on K_2:*

$$K_2(R) \cong K_2(M_n(R)).$$

Let $f: R \to S$ be a ring homomorphism such that the R-module S is finitely generated and projective. In chapter VI, we will construct a transfer map $f_*: K_n(R) \to K_n(S)$ for all n. However, it is easy to construct the transfer map directly on K_2, which is sometimes also called the *norm* homomorphism $N_{S/R}$.

COROLLARY 5.6.2 (FINITE TRANSFER). *Suppose that S is f.g. projective as an R-module. Then there is a natural transfer homomorphism $f_*: K_2(S) \to K_2(R)$.*

If R is commutative, so that $K_2(R)$ is a $K_0(R)$-module by Ex. 5.4, the composition $f_* f^*: K_2(R) \to K_2(S) \to K_2(R)$ is multiplication by $[S] \in K_0(R)$. In particular, if S is free of rank n, then $f_* f^*$ is multiplication by n.

PROOF. The construction of f_* is obtained by replacing $H_1(\mathrm{Aut})$ by $H_2'(\mathrm{Aut})$ in the proof of Lemma 1.7. The composite $f_* f^*$ is obtained by composing the functor $H_2([\mathrm{Aut}, \mathrm{Aut}])$ with $t\mathbf{P}(R) \xrightarrow{\otimes S} t\mathbf{P}(S) \to t\mathbf{P}(R)$. But this composite is the self-map $\otimes_R S$ on $t\mathbf{P}(R)$ giving rise to multiplication by $[S]$ on $K_2(R)$ in Ex. 5.5.

REMARK 5.6.3. We will see in chapter V that we can also define a transfer map $K_2(S) \to K_2(R)$ when S is a finite R-algebra of finite projective dimension over R.

Relative K_2 and relative Steinberg groups

Given an ideal I in a ring R, we may construct the augmented ring $R \oplus I$, with multiplication $(r,x)(s,y) = (rs, ry + xs + xy)$. This ring is equipped with two natural maps $pr, add: R \oplus I \to R$, defined by $pr(r,x) = r$ and $add(r,x) = r + x$. This ring was used to define the relative group $K_0(I)$ in Ex. II.2.3.

Let $St'(R,I)$ denote the normal subgroup of $St(R \oplus I)$ generated by all $x_{ij}(0,v)$ with $v \in I$. Clearly there is a map from $St'(R,I)$ to the subgroup $E(R \oplus I, 0 \oplus I)$ of $GL(R \oplus I)$ (see Lemma 2.1), and an exact sequence

$$1 \to St'(R,I) \to St(R \oplus I) \xrightarrow{pr} St(R) \to 1.$$

The following definition is taken from [Keune] and [Loday78].

Definition 5.7. The *relative Steinberg group $St(R,I)$* is defined to be the quotient of $St'(R,I)$ by the normal subgroup generated by all "cross-commutators" $[x_{ij}(0,u), x_{kl}(v,-v)]$ with $u, v \in I$.

Clearly the homomorphism $St(R \oplus I) \xrightarrow{add} St(R)$ sends these cross-commutators to 1, so it induces a homomorphism $St(R,I) \xrightarrow{add} St(R)$ whose image is the normal subgroup generated by the $x_{ij}(v)$, $v \in I$. By the definition of $E(R,I)$, the surjection $St(R) \to E(R)$ maps $St(R,I)$ onto $E(R,I)$. We define $K_2(R,I)$ to be the kernel of the map $St(R,I) \to E(R,I)$.

Theorem 5.7.1. *If I is an ideal of a ring R, then the exact sequence of Proposition 2.3 extends to an exact sequence*

$$K_2(R,I) \to K_2(R) \to K_2(R/I) \to K_1(R,I) \to K_1(R) \to K_1(R/I) \to K_0(I) \cdots$$

Proof. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
K_2(R,I) \arrow[r] \arrow[d] & St(R,I) \arrow[r] \arrow[d, "add"'] & GL(I) \arrow[r] \arrow[d, "into"'] & K_1(R,I) \arrow[d] \\
K_2(R) \arrow[r] \arrow[d] & St(R) \arrow[r] \arrow[d, "onto"'] & GL(R) \arrow[r] \arrow[d] & K_1(R) \arrow[d] \\
K_2(R/I) \arrow[r] & St(R/I) \arrow[r] & GL(R/I) \arrow[r] & K_1(R/I)
\end{tikzcd}$$

The exact sequence now follows from the Snake Lemma and Ex. 5.1.

If I and J are ideals in a ring R with $I \cap J = 0$, we may also consider I as an ideal of R/J. As in §1, these rings form a Milnor square:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R/J \arrow[d] \\
R/I \arrow[r] & R/(I+J).
\end{tikzcd}$$

THEOREM 5.8 (MAYER-VIETORIS). *If I and J are ideals of R with $I \cap J = 0$, then the Mayer-Vietoris sequence of Theorem 2.6 can be extended to K_2:*

$$\begin{aligned} &K_2(R) \xrightarrow{\Delta} K_2(R/I) \oplus K_2(R/J) \xrightarrow{\pm} K_2(R/I+J) \xrightarrow{\partial} \\ &K_1(R) \xrightarrow{\Delta} K_1(R/I) \oplus K_1(R/J) \xrightarrow{\pm} K_1(R/I+J) \xrightarrow{\partial} K_0(R) \to \cdots \end{aligned}$$

PROOF. By Ex. 5.10, we have the following commutative diagram:

$$\begin{tikzcd} K_2(R,I) \arrow[r] \arrow[d, "onto"'] & K_2(R) \arrow[r] \arrow[d] & K_2(R/I) \arrow[r] \arrow[d] & K_1(R,I) \arrow[r] \arrow[d, equal] & K_1(R) \arrow[r] \arrow[d] & K_1(R/I) \arrow[d] \\ K_2(R/J,I) \arrow[r] & K_2(R/J) \arrow[r] & K_2(\frac{R}{I+J}) \arrow[r] & K_1(R/J,I) \arrow[r] & K_1(R/J) \arrow[r] & K_1(\frac{R}{I+J}) \end{tikzcd}$$

By chasing this diagram, we obtain the exact Mayer-Vietoris sequence.

Commutative Banach Algebras

Let R be a commutative Banach algebra over the real or complex numbers. Just as $SK_1(R) = \pi_0 SL(R)$ and $K_1(R)$ surjects onto $\pi_0 GL(R)$ (by 1.5 and 1.5.1), there is a relation between $K_2(R)$ and $\pi_1 GL(R)$.

PROPOSITION 5.9. *Let R be a commutative Banach algebra. Then there is a surjection from $K_2(R)$ onto $\pi_1 SL(R) = \pi_1 E(R)$.*

PROOF. ([Milnor, p.59]) By Proposition 1.5, we know that $E_n(R)$ is the path component of the identity in the topological group $SL_n(R)$, so $\pi_1 SL(R) = \pi_1 E(R)$. Using the exponential map $M_n(R) \to GL_n(R)$, we see that $E_n(R)$ is locally contractible, so it has a universal covering space $\widetilde{E}_n$. The group map $\widetilde{E}_n \to E_n(R)$ is a central extension with kernel $\pi_1 E_n(R)$. Taking the direct limit as $n \to \infty$, we get a central extension $1 \to \pi_1 E(R) \to \widetilde{E} \to E(R) \to 1$. By universality, there is a unique homomorphism $\tilde{\phi}\colon St(R) \to \widetilde{E}$ over $E(R)$, and hence a unique map $K_2(R) \to \pi_1 E(R)$. Thus it suffices to show that $\tilde{\phi}$ is onto.

The map $\tilde{\phi}$ may be constructed explicitly as follows. Let $\tilde{e}_{ij}(r) \in \widetilde{E}$ be the endpoint of the path which starts at 1 and lifts the path $t \mapsto e_{ij}(tr)$ in $E(R)$. We claim that the map $\tilde{\phi}$ sends $x_{ij}(r)$ to $\tilde{e}_{ij}(r)$. To see this, it suffices to show that the Steinberg relations (5.1) are satisfied. But the paths $\tilde{e}_{ij}(tr)\tilde{e}_{ij}(ts)$ and $[\tilde{e}_{ij}(tr), \tilde{e}_{kl}(s)]$ cover the two paths $e_{ij}(tr)e_{ij}(s)$ and $[e_{ij}(tr), e_{kl}(s)]$ in $E(R)$. Evaluating at $t = 1$ yields the Steinberg relations.

By Proposition 1.5 there is a neighborhood U_n of 1 in $SL_n(R)$ in which we may express every matrix g as a product of elementary matrices $e_{ij}(r)$, where r depends continuously upon g. Replacing each $e_{ij}(r)$ with $\tilde{e}_{ij}(r)$ defines a continuous lifting of U_n to $\widetilde{E}_n$. Therefore the image of each map $\tilde{\phi}\colon St_n(R) \to \widetilde{E}_n$ contains a neighborhood $\widetilde{U}_n$ of 1. Since any open subset of a connected group (such as $\widetilde{E}_n$) generates the entire group, this proves that each $\tilde{\phi}_n$ is surjective. Passing to the limit as $n \to \infty$, we see that $\tilde{\phi}\colon St(R) \to \widetilde{E}$ is also surjective.

EXAMPLE 5.9.1. If $R=\mathbb{R}$ then $\pi_1 SL(\mathbb{R})\cong\pi_1 SO$ is cyclic of order 2. It follows that $K_2(\mathbb{R})$ has at least one nontrivial element. In fact, the symbol $\{-1,-1\}$ of Example 5.1.1 maps to the nonzero element of $\pi_1 SO$. We will see in 6.8.3 below that the kernel of $K_2(\mathbb{R})\to\pi_1 SO$ is a uniquely divisible abelian group with uncountably many elements.

EXAMPLE 5.9.2. Let X be a compact space with a nondegenerate basepoint. By Ex. II.3.11, we have $KO^{-2}(X)\cong[X,\Omega SO]=\pi_1 SL(\mathbb{R}^X)$, so $K_2(\mathbb{R}^X)$ maps onto the group $KO^{-2}(X)$.

Similarly, since $\Omega U\simeq\mathbb{Z}\times\Omega SU$, we see by Ex. II.3.11 that $KU^{-2}(X)\cong[X,\Omega U]=[X,\mathbb{Z}]\times[X,\Omega SU]$. Since $\pi_1 SL(\mathbb{C}^X)=\pi_1(SU^X)=[X,\Omega SU]$ and $[X,\mathbb{Z}]$ is a subgroup of $\mathbb{C}^X$, we can combine Proposition 5.9 with Example 1.5.3 to obtain the exact sequence

$$K_2(\mathbb{C}^X)\to KU^{-2}(X)\to\mathbb{C}^X\xrightarrow{\exp}K_1(\mathbb{C}^X)\to KU^{-1}(X)\to 0.$$

Steinberg symbols

If two matrices $A,B\in E(R)$ commute, we can construct an element in $K_2(R)$ by lifting their commutator to $St(R)$. To do this, choose $a,b\in St(R)$ with $\phi(a)=A$, $\phi(b)=B$ and define $A\bigstar B=[a,b]\in K_2(R)$. This definition is independent of the choice of a and b because any other lift will equal ac, bc' for central elements c,c', and $[ac,bc']=[a,b]$.

If $P\in GL(R)$ then $(PAP^{-1})\bigstar(PBP^{-1})=A\bigstar B$. To see this, suppose that $A,B,P\in GL_n(R)$ and let $g\in St_{2n}(R)$ be a lift of the block diagonal matrix $D=\operatorname{diag}(P,P^{-1})$. Since gag^{-1} and gbg^{-1} lift PAP^{-1} and PBP^{-1} and $[a,b]$ is central we have the desired relation: $[gag^{-1},gbg^{-1}]=g[a,b]g^{-1}=[a,b]$.

The $\bigstar$ symbol is also skew-symmetric and bilinear: $(A\bigstar B)(B\bigstar A)=1$ and $(A_1A_2)\bigstar B=(A_1\bigstar B)(A_2\bigstar B)$. These relations are immediate from the commutator identies $[a,b][b,a]=1$ and $[a_1a_2,b]=[a_1,[a_2,b]][a_2,b][a_1,b]$.

DEFINITION 5.10. If r,s are units in a commutative ring R, we define the Steinberg symbol $\{r,s\}\in K_2(R)$ to be

$$\{r,s\}=\begin{pmatrix}r&&\\&r^{-1}&\\&&1\end{pmatrix}\bigstar\begin{pmatrix}s&&\\&1&\\&&s^{-1}\end{pmatrix}=\begin{pmatrix}r&&\\&1&\\&&r^{-1}\end{pmatrix}\bigstar\begin{pmatrix}s&&\\&s^{-1}&\\&&1\end{pmatrix}.$$

Because the $\bigstar$ symbols are skew-symmetric and bilinear, so are the Steinberg symbols: $\{r,s\}\{s,r\}=1$ and $\{r_1r_2,s\}=\{r_1,s\}\{r_2,s\}$.

EXAMPLE 5.10.1. For any unit r of R we set $w_{ij}(r)=x_{ij}(r)x_{ji}(-r^{-1})x_{ij}(r)$ and $h_{ij}(r)=w_{ij}(r)w_{ij}(-1)$. In $GL(R)$, $\phi w_{ij}(r)$ is the monomial matrix with r and $-r^{-1}$ in the (i,j) and (j,i) places, while $\phi h_{ij}(r)$ is the diagonal matrix with r and r^{-1} in the i^{th} and j^{th} diagonal spots. By definition we then have:

$$\{r,s\}=[h_{12}(r),h_{13}(s)]=[h_{ij}(r),h_{ik}(s)].$$

LEMMA 5.10.2. *If both r and $1-r$ are units of R, then in $K_2(R)$ we have:*

$$\{r, 1-r\} = 1 \qquad \text{and} \qquad \{r, -r\} = 1.$$

PROOF. By Ex. 5.8, $w_{12}(-1) = w_{21}(1) = x_{21}(1)x_{12}(-1)x_{21}(1)$, $w_{12}(r)x_{21}(1) = x_{12}(-r^2)w_{12}(r)$ and $x_{21}(1)w_{12}(s) = w_{12}(s)x_{12}(-s^2)$. If $s = 1-r$ we can successively use the identities $r - r^2 = rs$, $r + s = 1$, $s - s^2 = rs$ and $\frac{1}{r} + \frac{1}{s} = \frac{1}{rs}$ to obtain:

$$\begin{aligned} w_{12}(r)w_{12}(-1)w_{12}(s) &= x_{12}(-r^2)w_{12}(r)x_{12}(-1)\; w_{12}(s)x_{12}(-s^2) \\ &= x_{12}(rs)x_{21}(-r^{-1})x_{12}(0)x_{21}(-s^{-1})x_{12}(rs) \\ &= x_{12}(rs)x_{21}(\frac{-1}{rs})x_{12}(rs) \\ &= w_{12}(rs). \end{aligned}$$

Multiplying by $w_{12}(-1)$ yields $h_{12}(r)h_{12}(s) = h_{12}(rs)$ when $r + s = 1$. By Ex. 5.9, this yields the first equation $\{r, s\} = 1$. Since $-r = (1-r)/(1-r^{-1})$, the first equation implies the second equation:

$$\{r, -r\} = \{r, 1-r\}\{r, 1-r^{-1}\}^{-1} = \{r^{-1}, 1-r^{-1}\} = 1. \tag{5.10.3}$$

REMARK 5.10.4. The equation $\{r, -r\} = 1$ holds more generally for every unit r, even if $1-r$ is not a unit. This follows from the fact that $K_2(\mathbb{Z}[r, \frac{1}{r}])$ injects into $K_2(\mathbb{Z}[r, \frac{1}{r}, \frac{1}{1-r}])$, a fact we shall establish in chapter V. For a direct proof, see [Milnor, 9.8].

The following useful result was proven for fields and division rings in §9 of [Milnor]. It was extended to commutative semilocal rings by Dennis and Stein [DS], and we cite it here for completeness.

THEOREM 5.10.5. *If R is a field, division ring, local ring, or even a semilocal ring, then $K_2(R)$ is generated by the Steinberg symbols $\{r, s\}$.*

DEFINITION 5.11 (DENNIS-STEIN SYMBOLS). If $r, s \in R$ commute and $1 - rs$ is a unit then the element

$$\langle r, s\rangle = x_{ji}\big(-s(1-rs)^{-1}\big)x_{ij}(-r)x_{ji}(s)x_{ij}\big((1-rs)^{-1}r\big)h_{ij}(1-rs)^{-1}$$

of $St(R)$ belongs to $K_2(R)$, because $\phi\langle r, s\rangle = 1$. By Ex. 5.11, it is independent of the choice of $i \neq j$, and if r is a unit of R then $\langle r, s\rangle = \{r, 1-rs\}$. If I is an ideal of R and $s \in I$ then we can even consider $\langle r, s\rangle$ as an element of $K_2(R, I)$; see 5.7. These elements are called *Dennis-Stein symbols* because they were first studied in [DS], where the following identities were established.

(D1) $\langle r, s\rangle\langle s, r\rangle = 1$
(D2) $\langle r, s\rangle\langle r, t\rangle = \langle r, s + t - rst\rangle$
(D3) $\langle r, st\rangle = \langle rs, t\rangle\langle tr, s\rangle$ (this holds in $K_2(R, I)$ if any of r, s, or t are in I.)

We warn the reader that the meaning of the symbol $\langle r, s\rangle$ changed circa 1980. We use the modern definition of this symbol, which equals $\langle -r, s\rangle^{-1}$ in the old literature, including that of *loc.cit.*.

The following result is essentially due to Maazen, Stienstra and van der Kallen. However, their work preceded the correct definition of $K_2(R, I)$ so the correct historical reference is [Keune].

THEOREM 5.11.1. *(a) Let R be a commutative local ring, or a field. Then $K_2(R)$ may be presented as the abelian group generated by the symbols $\langle r, s\rangle$ with $r, s \in R$ such that $1 - rs$ is a unit, subject only to the relations (D1), (D2) and (D3).*

(b) Let I be a radical ideal, contained in a commutative ring R. Then $K_2(R, I)$ may be presented as the abelian group generated by the symbols $\langle r, s\rangle$ with either $r \in R$ and $s \in I$, or else $r \in I$ and $s \in R$. These generators are subject only to the relations (D1), (D2), and the relation (D3) whenever r, s, or t is in I.

The product $K_1(R) \otimes K_1(R) \to K_2(R)$

Let R be a commutative ring, and suppose given two invertible matrices $g \in GL_m(R)$, $h \in GL_n(R)$. Identifying the tensor product $R^m \otimes R^n$ with R^{m+n}, then $g\otimes 1_n$ and $1_m\otimes h$ are commuting automorphisms of $R^m\otimes R^n$. Hence there is a ring homomorphism from $A = \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ to $E = \mathrm{End}_R(R^m \otimes R^n) \cong M_{m+n}(R)$ sending x and y to $g \otimes 1_n$ and $1_m \otimes h$. Recall that by Morita Invariance 5.6.1 the natural map $K_2(R) \to K_2(E)$ is an isomorphism.

DEFINITION 5.12. The element $\{g, h\}$ of $K_2(R)$ is defined to be the image of the Steinberg symbol $\{x, y\}$ under the homomorphism $K_2(A) \to K_2(E) \cong K_2(R)$.

Note that if $m = n = 1$ this agrees with the definition of the usual Steinberg symbol in 5.10, because $R = E$.

LEMMA 5.12.1. *The symbol $\{g, h\}$ is independent of the choice of m and n, and is skew-symmetric. Moreover, for each $\alpha \in GL_m(R)$ we have $\{g, h\} = \{\alpha g\alpha^{-1}, h\}$.*

PROOF. If we embed $GL_m(R)$ and $GL_n(R)$ in $GL_{m'}(R)$ and $GL_{n'}(R)$, respectively, then we embed E into the larger ring $E' = \mathrm{End}_R(R^{m'} \otimes R^{n'})$, which is also Morita equivalent to R. Since the natural maps $K_2(R) \to K_2(E) \to K_2(E')$ are isomorphisms, and $K_2(A) \to K_2(E) \to K_2(E') \cong K_2(R)$ defines the symbol with respect to the larger embedding, the symbol is independent of m and n.

Any linear automorphism of R^{m+n} induces an inner automorphism of E. Since the composition of $R \to E$ with such an automorphism is still $R \to E$, the symbol $\{g, h\}$ is unchanged by such an operation. Applying this to $\alpha \otimes 1_n$, the map $A \to E \to E$ sends x and y to $\alpha g\alpha^{-1} \otimes 1_n$ and $1_m\otimes h$, so $\{g, h\}$ must equal $\{\alpha g\alpha^{-1}, h\}$.

As another application, note that if $m = n$ the inner automorphism of E induced by $R^m\otimes R^n \cong R^n\otimes R^m$ sends $\{h, g\}$ to the image of $\{y, x\}$ under $K_2(A) \to K_2(E)$. This proves skew-symmetry, since $\{y, x\} = \{x, y\}^{-1}$.

THEOREM 5.12.2. *For every commutative ring R, there is a skew-symmetric bilinear pairing $K_1(R) \otimes K_1(R) \to K_2(R)$ induced by the symbol $\{g, h\}$.*

PROOF. We first show that the symbol is bimultiplicative when g and g' commute in $GL_m(R)$. Mapping $A[z, z^{-1}]$ into E by $z \mapsto g' \otimes 1_n$ allows us to deduce $\{gg', h\} = \{g, h\}\{g', h\}$ from the corresponding property of Steinberg symbols. If g and g' do not commute, the following trick establishes bimultiplicativity:

$$\{gg', h\} = \left\{\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & g'\end{pmatrix}, h\right\} = \left\{\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}, h\right\}\left\{\begin{pmatrix} 1 & 0 \\ 0 & g'\end{pmatrix}, h\right\} = \{g, h\}\{g', h\}.$$

If either g of h is a commutator, this implies that the symbol $\{g, h\}$ vanishes in the abelian group $K_2(R)$. Since the symbol $\{g, h\}$ is compatible with stabilization, it describes a function $K_1(R)\times K_1(R) \to K_2(R)$ which is multiplicative in each entry: $\{gg', h\} = \{g, h\}\{g', h\}$. If we write K_1 and K_2 additively the function is additive in each entry, i.e., bilinear.

EXERCISES

5.1 *Relative Steinberg groups.* Let I be an ideal in a ring R. Show that there is an exact sequence $St(R, I) \xrightarrow{add} St(R) \to St(R/I) \to 1$.

5.2 Consider the function $\rho_n: R^{n-1} \to St_n(R)$ sending $(r_1, ..., r_{n-1})$ to the product $x_{1n}(r_1)x_{2n}(r_2)\cdots x_{n-1,n}(r_{n-1})$. The Steinberg relations show that this is a group homomorphism. Show that ρ is an injection by showing that the composition $\phi\rho: R^{n-1} \to St_n(R) \to GL_n(R)$ is an injection. Then show that the elements $x_{ij}(r)$ with $i, j < n$ normalize the subgroup $P_n = \rho(R^n)$ of $St_n(R)$, *i.e.*, that $x_{ij}(r)P_nx_{ij}(-r) = P_n$.

5.2 Use the previous exercise and induction to show that there the subgroup T_n of $St_n(R)$ generated by the $x_{ij}(r)$ with $i < j$ maps isomorphically onto the subgroup of lower triangular matrices in $GL_n(R)$.

5.3 Let G be a group whose commutator group $[G, G]$ is abelian. Prove that the Jacobi identity holds for every $u, v, w \in G$:

$$[u, [v, w]][v, [w, u]][w, [u, v]] = 1.$$

If in addition $[u, w] = 1$ this implies that $[[u, v], w] = [u, [v, w]]$.

5.4 *Product with K_0.* Construct a product operation $K_0(R) \otimes K_2(A) \to K_2(A)$, assuming that R is commutative and A is an associative R-algebra. To do this, fix a f.g. projective R-module P. Each isomorphism $P \oplus Q = R^n$ gives rise to a homomorphism $h^P: GL_m(A) \to GL_{mn}(A) \subset GL(A)$ sending α to $\alpha \otimes 1$ and $E_m(A)$ to $E(A)$. Show that h^P is well-defined up to conjugation by an element of $E(A)$. Since conjugation acts trivially on homology, this implies that the induced map ${h^P}_*: H_2(E_m(A); \mathbb{Z}) \to H_2(E(A); \mathbb{Z}) = K_2(A)$ is well-defined. Then show that ${h^{P\oplus Q}}_* = {h^P}_* \oplus {h^Q}_*$ and pass to the limit as $m \to \infty$ to obtain the required endomorphism $[P]\cdot$ of $K_2(A)$.

5.5 If R is commutative and $P \in \mathbf{P}(R)$, show that $Q \mapsto Q \otimes_R P$ defines a functor from the translation category $t\mathbf{P}(A)$ to itself for every R-algebra A, and that the resulting endomorphism of $K_2(A) = \varinjlim H_2([\mathrm{Aut}(Q), \mathrm{Aut}(Q)])$ is the map h^P_* of the previous exercise. Use this description to show that the product makes $K_2(A)$ into a module over the ring $K_0(R)$.

5.6 *Projection Formula.* Suppose that $f: R \to S$ is a finite map of commutative rings, with $S \in \mathbf{P}(R)$. Show that for all $r \in K_i(R)$ and $s \in K_j(S)$ with $i + j = 2$ we have $f_*(f^*(r) \cdot s) = r \cdot f_*(s)$ in $K_2(R)$. The case $i = 0$ states that the transfer $f_*: K_2(S) \to K_2(R)$ is $K_0(R)$-linear, while the case $i = 1$ yields the useful formula $f_*\{r, s\} = \{r, Ns\}$ for Steinberg symbols in $K_2(R)$, where $r \in R^\times$, $s \in S^\times$ and $Ns = f_*(s) \in R^\times$ is the norm of s.

5.7 If $Y \xrightarrow{\rho} X$ and $X \xrightarrow{\pi} G$ are central extensions, show that the "composition" $Y \xrightarrow{\pi\rho} G$ is also a central extension. If X is a universal central extension of G, conclude that every central extension $Y \xrightarrow{\rho} X$ splits.

5.8 Show that the following identies hold in $St(R)$ (for i, j and k distinct).

(a) $w_{ij}(r)w_{ij}(-r) = 1$;
(b) $w_{ik}(r)x_{ij}(s)w_{ik}(-r) = x_{kj}(-r^{-1}s)$;
(c) $w_{ij}(r)x_{ij}(s)w_{ij}(-r) = x_{ji}(-r^{-1}sr^{-1})$;
(d) $w_{ij}(r)x_{ji}(s)w_{ij}(-r) = x_{ij}(-rsr)$;
(e) $w_{ij}(r)w_{ji}(r^{-1}) = 1$;

5.9 Use the previous exercise to show that $\{r, s\} = h_{ij}(rs)h_{ij}(s)^{-1}h_{ij}(r)^{-1}$. *Hint:* Conjugate $h_{ij}(s)$ by $w_{ik}(r)w_{ik}(-1)$.

5.10 *Excision.* If I and J are ideals in a ring R with $I \cap J = 0$, we may also consider I as an ideal of R/J. Show that $St(R, I)$ surjects onto $St(R/J, I)$, while the subgroups $E(R, I)$ and $E(R/J, I)$ of $GL(I)$ are equal. Use the 5-lemma to conclude that $K_1(R, I) \cong K_1(R/J, I)$ and that $K_2(R, I) \to K_2(R/J, I)$ is onto.

5.11 *Dennis-Stein symbols.* Let $\langle r, s\rangle_{ij}$ denote the element of $St(R)$ given in Definition 5.11. Show that this element is in $K_2(R)$. Then use Ex. 5.8 to show that if w is a product of elements $w_{ij}(1)$ such that the permutation matrix send i, j to k, ℓ then $w\langle r, s\rangle_{ij}w^{-1} = \langle r, s\rangle_{k\ell}$. This shows that the Dennis-Stein symbol is independent of the choice of indices i, j.

5.12 Let A be an abelian group and F a field. Show that, for all $n \geq 5$, homomorphisms $K_2(F) \xrightarrow{c} A$ are in 1–1 correspondence with central extensions of $SL_n(F)$ having kernel A.

5.13 If p is an odd prime, use Theorem 5.11.1 to show that $K_2(\mathbb{Z}/p^n) = 1$. If $n \geq 2$, show that $K_2(\mathbb{Z}/2^n) \cong K_2(\mathbb{Z}/4) \cong \{\pm 1\}$.

5.14 Let R be a commutative ring, and let Ω_R denote the module of Kähler differentials of R over $\mathbb{Z}$, as in Ex. 2.6.

(a) Show that there is a surjection from $K_2(R, I)$ onto $I \otimes_R \Omega_{R/I}$, sending $\langle x, r\rangle$ to $x \otimes dr$ ($r \in R$, $x \in I$).
(b) The *dual numbers* over R is the ring $R[\varepsilon]$ with $\varepsilon^2 = 0$. If $\frac{1}{2} \in R$, show that the map $K_2(R[\varepsilon], \varepsilon) \to \Omega_R$ of part (a) is an isomorphism.

§6. K_2 of fields

The following theorem was proven by Hideya Matsumoto in 1969. We refer the reader to [Milnor, §12] for a self-contained proof.

MATSUMOTO'S THEOREM 6.1. *If F is a field then $K_2(F)$ is the abelian group generated by the set of Steinberg symbols $\{x, y\}$ with $x, y \in F^\times$, subject only to the relations:*

(1) *(Bilinearity) $\{xx', y\} = \{x, y\}\{x', y\}$ and $\{x, yy'\} = \{x, y\}\{x, y'\}$;*
(2) *(Steinberg Relation) $\{x, 1-x\} = 1$ for all $x \neq 0, 1$.*

In other words, $K_2(F)$ is the quotient of $F^\times \otimes F^\times$ by the subgroup generated by the elements $x \otimes (1-x)$. Note that the calculation (5.10.3) implies that $\{x, -x\} = 1$ for all x, and this implies that the Steinberg symbols are skew-symmetric: $\{x, y\}\{y, x\} = \{x, -xy\}\{y, -xy\} = \{xy, -xy\} = 1$.

Corollary 6.1.1. *$K_2(\mathbb{F}_q) = 1$ for every finite field $\mathbb{F}_q$.*

Proof. If x generates the cyclic group $\mathbb{F}_q^\times$, we must show that the generator $x\otimes x$ of the cyclic group $\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times$ vanishes in K_2. If q is even, then $\{x,x\} = \{x,-x\} = 1$, so we may suppose that q is odd. Since $\{x,x\}^2 = 1$ by skew-symmetry, we have $\{x,x\} = \{x,x\}^{mn} = \{x^m,x^n\}$ for every odd m and n. Since odd powers of x are the same as non-squares, it suffices to find a non-square u such that $1-u$ is also a non-square. But such a u exists because $u \mapsto (1-u)$ is an involution on the set $\mathbb{F}_q - \{0,1\}$, and this set consists of $(q-1)/2$ non-squares but only $(q-3)/2$ squares.

Example 6.1.2. Let $F(t)$ be a rational function field in one variable t over F. Then $K_2(F)$ is a direct summand of $K_2F(t)$.

To see this, we construct a map $\lambda\colon K_2F(t) \to K_2(F)$ inverse to the natural map $K_2(F) \to K_2F(t)$. To this end, we define the *leading coefficient* of the rational function $f(t) = (a_0t^n + \cdots + a_n)/(b_0t^m + \cdots + b_m)$ to be $\mathrm{lead}(f) = a_0/b_0$ and set $\lambda(\{f,g\}) = \{\mathrm{lead}(f), \mathrm{lead}(g)\}$. To see that this defines a homomorphism $K_2F(t) \to K_2(F)$, we check the presentation in Matsumoto's Theorem. Bilinearity is clear from $\mathrm{lead}(f_1f_2) = \mathrm{lead}(f_1)\mathrm{lead}(f_2)$, and $\{\mathrm{lead}(f), \mathrm{lead}(1-f)\} = 1$ holds in $K_2(F)$ because $\mathrm{lead}(1-f)$ is either 1, $1-\mathrm{lead}(f)$ or $-\mathrm{lead}(f)$, according to whether $m > n$, $m = n$ or $m < n$.

Because K_2 commutes with filtered colimits, it follows that $K_2(F)$ injects into $K_2F(T)$ for *every* purely transcendental extension $F(T)$ of F.

Lemma 6.1.3. *For every field extension $F \subset E$, the kernel of $K_2(F) \to K_2(E)$ is a torsion subgroup.*

Proof. E is an algebraic extension of some purely transcendental extension $F(X)$ of F, and $K_2(F)$ injects into $K_2F(X)$ by Example 6.1.2. Thus we may assume that E is algebraic over F. Since E is the filtered union of finite extensions, we may even assume that E/F is a finite field extension. But in this case the result holds because (by 5.6.2) the composite $K_2(F) \to K_2(E) \to K_2(F)$ is multiplication by the integer $[E:F]$.

The next result is useful for manipulations with symbols.

Lemma 6.1.4 (Bass-Tate). *If $E = F(u)$ is a field extension of F, then every symbol of the form $\{b_1u - a_1, b_2u - a_2\}$ $(a_i, b_i \in F)$ is a product of symbols $\{c_i, d_i\}$ and $\{c_i, u - d_i\}$ with $c_i, d_i \in F$.*

Proof. Bilinearity allows us to assume that $b_1 = b_2 = 1$. Set $x = u - a_1$, $y = u - a_2$ and $a = a_2 - a_1$, so $x = a + y$. Then $1 = \frac{a}{x} + \frac{y}{x}$ yields the relation $1 = \{\frac{a}{x}, \frac{y}{x}\}$. Using $\{x,x\} = \{-1,x\}$, this expands to the desired expression: $\{x,y\} = \{a,y\}\{-1,x\}\{a^{-1},x\}$.

Together with the Projection Formula (Ex. 5.6), this yields:

Corollary 6.1.5. *If $E = F(u)$ is a quadratic field extension of F, then $K_2(E)$ is generated by elements coming from $K_2(F)$, together with elements of the form $\{c, u-d\}$. Thus the transfer map $N_{E/F}\colon K_2(E) \to K_2(F)$ is completely determined by the formulas $N_{E/F}\{c,d\} = \{c,d\}^2$, $N_{E/F}\{c,u-d\} = \{c, N(u-d)\}$ $(c, d \in F)$*

EXAMPLE 6.1.6. Since $\mathbb{C}$ is a quadratic extension of $\mathbb{R}$, every element of $K_2(\mathbb{C})$ is a product of symbols $\{r, s\}$ and $\{r, e^{i\theta}\}$ with $r, s, \theta \in \mathbb{R}$. Moreover, $N\{r, e^{i\theta}\} = 1$ in $K_2(\mathbb{R})$. Under the automorphism of $K_2(\mathbb{C})$ induced by complex conjugation, the symbols of the first kind are fixed and the symbols of the second kind are sent to their inverses. We will see in Theorem 6.4 below that $K_2(\mathbb{C})$ is uniquely divisible, *i.e.*, a vector space over $\mathbb{Q}$, and the decomposition of $K_2(\mathbb{C})$ into eigenspaces for ± 1 corresponds to symbols of the first and second kind.

EXAMPLE 6.1.7. Let F be an algebraically closed field. By Lemma 6.1.4, $K_2F(t)$ is generated by linear symbols of the form $\{a, b\}$ and $\{t - a, b\}$. It will follow from 6.5.2 below that every element u of $K_2F(t)$ uniquely determines finitely many elements $a_i \in F$, $b_i \in F^\times$ so that $u = \lambda(u) \prod \{t - a_i, b_i\}$, where $\lambda(u) \in K_2(F)$ was described in Example 6.1.2.

Steinberg symbols

DEFINITION 6.2. A *Steinberg symbol* on a field F with values in a multiplicative abelian group A is a bilinear map $c: F^\times \otimes F^\times \to A$ satisfying $c(r, 1 - r) = 1$. By Matsumoto's Theorem, these are in 1–1 correspondence with homomorphisms $K_2(F) \xrightarrow{c} A$.

EXAMPLE 6.2.1. There is a Steinberg symbol $(x, y)_\infty$ on the field $\mathbb{R}$ with values in the group $\{\pm 1\}$. Define $(x, y)_\infty$ to be: -1 if both x and y are negative, and $+1$ otherwise. The Steinberg relation $(x, 1-x)_\infty = +1$ holds because x and $1-x$ cannot be negative at the same time. The resulting map $K_2(\mathbb{R}) \to \{\pm 1\}$ is onto because $(-1, -1)_\infty = -1$. This shows that the symbol $\{-1, -1\}$ in $K_2(\mathbb{Z})$ is nontrivial, as promised in 5.2.2, and even shows that $K_2(\mathbb{Z})$ is a direct summand in $K_2(\mathbb{R})$.

For our next two examples, recall that a *local field* is a field F which is complete under a discrete valuation v, and whose residue field k_v is finite. Classically, every local field is either a finite extension of the p-adic rationals $\hat{\mathbb{Q}}_p$ or of $\mathbb{F}_p((t))$.

EXAMPLE 6.2.2 (HILBERT SYMBOLS). Let F be a local field containing $\frac{1}{2}$. The Hilbert (quadratic residue) symbol on F is defined by setting $c_F(r, s) \in \{\pm 1\}$ equal to $+1$ or -1, depending on whether or not the equation $rx^2 + sy^2 = 1$ has a solution in F. Bilinearity is classical when F is local; see [OMeara, p.164]. The Steinberg relation is trivial, because $x = y = 1$ is always a solution when $r + s = 1$.

Of course, the definition of $c_F(r, s)$ makes sense for any field of characteristic $\neq 2$, but it will not always be a Steinberg symbol because it can fail to be bilinear in r. It is a Steinberg symbol when $F = \mathbb{R}$, because the Hilbert symbol $c_\mathbb{R}(r, s)$ is the same as the symbol $(r, s)_\infty$ of the previous example.

EXAMPLE 6.2.3 (NORM RESIDUE SYMBOLS). The roots of unity in a local field F form a finite cyclic group μ, equal to the group μ_m of all m^{th} roots of unity for some integer m with $\frac{1}{m} \in F$. The classical m^{th} *power norm residue symbol* is a map $K_2(F) \to \mu_m$ defined as follows (see [S-LF] for more details).

Because $F^{\times m}$ has finite index in $F^\times$, there is a finite "Kummer" extension K containing the m^{th} roots of every element of F. The Galois group $G_F = Gal(K/F)$ is canonically isomorphic to $\mathrm{Hom}(F^\times, \mu_m)$, with the automorphism g of K corresponding to the homomorphism $\zeta: F^\times \to \mu_m$ sending $a \in F^\times$ to $\zeta(a) = g(x)/x$, where $x^m = a$. In addition, the cokernel of the norm map $K^\times \xrightarrow{N} F^\times$ is isomorphic to G_F by the "norm residue" isomorphism of local class field theory. The composite $F^\times \to F^\times/NK^\times \cong G_F \cong \mathrm{Hom}(F^\times, \mu_m)$, written as $x \mapsto (x, -)_F$, is adjoint to a nondegenerate bilinear map $(\ ,\ )_F \colon F^\times \otimes F^\times \to \mu_m$.

The Steinberg identity $(a, 1-a)_F = 1$ is proven by noting that $(1-a)$ is a norm from the intermediate field $E = F(x)$, $x^m = a$. Since $G_E \subset G_F$ corresponds to the norm map $E^\times/N_{K/E}K^\times \hookrightarrow F^\times/N_{K/F}K^\times$, the element g of $G_F = Gal(K/F)$ corresponding to the map $\zeta(a) = (a, 1-a)_F$ from $F^\times$ to μ_m must belong to G_E, *i.e.*, ζ must extend to a map $E^\times \to \mu_m$. But then $(a, 1-a)_F = \zeta(a) = \zeta(x)^m = 1$.

The name "norm residue" comes from the fact that for each x, the map $y \mapsto \{x, y\}$ is trivial iff $x \in NK^\times$. Since a primitive m^{th} root of unity ζ is not a norm from K, it follows that there is an $x \in F$ such that $(\zeta, x)_F \neq 1$. Therefore the norm residue symbol is a split surjection with inverse $\zeta^i \mapsto \{\zeta^i, x\}$.

The role of the norm residue symbol is explained by the following structural result, whose proof we cite from the literature.

MOORE'S THEOREM 6.2.4. *If F is a local field, then $K_2(F)$ is the direct sum of a uniquely divisible abelian group V and a finite cyclic group, isomorphic under the norm residue symbol to the group $\mu = \mu_n$ of roots of unity in F.*

PROOF. We have seen that the norm residue symbol is a split surjection. A proof that its kernel V is divisible, due to C. Moore, is given in the Appendix to [Milnor]. The fact that V is torsionfree (hence uniquely divisible) was proven by Tate [Tate] when $\mathrm{char}(F) = p$, and by Merkur'ev [Merk] when $\mathrm{char}(F) = 0$.

Tame symbols

Every discrete valuation v on a field F provides a Steinberg symbol. Recall that v is a homomorphism $F^\times \to \mathbb{Z}$ such that $v(r+s) \geq \min\{v(r), v(s)\}$. By convention, $v(0) = \infty$, so that the ring R of all r with $v(r) \geq 0$ is a discrete valuation ring (DVR). The units $R^\times$ form the set $v^{-1}(0)$, and the maximal ideal of R is generated by any $\pi \in R$ with $v(\pi) = 1$. The residue field k_v is defined to be $R/(\pi)$. If $u \in R$, we write $\bar{u}$ for the image of u under $R \to k_v$.

LEMMA 6.3. *For every discrete valuation v on F there is a Steinberg symbol $K_2(F) \xrightarrow{\partial_v} k_v^\times$, defined by*

$$\partial_v(\{r, s\}) = (-1)^{v(r)v(s)} \overline{\left(\frac{s^{v(r)}}{r^{v(s)}}\right)}.$$

This symbol is called the tame symbol of the valuation v. The tame symbol is onto, because if $u \in R^\times$ then $v(u) = 0$ and $\partial_v(\pi, u) = \bar{u}$.

PROOF. Writing $r = u_1\pi^{v_1}$ and $s = u_2\pi^{v_2}$ with $u_1, u_2 \in R^\times$, we must show that $\partial_v(r, s) = (-1)^{v_1v_2} \frac{\bar{u}_2^{v_1}}{\bar{u}_1^{v_2}}$ is a Steinberg symbol. By inspection, $\partial_v(r, s)$ is an element of $k_v^\times$, and ∂_v is bilinear. To see that $\partial_v(r, s) = 1$ when $r + s = 1$ we consider several cases. If $v_1 > 0$ then r is in the maximal ideal, so $s = 1 - r$ is a unit and $\partial_v(r, s) = \bar{s}^{v_1} = 1$. The proof when $v_2 > 0$ is the same, and the case $v_1 = v_2 = 0$

is trivial. If $v_1 < 0$ then $v(\frac{1}{r}) > 0$ and $\frac{1-r}{r} = -1 + \frac{1}{r}$ is congruent to $-1 \pmod{\pi}$. Since $v(r) = v(1-r)$, we have

$$\partial_v(r, 1-r) = (-1)^{v_1}\left(\frac{1-r}{r}\right)^{v_1} = (-1)^{v_1}(-1)^{v_1} = 1.$$

RAMIFICATION 6.3.1. Suppose that E is a finite extension of F, and that w is a valuation on E over the valuation v on F. Then there is an integer e, called the *ramification index*, such that $w(r) = e \cdot v(r)$ for every $r \in F$. The natural map $K_2(F) \to K_2(E)$ is compatible with the tame symbols in the sense that for every $r_1, r_2 \in F^\times$ we have $\partial_w(r_1, r_2) = \partial_v(r_1, r_2)^e$ in $k_w^\times$.

$$\begin{array}{ccc} K_2(F) & \xrightarrow{\partial_v} & k_v^\times \\ \downarrow & & e\downarrow x\mapsto x^e \\ K_2(E) & \xrightarrow{\partial_w} & k_w^\times \end{array}$$

Let S denote the integral closure of R in E. Then S has finitely many prime ideals $\mathfrak{p}_1, ..., \mathfrak{p}_n$ lying over $\mathfrak{p}$, with corresponding valuations $w_1, ..., w_n$ on E. We say that S is *unramified* over R if the ramification indices $e_1, ..., e_n$ are all 1; in this case the diagonal inclusion $\Delta\colon k_v^\times \hookrightarrow \prod_i k_{w_i}^\times$ is compatible with the tame symbols in the sense that $\Delta\partial_v(r_1, r_2)$ is the product of the $\partial_{w_i}(r_1, r_2)$.

COROLLARY 6.3.2. *If F contains the rational function field $\mathbb{Q}(t)$ or $\mathbb{F}_p(t_1, t_2)$, then $K_2(F)$ has the same cardinality as F. In particular, if F is uncountable then so is $K_2(F)$.*

PROOF. By hypothesis, F contains a transcendental element t. Choose a subset $X = \{x_\alpha\}$ of F so that $X \cup \{t\}$ is a transcendence basis for F over its ground field F_0, and set $k = F_0(X)$. Then the subfield $k(t)$ of F has a t-adic valuation with residue class field k. Hence $K_2(k(t))$ contains a subgroup $\{t, k^\times\}$ mapped isomorphically under the tame symbol to $k^\times$. By Lemma 6.1.3, the kernel of $k^\times \to K_2(k(t)) \to K_2(F)$ is contained in the torsion subgroup $\mu(k)$ of roots of unity in k. Thus the cardinality of $K_2(F)$ is bounded below by the cardinality of $k^\times/\mu(k)$. Since F is an algebraic extension of $k(t)$, and k contains either $\mathbb{Q}$ or $\mathbb{F}_p(t_2)$, we have the inequality $|F| = |k| = |k^\times/\mu(k)| \le |K_2(F)|$. The other inequality $|K_2(F)| \le |F|$ is immediate from Matsumoto's Theorem, since F is infinite.

THEOREM 6.4 (BASS-TATE). *When F is an algebraically closed field, $K_2(F)$ is a uniquely divisible abelian group.*

Recall that an abelian group is uniquely divisible when it is uniquely p-divisible for each prime p; a group is said to be *uniquely p-divisible* if it is p-divisible and has no p-torsion. Therefore, the theorem is an immediate consequence of our next proposition.

PROPOSITION 6.4.1 (BASS-TATE). *Let p be a prime number such that each polynomial $t^p - a$ ($a \in F$) splits in $F[t]$ into linear factors. Then $K_2(F)$ is uniquely p-divisible.*

PROOF. The hypothesis implies that $F^\times$ is p-divisible. Since the tensor product of p-divisible abelian groups is always uniquely p-divisible, $F^\times \otimes F^\times$ is uniquely p-divisible. Let R denote the kernel of the natural surjection $F^\times \otimes F^\times \to K_2(F)$. By inspection (or by the Snake Lemma), $K_2(F)$ is p-divisible and the p-torsion subgroup of $K_2(F)$ is isomorphic to R/pR.

Therefore it suffices to prove that R is p-divisible. Now R is generated by the elements $\psi(a) = (a)\otimes(1-a)$ of $F^\times \otimes F^\times$ ($a \in F - \{0,1\}$), so it suffices to show that each $\psi(a)$ is in pR. By hypothesis, there are $b_i \in F$ such that $t^p - a = \prod(t - b_i)$ in $F[t]$, so $1 - a = \prod(1 - b_i)$ and $b_i^p = a$ for each i. But then we compute in $F^\times \otimes F^\times$:

$$\psi(a) = (a) \otimes (1-a) = \sum (a) \otimes (1 - b_i) = \sum (b_i)^p \otimes (1 - b_i) = p \sum \psi(b_i).$$

COROLLARY 6.4.2. *If F is a perfect field of characteristic p, then $K_2(F)$ is uniquely p-divisible.*

The Localization Sequence for K_2

The following result will be proven in chapter V, but we find it useful to quote this result now. If $\mathfrak{p}$ is a nonzero prime ideal of a Dedekind domain R, the local ring $R_\mathfrak{p}$ is a discrete valuation ring, and hence determines a tame symbol.

LOCALIZATION THEOREM 6.5. *Let R be a Dedekind domain with field of fractions F. Then the tame symbols $K_2(F) \xrightarrow{\partial_\mathfrak{p}} (R/\mathfrak{p})^\times$ associated to the prime ideals of R fit into a long exact sequence*

$$\coprod_\mathfrak{p} K_2(R/\mathfrak{p}) \to K_2(R) \to K_2(F) \xrightarrow{\partial = \coprod \partial_\mathfrak{p}} \coprod_\mathfrak{p} (R/\mathfrak{p})^\times \to SK_1(R) \to 1$$

where the coproducts are over all nonzero prome ideals $\mathfrak{p}$ of R, and the maps from $(R/\mathfrak{p})^\times = K_1(R/\mathfrak{p})$ to $SK_1(R)$ are the transfer maps of Ex. 1.11. The transfer maps $K_2(R/\mathfrak{p}) \to K_2(R)$ will be defined in chapter V.

APPLICATION 6.5.1 ($K_2\mathbb{Q}$). If $R = \mathbb{Z}$ then, since $K_2(\mathbb{Z}/p) = 1$ and $SK_1(\mathbb{Z}) = 1$, we have an exact sequence $1 \to K_2(\mathbb{Z}) \to K_2(\mathbb{Q}) \xrightarrow{\partial} \coprod \mathbb{F}_p^\times \to 1$. As noted in Example 6.2.1, this sequence is split by the symbol $(r,s)_\infty$, so we have $K_2(\mathbb{Q}) \cong K_2(\mathbb{Z}) \oplus \coprod \mathbb{F}_p^\times$.

APPLICATION 6.5.2 (FUNCTION FIELDS). If R is the polynomial ring $F[t]$ for some field F, we know that $K_2(F[t]) = K_2(F)$ (see 5.2.3). Moreover, the natural map $K_2(F) \to K_2F(t)$ is split by the leading coefficient symbol λ of Example 6.1.2. Therefore we have a split exact sequence

$$1 \to K_2(F) \to K_2F(t) \xrightarrow{\partial} \coprod_\mathfrak{p} (F[t]/\mathfrak{p})^\times \to 1.$$

WEIL'S RECIPROCITY FORMULA 6.5.3. Just as in the case $R = \mathbb{Z}$, there is a valuation on $F(t)$ not arising from a prime ideal of $F[t]$. In this case, it is the valuation $v_\infty(f) = -deg(f)$ associated with the point at infinity, *i.e.*, with parameter t^{-1}. Since the symbol $(f, g)_\infty$ vanishes on $K_2(F)$, it must be expressable in terms of the tame symbols $\partial_\mathfrak{p}(f, g) = (f, g)_\mathfrak{p}$. The appropriate reciprocity formula first appeared in Weil's 1940 paper on the Riemann Hypothesis for curves:

$$(f, g)_\infty \cdot \prod_\mathfrak{p} N_\mathfrak{p}(f, g)_\mathfrak{p} = 1 \qquad \text{in } F^\times.$$

In Weil's formula "$N_\mathfrak{p}$" denotes the usual norm map $(F[t]/\mathfrak{p})^\times \to F^\times$. To establish this reciprocity formula, we observe that $K_2F(t)/K_2F = \coprod (F[t]/\mathfrak{p})^\times$ injects into $K_2\bar{F}(t)/K_2\bar{F}$, where $\bar{F}$ is the algebraic closure of F. Thus we may assume that F is algebraically closed. By Example 6.1.7, $K_2F(t)$ is generated by linear symbols of the form $\{a, t - b\}$. But $(a, t - b)_\infty = a$ and $\partial_{t-b}(a, t - b) = a^{-1}$, so the formula is clear.

Our next structural result was discovered by Merkur'ev and Suslin in 1981, and published in their landmark paper [MS]. Recall that an automorphism σ of a field E induces an automorphism of $K_2(E)$ sending $\{x, y\}$ to $\{\sigma x, \sigma y\}$.

THEOREM 6.6 (HILBERT'S THEOREM 90 FOR K_2). *Let E/F be a cyclic Galois field extension of prime degree p, and let σ be a generator of $Gal(E/F)$. Then the following sequence is exact, where N denotes the transfer map on K_2:*

$$K_2(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N} K_2(F).$$

Merkur'ev and Suslin gave this result the suggestive name "Hilbert's Theorem 90 for K_2," because of its formal similarity to the following result, which is universally called "Hilbert's Theorem 90 (for units)" because it was the 90^{th} theorem in Hibert's classical 1897 survey of algebraic number theory, *Theorie der Algebraische Zahlkörper.*

THEOREM 6.6.1 (HILBERT'S THEOREM 90 FOR UNITS). *Let E/F be a cyclic Galois field extension, and let σ be a generator of $Gal(E/F)$. If $1 - \sigma$ denotes the map $a \mapsto a/\sigma(a)$, then the following sequence is exact:*

$$1 \to F^\times \to E^\times \xrightarrow{1-\sigma} E^\times \xrightarrow{N} F^\times.$$

We postpone the proof of Hilbert's Theorem 90 for K_2 until chapter VI, since it uses higher algebraic K-theory in a crucial way to reduce to the following special case.

PROPOSITION 6.6.2. *Let F be a field containing a primitive n^{th} root of unity ζ, and let E be a cyclic field extension of degree n, with σ a generator of $Gal(E/F)$. Suppose in addition that the norm map $E^\times \xrightarrow{N} F^\times$ is onto, and that F has no extension fields of degree $< n$. Then the following sequence is exact:*

$$K_2(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N} K_2(F) \to 1.$$

PROOF. Since $N\zeta = 1$, Hilbert's Theorem 90 gives an $r \in E$ with $\sigma(r) = \zeta r$. Setting $c = N(r) \in F$, it is well-known and easy to see that $E = F(r)$, $r^n = c$.

Again by Hilbert's Theorem 90 for units and our assumption about norms, $E^\times \xrightarrow{1-\sigma} E^\times \xrightarrow{N} F^\times \to 1$ is an exact sequence of abelian groups. Applying the right exact functor $\otimes F^\times$ retains exactness. Therefore we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
E^\times \otimes F^\times & \xrightarrow{(1-\sigma)\otimes 1} & E^\times \otimes F^\times & \xrightarrow{N\otimes 1} & F^\times \otimes F^\times & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\
K_2(E) & \xrightarrow{1-\sigma} & K_2(E) & \longrightarrow & C & \longrightarrow & 1
\end{array}$$

in which C denotes the cokernel of $1-\sigma$.

Now every element of E is a polynomial $f(r)$ in r of degree $< n$, and $f(t)$ is a product of linear terms $b_it - a_i$ by our assumption. By Lemma 6.1.4, every element of $K_2(E)$ is a product of symbols of the form $\{a, b\}$ and $\{a, r-b\}$. Therefore the vertical maps $F^\times \otimes E^\times \to K_2(E)$ are onto in the above diagram. Hence γ is onto.

If $a \in F^\times$ and $x \in E^\times$ then the projection formula (Ex. 5.6) yields

$$N(1-\sigma)\{a, x\} = N\{a, x/(\sigma x)\} = \{a, Nx/N(\sigma x)\} = 1.$$

Hence the transfer map $K_2(E) \to K_2(F)$ factors through C. A diagram chase shows that it suffices to show that γ is a Steinberg symbol, so that it factors through $K_2(F)$. For this we must show that for all $y \in E$ we have $\gamma(Ny \otimes (1-Ny)) = 1$, i.e., that $\{y, 1-Ny\} \in (1-\sigma)K_2(E)$.

Fix $y \in E$ and set $z = N_{E/F}(y) \in F$. Factor $t^n - z = \prod f_i$ in $F[t]$, with the f_i irreducible, and let F_i denote the field $F(x_i)$, where $f_i(x_i) = 0$ and $x_i^n = z$. Setting $t = 1$, $1 - z = \prod f_i(1) = \prod N_{F_i/F}(1-x_i)$. Setting $E_i = E \otimes_F F_i$, so that $N_{F_i/F}(1-x_i) = N_{E_i/E}(1-x_i)$ and $\sigma(x_i) = x_i$, the projection formula (Ex. 5.6) gives

$$\{y, 1-z\} = \prod N_{E_i/E}\{y, 1-x_i\} = \prod N_{E_i/E}\{y/x_i, 1-x_i\}.$$

Thus it suffices to show that each $N_{E_i/E}\{y/x_i, 1-x_i\}$ is in $(1-\sigma)K_2(E)$. Now E_i/F_i is a cyclic extension whose norm $N = N_{E_i/F_i}$ satisfies $N(y/x_i) = N(y)/x_i^n = 1$. By Hilbert's Theorem 90 for units, $y/x_i = v_i/\sigma v_i$ for some $v_i \in E_i$. We now compute:

$$N_{E_i/E}\{y/x_i, 1-x_i\} = N_{E_i/E}\{v_i/\sigma v_i, 1-x_i\} = (1-\sigma)N_{E_i/E}\{v_i, 1-x_i\}.$$

Here are three pretty applications of Hilbert's Theorem 90 for K_2. When F is a perfect field, the first of these has already been proven in Proposition 6.4.1.

THEOREM 6.7. *If char$(F) = p \neq 0$, then the group $K_2(F)$ has no p-torsion.*

PROOF. Let x be an indeterminate and $y = x^p - x$; the field extension $F(x)/F(y)$ is an Artin-Schrier extension, and its Galois group is generated by an automorphism σ satisfying $\sigma(x) = x + 1$. By 6.5.2, $K_2(F)$ is a subgroup of both $K_2F(x)$ and $K_2F(y)$, and the projection formula shows that the norm $N\colon K_2F(x) \to K_2F(y)$ sends $u \in K_2(F)$ to u^p.

Now fix $u \in K_2(F)$ satisfying $u^p = 1$; we shall prove that $u = 1$. By Hilbert's Theorem 90 for K_2, $u = (1-\sigma)v = v(\sigma v)^{-1}$ for some $v \in K_2F(x)$.

Every prime ideal $\mathfrak{p}$ of $F[x]$ is unramified over $\mathfrak{p}_y = \mathfrak{p} \cap F[y]$, because $F[x]/\mathfrak{p}$ is either equal to, or an Artin-Schrier extension of, $F[y]/\mathfrak{p}_y$. By 6.3.1 and 6.5.2, we have a commutative diagram in which the vertical maps ∂ are surjective:

$$\begin{tikzcd}
K_2F(y) \arrow[r, "i^*"] \arrow[d, "\partial"'] & K_2F(x) \arrow[r, "1-\sigma"] \arrow[d, "\partial"'] & K_2F(x) \arrow[d, "\partial"'] \\
\coprod_{\mathfrak{p}_y}(F[y]/\mathfrak{p}_y)^\times \arrow[r, "\Delta"] & \coprod_{\mathfrak{p}}(F[x]/\mathfrak{p})^\times \arrow[r, "1-\sigma"] & \coprod_{\mathfrak{p}}(F[x]/\mathfrak{p})^\times
\end{tikzcd}$$

We claim that the bottom row is exact. By decomposing the row into subsequences invariant under σ, we see that there are two cases to consider. If a prime $\mathfrak{p}$ is not fixed by σ, then the fields $F[x]/\sigma^i\mathfrak{p}$ are all isomorphic to $E = F[y]/\mathfrak{p}_y$, and for $a_i \in E^\times$ we have

$$(1-\sigma)(a_0, a_1, \dots, a_{p-1}) = (a_0a_{p-1}^{-1}, a_1a_0^{-1}, \dots, a_{p-1}a_{p-2}^{-1})$$

in $\prod_{i=0}^{p-1}(F[x]/\sigma^i\mathfrak{p})^\times$. This vanishes iff the a_i agree, in which case $(a_0, \dots, a_{p-1})$ is the image of $a \in E^\times$. On the other hand, if σ fixes $\mathfrak{p}$ then $F[x]/\mathfrak{p}$ is a cyclic Galois extension of $E = F[y]/\mathfrak{p}_y$. Therefore if $a \in F[x]/\mathfrak{p}$ and $(1-\sigma)a = a/(\sigma a)^{-1}$ equals 1, then $a = \sigma(a)$, *i.e.*, $a \in E$. This establishes the claim.

A diagram chase shows that since $1 = \partial u = \partial(1-\sigma)v$, there is a v_0 in $K_2F(y)$ with $\partial(v) = \partial(i^*v_0)$. Since $i^* = \sigma i^*$, we have $(1-\sigma)i^*v_0 = 1$. Replacing v by $v(i^*v_0)^{-1}$, we may assume that $\partial(v) = 1$, *i.e.*, that v is in the subgroup $K_2(F)$ of $K_2F(x)$. Therefore we have $u = v(\sigma v)^{-1} = 1$. As u was any element of $K_2(F)$ satisfying $u^p = 1$, $K_2(F)$ has no p-torsion.

EXAMPLE 6.7.1. If $F = \mathbb{F}_q(t)$, $q = p^r$, we have $K_2(F) = \coprod(\mathbb{F}_q[t]/\mathfrak{p})^\times$. Since the units of each finite field $\mathbb{F}_q[t]/\mathfrak{p}$ form a cyclic group, and its order can be arbitrarily large (yet prime to p), $K_2\mathbb{F}_q(t)$ is a very large torsion group.

THEOREM 6.8. *If F contains a primitive n^{th} root of unity ζ, then every element of $K_2(F)$ of exponent n has the form $\{\zeta, x\}$ for some $x \in F^\times$.*

PROOF. We first suppose that n is a prime number p. Let x be an indeterminate and $y = x^p$; the Galois group of the field extension $F(x)/F(y)$ is generated by an automorphism σ satisfying $\sigma(x) = \zeta x$. By Application 6.5.2, $K_2(F)$ is a subgroup of $K_2F(x)$, and by the projection formula the norm $N\colon K_2F(x) \to K_2F(y)$ sends $u \in K_2(F)$ to u^p.

Fix $u \in K_2(F)$ satisfying $u^p = 1$. By Hilbert's Theorem 90 for K_2, if $u^p = 1$ then $u = (1-\sigma)v = v(\sigma v)^{-1}$ for some $v \in K_2F(x)$.

Now the extension $F[y] \subset F[x]$ is unramified at every prime ideal except $\mathfrak{p} = (x)$. As in the proof of Theorem 6.7, we have a commutative diagram whose bottom row is exact:

$$\begin{tikzcd}
K_2F(y) \arrow[r, "i^*"] \arrow[d, "\partial'"'] & K_2F(x) \arrow[r, "1-\sigma"] \arrow[d, "\partial'"'] & K_2F(x) \arrow[d, "\partial'"'] \\
\coprod_{\mathfrak{p}_y\neq(y)}(F[y]/\mathfrak{p}_y)^\times \arrow[r, "\Delta"] & \coprod_{\mathfrak{p}\neq(x)}(F[x]/\mathfrak{p})^\times \arrow[r, "1-\sigma"] & \coprod_{\mathfrak{p}\neq(x)}(F[x]/\mathfrak{p})^\times
\end{tikzcd}$$

As before, we may modify v by an element from $K_2F(y)$ to arrange that $\partial_{\mathfrak{p}}(v) = 1$ for all $\mathfrak{p} \neq (x)$. For $\mathfrak{p} = (x)$, let $a \in F = F[x]/(x)$ be such that $\partial_{(x)}(v) = a$ and set $v' = v\{a, x\}$. Then $\partial_{(x)}(v') = 1$ and $\partial_{\mathfrak{p}}(v') = \partial_{\mathfrak{p}}(v) = 1$ for every other $\mathfrak{p}$. It follows from 6.5.2 that v' is in $K_2(F)$. Therefore $(1 - \sigma)v' = 1$; since $v = v'\{a, x\}^{-1}$ this implies that u has the asserted form:

$$u = (1 - \sigma)\{a, x\}^{-1} = \{a, x\}^{-1}\{a, \zeta x\} = \{a, \zeta\}.$$

Now we proceed inductively, supposing that $n = mp$ and that the theorem has been proven for m (and p). If $u \in K_2(F)$ has exponent n then u^p has exponent m, so there is an $x \in F^\times$ so that $u^p = \{\zeta^p, x\}$. The element $u\{\zeta^p, x\}^{-1}$ has exponent p, so it equals $\{\zeta^m, y\} = \{\zeta, y^m\}$ for some $y \in F^\times$. Hence $u = \{\zeta, xy^m\}$, as required.

Remark 6.8.1. Suslin also proved the following result in [Su87]. Let F be a field containing a primitive p^{th} root of unity ζ, and let $F_0 \subset F$ be the subfield of constants. If $x \in F_0^\times$ and $\{\zeta, x\} = 1$ in $K_2(F)$ then $\{\zeta, x\} = 1$ in $K_2(F_0)$. If $\{\zeta, y\} = 1$ in $K_2(F)$ for some $y \in F^\times$ then $y = xz^p$ for some $x \in F_0^\times$ and $z \in F^\times$.

Application 6.8.2. We can use Theorem 6.8 to give another proof of Theorem 6.4, that when F is an algebraically closed field, the group $K_2(F)$ is uniquely divisible. Fix a prime p. For each $a \in F^\times$ there is an α with $\alpha^p = a$. Hence $\{a, b\} = \{\alpha, b\}^p$, so $K_2(F)$ is p-divisible. If $p \neq \text{char}(F)$ then there is no p-torsion because $\{\zeta, a\} = \{\zeta, \alpha\}^p = 1$. Finally, if $\text{char}(F) = p$, there is no p-torsion either by Theorem 6.7.

Application 6.8.3 ($K_2\mathbb{R}$). Theorem 6.8 states that $\{-1, -1\}$ is the only element of order 2 in $K_2\mathbb{R}$. Indeed, if r is a positive real number then:

$$\{-1, r\} = \{-1, \sqrt{r}\}^2 = 1, \quad \text{and} \quad \{-1, -r\} = \{-1, -1\}\{-1, r\} = \{-1, -1\}.$$

Note that $\{-1, -1\}$ is in the image of $K_2(\mathbb{Z})$, which is a summand by either Example 6.2.1 or Example 5.9.1. Recall from Example 6.1.6 that the image of $K_2\mathbb{R}$ in the uniquely divisible group $K_2\mathbb{C}$ is the eigenspace $K_2\mathbb{C}^+$, and that the composition $K_2\mathbb{R} \to K_2\mathbb{C} \xrightarrow{N} K_2\mathbb{R}$ is multiplication by 2, so its kernel is $K_2(\mathbb{Z})$. It follows that

$$K_2\mathbb{R} \cong K_2(\mathbb{Z}) \oplus K_2\mathbb{C}^+.$$

K_2 and the Brauer group

Let F be a field. Recall from II.5.4.3 that the Brauer group $Br(F)$ is generated by the classes of central simple algebras with two relations: $[A \otimes_F B] = [A] \cdot [B]$ and $[M_n(F)] = 0$. Here is one classical construction of elements in the Brauer group; it is a special case of the construction of crossed product algebras.

Cyclic algebras 6.9. Let ζ be a primitive n^{th} root of unity in F, and $\alpha, \beta \in F^\times$. The *cyclic algebra* $A = A_\zeta(\alpha, \beta)$ is defined to be the associative algebra with unit, which is generated by two elements x, y subject to the relations $x^n = \alpha \cdot 1$, $y^n = \beta \cdot 1$ and $yx = \zeta xy$. Thus A has dimension n^2 over F, a basis being the monomials $x^i y^j$ with $0 \le i, j < n$. The identity $(x+y)^n = (\alpha+\beta)\cdot 1$ is also easy to check.

When $n = 2$ (so $\zeta = -1$), cyclic algebras are called *quaternion algebras.* The name comes from the fact that the usual quaternions $\mathbb{H}$ are the cyclic algebra $A(-1,-1)$ over $\mathbb{R}$. Quaternion algebras arise in the Hasse invariant of quadratic forms.

It is classical, and not hard to prove, that A is a central simple algebra over F; see [BA, §8.5]. Moreover, the n-fold tensor product $A\otimes_F A\otimes_F\cdots\otimes_F A$ is a matrix algebra; see [BA, Theorem 8.12]. Thus we can consider $[A]\in Br(F)$ as an element of exponent n. We shall write ${}_nBr(F)$ for the subgroup of $Br(F)$ consisting of all elements x with $x^n = 1$, so that $[A]\in {}_nBr(F)$

For example, the following lemma shows that $A_\zeta(1,\beta)$ must be a matrix ring because $x^n = 1$. Thus $[A_\zeta(1,\beta)] = 1$ in $Br(F)$.

LEMMA 6.9.1. *Let A be a central simple algebra of dimension n^2 over a field F containing a primitive n^{th} root of unity ζ. If A contains an element $u\notin F$ such that $u^n = 1$, then $A\cong M_n(F)$.*

PROOF. The subalgebra $F[u]$ of A spanned by u is isomorphic to the commutative algebra $F[t]/(t^n-1)$. Since $t^n - 1 = \prod(t-\zeta^i)$, the Chinese Remainder Theorem yields $F[u]\cong F\times F\times\cdots\times F$. Hence $F[u]$ contains n idempotents e_i with $e_ie_j = 0$ for $i\ne j$. Therefore A splits as the direct sum $e_1A\oplus\cdots\oplus e_nA$ of right ideals. By the Artin-Wedderburn theorem, if $A = M_d(D)$ then A can be the direct sum of at most d right ideals. Hence $d = n$, and A must be isomorphic to $M_n(F)$.

PROPOSITION 6.9.2 (THE n^{th} POWER NORM RESIDUE SYMBOL). *If F contains a primitive n^{th} root of unity, there is a homomorphism $K_2(F)\to Br(F)$ sending $\{\alpha,\beta\}$ to the class of the cyclic algebra $A_\zeta(\alpha,\beta)$.*

Since the image is a subgroup of exponent n, we shall think of the power norm residue symbol as a map $K_2(F)/nK_2(F)\to {}_nBr(F)$.

This homomorphism is sometimes also called the *Galois symbol.*

PROOF. From Ex. 6.10 we see that in $Br(F)$ we have $[A_\zeta(\alpha,\beta)]\cdot[A_\zeta(\alpha,\gamma)] = [A_\zeta(\alpha,\beta\gamma)]$. Thus the map $F^\times\times F^\times\to Br(F)$ sending (α,β) to $[A_\zeta(\alpha,\beta)]$ is bilinear. To see that it is a Steinberg symbol we must check that $A = A_\zeta(\alpha,1-\alpha)$ is isomorphic to the matrix algebra $M_n(F)$. Since the element $x+y$ of A satisfies $(x+y)^n = 1$, Lemma 6.9.1 implies that A must be isomorphic to $M_n(F)$.

REMARK 6.9.3. Merkur'ev and Suslin proved in [MS] that $K_2(F)/nK_2(F)$ is isomorphic to the subgroup ${}_nBr(F)$ of elements of order n in $Br(F)$. By Matsumoto's Theorem, this implies that the n-torsion in the Brauer group is generated by cyclic algebras. We will describe the Merkur'ev-Suslin result in chapter VI.

The Galois symbol

We can generalize the power norm residue symbol to fields not containing enough roots of unity by introducing Galois cohomology. Here are the essential facts we shall need; see [WHomo] or [Milne].

SKETCH OF GALOIS COHOMOLOGY 6.10. Let F_{sep} denote the separable closure of a field F, and let $G = G_F$ denote the Galois group $Gal(F_{\text{sep}}/F)$. The family of subgroups $G_E = Gal(F_{\rm sep}/E)$, as E runs over all finite extensions of F, forms a basis for a topology of G. A G-module M is called *discrete* if the multiplication $G \times M \to M$ is continuous.

For example, the abelian group $\mathbf{G}_m = F_{\rm sep}^\times$ of units of $F_{\rm sep}$ is a discrete module, as is the subgroup μ_n of all n^{th} roots of unity. We can also make the tensor product of two discrete modules into a discrete module, with G acting diagonally. For example, the tensor product $\mu_n^{\otimes 2} = \mu_n \otimes \mu_n$ is also a G-discrete module. Note that the three G-modules $\mathbb{Z}/n$, μ_n and $\mu_n^{\otimes 2}$ have the same underlying abelian group, but are isomorphic G_F-modules only when $\mu_n \subset F$.

The G-invariant subgroup M^G of a discrete G-module M is a left exact functor on the category of discrete G_F-modules. The *Galois cohomology groups* $H^i_{et}(F;M)$ are defined to be its right derived functors. In particular, $H^0_{et}(F;M)$ is just M^G.

If E is a finite separable extension of F then $G_E \subset G_F$. Thus there is a forgetful functor from G_F-modules to G_E-modules, inducing maps $H^i_{et}(F;M) \to H^i_{et}(E;M)$. In the other direction, the induced module functor from G_E-modules to G_F-modules gives rise to cohomological transfer maps $tr_{E/F}\colon H^i_{et}(E;M) \to H^i_{et}(F;M)$; see [WHomo, 6.3.9 and 6.11.11].

Example 6.10.1 (Kummer Theory). The cohomology of the module $\mathbf{G}_m$ is of fundamental importance. Of course $H^0_{et}(F,\mathbf{G}_m) = F^\times$. By Hilbert's Theorem 90 for units, and a little homological algebra [WHomo, 6.11.16], we also have $H^1_{et}(F;\mathbf{G}_m) = 0$ and $H^2_{et}(F;\mathbf{G}_m) \cong Br(F)$.

If n is prime to char(F), the exact sequence of discrete modules

$$1 \to \mu_n \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 1$$

is refered to as the *Kummer sequence*. Writing $\mu_n(F)$ for the group μ_n^G of all n^{th} roots of unity in F, the corresponding cohomology sequence is called the *Kummer sequence.*

$$\begin{aligned} 1 \to &\mu_n(F) \to F^\times \xrightarrow{n} F^\times \to H^1_{et}(F;\mu_n) \to 1 \\ 1 \to &H^2_{et}(F;\mu_n) \to Br(F) \xrightarrow{n} Br(F) \end{aligned}$$

This yields isomorphisms $H^1_{et}(F;\mu_n) \cong F^\times/F^{\times n}$ and $H^2_{et}(F;\mu_n) \cong {}_nBr(F)$. If $\mu_n \subset F^\times$, this yields a natural isomorphism $H^2_{et}(F;\mu_n^{\otimes 2}) \cong {}_nBr(F) \otimes \mu_n(F)$.

There are also natural cup products in cohomology, such as the product

$$F^\times \otimes F^\times \to H^1_{et}(F;\mu_n) \otimes H^1_{et}(F;\mu_n) \xrightarrow{\cup} H^2_{et}(F;\mu_n^{\otimes 2}) \tag{6.10.2}$$

which satisfies the following *projection formula:* if E/F is a finite separable extension, $a \in F^\times$ and $b \in E^\times$, then $tr_{E/F}(a \cup b) = a \cup N_{E/F}(b)$.

Proposition 6.10.3 (Galois symbol). *The bilinear pairing (6.10.2) induces a Steinberg symbol $K_2(F) \to H^2_{et}(F;\mu_n^{\otimes 2})$ for every n prime to* char(F).

Proof. It suffices to show that $a \cup (1-a)$ vanishes for every $a \in F - \{0,1\}$. Fixing a, factor the separable polynomial $t^n - a = \prod f_i$ in $F[t]$ with the f_i irreducible, and let F_i denote the field $F(x_i)$ with $f_i(x_i) = 0$. Setting $t = 1$, $1-a = \prod_i N_{F_i/F}(1-x_i)$. Writing H^2_{et} additively, we have

$$a \cup (1-a) = \sum_i a \cup N_{F_i/F}(1-x_i) = \sum_i tr_{F_i/F}\big(a \cup (1-x_i)\big)$$
$$= n \sum_i tr_{F_i/F}\big(x_i \cup (1-x_i)\big).$$

Since the group $H^2_{et}(F;\mu_n^{\otimes 2})$ has exponent n, all these elements vanish, as desired.

Remark 6.10.4. Suppose that F contains a primitive n^{th} root of unity ζ. If we identify $\mathbb{Z}/n$ with μ_n via $1 \mapsto \zeta$, we have a natural isomorphism

$$_nBr(F) \cong {}_nBr(F) \otimes \mathbb{Z}/n \cong {}_nBr(F) \otimes \mu_n \cong H^2_{et}(F;\mu_n^{\otimes 2}).$$

Tate has shown in [Tate] that this isomorphism identifies the Galois symbol of Proposition 6.10.3 with the n^{th} power norm residue symbol of Proposition 6.9.2. The Merkur'ev-Suslin isomorphism of [MS] cited above in Remark 6.9.3 extends to a more general isomorphism $K_2(F)/nK_2(F) \cong H^2_{et}(F;\mu_n^{\otimes 2})$ for all fields F of characteristic prime to n. See chapter VI.

EXERCISES

6.1 Given a discrete valuation on a field F, with residue field k and parameter π, show that there is a surjection $\lambda: K_2(F) \to K_2(k)$ given by the formula $\lambda\{u\pi^i, v\pi^j\} = \{\bar{u}, \bar{v}\}$. Example 6.1.2 is a special case of this, in which $\pi = t^{-1}$.

6.2 (Bass-Tate) If $E = F(u)$ is a field extension of F, and $e_1, e_2 \in E$ are monic polynomials in u of some fixed degree $d > 0$, show that $\{e_1, e_2\}$ is a product of symbols $\{e_1, e_2'\}$ and $\{e, e_2''\}$ with e, e_2', e_2'' polynomials of degree $< d$. This generalizes Lemma 6.1.4.

6.3 If F is a number field with r_1 distinct embeddings $F \hookrightarrow \mathbb{R}$, show that the r_1 symbols $(,)_\infty$ on F define a surjection $K_2(F) \to \{\pm 1\}^{r_1}$.

6.4 If $\bar{F}$ denotes the algebraic closure of a field F, show that $K_2(\bar{\mathbb{Q}}) = K_2(\bar{\mathbb{F}}_p) = 1$.

6.5 *2-adic symbol on* $\mathbb{Q}$. Any nonzero rational number r can be written uniquely as $r = (-1)^i 2^j 5^k u$, where $i, k \in \{0,1\}$ and u is a quotient of integers congruent to 1 (mod 8). If $s = (-1)^{i'} 2^{j'} 5^{k'} u'$, set $(r,s)_2 = (-1)^{ii'+jj'+kk'}$. Show that this is a Steinberg symbol on $\mathbb{Q}$, with values in $\{\pm 1\}$.

6.6 Let $((r,s))_p$ denote the Hilbert symbol on $\hat{\mathbb{Q}}_p$ (6.2.2), and $(r,s)_p$ the tame symbol $K_2(\hat{\mathbb{Q}}_p) \to \mathbb{F}_p^\times$. Assume that p is odd, so that there is a unique surjection $\varepsilon: \mathbb{F}_p^\times \to \{\pm 1\}$. Show that $((r,s))_p = \varepsilon\ (r,s)_p$ for all $r, s \in \hat{\mathbb{Q}}_p^\times$.

6.7 *Quadratic Reciprocity.* If $r, s \in \mathbb{Q}^\times$, show that $(r,s)_\infty (r,s)_2 \prod_{p\neq 2}((r,s))_p = +1$. Here $(r,s)_2$ is the 2-adic symbol of Ex. 6.5.

Hint: From 6.5.1 and Ex. 6.6, the 2-adic symbol of Ex. 6.5 must satisfy some relation of the form

$$(r,s)_2 = (r,s)_\infty^{\varepsilon_\infty} \prod_{p\neq 2} ((r,s))_p^{\varepsilon_p},$$

where the exponents ε_p are either 0 or 1. Since $(-1,-1)_2 = (-1,-1)_\infty$ we have $\varepsilon_\infty = 1$. If p is a prime not congruent to 1 (mod 8), consider $\{2,p\}$ and $\{-1,p\}$. If p is a prime congruent to 1 (mod 8), Gauss proved that there is a prime $q < \sqrt{p}$ such that p is not a quadratic residue modulo q. Then $((p,q))_q = -1$, even though $(p,q)_\infty = (p,q)_2 = 1$. Since we may suppose inductively that ε_q equals 1, this implies that $\varepsilon_p \neq 0$.

6.8 (Suslin) Suppose that a field F is algebraically closed in a larger field E. Use Lemma 6.1.3 and Remark 6.8.1 to show that $K_2(F)$ injects into $K_2(E)$.

6.9 Let F be a field, and let $\Omega_F = \Omega_{F/\mathbb{Z}}$ denote the vector space of absolute Kähler differentials (see Ex. 2.6). The second exterior power of Ω_F is written as Ω^2_F. Show that there is a homomorphism $K_2(F) \to \Omega^2_F$ sending $\{x,y\}$ to $\frac{dx}{x} \wedge \frac{dy}{y}$.

6.10 Show that $A_\zeta(\alpha,\beta) \otimes A_\zeta(\alpha,\gamma) \cong M_n(A)$, where $A = A_\zeta(\alpha,\beta\gamma)$. *Hint:* Let x', y' generate $A_\zeta(\alpha,\beta)$ and x'', y'' generate $A_\zeta(\alpha,\gamma)$, and show that $x', y = y'y''$ generate A. Then show that $u = (x')^{-1}x'' + y''$ has $u^n = 1$. (For another proof, see [BA, Ex. 8.5.2].)

§7. Milnor K-theory of fields

Fix a field F, and consider the tensor algebra of the group $F^\times$,

$$T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus (F^\times \otimes F^\times \otimes F^\times) \oplus \cdots .$$

To keep notation straight, we write $l(x)$ for the element of degree one in $T(F^\times)$ corresponding to $x \in F^\times$.

Definition 7.1. The graded ring $K^M_*(F)$ is defined to be the quotient of $T(F^\times)$ by the ideal generated by the homogeneous elements $l(x) \otimes l(1-x)$ with $x \neq 0, 1$. The *Milnor K-group* $K^M_n(F)$ is defined to be the subgroup of elements of degree n. We shall write $\{x_1, \dots, x_n\}$ for the image of $l(x_1) \otimes \cdots \otimes l(x_n)$ in $K^M_n(F)$.

That is, $K^M_n(F)$ is presented as the group generated by symbols $\{x_1, \dots, x_n\}$ subject to two defining relations: $\{x_1, \dots, x_n\}$ is multiplicative in each x_i, and equals zero if $x_i + x_{i+1} = 1$ for some i.

The name comes from the fact that the ideas in this section first arose in Milnor's 1970 paper [M-QF]. Clearly we have $K^M_0(F) = \mathbb{Z}$, and $K^M_1 = F^\times$ (with the group operation written additively). By Matsumoto's Theorem 6.1 we also have $K^M_2(F) = K_2(F)$, the elements $\{x,y\}$ being the usual Steinberg symbols, except that the group operation in $K^M_2(F)$ is written additively.

Since $\{x_i, x_{i+1}\} + \{x_{i+1}, x_i\} = 0$ in $K^M_2(F)$, we see that interchanging two entries in $\{x_1, \dots, x_n\}$ yields the inverse. It follows that these symbols are alternating: for any permutation π with sign $(-1)^\pi$ we have

$$\{x_{\pi 1}, \dots, x_{\pi n}\} = (-1)^\pi \{x_1, \dots, x_n\}.$$

Examples 7.2. (a) If $\mathbb{F}_q$ is a finite field, then $K^M_n(\mathbb{F}_q) = 0$ for all $n \geq 2$, because $K^M_2(\mathbb{F}_q) = 0$ by Cor. 6.1.1. If F has transcendence degree 1 over a finite field (a global field of finite characteristic), Bass and Tate proved in [BT] that $K^M_n(F) = 0$ for all $n \geq 3$.

(b) If F is algebraically closed then $K_n^M(F)$ is uniquely divisible. Divisibility is clear because $F^\times$ is divisible. The proof that there is no p-torsion is the same as the proof for $n = 2$ given in Theorem 6.4, and is relegated to Ex. 7.3.

(c) When $F = \mathbb{R}$ we can define a symbol $K_n^M(\mathbb{R}) \to \{\pm 1\}$ by the following formula: $(x_1, \dots, x_n)_\infty$ equals -1 if all the x_i are negative, and equals $+1$ otherwise. When $n = 2$ this will be the symbol defined in Example 6.2.1.

To construct it, extend $\mathbb{Z} \to \mathbb{Z}/2$ to a ring homomorphism $T(\mathbb{R}^\times) \to (\mathbb{Z}/2)[t]$ by sending $l(x)$ to t if $x < 0$ and to 0 if $x > 0$. This sends the elements $l(x) \otimes l(1-x)$ to zero (as in 6.2.1), so it induces a graded ring homomorphism $K_*^M(\mathbb{R}) \to (\mathbb{Z}/2)[t]$. The symbol above is just the degree n part of this map.

By induction on n, it follows that $K_n^M(\mathbb{R})$ is the direct sum of a cyclic group of order 2 generated by $\{-1, \dots, -1\}$, and a divisible subgroup. In particular, this shows that $K_*^M(\mathbb{R})/2K_*^M(\mathbb{R})$ is the polynomial ring $(\mathbb{Z}/2)[\epsilon]$ on $\epsilon = l(-1)$. Using the norm map we shall see later that the divisible subgroup of $K_n^M(\mathbb{R})$ is in fact uniquely divisible. This gives a complete description of each $K_n^M(\mathbb{R})$ as an abelian group.

(d) When F is a number field, let r_1 be the number of embeddings of F into $\mathbb{R}$. Then we have a map from $K_n^M(F)$ to $K_n^M(\mathbb{R})^{r_1} \cong (\mathbb{Z}/2)^{r_1}$. Bass and Tate proved in [BT] that this map is an isomorphism for all $n \geq 3$: $K_n^M(F) \cong (\mathbb{Z}/2)^{r_1}$.

Tame symbols

Recall from Lemma 6.3 and Ex. 6.1 that every discrete valuation v on F induces a Steinberg symbol $K_2(F) \xrightarrow{\partial_v} k_v^\times$ and a map $K_2(F) \xrightarrow{\lambda} K_2(k_v)$. These symbols extend to all of Milnor K-theory.

Theorem 7.3 (Higher tame symbols). *For every discrete valuation v on F, there are two surjections*

$$K_n^M(F) \xrightarrow{\partial_v} K_{n-1}^M(k_v) \quad \textit{and} \quad K_n^M(F) \xrightarrow{\lambda} K_n^M(k_v)$$

satisfying the following conditions. Let $R = \{r \in F : v(r) \geq 0\}$ be the valuation ring, and π a parameter for v. If $u_i \in R^\times$, and $\bar{u}_i$ denotes the image of u_i in $k_v = R/(\pi)$ then

$$\lambda\{u_1\pi^{i_1}, \dots, u_n\pi^{i_n}\} = \{\bar{u}_1, \dots, \bar{u}_n\}, \qquad \partial_v\{\pi, u_2, \dots, u_n\} = \{\bar{u}_2, \dots, \bar{u}_n\}.$$

In particular, $\partial_v : K_2^M(F) \to k_v^\times$ is the tame symbol of Lemma 6.3, and $\lambda : K_2(F) \to K_2(k)$ is the map of Example 6.1.2 and Ex. 6.1.

Proof. (Serre) Let L denote the graded $K_*^M(k_v)$-algebra generated by an indeterminate Π in L_1, with the relation $\{\Pi, \Pi\} = \{-1, \Pi\}$. We claim that the group homomorphism

$$d : F^\times \to L_1 = l(k_v^\times) \oplus \mathbb{Z} \cdot \Pi, \qquad d(u\pi^i) = l(\bar{u}) + i\Pi$$

satisfies the relation: for $r \neq 0, 1$, $d(r)d(1-r) = 0$ in L_2. If so, the presentation of $K_*^M(F)$ shows that d extends to a graded ring homomorphism $d : K_*^M(F) \to L$. Since L_n is the direct sum of $K_n^M(k_v)$ and $K_{n-1}^M(k_v)$, we get two maps: $\lambda : K_n^M(F) \to$

$K_n^M(k_v)$ and $\partial_v: K_n^M(F) \to K_{n-1}^M(k_v)$. The verification of the relations is routine, and left to the reader.

If $1 \neq r \in R^\times$, then either $1-r \in R^\times$ and $d(r)d(1-r) = \{\bar{r}, 1-\bar{r}\} = 0$, or else $v(1-r) = i > 0$ and $d(r) = l(1) + 0 \cdot \Pi = 0$ so $d(r)d(1-r) = 0 \cdot d(1-r) = 0$. If $v(r) > 0$ then $1-r \in R^\times$ and the previous argument implies that $d(1-r)d(r) = 0$. If $r \notin R$ then $1/r \in R$, and we see from (5.10.3) and the above that $d(r)d(1-r) = d(1/r)d(-1/r)$. Therefore it suffices to show that $d(r)d(-r) = 0$ for every $r \in R$. If $r = \pi$ this is the given relation upon L, and if $r \in R^\times$ then $d(r)d(-r) = \{r, -r\} = 0$ by (5.10.3). Since the product in L is anticommutative, the general case $r = u\pi^i$ follows from this.

Corollary 7.3.1 (Rigidity). *Suppose that F is complete with respect to the valuation v, with residue field $k = k_v$. For every integer q prime to* char(k), *the maps $\lambda \oplus \partial_v: K_n^M(F)/q \to K_n^M(k)/q \oplus K_{n-1}^M(k)/q$ are isomorphisms for every n.*

Proof. Since the valuation ring R is complete, Hensel's Lemma implies that the group $1+\pi R$ is q-divisible. It follows that $l(1+\pi R)\cdot K_{n-1}^M(F)$ is also q-divisible. But by Ex. 7.2 this is the kernel of the map $d: K_n^M(F) \to L_n \cong K_n^M(k_v) \oplus K_{n-1}^M(k_v)$.

Leading Coefficients 7.3.2. As in Example 6.1.2, $K_n^M(F)$ is a direct summand of $K_n^M F(t)$. To see this, we consider the valuation $v_\infty(f) = -deg(f)$ on $F(t)$ of Example 6.5.3. Since t^{-1} is a parameter, each polynomial $f = ut^{-i}$ has lead$(f) = \bar{u}$. The map $\lambda: K_n^M F(t) \to K_n^M(F)$, given by $\lambda\{f_1, \dots, f_n\} = \{\text{lead}(f_1), \dots, \text{lead}(f_n)\}$, is clearly inverse to the natural map $K_n^M(F) \to K_n^M F(t)$.

Except for v_∞, every discrete valuation v on $F(t)$ which is trivial on F is the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$ associated to a prime ideal $\mathfrak{p}$ of $F[t]$. In this case k_v is the field $F[t]/\mathfrak{p}$, and we write $\partial_\mathfrak{p}$ for ∂_v.

Theorem 7.4 (Milnor). *There is a split exact sequence for each n, natural in the field F, and split by the map λ:*

$$0 \to K_n^M(F) \to K_n^M F(t) \xrightarrow{\partial = \coprod \partial_\mathfrak{p}} \coprod_\mathfrak{p} K_{n-1}^M(F[t]/\mathfrak{p}) \to 0.$$

Proof. Let L_d denote the subgroup of $K_n^M F(t)$ generated by those symbols $\{f_1, \dots, f_r\}$ such that all the polynomials f_i have degree $\leq d$. By Example 7.3.2, L_0 is a summand isomorphic to $K_n^M(F)$. Since $K_n^M F(t)$ is the union of the subgroups L_d, the theorem will follow from Lemma 7.4.2 below, using induction on d.

Let π be an irreducible polynomial of degree d and set $k = k_\pi = F[t]/(\pi)$. Then each element $\bar{a}$ of k is represented by a unique polynomial $a \in F[t]$ of degree $< d$.

Lemma 7.4.1. *There is a unique homomorphism $h = h_\pi: K_{n-1}^M(k) \to L_d/L_{d-1}$ carrying $\{\bar{a}_2, \dots, \bar{a}_n\}$ to the class of $\{\pi, a_2, \dots, a_n\}$ modulo L_{d-1}.*

Proof. The formula gives a well-defined set map h from $k^\times \times \cdots \times k^\times$ to L_d/L_{d-1}. To see that it is linear in $\bar{a}_2$, suppose that $\bar{a}_2 = \bar{a}_2'\bar{a}_2''$. If $a_2 \neq a_2'a_2''$ then there is a nonzero polynomial f of degree $< d$ with $a_2 = a_2'a_2'' + f\pi$. Since

$f\pi/a_2 = 1 - a_2'a_2''/a_2$ we have $\{f\pi/a_2, a_2'a_2''/a_2\} = 0$. Multiplying by $\{a_3, \dots, a_n\}$ gives

$$\{\pi, a_2'a_2''/a_2, a_3, \dots, a_n\} \equiv 0 \qquad \text{modulo } L_{d-1}.$$

Similarly, h is linear in $a_3, \dots, a_n$. To see that the multilinear map h factors through $K_{n-1}^M(k)$, we observe that if $\bar{a}_i + \bar{a}_{i+1} = 1$ in k then $a_i + a_{i+1} = 1$ in F.

LEMMA 7.4.2. *The homomorphisms $\partial_{(\pi)}$ and h_π induce an isomorphism between L_d/L_{d-1} and the direct sum $\oplus_\pi K_{n-1}^M(k_\pi)$ as π ranges over all monic irreducible polynomials of degree d in $F[t]$.*

PROOF. Since π cannot divide any polynomial of degree $< d$, the maps $\partial_{(\pi)}$ vanish on L_{d-1} and induce maps $\bar{\partial}_{(\pi)}: L_d/L_{d-1} \to K_{n-1}^M(k_\pi)$. By inspection, the composition of $\oplus h_\pi$ with the direct sum of the $\bar{\partial}_{(\pi)}$ is the identity on $\oplus_\pi K_{n-1}^M(k_\pi)$. Thus it suffices to show that $\oplus h_\pi$ maps onto L_d/L_{d-1}. By Ex. 6.2, L_d is generated by L_{d-1} and symbols $\{\pi, a_2, \dots, a_n\}$ where π has degree d and the a_i have degree $< d$. But each such symbol is h_π of an element of $K_{n-1}^M(k_\pi)$, so $\oplus h_\pi$ is onto.

The Transfer Map

Let v_∞ be the valuation on $F(t)$ with parameter t^{-1}. The formulas in Theorem 7.3 defining ∂_∞ show that it vanishes on $K_*^M(F)$. By Theorem 7.4, there are unique homomorphisms $N_\mathfrak{p}: K_n^M(F[t]/\mathfrak{p}) \to K_n^M(F)$ so that $-\partial_\infty = \sum_\mathfrak{p} N_\mathfrak{p}\partial_\mathfrak{p}$.

DEFINITION 7.5. Let E be a finite field extension of F generated by an element a. Then the *transfer map*, or *norm map* $N = N_{a/F}: K_*^M(E) \to K_*^M(F)$, is the unique map $N_\mathfrak{p}$ defined above, associated to the kernel $\mathfrak{p}$ of the map $F[t] \to E$ sending t to a.

We can calculate the norm of an element $x \in K_n^M(E)$ as $N_\mathfrak{p}(x) = -\partial_{v_\infty}(y)$, where $y \in K_{n+1}^M F(t)$ is such that $\partial_\mathfrak{p}(y) = x$ and $\partial_{\mathfrak{p}'}(y) = 0$ for all $\mathfrak{p}' \neq \mathfrak{p}$.

If $n = 0$, the transfer map $N: \mathbb{Z} \to \mathbb{Z}$ is multiplication by the degree $[E : F]$ of the field extension, while if $n = 1$ the map $N: E^\times \to F^\times$ is the usual norm map; see Ex. 7.5. We will show in 7.6 below that N is independent of the choice of $a \in E$ for all n. First we make two elementary observations.

If we let N_∞ denote the identity map on $K_n^M(F)$, and sum over the set of all discrete valuations on $F(t)$ which are trivial on F, the definition of the N_v yields:

WEIL RECIPROCITY FORMULA 7.5.1. $\sum_v N_v\partial_v(x) = 0$ for all $x \in K_n^M F(t)$.

PROJECTION FORMULA 7.5.2. *Let $E = F(a)$. Then for $x \in K_*^M(F)$ and $y \in K_*^M(E)$ the map $N = N_{a/F}$ satisfies $N\{x, y\} = \{x, N(y)\}$.*

PROOF. The inclusions of F in $F(t)$ and $F[t]/\mathfrak{p}$ allow us to view $K_*^M F(t)$ and $K_*^M(F[t]/\mathfrak{p})$ as graded modules over the ring $K_*^M(F)$. It follows from Theorem 7.4 that each $\partial_\mathfrak{p}$ is a graded module homomorphism of degree -1. This remark also applies to v_∞ and ∂_∞, because $F(t) = F(t^{-1})$. Therefore each $N_\mathfrak{p}$ is a graded module homomorphism of degree 0.

Taking $y = 1$ in $K_0^M(E) = \mathbb{Z}$, so $N(y) = [E : F]$ by Ex. 7.5, this yields

COROLLARY 7.5.3. *If the extension E/F has degree d, then the composition $K_*^M(F) \to K_*^M(E) \xrightarrow{N} K_*^M(F)$ is multiplication by d. In particular, the kernel of $K_*^M(F) \to K_*^M(E)$ is annihilated by d.*

DEFINITION 7.6. Let $E = F(a_1, \dots, a_r)$ be a finite field extension of F. The *transfer map* $N_{E/F}\colon K_*^M(E) \to K_*^M(F)$ is defined to be the composition of the transfer maps defined in 7.5:

$$K_n^M(E) \xrightarrow{N_{a_r}} K_n^M F(a_1, \dots, a_{r-1}) \xrightarrow{N_{a_{r-1}}} \cdots K_n^M(F(a_1) \xrightarrow{N_{a_1}} K_n^M(F).$$

The transfer map is well-defined by the following result of K. Kato.

THEOREM 7.6.1 (KATO). *The transfer map $N_{E/F}$ is independent of the choice of elements $a_1, \dots, a_r$ such that $E = F(a_1, \dots, a_r)$. In particular, if $F \subset F' \subset E$ then $N_{E/F} = N_{F'/F} N_{E/F'}$.*

The key trick used in the proof of this theorem is to fix a prime p and pass from F to the union F' of all finite extensions of F of degree prime to p. By Corollary 7.5.3 the kernel of $K_n^M(F) \to K_n^M(F')$ has no p-torsion, and the degree of every finite extension of F' is a power of p.

LEMMA 7.6.2. *(Kato) If E is a normal extension of F, and $[E : F]$ is a prime number p, then the map $N_{E/F} = N_{a/F}\colon K_*^M(E) \to K_*^M(F)$ does not depend upon the choice of a such that $E = F(a)$.*

PROOF. If also $E = F(b)$, then from Corollary 7.5.3 and Ex. 7.7 with $F' = E$ we see that $\delta(x) = N_{a/F}(x) - N_{b/F}(x)$ is annihilated by p. If $\delta(x) \neq 0$ for some $x \in K_n^M(E)$ then, again by Corollary 7.5.3, $\delta(x)$ must be nonzero in $K_n^M(F')$, where F' is the union of all finite extensions of F of degree prime to p. Again by Ex. 7.7, we see that we may replace F by F' and x by its image in $K_n^M(EF')$. Since the degree of every finite extension of F' is a power of p, the assertion for F' follows from Ex. 7.6, since the Projection Formula 7.5.2 yields $N_{a/F'}\{y, x_2, \dots, x_n\} = \{N(y), x_2, \dots, x_n\}$.

COROLLARY 7.6.3. *If in addition F is a complete discrete valuation field with residue field k_v, and the residue field of E is k_w, the following diagram commutes.*

$$\begin{tikzcd} K_n^M(E) \arrow[r, "\partial_w"] \arrow[d, "N"'] & K_{n-1}^M(k_w) \arrow[d, "N"] \\ K_n^M(F) \arrow[r, "\partial_v"] & K_{n-1}^M(k_v) \end{tikzcd}$$

PROOF. Ex. 7.6 implies that for each $u \in K_n^M(E)$ there is a finite field extension F' of F such that $[F' : F]$ is prime to p and the image of u in $K_n(EF')$ is generated by elements of the form $u' = \{y, x_2, \dots, x_n\}$ $(y \in EF',\ x_i \in F')$. By Ex. 7.7 and Ex. 7.8 it suffices to prove that $N_{k_w/k_v} \partial_w(u) = \partial_v(N_{EF'/F'} u)$ for every element u of this form. But this is an easy computation.

PROPOSITION 7.6.4 (KATO). *Let E and $F' = F(a)$ be extensions of F with E/F normal of prime degree p. If $E' = E(a)$ denotes the composite field, the following diagram commutes.*

$$\begin{array}{ccc} K_*^M(E') & \xrightarrow{N_{a/E}} & K_*^M(E) \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ K_*^M(F') & \xrightarrow{N_{a/F}} & K_*^M(F). \end{array}$$

PROOF. The vertical norm maps are well-defined by Lemma 7.6.2. Let $\pi \in F[t]$ and $\pi' \in E[t]$ be the minimal polynomials of a over F and E, respectively. Given $x \in K_n^M(E')$, we have $N_{a/E}(x) = -\partial_\infty(y)$, where $y \in K_{n+1}^M E(t)$ satisfies $\partial_{\pi'}(y) = x$ and $\partial_w(y) = 0$ if $w \neq w_{\pi'}$. If v is a valuation on $F(t)$, Ex. 7.9 gives:

$$\partial_v(N_{E(t)/F(t)}y) = \sum_{w|v} N_{E(w)/F(v)}(\partial_w y) = \begin{cases} N_{E'/F'}(x) & \text{if } v = v_\pi \\ N_{E/F}(\partial_\infty y) & \text{if } v = v_\infty \\ 0 & \text{else} \end{cases}$$

in $K_*^M(F')$. Two applications of Definition 7.5 give the desired calculation:

$$N_{a/F}(N_{E'/F'}x) = -\partial_\infty(N_{E(t)/F(t)}y) = -N_{E/F}(\partial_\infty y) = N_{E/F}(N_{a/F}x).$$

PROOF OF THEOREM 7.6.1. As in the proof of Lemma 7.6.2, we see from Corollary 7.5.3 and Ex. 7.7 with $F' = E$ that the indeterminacy is annihilated by $[E : F]$. Using the key trick of passing to a larger field, we may assume that the degree of every finite extension of F is a power of a fixed prime p.

Let us call a tower of intermediate fields $F = F_0 \subset F_1 \subset \cdots \subset F_r = E$ *maximal* if $[F_i : F_{i-1}] = p$ for all i. By Lemma 7.6.2, the transfer maps $N\colon K_*^M(F_i) \to K_*^M(F_{i-1})$ are independent of the choice of a such that $F_i = F_{i-1}(a)$. If $F \subset F_1 \subset E$ and $F \subset F' \subset E$ are maximal towers, Proposition 7.6.4 states that $N_{F'/F}N_{E/F'} = N_{F_1/F}N_{E/F_1}$, because if $F' \neq F_1$ then $E = F'F_1$. It follows by induction on $[E : F]$ that if $F = F_0 \subset F_1 \subset \cdots \subset F_r = E$ is a maximal tower then the composition of the norm maps

$$K_n^M(E) \xrightarrow{N} K_n^M(F_{r-1}) \xrightarrow{N} \cdots K_n^M(F_1) \xrightarrow{N} K_n^M(F)$$

is independent of the choice of maximal tower.

Comparing any tower to a maximal tower, we see that it suffices to prove that if $F \subset F_1 \subset F'$ is a maximal tower and $F' = F(a)$ then $N_{a/F} = N_{F_1/F}N_{F'/F_1}$. But this is just Proposition 7.6.4 with $E = F_1$ and $E' = F'$.

The dlog symbol and $\nu(n)_F$

For any field F, we write Ω_F^n for the nth exterior power of the vector space $\Omega_F = \Omega_{F/\mathbb{Z}}$ of Kähler differentials (Ex. 2.6). The direct sum over n forms a graded-commutative ring Ω_F^*, and the map $dlog\colon F^\times \to \Omega_F$ sending a to $\frac{da}{a}$ extends to a graded ring map from the tensor algebra $T(F^\times)$ to Ω_F^*. By Ex. 6.9, $l(a) \otimes l(1-a)$ maps to zero, so it factors through the quotient ring $K_*^M(F)$ of $T(F^\times)$. We record this observation for later reference.

LEMMA 7.7. *If F is any field, there is a graded ring homomorphism*

$$dlog\colon K_*^M(F) \to \Omega_F^*, \qquad dlog\{a_1, \dots, a_n\} = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}.$$

Now let F be a field of characteristic $p \neq 0$, so that $d(a^p) = p\,da = 0$. In fact, if $\{x_i\}$ is a p-basis of F over F^p then the symbols dx_i form a basis of the F-vector space Ω_F. Note that the set $d\Omega_F^{n-1}$ of all symbols $da_1 \wedge \cdots \wedge da_n$ forms an F^p-vector subspace of Ω_F^n.

DEFINITION 7.7.1. If $\mathrm{char}(F) = p \neq 0$, let $\nu(n)_F$ denote the kernel of the Artin-Schrier operator $\wp\colon \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1}$, which is defined by

$$\wp\left(x\,\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \right) = (x^p - x)\,\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}.$$

(In the literature, $\wp + 1$ is the inverse of the "Cartier" operator.)

Clearly $\wp(dlog\{a_1, \dots, a_n\}) = 0$, so the image of the *dlog* map lies in $\nu(n)_F$. The following theorem, which implies that these symbols span $\nu(n)_F$, was proven by Kato [K82] for $p = 2$, and for general p by Bloch, Kato and Gabber [BK, 2.1].

THEOREM 7.7.2. *(Bloch-Kato-Gabber) Let F be a field of characteristic $p \neq 0$. Then the dlog map induces an isomorphism $K_n^M(F)/pK_n^M(F) \cong \nu(n)_F$ for every $n \geq 0$.*

Using this result, Bloch and Kato also proved that the p-torsion subgroup of $K_n^M(F)$ is divisible [BK, 2.8]. Using this divisibility, Izhboldin found the following generalization of theorem 6.7; see [Izh].

IZHBOLDIN'S THEOREM 7.8. *If $\mathrm{char}(F) = p$, the group $K_n^M(F)$ has no p-torsion.*

PROOF. We proceed by induction on n, the case $n = 2$ being theorem 6.7. As in the proof of theorem 6.7, let x be an indeterminate and $y = x^p - x$; the field extension $F(x)/F(y)$ is an Artin-Schrier extension, and its Galois group is generated by an automorphism σ satisfying $\sigma(x) = x + 1$. By theorem 7.4, we can regard $K_n^M(F)$ as a subgroup of both $K_n^M F(x)$ and $K_n^M F(y)$.

For all field extensions E of $F(y)$ linearly disjoint from $F(x)$, i.e., with no root of $t^p - t - y$, write $E(x)$ for the field $E \otimes_{F(y)} F(x)$. Let $I(E)$ denote the set of all p-torsion elements in $K_n^M E(x)$ of the form $v - \sigma(v)$, $v \in K_n^M E(x)$, and let $P(E)$ denote the p-torsion subgroup of the kernel of the norm map $N_{x/E}\colon K_n^M E(x) \to K_n^M(E)$. Since $N\sigma(v) = N(v)$, $I(E) \subseteq P(E)$. Both $I(E)$ and $P(E)$ vary naturally with E, and are equal by proposition 7.8.2 below.

Fix $u \in K_n^M(F)$ with $pu = 0$. The projection formula 7.5.2 shows that the norm map $K_n^M F(x) \to K_n^M F(y)$ sends u to $pu = 0$. Hence $u \in P(F(y))$. By proposition 7.8.2, $u \in I(F(y))$, i.e., there is a $v \in K_n^M F(x)$ so that $u = v - \sigma(v)$ in $K_n^M F(x)$. Now apply the leading coefficient symbol λ of 7.3.2; since $\lambda(\sigma v) = \lambda(v)$ we have: $u = \lambda(u) = \lambda(v) - \lambda(\sigma v) = 0$. This proves Izhboldin's theorem.

Before proceeding to proposition 7.8.2, we need some facts about the group $I(E)$. We first claim that the transcendental extension $E \subset E(t)$ induces an isomorphism

$I(E) \cong I(E(t))$. Indeed, since $E(x,t)$ is purely transcendental over $E(x)$, theorem 7.4 and induction on n imply that $K_n^M E(x) \to K_n^M E(x,t)$ is an isomorphism on p-torsion subgroups, and the claim follows because the leading coefficient symbol 7.3.2 commutes with σ.

We next claim that if E/E' is a purely inseparable field extension then $I(E') \to I(E)$ is onto. For this we may assume that $E^p \subseteq E' \subset E$. The composition of the Frobenius map $E \to E^p$ with this inclusion induces the endomorphism of $K_n^M(E)$ sending $\{a_1, \dots, a_n\}$ to $\{a_1^p, \dots, a_n^p\} = p^n\{a_1, \dots, a_n\}$. Hence this claim follows from the following result.

Lemma 7.8.1. *The group $I(E)$ is p-divisible.*

Proof. Pick $v \in K_n^M E(x)$ so that $u = v - \sigma(v)$ is in $I(E)$. Now we invoke the Bloch-Kato result, mentioned above, that the p-torsion subgroup of $K_n^M(L)$ is divisible for every field L of characteristic p. By theorem 7.7.2, this implies that u vanishes in $K_n^M E(x)/p \cong \nu(n)_{E(x)}$. By Ex. 7.12 and theorem 7.7.2, the class of $v \bmod p$ comes from an element $w \in K_n^M(E)$, i.e., $v - w = pv'$ for some $v' \in K_n^M E(x)$. Then $u = v - \sigma(v) = pv' - p\sigma(v')$, it follows that $u' = v' - \sigma(v')$ is an element of $I(E)$ with $u = pu'$.

Proposition 7.8.2. *For all E containing $F(y)$, linearly disjoint from $F(x)$, $P(E) = I(E)$.*

Proof. We shall show that the obstruction $V(E) = P(E)/I(E)$ vanishes. This group has exponent p, because if $u \in P(E)$ then

$$\begin{aligned} pu &= pu - N_{EL/E}u = (p - 1 - \sigma - \cdots - \sigma^{p-1})u \\ &= \big((1-\sigma) + (1-\sigma^2) + \cdots + (1-\sigma^{p-1})\big)u \end{aligned}$$

is in $(1-\sigma)K_n^M E(x)$ and hence in $I(E)$. It follows that $V(E)$ injects into $I(E')$ whenever E'/E is an extension of degree prime to p.

Now we use the "Brauer-Severi" trick; this trick will be used again in chapter VI, where the name is explained. For each $b \in E$ we let E_b denote the field $E(t_1, \dots, t_{p-1}, \beta)$ with $t_1, \dots, t_{p-1}$ purely transcendental over E and $\beta^p - \beta - y + \sum b^i t_i^p = 0$. It is known that b is in the image of the norm map $E_b(x)^\times \to E_b^\times$; see [J37]. Since $E \cdot (E_b)^p$ is purely transcendental over E (on $\beta, t_2^p, \dots, t_{p-1}^p$), it follows that $I(E) \to I(E_b)$ is onto. Since $E_b(x)$ is purely transcendental over $E(x)$ (why?), $I(E(x)) = I(E_b(x))$ and $K_n^M E(x)$ embeds in $K_n^M E_b(x)$ by theorem 7.4. Hence $K_n^M(E(x))/I(E)$ embeds in $K_n^M E_b(x)/I(E_b)$. Since $V(E) \subset K_n^M E(x)/I(E)$ by definition, we see that $V(E)$ also embeds into $V(E_b)$.

Now if we take the composite of all the fields E_b, $b \in E$, and then form its maximal algebraic extension E' of degree prime to p, it follows that $V(E)$ embeds into $V(E')$. Repeating this construction a countable number of times yields an extension field E'' of E such that $V(E)$ embeds into $V(E'')$ and every element of E'' is a norm from $E''(x)$. Hence it suffices to prove that $V(E'') = 0$. The proof in this special case is completely parallel to the proof of proposition 6.6.2, and we leave the details to Ex. 7.13.

This completes the proof of Izhboldin's Theorem 7.8.

Corollary 7.8.3 (Hilbert's Theorem 90 for K_*^M). *Let $j\colon F \subset L$ be a degree p field extension, with* $\operatorname{char}(F) = p$*, and let σ be a generator of $G = Gal(L/F)$. Then $K_n^M(F) \cong K_n^M(L)^G$, and the following sequence is exact for all $n > 0$:*

$$0 \to K_n^M(F) \xrightarrow{j^*} K_n^M(L) \xrightarrow{1-\sigma} K_n^M(L) \xrightarrow{N} K_n^M(F).$$

Proof. Since $K_n^M(F)$ has no p-torsion, Corollary 7.5.3 implies that j^* is an injection. To prove exactness at the next spot, suppose that $v \in K_n^M(L)$ has $\sigma(v) = v$. By Ex. 7.12 and theorem 7.7.2, the class of $v \bmod p$ comes from an element $w \in K_n^M(E)$, i.e., $v - j^*(w) = pv'$ for some $v' \in K_n^M E(x)$. Hence $p\sigma(v') = \sigma(pv') = pv'$. Since $K_n^M(L)$ has no p-torsion, $\sigma(v') = v'$. But then pv' equals $j^*N(v') = \sum \sigma^i(v')$, and hence $v = j^*(w) + j^*(Nv')$. In particular, this proves that $K_n^M(F) \cong K_n^M(L)^G$.

To prove exactness at the final spot, note that G acts on $K_n^M(L)$, and that $\ker(N)/\operatorname{im}(1-\sigma)$ is isomorphic to the cohomology group $H^1(G, K_n^M(L))$; see [WHomo, 6.2.2]. Now consider the exact sequence of $Gal(L/F)$-modules

$$0 \to K_n^M(L) \xrightarrow{p} K_n^M(L) \xrightarrow{7.7.2} \nu(n)_L \to 0.$$

Using Ex. 7.12, the long exact sequence for group cohomology begins

$$0 \to K_n^M(F) \xrightarrow{p} K_n^M(F) \to \nu(n)_F \to H^1(G, K_n^M(L)) \xrightarrow{p} H^1(G, K_n^M(L)).$$

But $K_n^M(F)$ maps onto $\nu(n)_F$ by theorem 7.7.2, and the group $H^1(G, A)$ has exponent p for all G-modules A [WHomo, 6.5.8]. It follows that $H^1(G, K_n^M(L)) = 0$, so $\ker(N) = \operatorname{im}(1-\sigma)$, as desired.

Relation to the Witt ring

Let F be a field of characteristic $\neq 2$. Recall from §5.6 of chapter II that the *Witt ring* $W(F)$ is the quotient of the Grothendieck group $K_0\mathbf{SBil}(F)$ of symmetric inner product spaces over F by the subgroup $\{nH\}$ generated by the hyperbolic form $\langle 1\rangle \oplus \langle -1\rangle$. The dimension of the underlying vector space induces an augmentation $K_0\mathbf{SBil}(F) \to \mathbb{Z}$, sending $\{nH\}$ isomorphically onto $2\mathbb{Z}$, so it induces an augmentation $\varepsilon\colon W(F) \to \mathbb{Z}/2$.

We shall be interested in the augmentation ideals $I = \ker(\varepsilon)$ of $W(F)$ and $\hat{I}$ of $K_0\mathbf{SBil}(F)$. Since $H \cap \hat{I} = 0$, we have $\hat{I} \cong I$. Now I is generated by the classes $\langle a\rangle - 1$, $a \in F - \{0, 1\}$. The powers I^n of I form a decreasing chain of ideals $W(F) \supset I \supset I^2 \supset \cdots$.

For convenience, we shall write $K_n^M(F)/2$ for $K_n^M(F)/2K_n^M(F)$.

Theorem 7.9 (Milnor). *There is a unique surjective homomorphism*

$$s_n\colon K_n^M(F)/2 \to I^n/I^{n+1}$$

sending each product $\{a_1, \dots, a_n\}$ in $K_n^M(F)$ to the product $\prod_{i=1}^n \big(\langle a_i\rangle - 1\big)$ modulo I^{n+1}. The homomorphisms s_1 and s_2 are isomorphisms.

Proof. Because $\big(\langle a\rangle - 1\big) + \big(\langle b\rangle - 1\big) \equiv \langle ab\rangle - 1$ modulo I^2 (II.5.6.5), the map $l(a_1)\times\cdots\times l(a_n) \mapsto \prod\big(\langle a_i\rangle - 1\big)$ is a multilinear map from $F^\times$ to I^n/I^{n+1}. Moreover, if $a_i + a_{i+1} = 1$ for any i, we know from Ex. II.5.11 that $(\langle a_i\rangle - 1)(\langle a_{i+1}\rangle - 1) = 0$. By the presentation of $K_*^M(F)$, this gives rise to a group homomorphism from $K_n^M(F)$ to I^n/I^{n+1}. It annihilates $2K_*^M(F)$ because $\langle a^2\rangle = 1$:

$$2s_n\{a_1, \dots, a_n\} = s_n\{a_1^2, a_2 \dots\} = \left(\langle a_1^2\rangle - 1\right) \prod_{i=2}^{n} \left(\langle a_i\rangle - 1\right) = 0.$$

It is surjective because I is generated by the $(\langle a\rangle - 1)$. When $n = 1$ the map is the isomorphism $F^\times/F^{\times 2} \cong I/I^2$ of chapter II. We will see that s_2 is an isomorphism in Corollary 7.10.3 below, using the Hasse invariant w_2.

EXAMPLE 7.9.1. For the real numbers $\mathbb{R}$, we have $W(\mathbb{R}) = \mathbb{Z}$ and $I = 2\mathbb{Z}$ on $s_1(-1) = \langle -1\rangle - 1 = 2\langle -1\rangle$. On the other hand, we saw in Example 7.2(c) that $K_n^M(\mathbb{R})/2 \cong \mathbb{Z}/2$ on $\{-1, \dots, -1\}$. In this case each s_n is the isomorphism $\mathbb{Z}/2 \cong 2^n\mathbb{Z}/2^{n+1}\mathbb{Z}$.

At the other extreme, if F is algebraically closed then $W(F) = \mathbb{Z}/2$. Since $K_n^M(F)$ is divisible, $K_n^M(F)/2 = 0$ for all $n \geq 1$. Here s_n is the isomorphism $0 = 0$.

REMARK 7.9.2. In 1970, Milnor asked if the surjection $s_n \colon K_n^M(F)/2 \to I^n/I^{n+1}$ is an isomorphism for all n and F, $\mathrm{char}(F) \neq 2$ (on p. 332 of [M-QF]). Milnor proved this was so for local and global fields. In 1996, Voevodsky proved this result for all fields and all n; see [V-MC].

DEFINITION 7.10 (STIEFEL-WHITNEY INVARIANT). The (total) *Stiefel-Whitney invariant* $w(M)$ of the symmetric inner product space $M = \langle a_1\rangle \oplus \cdots \oplus \langle a_n\rangle$ is the element of $\prod_{i=0}^{\infty} K_i^M(F)/2$ defined by the formula

$$w(M) = \prod_{i=1}^{n} \left(1 + l(a_i)\right) = 1 + l(a_1 \cdots a_n) + \cdots + \{a_1, \dots, a_n\}$$

The lemma below shows that $w(M)$ is independent of the representation of M as a direct sum of 1-dimensional forms. We write $w(M) = 1 + w_1(M) + w_2(M) + \cdots$, where the i^{th} *Stiefel-Whitney invariant* $w_i(M) \in K_i^M(F)/2$ equals the i^{th} elementary symmetric function of $l(a_1), \dots, l(a_n)$. For example, $w_1(M) = a_1 \cdots a_n \in F^\times/F^{\times 2}$ is just the classical "discriminant" of M defined in II.5.6.3, while the second elementary symmetric function $w_2(M) = \sum_{i<j}\{a_i, a_j\}$ lies in $K_2(F)/2$ and is called the *Hasse invariant* of M; see [M-SBF].

For $M = \langle a\rangle \oplus \langle b\rangle$ we have $w_1(M) = ab$ and $w_2(M) = \{a, b\}$, with $w_i(M) = 0$ for $i \geq 3$. In particular, the hyperbolic plane H has $w_i(H) = 0$ for all $i \geq 2$.

LEMMA 7.10.1. *$w(M)$ is a well-defined unit in the ring $\prod_{i=1}^{\infty} K_i^M(F)/2$. It satisfies the Whitney sum formula*

$$w(M \oplus N) = w(M)w(N),$$

so w extends to a function on $K_0\mathbf{SBil}(F)$. Hence each Stiefel-Whitney invariant w_i extends to a function $K_0\mathbf{SBil}(F) \xrightarrow{w_i} K_i^M(F)/2$.

PROOF. To show that $w(M)$ is well defined, it suffices to consider the rank two case. Suppose that $\langle a\rangle \oplus \langle b\rangle \cong \langle \alpha\rangle \oplus \langle \beta\rangle$. Then the equation $ax^2 + by^2 = \alpha$

must have a solution x, y in F. The case $y = 0$ (or $x = 0$) is straightforward, since $\langle \alpha \rangle = \langle ax^2 \rangle = \langle a \rangle$, so we may assume that x and y are nonzero. Since the discriminant w_1 is an invariant, we have $ab = \alpha\beta u^2$ for some $u \in F$, and all we must show is that $\{a, b\} = \{\alpha, \beta\}$ in $K_2(F)/2$. The equation $1 = ax^2/\alpha + by^2/\alpha$ yields the equation

$$0 = \{ax^2/\alpha, by^2/\alpha\} \equiv \{a, b\} + \{\alpha, \alpha\} - \{a, \alpha\} - \{b, \alpha\} \equiv \{a, b\} - \{\alpha, ab/\alpha\}$$

in $K_2(F)/2K_2(F)$. Substituting $ab = \alpha\beta u^2$, this implies that $\{a, b\} \equiv \{\alpha, \beta\}$ modulo $2K_2(F)$, as desired.

EXAMPLE 7.10.2. Since $I \cong \hat{I}$, we may consider the w_i as functions on $I \subseteq W(F)$. However, care must be taken as $w_i(M)$ need not equal $w_2(M \oplus H)$. For example, $w_2(M \oplus H) = w_2(M) + \{w_1(M), -1\}$. In particular, $w_2(H \oplus H) = \{-1, -1\}$ can be nontrivial. The *Hasse-Witt invariant* of an element $x \in I \subset W(F)$ is defined to be $h(x) = w_2(V, B)$, where (V, B) is an inner product space representing x so that $\dim(V) \equiv 0 \bmod 8$.

COROLLARY 7.10.3. *The Hasse invariant $w_2\colon \hat{I} \to K_2(F)/2$ induces an isomorphism from $\hat{I}^2/\hat{I}^3 \cong I^2/I^3$ to $K_2^M(F)/2$, inverse to the map s_2 of Theorem 7.9.*

PROOF. By Ex. 7.11, w_2 vanishes on the ideal $\hat{I}^3 \cong I^3$, and hence defines a function from $\hat{I}^2/\hat{I}^3$ to $K_2(F)/2$. Since the total Stiefel-Whitney invariant of $s_2\{a, b\} = \big(\langle a \rangle - 1\big)\big(\langle b \rangle - 1\big)$ is $1 + \{a, b\}$, this function provides an inverse to the function s_2 of Theorem 7.9.

If $\mathrm{char}(F) = 2$, there is an elegant formula for the filtration quotients of the Witt ring $W(F)$ and the $W(F)$-module $WQ(F)$ (see II.5) due to K. Kato [K82]. Recall from 7.7.2 that $K_n^M(F)/2 \cong \nu(n)_F$, where $\nu(n)_F$ is the kernel of the operator $\wp$. The case $n = 0$ of Kato's result was described in Ex. II.5.12(d).

THEOREM 7.10.4 (KATO [K82]). *Let F be a field of characteristic 2. Then the map s_n of Theorem 7.9 induces an isomorphism $K_n^M(F)/2 \cong \nu(n)_F \cong I^n/I^{n+1}$, and there is a short exact sequence*

$$0 \to I^n/I^{n+1} \to \Omega_F^n \xrightarrow{\wp} \Omega_F^n/d\Omega_F^{n-1} \to I^n\ WQ(F)/I^{n+1}\ WQ(F) \to 0.$$

The Galois symbol

For the next result, we need some facts about Galois cohomology, expanding slightly upon the facts mentioned in 6.10. Assuming that n is prime to $\mathrm{char}(F)$, there are natural cohomology cup products $H^i_{et}(F; M) \otimes H^j_{et}(F; N) \xrightarrow{\cup} H^{i+j}_{et}(F; M \otimes N)$ which are associative in M and N. This makes the direct sum $H^*_{et}(F; M^{\otimes *}) = \oplus_{i=0}^{\infty} H^i_{et}(F; M^{\otimes i})$ into a graded-commutative ring for every $\mathbb{Z}/n$-module M over the Galois group $Gal(F_{\mathrm{sep}}/F)$. (By convention, $M^{\otimes 0}$ is $\mathbb{Z}/n$.) In particular, both $H^*_{et}(F; \mathbb{Z}/n)$ and $H^*_{et}(F; \mu_n^{\otimes *})$ are rings, and are isomorphic only when F contains a primitive n^{th} root of unity.

THEOREM 7.11 (GALOIS SYMBOLS). *(Bass-Tate) Fix a field F and an integer n prime to* char(F)*.*

(1) *If F contains a primitive n^{th} root of unity, the Kummer isomorphism from $F^\times/F^{\times n}$ to $H^1_{et}(F;\mathbb{Z}/n)$ extends uniquely to a graded ring homomorphism*

$$h_F\colon K^M_*(F)/n \to H^*(F;\mathbb{Z}/n).$$

(2) *More generally, the Kummer isomorphism from $F^\times/F^{\times n}$ to $H^1(F;\mu_n)$ extends uniquely to a graded ring homomorphism*

$$h_F\colon K^M_*(F)/n \to H^*_{et}(F;\mu_n^{\otimes *}) = \oplus_{i=0}^{\infty} H^i_{et}(F;\mu_n^{\otimes i}).$$

The individual maps $K^M_i(F) \to H^i_{et}(F;\mu_n^{\otimes i})$ are called the higher Galois symbols.

PROOF. The first assertion is just a special case of the second assertion. As in (6.10.2), the Kummer isomorphism induces a map from the tensor algebra $T(F^\times)$ to $H^*_{et}(F;\mu_n^{\otimes *})$, which in degree i is the iterated cup product

$$F^\times \otimes \cdots F^\times = (F^\times)^{\otimes n} \cong \big(H^1_{et}(F;\mu_n)\big)^{\otimes i} \xrightarrow{\cup} H^i_{et}(F;\mu_n^{\otimes i}).$$

By Proposition 6.10.3, the Steinberg Relation is satisfied in $H^2_{et}(F,\mu_n^{\otimes 2})$. Hence the presentation of $K^M_*(F)$ yields a ring homomorphism from $K^M_*(F)$ to $H^*_{et}(F;\mu_n^{\otimes *})$.

REMARK 7.11.1. Milnor studied the Galois symbol for $n = 2$ and stated (on p.340 of [M-QF]) that, "I do not know any examples for which the homomorphism h_F fails to be bijective." In 1996, Voevodsky announced a proof of this result for $n = 2$; see [V-MC]. The following partial had been obtained a decade earlier, by Rost and Merkurjev-Suslin [MS2].

THEOREM 7.11.2. *If* char$(F) \neq 2$*, the Galois symbol*

$$K^M_3(F)/2 \to H^3_{et}(F;\mathbb{Z}/2) = H^3_{et}(F;\mu_2^{\otimes 3})$$

is an isomorphism

EXERCISES

7.1 Let v be a discrete valuation on a field F. Show that the maps $\lambda\colon K^M_n(F) \to K^M_n(k_v)$ and $\partial_v\colon K^M_n(F) \to K^M_{n-1}(k_v)$ of Theorem 7.3 are independent of the choice of parameter π, and that they vanish on $l(u)\cdot K^M_{n-1}(F)$ whenever $u \in (1+\pi R)$. Show that the map λ also vanishes on $l(\pi)\cdot K^M_{n-1}(F)$.

7.2 Continuing Exercise 7.1, show that the kernel of the map $d\colon K^M_n(F) \to L_n$ of Theorem 7.3 is exactly $l(1+\pi R)\cdot K^M_{n-1}(F)$. Conclude that the kernel of the map λ is exactly $l(1+\pi R)\cdot K^M_{n-1}(F) + l(\pi)\cdot K^M_{n-1}(F)$.

7.3 (Bass-Tate) Generalize Theorem 6.4 to show that for all $n \geq 2$:
(a) If F is an algebraically closed field, then $K^M_n(F)$ is uniquely divisible.
(b) If F is a perfect field of characteristic p then $K^M_n(F)$ is uniquely p-divisible.

7.4 Let F be a local field with valuation v and finite residue field k. Show that $K_n^M(F)$ is divisible for all $n \geq 3$. *Hint:* By Moore's Theorem 6.2.4, $K_n^M(F)$ is ℓ-divisible unless F has a ℓ^{th} root of unity. Moreover, for every $x \notin F^{\times\ell}$ there is a $y \notin F^{\times\ell}$ so that $\{x, y\}$ generates $K_2(F)/\ell$. Given a, b, c with $\{b, c\} \notin \ell K_2(F)$, find $a', b' \notin F^{\times\ell}$ so that $\{b', c\} \equiv 0$ and $\{a', b'\} \equiv \{a, b\}$ modulo $\ell K_2(F)$, and observe that $\{a, b, c\} \equiv \{a', b', c\} \equiv 0$.

In fact, I. Sivitskii has shown that $K_n^M(F)$ is uniquely divisible for $n \geq 3$ when F is a local field. See [Siv].

7.5 Let $E = F(a)$ be a finite extension of F, and consider the transfer map $N = N_{a/F}: K_n^M(E) \to K_n^M(F)$ in definition 7.5. Use Weil's Formula (7.5.1) to show that when $n = 0$ the transfer map $N: \mathbb{Z} \to \mathbb{Z}$ is multiplication by $[E : F]$, and that when $n = 1$ the transfer map $N: E^\times \to F^\times$ is the usual norm map.

7.6 Suppose that the degree of every finite extension of a field F is a power of some fixed prime p. If E is an extension of degree p and $n > 0$, use Ex. 6.2 to show that $K_n^M(E)$ is generated by elements of the form $\{y, x_2, \dots, x_n\}$, where $y \in E^\times$ and the x_i are in $F^\times$.

7.7 *Ramification and the transfer.* Let F' and $E = F(a)$ be finite field extensions of F, and suppose that the irreducible polynomial $\pi \in F[t]$ of a has a decomposition $\pi = \prod \pi^{e_i}$ in $F'[t]$. Let E_i denote $F'(a_i)$, where each a_i has minimal polynomial π_i. Show that the following diagram commutes.

$$\begin{array}{ccc} K_n^M(E) & \xrightarrow{e_1,\dots,e_r} & \oplus K_n^M(E_i) \\ {\scriptstyle N_{a/F}}\downarrow & & \downarrow{\scriptstyle \sum N_{a_i/F'}} \\ K_n^M(F) & \longrightarrow & K_n^M(F') \end{array}$$

7.8 *Ramification and ∂_v.* Suppose that E is a finite extension of F, and that w is a valuation on E over the valuation v on F, with ramification index e. (See 6.3.1.) Use the formulas for ∂_v and ∂_w in Theorem 7.3 to show that for every $x \in K_n^M(F)$ we have $\partial_w(x) = e \cdot \partial_v(x)$ in $K_{n-1}^M(k_w)$

7.9 If E/F is a normal extension of prime degree p, and v is a valuation on $F(t)$ trivial on F, show that $\partial_v N_{E(t)/F(t)} = \sum_w N_{E(w)/F(v)} \partial_w$, where the sum is over all the valuations w of $E(t)$ over v. *Hint:* If $F(t)_v$ and $E(t)_w$ denote the completions of $F(t)$ and $E(t)$ at v and w, respectively, use Ex. 7.7 and Lemma 7.6.3 to show that the following diagram commutes.

$$\begin{array}{ccccc} K_{n+1}^M E(t) & \longrightarrow & \bigoplus_w K_{n+1}^M E(t)_w & \xrightarrow{\partial} & \bigoplus_w K_n^M E(w) \\ {\scriptstyle N_{E(t)/F(t)}}\downarrow & & \downarrow{\scriptstyle \sum_w N_{E(t)_w/F(t)_v}} & & \downarrow{\scriptstyle \sum_w N_{e(w)/F(v)}} \\ K_n^M F(t) & \longrightarrow & K_n^M F(t)_w & \xrightarrow{\partial} & K_n^M F(v) \end{array}$$

7.10 If v is a valuation on F, and $x \in K_i^M(F)$, $y \in K_j^M(F)$, show that

$$\partial_v(xy) = \lambda(x)\partial_v(y) + (-1)^j \partial_v(x)\rho(y)$$

where $\rho: K_*^M(F) \to K_*^M(k_v)$ is a ring homomorphism characterized by the formula $\rho(l(u\pi^i)) = l((-1)^i \bar{u})$.

7.11 Let $t = 2^{n-1}$ and set $z = \prod_{i=1}^n (\langle a_i \rangle - 1)$; this is a generator of the ideal $\hat{I}^n$ in $K_0\mathbf{SBil}(F)$. Show that the Stiefel-Whitney invariant $w(z)$ is equal to: $1 + \{a_n, \dots, a_n, -1, -1, \dots, -1\}$ if n is odd, and to $1 + \{a_n, \dots, a_n, -1, -1, \dots, -1\}$ if n is even. This shows that the invariants w_i vanish on the ideal $\hat{I}^n$ if $i < t = 2^{n-1}$, and that w_t induces a homomorphism from $I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1}$ to $K_t^M(F)/2$.

For example, this implies that w_1 vanishes on $\hat{I}^2$, while w_2 and w_3 vanish on $\hat{I}^3$.

7.12 (Izhboldin) Let L/F be a field extension of degree $p = \mathrm{char}(F)$, with Galois group G. Show that Ω_F^n is isomorphic to $(\Omega_L^n)^G$, and that $\Omega_F^n/d\Omega_F^{n-1}$ is isomorphic to $(\Omega_L^n/d\Omega_L^{n-1})^G$. Conclude that $\nu(n)_F \cong \nu(n)_L^G$.

7.13 In this exercise we complete the proof of proposition 7.8.2, and extablish a special case of 7.8.3. Suppose that $E(x)$ is a degree p field extension of E, $\mathrm{char}(E) = p$, and that σ is a generator of $Gal(E(x)/E)$. Suppose in addition that the norm map $E(x)^\times \to E^\times$ is onto, and that E has no extensions of degree $< p$. Modify the proof of proposition 6.6.2 to show that the following sequence is exact:

$$K_n^M E(x) \xrightarrow{1-\sigma} K_n^M E(x) \xrightarrow{N} K_n^M E \to 0.$$

CHAPTER IV

DEFINITIONS OF HIGHER K-THEORY

The higher algebraic K-groups of a ring R are defined to be the homotopy groups $K_n(R) = \pi_n K(R)$ of a certain topological space $K(R)$, which we shall construct in this chapter. Of course, the space $K(R)$ is rigged so that if $n = 0, 1, 2$ then $\pi_n K(R)$ agrees with the groups $K_n(R)$ constructed in chapters II and III.

We shall also define the higher K-theory of a category $\mathcal{A}$ in each of the three settings where $K_0(\mathcal{A})$ was defined in chapter II: when $\mathcal{A}$ is a symmetric monoidal category (§3), an exact category (§4) and a Waldhausen category (§6). In each case we build a "K-theory space" $K\mathcal{A}$ and define the group $K_n\mathcal{A}$ to be its homotopy groups: $K_n\mathcal{A} = \pi_n K\mathcal{A}$. Of course the group $\pi_0 K\mathcal{A}$ will agree with the corresponding group $K_0\mathcal{A}$ defined in chapter II.

We will show these definitions of $K_n\mathcal{A}$ coincide whenever they coincide for K_0. For example, the group $K_0(R)$ of a ring R was defined in §II.2 as K_0 of the category $\mathbf{P}(R)$ of f.g. projective R-modules, but to define $K_0\mathbf{P}(R)$ we could also regard the category $\mathbf{P}(R)$ as being either a symmetric monoidal category (II.5.2), an exact category (II.7.1) or a Waldhausen category (II.9.1.3). We will show that the different constructions give homotopy equivalent spaces $K\mathbf{P}(R)$, and hence the same homotopy groups. Thus the groups $K_n(R) = \pi_n K\mathbf{P}(R)$ will be independent of the construction used.

Many readers will not be interested in the topological details, so we have designed this chapter to allow "surfing." Since the most non-technical way to construct $K(R)$ is to use the "+"-construction, we will do this in §1 below.

In §2, we summarize the basic facts about the geometric realization BC of a category C, and the basic connection between category theory and homotopy theory needed for the rest of the constructions. Indeed, the K-theory space $K\mathcal{A}$ is constructed in each setting using the geometric realization BC of some category C, concocted out of $\mathcal{A}$. For this, we assume only that the reader has a slight familiarity with cell complexes, or *CW complexes*, which are spaces obtained by successive attachment of cells, with the weak topology.

Sections 3–6 give the construction of the K-theory spaces. Thus in §3 we have group completion constructions for symmetric monoidal categories (and the connection to the +-construction). Quillen's Q-construction for abelian and exact categories is given in §4; in §5 we prove the "$+ = Q$" theorem, that the Q-construction and group completion constructions agree for split exact categories (II.7.1.2). The $wS.$ construction for Waldhausen categories is in §6, along with its connection to the Q-construction.

We conclude with a short section (§7) on homotopy groups with finite coefficients. These have shown to be remarkably useful in describing the structure of the groups $K_n(\mathcal{A})$, especially as related to étale cohomology.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§1. The BGL^+ definition for Rings

Let R be an associative ring with unit. Recall from chapter III that the *infinite general linear group* $GL(R)$ is the union of the finite groups $GL_n(R)$, and that its commutator subgroup is the perfect group $E(R)$ generated by the elementary matrices $e_{ij}(r)$. Moreover the group $K_1(R)$ is defined to be the quotient $GL(R)/E(R)$.

In 1969, Quillen proposed defining the higher K-theory of a ring R to be the homotopy groups of a certain topological space, which he called "$BGL(R)^+$." Before describing the elementary properties of Quillen's construction, and the related subject of acyclic maps, we present Quillen's description of $BGL(R)^+$ and define the groups $K_n(R)$ for $n \geq 1$.

For any group G, we can naturally construct a connected topological space BG whose fundamental group is G, but whose higher homotopy groups are zero. Moreover, the homology of the topological space BG coincides with the algebraic homology of the group G. Details of this construction are in §2 below. For $G = GL(R)$ we obtain the space $BGL(R)$, which is central to the following definition.

Definition 1.1. The notation $BGL(R)^+$ will denote any CW complex X which has a distinguished map $BGL(R) \to BGL(R)^+$ such that

(1) $\pi_1 BGL(R)^+ = K_1(R)$, and the natural map from $GL(R) = \pi_1 BGL(R)$ to $\pi_1 BGL(R)^+$ is onto with kernel $E(R)$;

(2) $H_*(BGL(R); M) \xrightarrow{\cong} H_*(BGL(R)^+; M)$ for every $GL(R)$-module M.

We will sometimes say that X is a *model* for $BGL(R)^+$.

For $n \geq 1$, $K_n(R)$ is defined to be the homotopy group $\pi_n BGL(R)^+$.

By Theorem 1.4 below, any two models are homotopy equivalent, *i.e.,* the space $BGL(R)^+$ is uniquely defined up to homotopy. Hence the homotopy groups $K_n(R)$ of $BGL(R)^+$ are well-defined.

By construction, $K_1(R)$ agrees with the group $K_1(R) = GL(R)/E(R)$ defined in chapter III. We will see in 1.6.1 below that $K_2(R) = \pi_2 BGL^+(R)$ agrees with the group $K_2(R)$ defined in chapter III.

Several different models for $BGL(R)^+$ are described in 1.8 below. We will construct even more models for $BGL(R)^+$ in the rest of this chapter: the space $\mathbf{P}^{-1}\mathbf{P}(R)$ of §3, the space $\Omega BQ\mathbf{P}(R)$ of §4 and the space $\Omega(\mathrm{iso}\, S.S)$ arising from the Waldhausen construction in §6.

Definition 1.1.1. Write $K(R)$ for the product $K_0(R) \times BGL(R)^+$. That is, $K(R)$ is the disjoint union of copies of the connected space $BGL(R)^+$, one for each element of $K_0(R)$. By construction, $K_0(R) = \pi_0 K(R)$. Moreover, it is clear that $\pi_n K(R) = \pi_n BGL(R)^+ = K_n(R)$ for $n \geq 1$.

Functoriality 1.1.2. Each K_n is a functor from rings to abelian groups, while the topological spaces $BGL(R)^+$ and $K(R)$ are functors from rings to the homotopy category of topological spaces. However, without more information about the models used, the topological maps $BGL(R)^+ \to BGL(R')^+$ are only well-defined up to homotopy.

To see this, note that any ring map $R \to R'$ induces a natural group map $GL(R) \to GL(R')$, and hence a natural map $BGL(R) \to BGL(R')$. This induces a map $BGL(R)^+ \to BGL(R')^+$, unique up to homotopy, by Theorem 1.4 below. Thus the group maps $K_n(R) \to K_n(R')$ are well defined. Since the identity of R induces the identity on $BGL(R)^+$, only composition remains to be considered.

Given a second map $R' \to R''$, the composition $BGL(R) \to BGL(R') \to BGL(R'')$ is induced by $R \to R''$ because BGL is natural. By uniqueness in Theorem 1.4, the composition $BGL(R)^+ \to BGL(R')^+ \to BGL(R'')^+$ must be homotopy equivalent to any a priori map $BGL(R)^+ \to BGL(R'')^+$.

HOMOTOPY FIBER 1.1.3. The maps $\pi_* X \to \pi_* Y$ induced by a continuous map $X \xrightarrow{f} Y$ can always be made to fit into a long exact sequence, in a natural way. The *homotopy fiber* $F(f)$ of a f, relative to a basepoint $*_Y$ of Y, is the space of pairs (x, γ), where $x \in X$ and $\gamma\colon [0,1] \to Y$ is a path in Y starting at the baspoint $\gamma(0) = *_Y$, and ending at $\gamma(1) = f(x)$. The key property of the homotopy fiber is that (given a basepoint $*_X$ with $f(*_X) = *_Y$) there is a long exact sequence of homotopy groups/pointed sets

$$\cdots \pi_{n+1} Y \xrightarrow{\partial} \pi_n F(f) \to \pi_n X \to \pi_n Y \xrightarrow{\partial} \pi_{n-1} F(f) \to \cdots$$
$$\cdots \xrightarrow{\partial} \pi_1 F(f) \to \pi_1 X \to \pi_1 Y \xrightarrow{\partial} \pi_0 F(f) \to \pi_0 X \to \pi_0 Y.$$

RELATIVE GROUPS 1.1.4. Given a ring homomorphism $f\colon R \to R'$, let $K(f)$ be the homotopy fiber of $K(R) \to K(R')$, and set $K_n(f) = \pi_n K(f)$. This construction is designed so that these relative groups fit into a long exact sequence:

$$\cdots K_{n+1}(R') \xrightarrow{\partial} K_n(f) \to K_n(R) \to K_n(R') \xrightarrow{\partial} \cdots$$
$$\cdots K_1(R, I) \to K_1(R) to K_1(R') \xrightarrow{\partial} K_0(f) \to K_0(R) \to K_0(R').$$

When $R' = R/I$ for some ideal I, we write $K(R, I)$ for $K(R \to R/I)$. It is easy to see (Ex. 1.10) that $K_0(R, I)$ and $K_1(R, I)$ agree with the relative groups defined in Ex. II.2.3 and III.2.2, and that the ending of this sequence is the exact sequence of III, 2.3 and 5.7. Keune and Loday have shown that $K_2(R, I)$ agrees with the defined in III.5.7.

Acyclic Spaces and Acyclic Maps

The definition of $BGL(R)^+$ fits into the general framework of acyclic maps, which we now discuss. Our discussion of acyclicity is taken from [HH] and [Berrick].

DEFINITION 1.2 (ACYCLIC SPACES). We call a topological space F *acyclic* if it has the homology of a point, that is, if $\widetilde{H}_*(F; \mathbb{Z}) = 0$.

LEMMA 1.2.1. *Let F be an acyclic space. Then F is connected, its fundamental group $G = \pi_1(F)$ must be perfect, and $H_2(G; \mathbb{Z}) = 0$ as well.*

PROOF. The acyclic space F must be connected, as $H_0(F) = \mathbb{Z}$. Because $G/[G, G] = H_1(F; \mathbb{Z}) = 0$, we have $G = [G, G]$, *i.e.,* G is a perfect group. To calculate $H_2(G)$, observe that the universal covering space $\tilde{F}$ has $H_1(\tilde{F}; \mathbb{Z}) = 0$. Moreover, the homotopy fiber of the canonical map $F \to BG$ is homotopy equivalent to $\tilde{F}$ (consider the long exact sequence of homotopy groups 1.1.3 to see this). The Serre Spectral Sequence for this homotopy fibration is $E^2_{pq} = H_p(G; H_q(\tilde{F}; \mathbb{Z})) \Rightarrow H_{p+q}(F; \mathbb{Z})$ and the conclusion that $H_2(G; \mathbb{Z}) = 0$ follows from the associated exact sequence of low degree terms:

$$H_2(F; \mathbb{Z}) \to H_2(G; \mathbb{Z}) \xrightarrow{d^2} H_1(\tilde{F}; \mathbb{Z})^G \to H_1(F; \mathbb{Z}) \to H_1(G; \mathbb{Z}).$$

Example 1.2.2 (Volodin Spaces). The Volodin space $X(R)$ is an acyclic subspace of $BGL(R)$, constructed as follows. For each n, let $T_n(R)$ denote the subgroup of $GL_n(R)$ consisting of upper triangular matrices. As n varies, the union of these groups forms a subgroup $T(R)$ of $GL(R)$. Similarly we may regard the permutation groups Σ_n as subgroups of $GL_n(R)$ by their representation as permutation matrices, and their union is the infinite permutation group $\Sigma \subset GL(R)$. For each $\sigma \in \Sigma_n$, let $T_n^\sigma(R)$ denote the subgroup of $GL_n(R)$ obtained by conjugating $T_n(R)$ by σ. For example, if $\sigma = (n \dots 1)$ then $T_n^\sigma(R)$ is the subgroup of lower triangular matrices.

Since the classifying spaces $BT_n(R)$ and $BT_n(R)^\sigma$ are subspaces of $BGL_n(R)$, and hence of $BGL(R)$, we may form their union over all n and σ: $X(R) = \bigcup_{n,\sigma} BT_n(R)^\sigma$. The space $X(R)$ is acyclic (see [Suslin]). Since $X(R)$ was first described by Volodin in 1971, it is usually called the *Volodin space* of R.

The image of the map $\pi_1 X(R) \to GL(R) = \pi_1 BGL(R)$ is the group $E(R)$. To see this, note that the maps $BT_n^\sigma(R) \to BGL(R)$ factor through $X(R)$; applying π_1 shows that the image of $\pi_1(X)$ contains all the subgroups $T_n^\sigma(R)$ of $GL(R)$. Hence these subgroups contain the generators $e_{ij}(r)$ of $E(R)$. Since the image of the perfect group $\pi_1 X(R)$ is contained in the commutator subgroup $E(R)$ of $GL(R)$, the image must be $E(R)$.

Definition 1.3 (Acyclic maps). Let X and Y be based connected CW complexes. A cellular map $f\colon X \to Y$ is called *acyclic* if the homotopy fiber $F(f)$ of f is acyclic (has the homology of a point).

From the long exact sequence of homotopy groups/pointed sets, we see that if $X \to Y$ is acyclic, then the map $\pi_1(X) \to \pi_1(Y)$ is onto, and its kernel P is a perfect normal subgroup of $\pi_1(X)$.

Definition 1.3.1. Let P be a perfect normal subgroup of $\pi_1(X)$, where X is a based connected CW complex. An acyclic map $f\colon X \to Y$ is called a *$+$–construction* on X (relative to P) if P is the kernel of $\pi_1(X) \to \pi_1(Y)$.

When Quillen introduced this definition in 1969, he observed that both Y and the map f are determined up to homotopy by the subgroup P. This is the content of the following theorem; its proof uses topological obstruction theory. Part (a) is proven in Ex. 1.3; an explicit proof may be found in §5 of [Berrick].

Theorem 1.4 (Quillen). *Let P be a perfect normal subgroup of $\pi_1(X)$. Then*

(1) *There is a $+$–construction $f\colon X \to Y$ relative to P*
(2) *Let $f\colon X \to Y$ be a $+$–construction relative to P, and $g\colon X \to Z$ a map such that P vanishes in $\pi_1(Z)$. Then there is a map $h\colon Y \to Z$, unique up to homotopy, such that $g = hf$.*
(3) *In particular, if g is another $+$–construction relative to P, then the map h in (2) is a homotopy equivalence: $h\colon Y \xrightarrow{\sim} Z$.*

Remark. Every group G has a unique largest perfect subgroup P, called the *perfect radical* of G; see Ex. 1.4. If no mention is made to the contrary, the notation X^+ will always denote the $+$–construction relative to the perfect radical of $\pi_1(X)$.

The first construction along these lines was announced by Quillen in 1969, so we have adopted Quillen's term "$+$–construction" as well as his notation. A good description of his approach may be found in [HH] or [Berrick].

LEMMA 1.5. *A map $f: X \to Y$ is acyclic iff $H_*(X, M) \cong H_*(Y, M)$ for every $\pi_1(Y)$-module M.*

PROOF. Suppose first that f is acyclic, with homotopy fiber $F(f)$. Since the map $\pi_1 F(f) \to \pi_1 Y$ is trivial, $\pi_1 F(f)$ acts trivially upon M. By the Universal Coefficient Theorem, $H_q(F(q); M) = 0$ for $q \neq 0$ and $H_0(F; M) = M$. Therefore $E^2_{pq} = 0$ for $q \neq 0$ in the Serre Spectral Sequence for f:

$$E^2_{pq} = H_p(Y; H_q(F(f); M)) \Rightarrow H_{p+q}(X; M).$$

Hence the spectral sequence collapses to yield $H_p(X; M) \xrightarrow{\cong} H_p(Y; M)$ for all p.

Conversely, we suppose first that $\pi_1 Y = 0$ and $H_*(X; \mathbb{Z}) \cong H_*(Y; \mathbb{Z})$. By the Comparison Theorem for the Serre Spectral Sequences for $F(f) \to X \xrightarrow{f} Y$ and $* \to Y \xrightarrow{=} Y$, we have $H_*(F(f); \mathbb{Z}) = 0$. Hence $F(f)$ and f are acyclic.

The general case reduces to this by the following trick. Let $\tilde{Y}$ denote the universal covering space of Y, and $\tilde{X} = X \times_Y \tilde{Y}$ the corresponding covering space of X. Then there are natural isomorphisms $H_*(\tilde{Y}; \mathbb{Z}) \cong H_*(Y; M)$ and $H_*(\tilde{X}; \mathbb{Z}) \cong H_*(X; M)$, where $M = \mathbb{Z}[\pi_1(Y)]$. The assumption that $H_*(X; M) \cong H_*(Y; M)$ implies that the map $\tilde{f}: \tilde{X} \to \tilde{Y}$ induces isomorphisms on integral homology. But $\pi_1(\tilde{Y}) = 0$, so by the special case above the homotopy fiber $F(\tilde{f})$ of $\tilde{f}$ is an acyclic space. But by path lifting we have $F(\tilde{f}) \cong F(f)$, so $F(f)$ is acyclic. Thus f is an acyclic map.

Recall from III.3.3.3 that every perfect group P has a universal central extension $E \to P$, and that the kernel of this extension is the abelian group $H_2(P; \mathbb{Z})$.

PROPOSITION 1.6. *Let P be a perfect normal subgroup of a group G, with corresponding $+$–construction $f: BG \to BG^+$. If $F(f)$ is the homotopy fiber of f then $\pi_1 F(f)$ is the universal central extension of P, and $\pi_2(BG^+) \cong H_2(P; \mathbb{Z})$.*

PROOF. We have an exact sequence $\pi_2(BG) \to \pi_2(BG^+) \to \pi_1 F(f) \to G \to G/P \to 1$. But $\pi_2(BG) = 0$, and $\pi_2(BG^+)$ is in the center of $\pi_1 F(f)$ by [Wh, IV.3.5]. Thus $\pi_1 F(f)$ is a central extension of P with kernel $\pi_2(BG^+)$. But $F(f)$ is acyclic, so $\pi_1 F(f)$ is perfect and $H_2(F; \mathbb{Z}) = 0$ by 1.2. By the Recognition Theorem III.3.4, $\pi_1 F(f)$ is the universal central extension of P.

Recall from Theorem III.3.5 that the Steinberg group $St(R)$ is the universal central extension of the perfect group $E(R)$. Thus we have:

COROLLARY 1.6.1. *The group $K_2(R) = \pi_2 BGL(R)^+$ is isomorphic to the group $K_2(R) \cong H_2(E(R); \mathbb{Z})$ of chapter III.*

In fact, we will see in Ex. 1.6 and 1.7 that $K_n(R) \cong \pi_n(BE(R)^+$ for all $n \geq 2$, and $K_n(R) \cong \pi_n(BSt(R)^+$ for all $n \geq 3$, with $K_3(R) \cong H_3(St(R); \mathbb{Z})$.

COROLLARY 1.6.2. *The fundamental group $\pi_1 X(R)$ of the Volodin space (1.2.2) is the Steinberg group $St(R)$.*

Construction Techniques

One problem with the +construction approach is the fact that $BGL(R)^+$ is not a uniquely defined space. It is not hard to see that $BGL(R)^+$ is an H-space (see Ex. 1.9). Quillen proved that that it is also an infinite loop space. We omit the proof here, because it will follow from the $+ = Q$ theorem in section 5.

Here is one of the most useful recognition criteria, due to Quillen. The proof is is an application of obstruction theory, which we omit (but see [Ger72, 1.5].)

THEOREM 1.7. *The map $i\colon BGL(R) \to BGL(R)^+$ is universal for maps into H-spaces. That is, for each map $f\colon BGL(R) \to H$, where H is an H-space, there is a map $g\colon BGL(R)^+ \to H$ so that $f = gi$, and such that the induced map $\pi_i(BGL(R)^+) \to \pi_i(H)$ is independent of g.*

CONSTRUCTIONS 1.8. Here are some ways that $BGL(R)^+$ may be constructed:

(i) Using point-set topology, *e.g.*, by attaching 2-cells and 3-cells to $BGL(R)$. This method is described Ex. 1.3, and in the books [Berrick] and [Rosenberg].

(ii) By the Bousfield-Kan integral completion functor $\mathbb{Z}_\infty$: we set $BGL(R)^+ = \mathbb{Z}_\infty BGL(R)$. This approach has the advantage of being absolutely functorial in R, and is used in [Dror] and [Gersten].

(iii) "Group completing" the H-space $\coprod_{n=0}^{\infty} BGL_n(R)$ yields an infinite loop space whose basepoint component is $BGL(R)^+$. This method will be discussed more in section 3, and is due to G. Segal [Segal].

(iv) By taking BGL of a free simplicial ring $F_*(R)$, as in [Swan].

(v) *Volodin's construction.* Let $X(R)$ denote the acyclic Volodin space of Example 1.2.2. By Ex. 1.5, the quotient group $BGL(R)/X(R)$ is a model for $BGL(R)^+$.

An excellent survey of these constructions (excluding Volodin's) may be found in [Ger72].

We conclude this section with a description of Quillen's construction for the K-theory of finite fields, arising from his work on the Adams Conjecture [Q70]. Adams had shown that the Adams operations ψ^k on topological K-theory (II.4.4) were represented by maps $\psi^k\colon BU \to BU$ in the sense that for each X the induced map

$$\widetilde{KU}(X) = [X, BU] \xrightarrow{\psi^k} [X, BU] = \widetilde{KU}(X)$$

is the Adams operation.

Fix a finite field $\mathbb{F}_q$ with q elements. For each n, the Brauer lifting of the trivial and standard n-dimensional representations of $GL_n(\mathbb{F}_q)$ are n-dimensional complex representations, given by homomorphisms $1_n, \rho_n\colon GL_n(\mathbb{F}_q) \to U$. Since BU is an H-space, we can form the difference $b^n = B(\rho_n) - B(1_n)$ as a map $BGL_n(\mathbb{F}_q) \to BU$. Quillen observed that b^n and b^{n+1} are compatible up to homotopy with the inclusion of $BGL_n(\mathbb{F}_q)$ in $BGL_{n+1}(\mathbb{F}_q)$. Hence there is a map $b\colon BGL(\mathbb{F}_q) \to BU$, well defined up to homotopy. By Theorem 1.7, b induces a map from $BGL(\mathbb{F}_q)^+$ to BU.

THEOREM 1.9. *(Quillen) The map $BGL(\mathbb{F}_q)^+ \to BU$ identifies $BGL(\mathbb{F}_q)^+$ with the homotopy fiber of $\psi^q - 1$. That is, the following is a homotopy fibration.*

$$BGL(\mathbb{F}_q)^+ \xrightarrow{b} BU \xrightarrow{\psi^q - 1} BU$$

On homotopy groups, II.4.4.1 shows that ψ^q is multiplication by q^i on $\pi_{2i}BU = \widetilde{KU}(S^{2i})$. Using the homotopy sequence 1.1.3 we immediately deduce:

COROLLARY 1.9.1. *For every finite field $\mathbb{F}_q$, and $n \geq 1$, we have*

$$K_n(\mathbb{F}_q) = \pi_n BGL(\mathbb{F}_q)^+ \cong \begin{cases} \mathbb{Z}/(q^i - 1) & n = 2i - 1, \\ 0 & n \text{ even.} \end{cases}$$

Moreover, if $\mathbb{F}_q \subset \mathbb{F}_{q'}$ then $K_n(\mathbb{F}_q) \subset K_n(\mathbb{F}_{q'})$ for all n.

EXERCISES

1.1. (Kervaire) Let X be a homology n-sphere, *i.e.*, a space with $H_*(X) = H_*(S^n)$. Show that there is a homotopy equivalence $S^n \to X^+$. *Hint:* Show that $\pi_1(X)$ is perfect if $n \neq 1$, so X^+ is simply connected, and use the Hurewicz theorem.

1.2. a) If F is acyclic space, show that F^+ is contractible.

b) If $X \xrightarrow{f} Y$ is acyclic and $\pi_1(X) = \pi_1(Y)$, show that f is a homotopy equivalence.

1.3 Here is a point-set construction of X^+ relative to a perfect normal subgroup P. Form Y by attaching one 2-cell e_p for each element of P, so that $\pi_1(Y) = \pi_1(X)/P$. Show that $H_2(Y;\mathbb{Z})$ is the direct sum of $H_2(X;\mathbb{Z})$ and the free abelian group $\mathbb{Z}[P]$ on the $[e_p]$. Next, prove that the homology classes $[e_p]$ are represented by maps $h_p\colon S^2 \to Y$, and form Z by attaching 3-cells to Y (one for each $p \in P$) using the h_p. Finally, prove that Z is a model for X^+.

1.4 *Perfect Radicals.* Show that the union of two perfect subgroups of any group G is itself a perfect subgroup. Conclude that G has a largest perfect subgroup P, called the *perfect radical* of G, and that it is normal subgroup of G.

1.5 Let cone(i) denote the mapping cone of a map $F \xrightarrow{i} X$. If F is an acyclic space, show that the map $X \to \text{cone}(i)$ is acyclic. If F is a subcomplex of X then cone(i) is homotopy equivalent to the quotient space X/F, so it too is acyclic. Conclude that if $X(R)$ is the Volodin space of Example 1.2.2 then $BGL(R)/X(R)$ is a model for $BGL(R)^+$. *Hint:* Consider long exact sequences in homology.

1.6 Let P be a perfect normal subgroup of G. Show that BP^+ is homotopy equivalent to the universal covering space of BG^+. Hence $\pi_n(BP^+) \cong \pi_n(BG^+)$ for all $n \geq 2$. *Hint:* BE is homotopy equivalent to a covering space of BG.

For $G = GL(R)$ and $P = E(R)$, This shows that $BE(R)^+$ is homotopy equivalent to the universal covering space of $BGL(R)^+$. Thus $K_n(R) \cong \pi_n BE(R)^+$ for $n \geq 2$.

If R is a commutative ring, show that the group map $SL(R) \to GL(R)$ induces isomorphisms $\pi_n BSL(R)^+ \cong K_n(R)$ for $n \geq 2$, and that $\pi_1 BSL(R)^+ \cong SK_1(R)$.

1.7 Suppose that $A \to S \to P$ is a universal central extension (III.5.3). In particular, S and P are perfect groups. Show that there is a homotopy fibration $BA \to BS^+ \to BP^+$. Conclude that $\pi_n(BS^+) = 0$ for $n \leq 2$, and that $\pi_n(BS^+) \cong \pi_n(BP^+) \cong \pi_n(BG^+)$ for all $n \geq 3$. In particular, $\pi_3(BP^+) \cong H_3(S;\mathbb{Z})$.

Since the Steinberg group $St(R)$ is the universal central extension of $E(R)$, this shows that $K_n(R) \cong \pi_n St(R)^+$ for all $n \geq 3$, and that $K_3(R) \cong H_3(St(R);\mathbb{Z})$.

1.8 For $n \geq 3$, let P_n denote the normal closure of the perfect group $E_n(R)$ in $GL_n(R)$, and let $BGL_n(R)^+$ denote the $+$–construction on $BGL_n(R)$ relative to P_n. Corresponding to the inclusions $GL_n \subset GL_{n+1}$ we can choose a sequence of maps $BGL_n(R)^+ \to BGL_{n+1}(R)^+$. Show that $\varinjlim BGL_n(R)^+$ is $BGL(R)^+$.

1.9 For each m and n, the group map $\Box\colon GL_m(R) \times GL_n(R) \to GL_{m+n}(R) \subset GL(R)$ induces a map $BGL_m(R) \times BGL_n(R) \to BGL(R) \to BGL(R)^+$. Show that these maps induce an H-space structure on $BGL(R)^+$.

1.10 Let I be an ideal in a ring R. Show that $\pi_0 K(R \to R/I)$ is isomorphic to the group $K_0(I)$ defined in Ex. II.2.3, and that the maps $K_1(R/I) \to K_0(I) \to K_0(R)$ in *loc. cit.* agree with those of 1.1.4. *Hint:* Use excision.

Use Ex. III.2.8 to show that $\pi_1 K(R \to R/I)$ is isomorphic to the group $K_1(R, I)$ of III.2.2, and that the maps $K_2(R/I) \to K_1(R, I) \to K_1(R)$ in III.2.3 agree with those of 1.1.4..

§2. Geometric realization of a small category

Recall (II.6.1.3) that a "small" category is a category whose objects form a set. If C is a small category, its *geometric realization* BC is a CW complex constructed naturally out of C. By definition, BC is the geometric realization $|NC|$ of the nerve NC of C; see 2.1.3 below. However, it is characterized in a simple way.

Characterization 2.1. The realization BC of a small category C is the CW complex uniquely characterized up to homeomorphism by the following properties:

(1) (Naturality) A functor $F\colon C \to D$ induces a cellular map $BF\colon BC \to BD$, and $BF \circ BG = B(FG)$;
(2) If C is a subcategory of D, BC is a subcomplex of BD;
(3) If C is the colimit of categories $\{C_\alpha\}$ then $BC = \bigcup BC_\alpha$ is the colimit of the BC_α;
(4) $B(C \times D)$ is homeomorphic to $(BC) \times (BD)$.
(5) $B\mathbf{n}$ is the standard $(n-1)$–simplex, where $\mathbf{n}$ denotes the category with n objects $\{0, 1, \cdots, n-1\}$, with exactly one morphism $i \to j$ for each $i \le j$.

Here are some useful special cases of (4) for small n:
$B\mathbf{0} = \emptyset$ is the empty set, because $\mathbf{0}$ is the empty category.
$B\mathbf{1} = \{0\}$ is a one-point space, since $\mathbf{1}$ is the one object-one morphism category.
$B\mathbf{2} = [0, 1]$ is the unit interval, whose picture is: $0 \cdot \longrightarrow \cdot 1$.
$B\mathbf{3}$ is the 2-simplex, whose picture is:

$$\begin{tikzcd} & \overset{1}{\cdot} \arrow[dr, "f"] & \\ 0\cdot \arrow[ur, "g"] \arrow[rr, "f\circ g"'] & & \cdot 2 \end{tikzcd}$$

The small categories form the objects of a category CAT, whose morphisms are functors. By (1), we see that geometric realization is a functor from CAT to the category of CW complexes and cellular maps.

Recipe 2.1.1. The above characterization of the CW complex BC gives it the following explicit cellular decomposition. The 0-cells (vertices) are the objects of C. The 1-cells (edges) are the morphisms in C, excluding all identity morphisms, and they are attached to their source and target. For each pair (f, g) of composable maps in C, attach a 2-simplex, using the above picture of $B\mathbf{3}$ as the model. (Ignore pairs (f, g) where either f or g is an identity.) Inductively, given an n-tuple of composable maps in C (none an identity map), attach an n-simplex, using $B(\mathbf{n+1})$ as the model. By (2), BC is the union of these spaces, equipped with the weak topology.

Notice that this recipe implies a canonical cellular homeomorphism between BC and the realization BC^{op} of the opposite category C^{op}. In effect, the recipe doesn't notice which way the arrows run.

Example 2.1.2. Let C_2 be the category with one object and one nontrivial morphism σ satisfying $\sigma^2 = 1$. The recipe tells us that BC_2 has exactly one n-cell for each n, attached to the $(n-1)$-cell by a map of degree 2 (corresponding to the first and last faces of the n-simplex). Therefore the n-skeleton of BC_2 is the projective n-space $\mathbb{RP}^n$, and their union BC_2 is the infinite projective space $\mathbb{RP}^\infty$.

Although the above recipe gives an explicit description of the cell decomposition of BC, we were a bit vague about the attaching maps. To be more precise, we shall assume that the reader has a slight familiarity with the basic notions in the theory of simplicial sets, as found for example in [WHomo] or [May]. For example, a simplicial set X is a sequence of sets $X_0, X_1, \dots,$ together with "face" maps $\partial_i \colon X_n \to X_{n-1}$ and "degeneracy maps" $\sigma_i \colon X_n \to X_{n+1}$ $(0 \le i \le n)$, subject to certain identities for the compositions of these maps.

The above recipe for BC is broken down into two steps. First we constructs a simplicial set NC, called the nerve of the category C, and then we set $BC = |NC|$.

Definition 2.1.3 (The nerve of C). The *nerve* NC of a small category C is the simplicial set defined by the following data. Its n-simplices are functors $c \colon \mathbf{n+1} \to C$, *i.e.*, diagrams in C of the form

$$c_0 \to c_1 \to \cdots \to c_n.$$

The i^{th} face $\partial_i(c)$ of this simplex is obtained by deleting c_i in the evident way; to get the i^{th} degeneracy $\sigma_i(c)$, one replaces c_i by $c_i \xrightarrow{=} c_i$.

The geometric realization $|X.|$ of a simplicial set $X.$ is defined to be the CW complex obtained by following the recipe 2.1.1 above, attaching an n-cell for each nondegenerate n-simplex x, identifying the boundary faces of the simplex with the $(n-1)$-simplices indexed by the $\partial_i x$. See [WHomo, 8.1.6] or [May, §14] for more details.

BC is defined as the geometric realization $|NC|$ of the nerve of C. From this prescription, it is clear that BC is given by recipe 2.1.1 above.

By abuse of notation, we will say that a category is contractible, or connected, or has any other topological property if its geometric realization has that property. Similarly, we will say that a functor $F \colon C \to D$ is a homotopy equivalence if BF is a homotopy equivalence $BC \simeq BD$.

Homotopy-theoretic properties 2.2. A natural transformation $\eta \colon F_0 \Rightarrow F_1$ between two functors $F_i \colon C \to D$ gives a homotopy $BC \times [0,1] \to BD$ between the maps BF_0 and BF_1. This follows from (1) and (3), because η may be viewed as a functor from $C \times \mathbf{2}$ to D whose restriction to $C \times \{i\}$ is F_i.

As a consequence, any adjoint pair of functors $L \colon C \to D$, $R \colon D \to C$ induces a homotopy equivalence between BC and BD, because there are natural transformations $LR \Rightarrow id_D$ and $id_C \Rightarrow RL$.

Example 2.2.1 (Skeleta). Any equivalence $C_0 \xrightarrow{F} C$ between small categories induces a homotopy equivalence $BC_0 \xrightarrow{\sim} BC$, because F has an adjoint.

In practice, we will often work with a category C, such as $\mathbf{P}(R)$ or $\mathbf{M}(R)$, which is not actually a small category, but which is *skeletally small* (II.6.1.3). This means that C is equivalent to a small category, say to C_0. In this case, we can use BC_0 instead of the mythical BC, because any other choice for C_0 will have a homotopy equivalent geometric realization. We shall usually overlook this fine set-theoretic point in practice, just as we did in defining K_0 in chapter II.

Example 2.2.2 (Initial objects). Any category with an initial object is contractible, because then the natural functor $C \to \mathbf{1}$ has a left adjoint. Similarly, any category with a terminal object is contractible.

For example, suppose given an object d of a category C. The *comma category C/d of objects over d* has as its objects the morphisms $f: c \to d$ in C with target d. A morphism in the comma category from f to $f': c' \to d$ is a morphism $h: c \to c'$ so that $f = f'h$. The comma category C/d is contractible because it has a terminal object, namely the identity map $\mathrm{id}_d: d \xrightarrow{=} d$. The dual comma category $d\backslash C$ with objects $d \to c$ is similar, and left to the reader.

Example 2.2.3 (Comma categories). Suppose given a functor $F: C \to D$ and an object d of D. The comma category F/d has as its objects all pairs (c, f) with c an object in C and f a morphism in D from $F(c)$ to d. By abuse of notation, we shall write such objects as $F(c) \xrightarrow{f} d$. A morphism in F/d from this object to $F(c') \xrightarrow{f'} d$ is a morphism $h: c \to c'$ in C so that the following diagram commutes in D.

$$\begin{tikzcd} F(c) \arrow[rr, "F(h)"] \arrow[dr, "f"'] & & F(c') \arrow[dl, "f'"] \\ & d & \end{tikzcd}$$

There is a canonical forgetful functor $j: F/d \to C$, $j(c, f) = c$, and there is a natural transformation $\eta_{(c,f)} = f$ from the composite $F \circ j: F/d \to D$ to the constant functor with image d. So $B(F \circ j)$ is a contractible map. It follows that there is a natural continuous map from $B(F/d)$ to the homotopy fiber of $BC \to BD$.

There is a dual comma category $d\backslash F$, whose objects are written as $d \longrightarrow F(c)$, and morphisms are morphisms $h: c \to c'$ in C. It also has a forgetful functor to C, and a map from $B(d\backslash F)$ to the homotopy fiber of $BC \to BD$. In fact, $d\backslash F = (d/F^{op})^{op}$.

The set π_0 of components of a category

The set $\pi_0(X)$ of connected components of any CW complex X can be described as the set of vertices modulo the incidence relation of edges. For BC this takes the following form. Let $\mathrm{obj}(C)$ denote the set of objects of C.

Lemma 2.3. *Let $\sim$ be the equivalence relation on obj(C) which is generated by the relation that $c \sim c'$ if there is a morphism in C between c and c'. Then*

$$\pi_0(BC) = obj(C)/\sim .$$

Translation categories 2.3.1. Suppose that G is a group, or even a monoid, acting on a set X. The *translation category* $G\!\int\! X$ is defined as the category whose objects are the elements of X, with $\mathrm{Hom}(x, x') = \{g \in G | g\cdot x = x'\}$. By Lemma 2.3, $\pi_0(G\!\int\! X)$ is the orbit space X/G. The components of $G\!\int\! X$ are described in Ex. 2.2.

Thinking of a G-set X as a functor $G \to CAT$, the translation category becomes a special case of the following construction, due to Grothendieck.

Example 2.3.2. Let I be a small category. Given a functor $X: I \to \mathbf{Sets}$, let $I\!\int\! X$ denote the category of pairs (i, x) with i an object of I and $x \in X(i)$, in which a morphism $(i, x) \to (i', x')$ is a morphism $f: i \to i'$ in I with $X(f)(x) = x'$. By Lemma 2.3 we have $\pi_0(I\!\int\! X) = \mathrm{colim}_{i\in I}\, X(i)$.

More generally, given a functor $X: I \to CAT$, let $I\!\int\! X$ denote the category of pairs (i, x) with i an object of I and x an object of $X(i)$, in which a morphism $(f, \phi): (i, x) \to (i', x')$ is given by a morphism $f: i \to i'$ in I and a morphism $\phi: X(f)(x) \to x'$ in $X(i')$. Using Lemma 2.3, it is not hard to show that $\pi_0(I\!\int\! X) = \mathrm{colim}_{i\in I}\, \pi_0 X(i)$.

The fundamental group π_1 of a category

Suppose that T is a set of morphisms in a category C. The *graph of* T is the 1-dimensional subcomplex of BC consisting of the edges corresponding to T and their incident vertices. We say that T is a *tree* in C if its graph is contractible (*i.e.*, a tree in the sense of graph theory). If C is connected then (by Zorn's Lemma) a tree T is maximal (a *maximal tree*) just in case every object of C is either the source or target of a morphism in T. The following well-known formula for the fundamental group of BC is a straight-forward application of Van Kampen's Theorem.

LEMMA 2.4. *Suppose that T is a maximal tree in a small connected category C. Then the group $\pi_1(BC)$ has the following presentation: it is generated by symbols $[f]$, one for every morphism in C, modulo the relations that*

(1) $[t] = 1$ *for every* $t \in T$, *and* $[id_c] = 1$ *for the identity morphism* id_c *of each object* c.
(2) $[f] \cdot [g] = [f \circ g]$ *for every pair* (f, g) *of composable morphisms in* C.

This presentation does not depend upon the choice of the object c_0 of C chosen as the basepoint. Geometrically, the class of $f\colon c_1 \to c_2$ is represented by the unique path in T from c_0 to c_1, followed by the edge f, followed by the unique path in T from c_2 back to c_0.

APPLICATION 2.4.1 (GROUPS). Let G be a group, considered as a category with one object. Since BG has only one vertex, BG is connected. By Lemma 2.4 (with T empty) we see that $\pi_1(BG) = G$. In fact, $\pi_i(BG) = 0$ for all $i \geq 2$. (See Ex. 2.2.) BG is often called the *classifying space* of the group G, for reasons discussed in Examples 2.8.2 and 2.8.3 below.

APPLICATION 2.4.2 (MONOIDS). If M is a monoid then BM has only one vertex. This time, Lemma 2.4 shows that the group $\pi = \pi_1(BM)$ is the group completion (Ex. II.1.1) of the monoid M.

For our purposes, one important thing about BG is that its homology is the same as the ordinary Eilenberg-MacLane homology of the group G (see [WHomo, 6.10.5 or 8.2.3]). In fact, if M is any G-module then we may consider M as a local coefficient system on BG (see 2.5.1). The chain complex used to form the simplicial homology with coefficients in M is the same as the canonical chain complex used to compute the homology of G, so we have $H_*(BG; M) = H_*(G; M)$. As a special case, we have $H_1(BG; \mathbb{Z}) = H_1(G; \mathbb{Z}) = G/[G, G]$, where $[G, G]$ denotes the commutator subgroup of G, *i.e.*, the subgroup of G generated by all commutators $[g, h] = ghg^{-1}h^{-1}$ $(g, h \in G)$.

(2.5) THE HOMOLOGY OF C AND BC. The ith homology of a CW complex X such as BC is given by the homology of the *cellular chain complex* $C_*(X)$. By definition, $C_n(X)$ is the free abelian group on the n-cells of X. If e is an $n+1$-cell and f is an n-cell, then the coefficient of $[f]$ in the boundary of $[e]$ is the degree of the attaching map of e, followed by projection onto f: $S^n \to X^{(n)} \xrightarrow{f} S^n$. attached

For example, $H_*(BC; \mathbb{Z})$ is the homology of the cellular chain complex $C_*(BC)$, which in degree n is the free abelian group on the set of all n-tuples $(f_1, ..., f_n)$ of composable morphisms in C, composable in the order $c_n \xrightarrow{f_n} \cdots \to c_1 \xrightarrow{f_1} c_0$. The boundary map in this complex sends the generator $(f_1, ..., f_n)$ to the alternating sum obtained by succesively deleting the c_i in the evident way:

$$(f_2, ..., f_n) - (f_1 f_2, f_3, ..., f_n) + \cdots \pm (..., f_i f_{i+1}, ...) \mp \cdots \pm (..., f_{n-1} f_n) \mp (..., f_{n-1}).$$

More generally, for each functor $M\colon C \to \mathbf{Ab}$ we let $H_i(C; M)$ denote the i^{th} homology of the chain complex

$$\cdots \to \coprod_{c_n \to \cdots \to c_0} M(c_n) \to \cdots \to \coprod_{c_1 \to c_0} M(c_1) \to \coprod_{c_0} M(c_0).$$

The final boundary map sends the copy of $M(c_1)$ indexed by $c_1 \xrightarrow{f} c_0$ to $M(c_0) \oplus M(c_1)$ by $x \mapsto (fx, -x)$. The cokernel of this map is the usual description for the colimit of the functor M, so $H_0(C; M) = \operatorname{colim}_{c \in C} M(c)$.

Local coefficients 2.5.1. A functor $C \to \mathbf{Sets}$ is said to be *morphism-inverting* if it carries all morphisms of C into isomorphisms. By Ex. 2.1, morphism-inverting functors are in 1–1 correspondence with covering spaces of BC. Therefore the morphism-inverting functors $M\colon C \to \mathbf{Ab}$ are in 1–1 correspondence with local coefficient systems on the topological space BC. In this case, the groups $H_i(C; M)$ are canonically isomorphic to $H_i(BC; M)$, the topologist's homology groups of BC with local coefficients M. The isomorphism is given in [Wh, VI.4.8].

Homotopy Fibers of Functors

If $F\colon C \to D$ is a functor, it is useful to study the realization map $BF\colon BC \to BD$ in terms of homotopy groups, and for this we want a category-theoretic interpretation of the homotopy fiber (1.1.3). The naïve approximations to the homotopy fiber are the realization of the comma categories F/d and its dual $d\backslash F$. Indeed, we saw in 2.2.3 that there are continuous maps from both $B(F/d)$ and $B(d\backslash F)$ to the homotopy fiber.

Here is the fundamental theorem used to prove that two categories are homotopy equivalent. We cite it without proof from [Q341].

2.6 Quillen's Theorem A. *Let $F\colon C \to D$ be a functor such that F/d is contractible for every d in D. Then $BF\colon BC \xrightarrow{\simeq} BD$ is a homotopy equivalence.*

Example 2.6.1. If $F\colon C \to D$ has a right adjoint G, then F/d is isomorphic to the comma category $C/G(d)$, which is contractible by Example 2.2.2. In this case, Quillen's Theorem A recovers the observation in 2.2 that C and D are homotopy equivalent.

Example 2.6.2. Consider the inclusion of monoids $i\colon \mathbb{N} \hookrightarrow \mathbb{Z}$ as a functor between categories with one object $*$. Then $*\backslash i$ isomorphic to the translation category $\mathbb{N}\int\mathbb{Z}$, which is contractible (why?). Quillen's Theorem A shows that $B\mathbb{N} \simeq B\mathbb{Z} \simeq S^1$.

The *inverse image* $F^{-1}(d)$ of an object d is the subcategory of C consisting of all objects c with $F(c) = d$, and all morphisms h in C mapping to the identity of d. It is isomorphic to the full subcategory of F/d consisting of pairs $(c, F(c) \xrightarrow{=} d)$, and also to the full subcategory of pairs $(d \xrightarrow{=} F(c), c)$ of $d\backslash F$. It will usually not be homotopy equivalent to either entire comma category.

One way to ensure that $F^{-1}(d)$ is homotopic to a comma category is to assume that F is either pre-fibered or pre-cofibered in the following sense.

Fibered and Cofibered functors 2.6.2. (Cf. [SGA 1, Exp. VI]) We say that a functor $F\colon C \to D$ is *pre-fibered* if for every d in D the inclusion $F^{-1}(d) \hookrightarrow d\backslash F$ has a right adjoint. This implies that $BF^{-1}(d) \simeq B(d\backslash F)$, and the *base-change* functor $f^*\colon F^{-1}(d') \to F^{-1}(d)$ associated to a morphism $f\colon d \to d'$ in D is defined as the composite $F^{-1}(d') \hookrightarrow (d\backslash F) \to F^{-1}(d)$. F is called *fibered* if it is pre-fibered and $g^*f^* = (fg)^*$ for every pair of composable maps f, g, so that F^{-1} gives a contravariant functor from D to CAT.

Dually, we say that F is *pre-cofibered* if for every d the inclusion $F^{-1}(d) \hookrightarrow F/d$ has a left adjoint. In this case we have $BF^{-1}(d) \simeq B(F/d)$. The *cobase-change* functor $f_*\colon F^{-1}(d) \to F^{-1}(d')$ associated to a morphism $f\colon d \to d'$ in D is defined as the composite $F^{-1}(d) \hookrightarrow (F/d') \to F^{-1}(d')$. F is called *cofibered* if it is pre-cofibered and $(fg)_* = f_*g_*$ for every pair of composable maps f, g, so that F^{-1} gives a covariant functor from D to CAT.

These notions allow us to state a variation on Quillen's Theorem A.

Corollary 2.6.3. *Suppose that $F\colon C \to D$ is either pre-fibered or pre-cofibered, and that $F^{-1}(d)$ is contractible for each d in D. Then BF is a homotopy equivalence $BC \simeq BD$.*

Example 2.6.4. Cofibered functors over D are in 1–1 correspondence with functors $D \to CAT$. We have already mentioned one direction: if $F\colon C \to D$ is cofibered, F^{-1} is a functor from D to CAT. Conversely, for each functor $X\colon D \to CAT$, the category $D\int X$ of Example 2.3.2 is cofibered over D by the forgetful functor $(d, x) \mapsto d$. It is easy to check that these are inverses: C is equivalent to $D\int F^{-1}$.

Here is the fundamental theorem use to construct homotopy fibration sequences of categories. We cite it without proof from [Q341], noting that it has a dual formulation in which F/d is replaced by $d\backslash F$; see Ex. 2.6.

2.7 Quillen's Theorem B. *Let $F\colon C \to D$ be a functor such that for every morphism $d \to d'$ in D the induced functor $F/d \to F/d'$ is a homotopy equivalence. Then for each d in D the geometric realization of the sequence*

$$F/d \xrightarrow{j} C \xrightarrow{F} D$$

is a homotopy fibration sequence. Thus there is a long exact sequence

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i B(F/d) \xrightarrow{j} \pi_i(BC) \xrightarrow{F} \pi_i(BD) \xrightarrow{\partial} \cdots .$$

Corollary 2.7.1. *Suppose that F is pre-fibered, and for every $f\colon d \to d'$ in D the base-change f^* is a homotopy equivalence. Then for each d in D the geometric realization of the sequence*

$$F^{-1}(d) \xrightarrow{j} C \xrightarrow{F} D$$

is a homotopy fibration sequence. Thus there is a long exact sequence

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i BF^{-1}(d) \xrightarrow{j} \pi_i(BC) \xrightarrow{F} \pi_i(BD) \xrightarrow{\partial} \cdots .$$

Topological Categories 2.8. If $C = C^{top}$ is a *topological* category (*i.e.*, the object and morphism sets form topological spaces), then the nerve of C^{top} is a simplicial topological space. Using the appropriate geometric realization of simplicial spaces, we can form the topological space $BC^{top} = |NC^{top}|$. It has the same underlying set as our previous realization BC^{δ} (the δ standing for "discrete," *i.e.,* no topology), but the topology of BC^{top} is more intricate. Since the identity may be viewed as a continuous functor $C^{\delta} \to C^{top}$ between topological categories, it induces a continuous map $BC^{\delta} \to BC^{top}$.

For example, any topological group $G = G^{top}$ is a topological category, so we need to distinguish between the two connected spaces BG^{δ} and BG^{top}. It is traditional to write BG for BG^{top}, reserving the notation BG^{δ} for the less structured space. As noted above, BG^{δ} has only one nonzero homotopy group: $\pi_1(BG^{\delta}) = G^{\delta}$. In contrast, the loop space $\Omega(BG^{top})$ is G^{top}, so $\pi_i BG^{top} = \pi_{i-1}G^{top}$ for $i > 0$.

Example 2.8.1. Let $G = \mathbb{R}$ be the topological group of real numbers under addition. Then $B\mathbb{R}^{top}$ is contractible because $\mathbb{R}^{top}$ is, but $B\mathbb{R}^{\delta}$ is not contractible because $\pi_1(B\mathbb{R}^{\delta}) = \mathbb{R}$.

Example 2.8.2 (BU). The unitary groups U_n are topological groups, and we see from I.4.10.1 that BU_n is homotopy equivalent to the infinite complex Grassmannian manifold G_n, which classifies n-dimensional complex vector bundles by Theorem I.4.10. The unitary group U_n is a deformation retract of the complex general linear group $GL_n(\mathbb{C})^{top}$. Thus BU_n and $BGL_n(\mathbb{C})^{top}$ are homotopy equivalent spaces. Taking the limit as $n \to \infty$, we have a homotopy equivalence $BU \simeq BGL(\mathbb{C})^{top}$.

By Theorem II.3.2, $KU(X) \cong [X, \mathbb{Z} \times BU]$ and $\widetilde{KU}(X) \cong [X, BU]$ for every compact space X. By Ex. II.3.11 we also have $KU^{-n}(X) \cong [X, \Omega^n(\mathbb{Z} \times BU)]$ for all $n \geq 0$. In particular, for the one-point space $*$ the groups $KU^{-n}(*) = \pi_n(\mathbb{Z} \times BU)$ are periodic of order 2: $\mathbb{Z}$ if n is even, 0 if not. This follow from the observation in II.3.2 that the homotopy groups of BU are periodic — except for $\pi_0(BU)$, which is zero as BU is connected.

A refinement of Bott periodicity states that $\Omega U \simeq \mathbb{Z} \times BU$. Since $\Omega(BU) \simeq U$, we have $\Omega^2(\mathbb{Z} \times BU) \simeq \Omega^2 BU \simeq \mathbb{Z} \times BU$ and $\Omega^2 U \simeq U$. This yields the periodicity formula: $KU^{-n}(X) = KU^{-n-2}(X)$.

Example 2.8.3 (BO). The orthogonal group O_n is a deformation retract of the real general linear group $GL_n(\mathbb{R})^{top}$. Thus the spaces BO_n and $BGL_n(\mathbb{R})^{top}$ are homotopy equivalent, and we see from I.4.10.1 that they are also homotopy equivalent to the infinite real Grassmannian manifold G_n. In particular, they classify n-dimensional real vector bundles by Theorem I.4.10. Taking the limit as $n \to \infty$, we have a homotopy equivalence $BO \simeq BGL(\mathbb{R})^{top}$.

Bott periodicity states that the homotopy groups of BO are periodic of order 8 — except for $\pi_0(BO) = 0$, and that the homotopy groups of $\mathbb{Z} \times BO$ are actually periodic of order 8. These homotopy groups are tabulated in I.3.1.1. A refinement of Bott periodicity states that $\Omega^7 O \simeq \mathbb{Z} \times BO$. Since $\Omega(BO) \simeq O$, we have $\Omega^8(\mathbb{Z} \times BO) \simeq \Omega^8(BO) \simeq \mathbb{Z} \times BO$ and $\Omega^8 O \simeq O$.

By Definition II.3.5 and Ex. II.3.11, the (real) topological K-theory of a compact space X is given by the formula $KO^{-n}(X) = [X, \Omega^n(\mathbb{Z} \times BO)], \quad n \geq 0$. This yields the periodicity formula: $KO^{-n}(X) = KO^{-n-8}(X)$.

EXERCISES

2.1 *Covering spaces.* If $X: I \to \mathbf{Sets}$ is a morphism-inverting functor, use the recipe 2.1.1 to show that the forgetful functor $I\int X \to I$ of Example 2.3.1 makes $B(I\int X)$ into a covering space of BI with fiber $X(i)$ over each vertex i of BI.

Conversely, if $E \xrightarrow{\pi} BI$ is a covering space, show that $X(i) = \pi^{-1}(i)$ defines a morphism-inverting functor on I, where i is considered as a 0-cell of BI. Conclude that these constructions give a 1–1 correspondence between covering spaces of BI and morphism-inverting functors.

2.2 *Translation categories.* Suppose that a group G acts on a set X, and form the translation category $G\int X$. Show that $B(G\int X)$ is homotopy equivalent to the disjoint union of the classifying spaces BG_x of the stabilizer subgroups G_x, one space for each orbit in X. For example, if X is the coset space G/H then $B(G\int X) \simeq BH$.

In particular, if $X = G$ is given the G-set structure $g \cdot g' = gg'$, this shows that $B(G\int G)$ is contractible, *i.e.*, the universal covering space of BG. Use this to calculate the homotopy groups of BG, as described in Example 2.4.1.

2.3 Let H be a subgroup of G, and $\iota: H \hookrightarrow G$ the inclusion as a subcategory.

(a) Show that $\iota/*$ is the category $H\int G$ of Ex. 2.1. Conclude that the homotopy fiber of $BH \to BG$ is the discrete set G/H, while $B\iota^{-1}(*)$ is a point.

(b) Use Ex. 2.2 to give another proof of (a).

2.4 If C is a filtering category [WHomo, 2.6.13], show that BC is contractible. *Hint:* By [Wh, V.3.5] it suffices to show that all homotopy groups are trivial. But any map from a sphere into a CW complex lands in a finite subcomplex, and every finite subcomplex of BC lands in the realization BD of a finite subcategory D of C. Show that D lies in another subcategory D' of C which has a terminal object.

2.5 *Mapping telescopes.* If $\cup\mathbf{n}$ denotes the union of the categories $\mathbf{n}$ of 2.1, then a functor $\cup\mathbf{n} \xrightarrow{C} CAT$ is just a sequence $C_1 \to C_2 \to C_3 \to \cdots$ of categories. Show that the geometric realization of the category $L = (\cup\mathbf{n})\int C$ of Example 2.3.2 is homotopy equivalent to BC, where C is the colimit of the C_n. In particular, this shows that $BL \simeq \lim_{n\to\infty} BC_n$. *Hint:* $C_n \simeq \mathbf{n}\int C$.

2.6 Suppose that $F: C \to D$ is pre-cofibered (definition 2.6.2).

(a) Show that $F^{op}: C^{op} \to D^{op}$ is pre-fibered. If F is cofibered, F^{op} is fibered.

(b) Derive the dual formulation of Quillen's theorem B, using $d\backslash F$, from 2.7.

(c) If each cobase-change functor f_* is a homotopy equivalence, show that the geometric realization of $F^{-1}(d) \to C \xrightarrow{F} D$ is a homotopy fibration sequence for each d in D, and there is a long exact sequence:

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i BF^{-1}(d) \to \pi_i(BC) \xrightarrow{F} \pi_i(BD) \xrightarrow{\partial} \cdots .$$

2.7 Let $F: C \to D$ be a cofibered functor (2.6.2). Construct a first quadrant double complex E^0 in which E^0_{pq} is the free abelian group on the pairs $(c_0 \to \cdots \to c_q, F(c_q) \to d_0 \to \cdots \to d_p)$ of sequences of composable maps in C and D. By filtering the double complex by columns, show that the homology of the total complex Tot E^0 is $H_q(\mathrm{Tot}\ E^0) \cong H_q(C;\mathbb{Z})$. Then show that the row filtration yields a spectral sequence converging to $H_*(C;\mathbb{Z})$ with $E^2_{pq} = H_p(D; H_qF^{-1})$, the homology of D with coefficients in the functor $d \mapsto H_q(F^{-1}(d);\mathbb{Z})$ described in 2.5.

2.8 A *lax functor* $\mathbf{M}: I \to CAT$ consists of functions assigning: (1) a category $\mathbf{M}(i)$ to each object i; (2) a functor $f_*: \mathbf{M}(i) \to \mathbf{M}(j)$ to every map $i \xrightarrow{f} j$ in I; (3) a natural transformation $(\mathrm{id}_i)_* \Rightarrow \mathrm{id}_{\mathbf{M}(i)}$ for each i; (4) a natural transformation $(fg)_* \Rightarrow f_*g_*$ for every pair of composable maps in I. This data is required to be "coherent" in the sense that the two transformations $(fgh)_* \Rightarrow f_*g_*h_*$ agree, and so do the various transformations $f_* \Rightarrow f_*$. For example, a functor is a lax functor in which (3) and (4) are identities.

Show that the definitions of objects and morphisms in Example 2.3.2 define a category $I\int\mathbf{M}$, where the map ϕ'' in the composition $(f'f, \phi'')$ of (f, ϕ) and (f', ϕ') is $(f'f)_*(x) \to f'_*f_*(x) \to f'_*(x') \to x''$. Show that the projection functor $\pi: I\int M \to I$ is pre-cofibered.

2.9 *Subdivision.* If $\mathcal{C}$ is a category, its *Segal subdivision* $Sub(\mathcal{C})$ is the category whose objects are the morphisms in $\mathcal{C}$; a morphism from $i: A \to B$ to $i': A' \to B'$ is a pair of maps $(A' \to A, B \to B')$ so that i' is $A' \to A \xrightarrow{i} B \to B'$.

(a) Draw the Segal subdivisions of the unit interval **2** and the 2-simplex **3**.

(b) Show that the source and target functors $\mathcal{C}^{op} \leftarrow Sub(\mathcal{C}) \to \mathcal{C}$ are homotopy equivalences. *Hint:* Use Quillen's Theorem A and 2.2.2.

2.10 Given a simplicial set X, its *Segal subdivision* $Sub(X)$ is the sequence of sets $X_1, X_3, X_5, \dots$, made into a simplicial set by declaring the face maps $\partial'_i: X_{2n+1} \to X_{2n-1}$ to be $\partial_i\partial_{2n+2-i}$ and $\sigma'_i: X_{2n+1} \to X_{2n+3}$ to be $\sigma_i\sigma_{2n+2-i}$.

If X is the nerve of a category $\mathcal{C}$, show that $Sub(X)$ is the nerve of the Segal subdivision category $Sub(\mathcal{C})$ of Ex. 2.9.

2.11 If $\mathcal{C}$ is a category, its *arrow category* $\mathcal{C}/\mathcal{C}$ has the morphisms of $\mathcal{C}$ as its objects, and a map $(a, b): f \to f'$ in $Ar(\mathcal{C})$ is a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

If $f: A \to B$ then the source $s(f) = A$ and target $t(f) = B$ of f define functors $\mathcal{C}/\mathcal{C} \to \mathcal{C}$.

Show that s is a fibered functor, and that t is a cofibered functor. Then show that both s and t are homotopy equivalences.

§3. Symmetric Monoidal Categories

The geometric realization BS of a symmetric monoidal category is an H-space with a homotopy-commutative, homotopy-associative product. To see this, recall from Definition I.5.1 that a symmetric monoidal category is a category S with a functor $\square\colon S \times S \to S$ which has a unit object "e" and is associative and is commutative, all up to coherent natural isomorphism. By 2.1(4) the geometric realization of $\square$ is the "product" map $(BS) \times (BS) \cong B(S \times S) \to BS$. The natural isomorphisms $s\square e \cong s \cong e\square s$ imply that the vertex e is an identity up to homotopy, *i.e.*, that BS is an H-space. The other axioms imply that the product on BS is homotopy commutative and homotopy associative.

In many cases e is an initial object of S, and therefore the H-space BS is contractible by Example 2.2.2. For example, any additive category $\mathcal{A}$ is symmetric monoidal category (with $\square = \oplus$), and $e = 0$ is an initial object, so $B\mathcal{A}$ is contractible. Similarly, the category $\mathbf{Sets}_f$ of finite sets is symmetric monoidal ($\square$ being disjoint union) by I.5.2, and $e = \emptyset$ is initial, so $B\mathbf{Sets}_f$ is contractible.

Here is an easy way to modify S in order to get an interesting H-space.

Definition 3.1. Let $\operatorname{iso} S$ denote the subcategory of isomorphisms in S. It has the same objects as S, but its morphisms are the isomorphisms in S. Because $\operatorname{iso} S$ is also symmetric monoidal, $B(\operatorname{iso} S)$ is an H-space.

By Lemma 2.3, the abelian monoid $\pi_0(\operatorname{iso} S)$ is just the set of isomorphism classes of objects in S — the monoid S^{iso} considered in §II.5. In fact, $\operatorname{iso} S$ is equivalent to the disjoint union $\coprod \operatorname{Aut}_S(s)$ of the one-object categories $\operatorname{Aut}_S(s)$, and $B(\operatorname{iso} S)$ is homotopy equivalent to the disjoint union of the classifying spaces $B\operatorname{Aut}(s)$, $s \in S^{\text{iso}}$.

Examples 3.1.1. $B(\operatorname{iso} S)$ is often an interesting H-space.

(a) In the category $\mathbf{Sets}_f$ of finite sets, the group of automorphisms of any n-element set is isomorphic to the permutation group Σ_n. Thus the subcategory $\operatorname{iso}\mathbf{Sets}_f$ is equivalent to $\coprod \Sigma_n$, the disjoint union of the one-object categories Σ_n. Thus the classifying space $B(\operatorname{iso}\mathbf{Sets}_f)$ is homotopy equivalent to the disjoint union of the classifying spaces $B\Sigma_n$, $n \geq 0$.

(b) The additive category $\mathbf{P}(R)$ of f.g. projective R-modules has 0 as an initial object, so $B\mathbf{P}(R)$ is a contractible space. However, its subcategory $\mathbf{P} = \operatorname{iso}\mathbf{P}(R)$ of isomorphisms is more interesting. The topological space $B\mathbf{P}$ is equivalent to the disjoint union of the classifying spaces $B\operatorname{Aut}(P)$ as P runs over the set of isomorphism classes of f.g. projective R-modules.

(c) Fix a ring R, and let $\mathbf{F} = \mathbf{F}(R)$ be the category $\coprod GL_n(R)$ whose objects are the based free R-modules $\{0, R, R^2, \cdots, R^n, \cdots\}$. There are no maps in $\mathbf{F}$ between R^m and R^n if $m \neq n$, and the self-maps of R^n form the group $GL_n(R)$. This is a symmetric monoidal category: $R^m\square R^n = R^{m+n}$ by concatenation of bases. The space $B\mathbf{F}(R)$ is equivalent to the disjoint union of the classifying spaces $BGL_n(R)$.

If R satisfies the Invariant Basis Property (I.1.1), then $\mathbf{F}(R)$ is a full subcategory of $\operatorname{iso}\mathbf{P}(R)$. In this case, we saw in II.5.4.1 that $\mathbf{F}(R)$ is cofinal in $\operatorname{iso}\mathbf{P}(R)$.

(d) Fix a commutative ring R, and let $S = \mathbf{Pic}(R)$ be the category of invertible R-modules and their isomorphisms. This is a symmetric monoidal category in which $\square$ is tensor product and e is R; see II.5.2(5). In this case, $S = \operatorname{iso} S$ and S^{iso} is the Picard group $\operatorname{Pic}(R)$ discussed in §I.3. By Lemma I.3.3, $\operatorname{Aut}(L) = R^\times$ for every L. Thus $\mathbf{Pic}(R)$ is equivalent to a disjoint union of copies of $R^\times$, and $B(\mathbf{Pic})$ is homotopy equivalent to the product $\operatorname{Pic}(R) \times B(R^\times)$.

(e) If F is a field, we saw in II.5.7 that the categories $\mathbf{SBil}(F)$ and $\mathbf{Quad}(F) = \mathbf{Quad}^+(F)$ of symmetric inner product spaces and quadratic spaces are symmetric monoidal categories. More generally, let A be any ring with involution, and $\epsilon = \pm 1$. Then the category $\mathbf{Quad}^\epsilon(A)$ of nonsingular ϵ-quadratic A-modules is a symmetric monoidal category with $\Box = \oplus$ and $e = 0$. See [B72, Bak] for more details.

The $S^{-1}S$ Construction

In [GQ], Quillen gave a construction of a category $S^{-1}S$ such that $K(S) = B(S^{-1}S)$ is a "group completion" of BS (see 3.4 below), provided that every map in S is an isomorphism and every translation $s\Box\colon \mathrm{Aut}_S(t) \to \mathrm{Aut}_S(s\Box t)$ is an injection. The motivation for this construction comes from the construction of the universal abelian group completion of an abelian monoid given in Chapter II, §1.

Definition 3.2 ($S^{-1}S$). The objects of $S^{-1}S$ are pairs (m,n) of objects of S. A morphism in $S^{-1}S$ is an equivalence class of composites

$$(m_1, m_2) \xrightarrow{s\Box} (s\Box m_1, s\Box m_2) \xrightarrow{(f,g)} (n_1, n_2).$$

This composite is equivalent to

$$(m_1, m_2) \xrightarrow{t\Box} (t\Box m_1, t\Box m_2) \xrightarrow{(f',g')} (n_1, n_2)$$

exactly when there is an isomorphism $\alpha\colon s \xrightarrow{\simeq} t$ in S so that composition with $\alpha\Box m_i$ sends f' and g' to f and g.

Explanation 3.2.1. There are two basic types of morphisms in $S^{-1}S$. The first type is a pair of maps $(f_1, f_2)\colon (m_1, m_2) \to (n_1, n_2)$ with $f_i\colon m_i \to n_i$ in S, arising from the inclusion of $S \times S$ in $S^{-1}S$. The second type is a formal map $s\Box\colon (m,n) \to (s\Box m, s\Box n)$.

We shall say that "translations are faithful" in S if every translation $\mathrm{Aut}(s) \to \mathrm{Aut}(s\Box t)$ in S is an injection. In this case every map in $S^{-1}S$ determines its object s up to unique isomorphism.

Remark 3.2.2. $S^{-1}S$ is a symmetric monoidal category, with $(m,n)\Box(m',n') = (m\Box m', n\Box n')$, and the functor $S \to S^{-1}S$ sending m to (e,m) is monoidal. Hence the natural map $BS \to B(S^{-1}S)$ is an H-space map, and $\pi_0(S) \to \pi_0(S^{-1}S)$ is a map of abelian monoids.

In fact $\pi_0(S^{-1}S)$ is an abelian group, the inverse of (m,n) being (n,m). This follows from the existence of a morphism η in $S^{-1}S$ from (e,e) to $(m,n)\Box(n,m) = (m\Box n, n\Box m)$. Warning: η is not a natural transformation! See Ex. 3.3.

Definition 3.3. Let S be a symmetric monoidal category in which every morphism is an isomorphism. Its K-groups are the homotopy groups of $B(S^{-1}S)$:

$$K_n^\Box(S) = \pi_n(BS^{-1}S).$$

It is sometimes convenient to write $K^\Box(S)$ for the geometric realization $B(S^{-1}S)$, and call it the *K-theory space* of S, so that $K_n^\Box(S) = \pi_n K^\Box(S)$.

In order to connect this definition up with the definition of $K_0^{\square}(S)$ given in section II.5, we recall from 3.2.2 that the functor $S \to S^{-1}S$ induces a map of abelian monoids from $\pi_0(S) = S^{\text{iso}}$ to $\pi_0(S^{-1}S)$.

LEMMA 3.3.1. *The abelian group $K_0^{\square}(S) = \pi_0(S^{-1}S)$ is the group completion of the abelian monoid $\pi_0(S) = S^{\text{iso}}$. Thus definition 3.3 agrees with the definition of $K_0^{\square}(S)$ given in II.5.1.2.*

PROOF. Let A denote the group completion of $\pi_0(S)$, and consider the function $\alpha(m,n) = [m] - [n]$ from the objects of $S^{-1}S$ to A. If $s \in S$ and $f_i : m_i \to n_i$ are morphisms in S then in A we have $\alpha(m,n) = \alpha(s\square m, s\square n)$ and $\alpha(m_1,m_2) = [m_1] - [m_2] = [n_1] - [n_2] = \alpha(n_1,n_2)$. By Lemma 2.3, α induces a set map $\pi_0(S^{-1}S) \to A$. By construction, α is an inverse to the universal homomorphism $A \to \pi_0(S^{-1}S)$.

Group Completions

Group completion constructions for K-theory were developed in the early 1970's by topologists studying infinite loop spaces. These constructions all apply to symmetric monoidal categories.

Any discussion of group completions depends upon the following well-known facts (see [Wh, III.7]). Let X be a homotopy commutative, homotopy associative H-space. Its set of components $\pi_0 X$ is an abelian monoid, and $H_0(X;\mathbb{Z})$ is the monoid ring $\mathbb{Z}[\pi_0(X)]$. Moreover, the integral homology $H_*(X;\mathbb{Z})$ is an associative graded-commutative ring with unit.

We say that a homotopy associative H-space X is *group-like* if it has a homotopy inverse; see [Wh, III.4]. Of course this implies that $\pi_0(X)$ is a group. When X is a CW complex, the converse holds [Wh, X.2.2]: if the monoid $\pi_0(X)$ is a group, then X is group-like.

For example, if $S = \operatorname{iso} S$ then $\pi_0(BS)$ is the abelian monoid S^{iso} of isomorphism classes, and $H_0(BS;\mathbb{Z})$ is the monoid ring $\mathbb{Z}[S^{\text{iso}}]$. In this case, the above remarks show that BS is grouplike if and only S^{iso} is an abelian group under $\square$.

DEFINITION 3.4 (GROUP COMPLETION). Let X be a homotopy commutative, homotopy associative H-space. A *group completion* of X is an H-space Y, together with an H-space map $X \to Y$, such that $\pi_0(Y)$ is the group completion of the abelian monoid $\pi_0(X)$ (in the sense of §I.1), and the homology ring $H_*(Y;k)$ is isomorphic to the localization $\pi_0(X)^{-1}H_*(X;k)$ of $H_*(X;k)$ by the natural map, for all commutative rings k.

If X is a CW complex (such as $X = BS$), we shall assume that Y is also a CW complex. This hypothesis implies that the group completion Y is group-like.

It is possible to show directly that $\mathbb{Z} \times BGL(R)^+$ is a group completion of BS when $S = \coprod GL_n(R)$; see Ex. 3.9. We will see in theorems 3.6 and 3.7 below that the K-theory space $B(S^{-1}S)$ is a group completion of BS, and that $B(S^{-1}S)$ is homotopy equivalent to $\mathbb{Z} \times BGL(R)^+$.

LEMMA 3.4.1. *If X is a group-like H-space then X its own group completion, and any other group completion $f : X \to Y$ is a homotopy equivalence.*

PROOF. Since f is a homology isomorphism, it is an isomorphism on π_0 and π_1. Therefore the map of basepoint components is a $+$–construction relative to the subgroup 1 of $\pi_1(X)$, and Theorem 1.4 implies that $X \simeq Y$.

SEGAL'S ΩB METHOD 3.4.2. If X is a topological *monoid*, such as $\coprod BGL_n(R)$ or $\coprod B\Sigma_n$, then we can form BX, the geometric realization of the (one-object) topological category X. In this case, ΩBX is an infinite loop space and the natural map $X \to \Omega BX$ is a group completion. For example, if X is the one-object monoid $\mathbb{N}$ then $B\mathbb{N} \simeq S^1$, $\pi_0(\Omega B\mathbb{N})$ is $\mathbb{Z}$, and every component of $\Omega B\mathbb{N}$ is contractible. See [Adams] for more details.

MACHINE METHODS 3.4.3. (See [Adams].) If X isn't quite a monoid, but the homotopy associativity of its product is nice enough, then there are constructions called "infinite loop space machines" which can construct a group completion Y of X, and give Y the structure of an infinite loop space. All machines produce the same infinite loop space Y (up to homotopy); see [MT]. Some typical machines are described in [Segal], and [May74].

The realization $X = BS$ of a symmetric monoidal category S is nice enough to be used by infinite loop space machines. These machines produce an infinite loop space $K(S)$ and a map $BS \to K(S)$ which is a group completion. Most infinite loop machines will also produce explicit deloopings of $K(S)$ in the form of an Ω-spectrum $\mathbf{K}(S)$, the *K-theory spectrum* of S. The production of $\mathbf{K}(S)$ is natural enough that monoidal functors between symmetric monoidal categories induce maps of the corresponding spectra.

Actions on other categories

To show that $B(S^{-1}S)$ is a group completion of BS, we need to fit the definition of $S^{-1}S$ into a more general framework.

DEFINITION 3.5. A monoidal category S is said to *act* upon a category X by a functor $\Box\colon S \times X \to X$ if there are natural isomorphisms $s\Box(t\Box x) \cong (s\Box t)\Box x$ and $e\Box x \cong x$ for $s, t \in S$ and $x \in X$, satisfying coherence conditions for the products $s\Box t\Box u\Box x$ and $s\Box e\Box x$ analogous to the coherence conditions defining S.

DEFINITION 3.5.1. If S acts upon X, the category $\langle S, X\rangle$ has the same objects as X. A morphism from x to y in $\langle S, X\rangle$ is an equivalence class of pairs $(s, s\Box x \xrightarrow{\phi} y)$, where $s \in S$ and ϕ is a morphism in X. Two pairs (s, ϕ) and (s', ϕ') are equivalent in case there is an isomorphism $s \cong s'$ identifying ϕ' with $s'\Box x \cong s\Box x \xrightarrow{\phi} y$.

We shall write $S^{-1}X$ for $\langle S, S \times X\rangle$, where S acts on both factors of $S \times X$. Note that when $X = S$ this definition recovers the definition of $S^{-1}S$ given in 3.2 above. If S is symmetric monoidal, then the formula $s\Box(t\Box x) = (s\Box t, x)$ defines an action of S on $S^{-1}X$.

For example, if every arrow in S is an isomorphism, then e is an initial object of $\langle S, S\rangle$ and therefore the space $B\langle S, S\rangle$ is contractible.

We say that S acts *invertibly* upon X if each translation functor $s\Box\colon X \to X$ is a homotopy equivalence. For example, S acts invertibly on $S^{-1}X$ (if S is symmetric) by the formula $s\Box(t, x) = (s\Box t, x)$, the homotopy inverse of the translation $(t, x) \mapsto (s\Box t, x)$ being the translation $(t, x) \mapsto (t, s\Box x)$, because of the natural transformation $(t, x) \mapsto (s\Box t, s\Box x)$.

Now $\pi_0 S$ is a multiplicatively closed subset of the ring $H_0(S) = \mathbb{Z}[\pi_0 S]$, so it acts on $H_*(X)$ and acts invertibly upon $H_*(S^{-1}X)$. Thus the functor $X \to S^{-1}X$ sending x to $(0, x)$ induces a map

$$(\pi_0 S)^{-1} H_q(X) \to H_q(S^{-1}X). \tag{3.5.2}$$

THEOREM 3.6 (QUILLEN). *If every map in S is an isomorphism and translations are faithful in S, then the map (3.5.2) is an isomorphism for all X and q.*

In particular, $B(S^{-1}S)$ is a group completion of the H-space BS.

PROOF. (See [GQ, p. 221].) By Ex. 3.4, the projection functor $\rho: S^{-1}X \to \langle S, S\rangle$ is cofibered with fiber X. By Ex. 2.7 there is an associated spectral sequence $E^2_{pq} = H_p(\langle S, S\rangle; H_q(X)) \Rightarrow H_{p+q}(S^{-1}X)$. Localizing this at the multiplicatively closed subset $\pi_0 S$ of $H_0(S)$ is exact, and $\pi_0 S$ already acts invertibly on $H_*(S^{-1}X)$ by Ex. 2.7, so there is also a spectral sequence $E^2_{pq} = H_p(\langle S, S\rangle; M_q) \Rightarrow H_{p+q}(S^{-1}X)$, where $M_q = (\pi_0 S)^{-1}H_q(X)$. But the functors M_q are morphism-inverting, so by Ex. 2.1 and the contractibility of $\langle S, S\rangle$, the group $H_p(\langle S, S\rangle; M_q)$ is zero for $p \neq 0$, and equals M_q for $p = 0$. Thus the spectral sequence degenerates to the claimed isomorphism (3.5.2).

The final assertion is immediate from this and definition 3.4, given remark 3.2.2 and lemma 3.3.1.

Bass gave a classical definition of $K_1(S)$ and $K_2(S)$ in [B72]; we gave them implicitly in III.1.6.3 and III.5.6. We can now state these classical definitions, and show that they coincide with the K-groups defined in this section.

COROLLARY 3.6.1. *If $S = \operatorname{iso} S$ and translations are faithful in S, then:*

$$K_1(S) = \varinjlim_{s\in S} H_1(\mathrm{Aut}(s); \mathbb{Z}),$$

$$K_2(S) = \varinjlim_{s\in S} H_2([\mathrm{Aut}(s), \mathrm{Aut}(s)]; \mathbb{Z}).$$

PROOF. ([We81]) The localization of $H_q(BS) = \oplus_{s\in S} H_q(\mathrm{Aut}(s)$ at $\pi_0(X) = S^{\mathrm{iso}}$ is the direct limit of the groups $H_q(\mathrm{Aut}(s)$, taken over the translation category of all $s \in S$. Since $\pi_1(X) = H_1(X; \mathbb{Z})$ for every H-space X, this gives the formula for $K_1(S) = \pi_1 B(S^{-1}S)$.

For K_2 we observe that any monoidal category S is the filtered colimit of its monoidal subcategories having countably many objects. Since $K_2(S)$ and Bass' H_2 definition commute with filtered colimits, we may assume that S has countably many objects. In this case the proof is relegated to Ex. 3.10, which is worked out in [We81].

Relation to the +–construction

Let $S = \mathbf{F}(R) = \coprod GL_n(R)$ be the monoidal category of free R-modules, as in example 3.1.1(c). In this section, we shall identify the +–construction on $BGL(R)$ with the basepoint component of $K(S) = B(S^{-1}S)$. As theorems 3.6 and 1.7 suggest, we need to find an acyclic map from $BGL(R)$ to a connected component of $B(S^{-1}S)$.

Any group map η from $GL_n(R)$ to $\mathrm{Aut}_{S^{-1}S}(R^n, R^n)$ gives a map from $BGL_n(R)$ to $B(S^{-1}S)$. Consider the maps $\eta = \eta_n$ defined by $\eta_n(g) = (g, 1)$. Because

$$\begin{array}{ccc}
GL_n(R) & \xrightarrow{\eta} & \mathrm{Aut}(R^n, R^n) \\
{\scriptstyle \Box R}\big\downarrow & & {\scriptstyle \Box(R,R)}\big\downarrow \\
GL_{n+1}(R) & \xrightarrow{\eta} & \mathrm{Aut}(R^{n+1}, R^{n+1})
\end{array}$$

commutes, there is a natural transformation from η to $\eta(\Box R)$. The resulting homotopy of maps $BGL_n(R) \to B(S^{-1}S)$ gives a map from the "mapping telescope" construction of $BGL(R)$ to $B(S^{-1}S)$; see Ex. 2.5. In fact, this map lands in the connected component Y_S of the identity in $B(S^{-1}S)$. Since $B(S^{-1}S)$ is an H-space, so is the connected component Y_S of the identity.

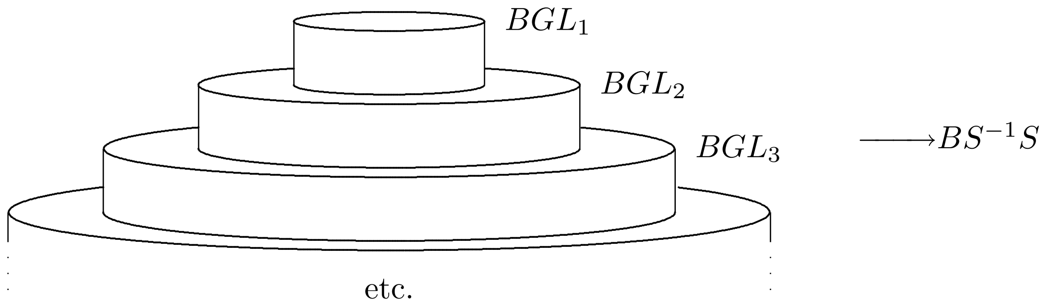

Figure 1. The mapping telescope of $BGL(R)$ and $B(S^{-1}S)$.

THEOREM 3.7. *When S is $\coprod GL_n(R)$,*

$$B(S^{-1}S) \simeq \mathbb{Z} \times BGL(R)^+.$$

PROOF. (Quillen) We shall show that the map $BGL(R) \to Y_S$ is acyclic. By theorem 1.7, this will induce an acyclic map $BGL(R)^+ \to Y_S$. This must be a homotopy equivalence by lemma 3.4.1, and the theorem will then follow.

Let $e \in \pi_0 BS$ be the class of R. By theorem 3.6, $H_*B(S^{-1}S)$ is the localization of the ring $H_*(BS)$ at $\pi_0(S) = \{e^n\}$. But this localization is the colimit of the maps $H_*(BS) \to H_*(BS)$ coming from the translation $\oplus R\colon S \to S$. Hence $H_*B(S^{-1}S) \cong H_*(Y_S) \otimes \mathbb{Z}[e, e^{-1}]$, where Y_S denotes the basepoint component of $B(S^{-1}S)$, and $H_*(Y_S) \cong \operatorname{colim} H_*(BGL_n(R)) = H_*(BGL(R))$. This means that the map $BGL(R) \to Y_S$ is acyclic, as required.

EXAMPLE 3.7.1. (Segal) Consider the symmetric monoidal category $S = \coprod \Sigma_n$, equivalent to the category $\mathbf{Sets}_f$ of example 3.1.1(a). The infinite symmetric group Σ is the union of the symmetric groups Σ_n along the inclusions $\Box 1$ from Σ_n to Σ_{n+1}, and these inclusions assemble to give a map from the mapping telescope construction of $B\Sigma$ to $B(S^{-1}S)$, just as they did for $GL(R)$. Moreover the proof of theorem 3.7 formally goes through to prove that $B(S^{-1}S) \simeq K(\mathbf{Sets}_f)$ is homotopy equivalent to $\mathbb{Z} \times B\Sigma^+$. This is the equivalence of parts (a) and (b) in the following result. We refer the reader to [BP71] and [Adams, §3.2] for the equivalence of parts (b) and (c).

THE BARRATT-PRIDDY-QUILLEN-SEGAL THEOREM 3.7.2. *The following three infinite loop spaces are the same:*
(a) the group completion $K(\mathbf{Sets}_f)$ of $B\mathbf{Sets}_f$;
(b) $\mathbb{Z} \times B\Sigma^+$, where Σ is the union of the symmetric groups Σ_n; and
(c) The infinite loop space $\Omega^\infty S^\infty = \lim_{n\to\infty} \Omega^n S^n$.
Hence the groups $K_n(\mathbf{Sets}_f)$ are the stable homotopy groups of spheres, π_n^s.

More generally, suppose that S has a countable sequence of objects $s_1, \dots$ such that $s_{n+1} = s_n \Box a_n$ for some $a_n \in S$, and satisfying the cofinality condition that for every $s \in S$ there is an s' and an n so that $s \Box s' \cong s_n$. In this case we can form the group $\operatorname{Aut}(S) = \operatorname{colim}_{n\to\infty} \operatorname{Aut}_S(s_n)$.

THEOREM 3.7.3. *Let $S = \operatorname{iso} S$ be a symmetric monoidal whose translations are faithful, and suppose the above condition is satisfied, so that the group $Aut(S)$ exists. Then the commutator subgroup E of $Aut(S)$ is a perfect normal subgroup, $K_1(S) = \mathrm{Aut}(S)/E$, and the $+$–construction on $B\,\mathrm{Aut}(S)$ is the connected component of the identity in the group completion $K(S)$. Thus*

$$K(S) \simeq K_0(S) \times B\,\mathrm{Aut}(S)^+.$$

PROOF. ([We81]) The assertions about E are essentially on p. 355 of [Bass]. On the other hand, the mapping telescope construction mentioned above gives an acyclic map from $B\,\mathrm{Aut}(S)$ to the basepoint component of $B(S^{-1}S)$, and such a map is by definition a $+$–construction.

Cofinality

A monoidal functor $f\colon S \to T$ is called *cofinal* if for every t in T there is a t' and an s in S so that $t\Box t' \cong f(s)$; cf. II.5.3. For example, the functor $\mathbf{F}(R) \to \mathbf{P}(R)$ of example 3.1.1(c) is cofinal, because every projective module is a summand of a free one. For $\mathbf{Pic}(R)$, the one-object subcategory $R^\times$ is cofinal.

COFINALITY THEOREM 3.8. *Suppose that $f\colon S \to T$ is cofinal. Then*
(a) If T acts on X then $S^{-1}X \simeq T^{-1}X$.
(b) If $\mathrm{Aut}_S(s) \cong \mathrm{Aut}_T(fs)$ for all s in S then the basepoint components of $K(S)$ and $K(T)$ are homotopy equivalent. Consequently, for all $n \geq 1$ we have $K_n(S) \cong K_n(T)$.

PROOF. By cofinality, S acts invertibly on X if and only if T acts invertibly on X. Hence Ex. 3.5 yields

$$S^{-1}X \xrightarrow{\simeq} T^{-1}(S^{-1}X) \cong S^{-1}(T^{-1}X) \xleftarrow{\simeq} T^{-1}X.$$

An alternate proof of part (a) is sketched in Ex. 3.7.

For part (b), let Y_S and Y_T denote the connected components of $B(S^{-1}S)$ and $B(T^{-1}T)$. Writing the subscript $s \in S$ to indicate a colimit over the translation category 2.3.1 of $\pi_0(S)$, and similarly for the subscript $t \in T$, theorem 3.6 yields:

$$\begin{aligned} H_*(Y_S) &= \operatorname{colim}_{s\in S} H_*(B\,\mathrm{Aut}(s)) = \operatorname{colim}_{s\in S} H_*(B\,\mathrm{Aut}(fs)) \\ &\cong \operatorname{colim}_{t\in T} H_*(B\,\mathrm{Aut}(t)) = H_*(Y_T). \end{aligned}$$

Hence the connected H-spaces Y_S and Y_T have the same homology, and this implies that they are homotopy equivalent.

Note that $K_0(\mathbf{F}) = \mathbb{Z}$ is not the same as $K_0(\mathbf{P}) = K_0(R)$ in general, although $K_n(\mathbf{F}) \cong K_n(\mathbf{P})$ for $n \geq 1$ by the Cofinality Theorem 3.8(b). By theorem 3.7 this establishes the following important result.

COROLLARY 3.8.1. *Let $S = \operatorname{iso}\mathbf{P}(R)$ be the category of f.g. projective R-modules and their isomorphisms. Then*

$$B(S^{-1}S) \simeq K_0(R) \times BGL(R)^+.$$

Let's conclude with a look back at the other motivating examples in 3.1.1. In each of these examples, every morphism is an isomorphism and the translations are faithful, so the classifying space of $S^{-1}S$ is a group completion of BS.

EXAMPLE 3.9.1 (STABLE HOMOTOPY). The "free R-module" on a finite set determines a functor from $\mathbf{Sets}_f$ to $\mathbf{P}(R)$, or from the subcategory $\coprod \Sigma_n$ of $\mathbf{Sets}_f$ to $\coprod GL_n(R)$. This functor identifies the symmetric group Σ_n maps with the permutation matrices in $GL_n(R)$. Applying group completions, theorem 3.7 and 3.7.1 show that this gives a map from $\Omega^\infty S^\infty$ to $K(R)$, hence maps $\pi_n^s \to K_n(R)$.

EXAMPLE 3.9.2 (PICARD GROUPS). Let R be a commutative ring, and consider the symmetric monoidal category $S = \mathbf{Pic}(R)$ of Example II.5.2(5). Because $\pi_0(S)$ is already a group, S and $S^{-1}S$ are homotopy equivalent (by lemma 3.4.1). Therefore we get

$$K_0\mathbf{Pic}(R) = \mathrm{Pic}(R), \quad K_1\mathbf{Pic}(R) = U(R) \quad \text{and } K_n\mathbf{Pic}(R) = 0 \quad \text{for } n \geq 2.$$

The determinant functor from $\mathbf{P} = \mathrm{iso}\,\mathbf{P}(R)$ to $\mathbf{Pic}(R)$ constructed in §I.3 gives a map from $K(R) = K(\mathbf{P})$ to $K\mathbf{Pic}(R)$. Upon taking homotopy groups, this yields the familiar maps $\det\colon K_0(R) \to \mathrm{Pic}(R)$ of II.2.6 and $\det\colon K_1(R) \to R^\times$ of III.1.1.1.

EXAMPLE 3.9.3 (L-THEORY). Let $S = \mathbf{Quad}^\epsilon(A)$ denote the category of nonsingular ϵ-quadratic A-modules, where $\epsilon = \pm 1$ and A is any ring with involution [B72, Bak]. The K-groups of this category are the L-groups ${}_\epsilon L_n(A)$ of Karoubi and others. For this category, the sequence of hyperbolic spaces H^n is cofinal (by Ex. II.5.10), and the automorphism group of H^n is the *orthogonal group* ${}_\epsilon O_n$. The infinite orthogonal group ${}_\epsilon O = {}_\epsilon O(A)$, which is the direct limit of the groups ${}_\epsilon O_n$, is the group $\mathrm{Aut}(S)$ in this case. By theorem 3.7.3, we have

$$K(\mathbf{Quad}^\epsilon(A)) \simeq {}_\epsilon L_0 \times B_\epsilon O^+.$$

When $A = \mathbb{R}$, the classical orthogonal group O is ${}_{+1}O$. When $A = \mathbb{C}$ and the involution is complex conjugation, the classical unitary group U is ${}_{+1}O(\mathbb{C})$. The description of $K_1(S)$ and $K_2(S)$ in 3.6.1 is given in [B72, p. 197]. For more bells and whistles, and classical details, we refer the reader to [Bak].

EXAMPLE 3.9.4. When R is a topological ring (such as $\mathbb{R}$ or $\mathbb{C}$), we can think of $\mathbf{P}(R)$ as a *topological* symmetric monoidal category. Infinite loop space machines (3.4.3) also accept topological symmetric monoidal categories, and we write $K(R^{top})$ for $K(\mathbf{P}(R)^{top})$. This gives natural maps from the infinite loop spaces $K(R^\delta)$ to $K(R^{top})$. The naturality of these maps allows us to utilize infinite loop space machinery. As an example of the usefulness, we remark that

$$K(\mathbb{R}^{top}) \simeq \mathbb{Z} \times BO \qquad \text{and} \qquad K(\mathbb{C}^{top}) \simeq \mathbb{Z} \times BU.$$

EXERCISES

3.1 Let $\mathbb{N}$ be the additive monoid $\{0, 1, ...\}$, considered as a symmetric monoidal category with one object. Show that $\langle \mathbb{N}, \mathbb{N} \rangle$ is the union $\cup\mathbf{n}$ of the ordered categories $\mathbf{n}$, and that $\mathbb{N}^{-1}\mathbb{N}$ is a poset, each component being isomorphic to $\cup\mathbf{n}$.

3.2 Show that a sequence $X_0 \to X_1 \to \cdots$ of categories determines an action of $\mathbb{N}$ on the disjoint union $X = \coprod X_n$, and that $\langle \mathbb{N}, X \rangle$ is the mapping telescope category $\cup\mathbf{n} \textstyle\int X$ of Ex. 2.5.

3.3 (Thomason) Let S be symmetric monoidal, and let $\iota\colon S^{-1}S \to S^{-1}S$ be the functor sending (m,n) to (n,m) and (f_1,f_2) to (f_2,f_1). Show that there is no natural transformation $0 \Rightarrow \mathrm{id}\square\iota$. *Hint:* The obvious candidate is given in 3.2.2.

Thomason has shown that $B\iota$ is the homotopy inverse for the H-space structure on $B(S^{-1}S)$, but for subtle reasons.

3.4 (Quillen) Suppose that $S = \mathrm{iso}S$, and that the translations in S are faithful. Show that the projection $S^{-1}X \xrightarrow{\rho} \langle S,S\rangle$ is cofibered, where $\rho(s,x)=s$.

3.5 Let $S = \mathrm{iso}\,S$ be a monoidal category whose translations are faithful (3.2.1). Suppose that S acts invertibly upon a category X. Show that the functors $X \to S^{-1}X$ $(x \mapsto (s,x))$ are homotopy equivalences for every s in S. If S acts upon a category Y, then S always acts invertibly upon $S^{-1}Y$, so this shows that $S^{-1}Y \simeq S^{-1}(S^{-1}Y)$. *Hint:* Use exercises 2.6 and 3.4, and the contractibility of $\langle S,S\rangle$.

3.6 Suppose that every map in X is monic, and that each translation $\mathrm{Aut}_S(s) \xrightarrow{\square x} \mathrm{Aut}_X(s\square x)$ is an injection. Show that the sequence $S^{-1}S \xrightarrow{\square x} S^{-1}X \xrightarrow{\pi} \langle S,X\rangle$ is a homotopy fibration for each x in X, where π is projection onto the second factor. In particular, if $\langle S,X\rangle$ is contractible, this proves that $S^{-1}S \xrightarrow{\square x} S^{-1}X$ is a homotopy equivalence. *Hint:* Show that π and $S^{-1}\pi\colon S^{-1}(S^{-1}X) \to \langle S,X\rangle$ are cofibered, and use the previous exercise.

3.7 Use exercises 3.4 and 3.5 to give another proof of the Cofinality Theorem 3.8(b).

3.8 If S is a symmetrical monoidal category, so is its opposite category S^{op}. Show that the group completions $K(S)$ and $K(S^{op})$ are homotopy equivalent.

3.9 Fix a ring R and set $S = \coprod GL_n(R)$. The maps $BGL_n(R) \to BGL(R) \to \{n\}\times BGL(R)^+$ assemble to give a map from BS to $\mathbb{Z}\times BGL(R)^+$. Use Ex. 1.9 to show that it is an H-space map. Then show directly that this makes $\mathbb{Z}\times BGL(R)^+$ into a group completion of BS.

3.10 Let S be a symmetric monoidal category with countably many objects, so that the group $\mathrm{Aut}(S)$ exists and its commutator subgroup E is perfect, as in 3.7.3. Let F denote the homotopy fiber of the H-space map $B\,\mathrm{Aut}(S)^+ \to B(K_1S)$.

(a) Show that $\pi_1(F)=0$ and $H_2(F;\mathbb{Z}) \cong \pi_2(F) \cong K_2(S)$.

(b) ([We81]) Show that the natural map $BE \to F$ induces $H_*(BE) \cong H_*(F)$, so that $F = BE^+$. *Hint:* Show that K_1S acts trivially upon the homology of BE and F, and apply the comparison theorem for spectral sequences.

(c) Conclude that $K_2(S) \cong H_2(E) \cong \varinjlim_{s\in S} H_2([\mathrm{Aut}(s),\mathrm{Aut}(s)];\mathbb{Z})$.

3.11 If $f\colon X \to Y$ is a functor, we say that an action of S on X is *fiberwise* if $S\times X \xrightarrow{\square} X \xrightarrow{f} Y$ equals the projection $S\times X \to X$ followed by f.

(a) Show that a fiberwise action on X restricts to an action of S on each fiber category $X_y = f^{-1}(y)$, and that f induces a functor $S^{-1}X \to Y$ whose fibers are the categories $S^{-1}(X_y)$.

(b) If f is a fibered functor (2.6.2), we say that a fiberwise action is *cartesian* if the basechange maps commute with the action of S on the fibers. Show that in this case $S^{-1}X \to Y$ is a fibered functor.

§4. Quillen's Q-construction

The higher K-theory groups of a small exact category $\mathcal{A}$ are defined to be the homotopy groups $K_n(\mathcal{A}) = \pi_{n+1}(BQ\mathcal{A})$ of the geometric realization of a certain auxiliary category $Q\mathcal{A}$, which we now define. This category has the same objects as $\mathcal{A}$, but morphisms are harder to describe. Here is the formal definition; we refer the reader to Ex. 4.1 for a more intuitive interpretation of morphisms in terms of subquotients.

Definition 4.1. A morphism from A to B in $Q\mathcal{A}$ is an equivalence class of diagrams

$$A \overset{j}{\twoheadleftarrow} B_2 \overset{i}{\rightarrowtail} B \tag{4.1.1}$$

where j is an admissible epimorphism and i is an admissible monomorphism in $\mathcal{A}$. Two such diagrams are equivalent if there is an isomorphism between them which is the identity on A and B. The composition of the above morphism with a morphism $B \twoheadleftarrow C_2 \rightarrowtail C$ is $A \twoheadleftarrow C_1 \rightarrowtail C$, where $C_1 = B_2 \times_B C_2$.

$$\begin{tikzcd}
 & C_1 \arrow[r, tail] \arrow[d, two heads] & C_2 \arrow[r, tail] \arrow[d, two heads] & C \\
A & B_2 \arrow[l, two heads] \arrow[r, tail] & B &
\end{tikzcd}$$

Two distinguished types of morphisms play a special role in $Q\mathcal{A}$: the admissible monics $A \rightarrowtail B$ (take $B_2 = A$) and the oppositely oriented admissible epis $A \twoheadleftarrow B$ (take $B_2 = B$). Both types are closed under composition, and the composition of $A \twoheadleftarrow B_2$ with $B_2 \rightarrowtail B$ is the morphism (4.1.1). In fact, every morphism in $Q\mathcal{A}$ factors as such a composition in a way that is unique up to isomorphism.

Subobjects 4.1.2. Recall from [Mac] that (in any category) a *subobject* of an object B is an equivalence class of monics $B_2 \rightarrowtail B$, two monics being equivalent if they factor through each other. In an exact category $\mathcal{A}$, we call a subobject *admissible* if any (hence every) representative $B_2 \rightarrowtail B$ is an admissible monic.

By definition, every morphism from A to B in $Q\mathcal{A}$ determines a unique admissible subobject of B in $\mathcal{A}$. If we fix a representative $B_2 \rightarrowtail B$ for each subobject in $\mathcal{A}$, then a morphism in $Q\mathcal{A}$ from A to B is a pair consisting of an admissible subobject B_2 of B and an admissible epi $B_2 \twoheadrightarrow A$.

In particular, this shows that morphisms from 0 to B in $Q\mathcal{A}$ are in 1-1 correspondence with admissible subobjects of B.

Isomorphisms in $Q\mathcal{A}$ are in 1-1 correspondence with isomorphisms in $\mathcal{A}$. To see this, note that every isomorphism $i\colon A \cong B$ in $\mathcal{A}$ gives rise to an isomorphism in $Q\mathcal{A}$, represented either by $A \overset{i}{\rightarrowtail} B$ or by $A \overset{i^{-1}}{\twoheadleftarrow} B$. Conversely, since the subobject determined by an isomorphism in $Q\mathcal{A}$ must be the maximal subobject $B \overset{=}{\rightarrowtail} B$, every isomorphism in $Q\mathcal{A}$ arises in this way.

Remark 4.1.3. Some set-theoretic restriction is necessary for $Q\mathcal{A}$ to be a category in our universe. It suffices for $\mathcal{A}$ to be *well-powered*, *i.e.*, for each object of $\mathcal{A}$ to have a set of subobjects. We shall tacitly assume this, since we will soon need the stronger assumption that $\mathcal{A}$ is a small category.

We now consider the geometric realization $BQ\mathcal{A}$ as a based topological space, the basepoint being the vertex corresponding to the object 0. In fact, $BQ\mathcal{A}$ is a connected CW complex, because the morphisms $0 \rightarrowtail A$ in $Q\mathcal{A}$ give paths in $BQ\mathcal{A}$ from the basepoint 0 to every vertex A. (See Lemma 2.3.)

PROPOSITION 4.2. *The geometric realization $BQ\mathcal{A}$ is a connected CW complex with $\pi_1(BQ\mathcal{A}) \cong K_0(\mathcal{A})$. The element of $\pi_1(BQ\mathcal{A})$ corresponding to $[A] \in K_0(\mathcal{A})$ is represented by the based loop composed of the two edges $0 \rightarrowtail A$ and $0 \twoheadleftarrow A$:*

$$0 \rightarrowtail A \twoheadrightarrow 0. \tag{4.2.1}$$

PROOF. Let T denote the family of all morphisms $0 \rightarrowtail A$ in $Q\mathcal{A}$. Since each nonzero vertex occurs exactly once, T is a maximal tree. By Lemma 2.4, $\pi_1(BQ\mathcal{A})$ has the following presentation: it is generated by the morphisms in $Q\mathcal{A}$, modulo the relations that $[0 \rightarrowtail A] = 1$ and $[f] \cdot [g] = [f \circ g]$ for every pair of composable arrows in $Q\mathcal{A}$. Moreover, the element of $\pi_1(BQ\mathcal{A})$ corresponding to a morphism from A to B is the based loop following the edges $0 \rightarrowtail A \to B \leftarrowtail 0$.

Since the composition $0 \rightarrowtail B_2 \rightarrowtail B$ is in T, this shows that $[B_2 \rightarrowtail B] = 1$ in $\pi_1(BQ\mathcal{A})$. Therefore $[A \twoheadleftarrow B_2 \rightarrowtail B] = [A \twoheadleftarrow B_2]$. Similarly, the composition $0 \twoheadleftarrow A \twoheadleftarrow B$ yields the relation $[A \twoheadleftarrow B][0 \twoheadleftarrow A] = [0 \twoheadleftarrow B]$. Since every morphism (4.1.1) factors, this shows that $\pi_1(BQ\mathcal{A})$ is generated by the morphisms $[0 \twoheadleftarrow A]$.

If $A \rightarrowtail B \twoheadrightarrow C$ is an exact sequence in A, then the composition $0 \rightarrowtail C \twoheadleftarrow B$ in $Q\mathcal{A}$ is $0 \twoheadleftarrow A \rightarrowtail B$. This yields the additivity relation

$$[0 \twoheadleftarrow B] = [C \twoheadleftarrow B][0 \twoheadleftarrow C] = [0 \twoheadleftarrow A][0 \twoheadleftarrow C] \tag{4.2.2}$$

in $\pi_1(BQ\mathcal{A})$, represented by the following picture in $BQ\mathcal{A}$:

$$\begin{array}{ccccccc} 0 & \rightarrowtail & A & \rightarrowtail & B & & \\ & & \downarrow & \nearrow & \downarrow & \searrow & \\ & & 0 & \rightarrowtail & C & \twoheadrightarrow & 0 \end{array}$$

Since every relation $[f] \cdot [g] = [f \circ g]$ may be rewritten in terms of the additivity relation, $\pi_1(BQ\mathcal{A})$ is generated by the $[0 \twoheadleftarrow A]$ with (4.2.2) as the only relation. Therefore $K_0(\mathcal{A}) \cong \pi_1(BQ\mathcal{A})$.

DEFINITION 4.3. Let $\mathcal{A}$ be a small exact category. Then $K\mathcal{A}$ denotes the space $\Omega BQ\mathcal{A}$, and we set

$$K_n(\mathcal{A}) = \pi_n K\mathcal{A} = \pi_{n+1}(BQ\mathcal{A}) \qquad \text{for } n \geq 0.$$

Proposition 4.2 shows that this definition of $K_0(\mathcal{A})$ agrees with the one given in chapter II. Note that any exact functor $F: \mathcal{A} \to \mathcal{B}$ induces a functor $Q\mathcal{A} \to Q\mathcal{B}$, hence maps $BQ\mathcal{A} \to BQ\mathcal{B}$ and $K_n(\mathcal{A}) \to K_n(\mathcal{B})$. Thus the space $K\mathcal{A} = \Omega\, BQ\mathcal{A}$ and all the groups $K_n(\mathcal{A})$ are functors from exact categories and exact functors to spaces and abelian groups, respectively. Moreover, isomorphic functors induce the same map on K-groups, because they induce isomorphic functors $Q\mathcal{A} \to Q\mathcal{A}'$.

REMARK 4.3.1. If an exact category $\mathcal{A}$ is not small but has a set of isomorphism classes of objects then we define $K_n(\mathcal{A})$ to be $K_n(\mathcal{A}')$, where $\mathcal{A}'$ is a small subcategory equivalent to $\mathcal{A}$. By Ex. 4.2 this is independent of the choice of $\mathcal{A}'$. From now on, whenever we talk about the K-theory of a large exact category $\mathcal{A}$ we will use this device, assuming tacitly that we have replaced it by a small $\mathcal{A}'$. For example, this is the case in the following definitions.

DEFINITION 4.3.2. Let R be a ring with unit, and let $\mathbf{P}(R)$ denote the exact category of f.g. projective R-modules. We define the K-groups of R by $K_n(R) = K_n\mathbf{P}(R)$. For $n = 0$, lemma 4.2 shows that this agrees with the definition of $K_0(R)$ in chapter II. For $n \geq 1$, agreement with the $+$–construcion definition 1.1.1 will have to wait until section 5.

If R is noetherian, let $\mathbf{M}(R)$ denote the category of f.g. R-modules. Otherwise, $\mathbf{M}(R)$ is the category of pseudo-coherent modules defined in II.7.1.4. We define the G-groups of R by $G_n(R) = K_n\mathbf{M}(R)$. For $n = 0$, this also agrees with the definition in chapter II.

Similarly, if X is a scheme which is quasi-projective (over a commutative ring), we define $K_n(X) = K_n\mathbf{VB}(X)$. If X is noetherian, we define $G_n(X) = K_n\mathbf{M}(X)$. For $n = 0$, this agrees with the definition of $K_0(X)$ and $G_0(X)$ in chapter II.

MORITA INVARIANCE 4.3.3. Recall from II.2.7 that if two rings R and S are Morita equivalent then there are equivalences $\mathbf{P}(R) \cong \mathbf{P}(S)$ and $\mathbf{M}(R) \cong \mathbf{M}(S)$. It follows that $K_n(R) \cong K_n(S)$ and $G_n(R) \cong G_n(S)$ for all n.

ELEMENTARY PROPERTIES 4.3.4. Here are some elementary properties of the above definition.

If $\mathcal{A}^{op}$ denotes the opposite category of $\mathcal{A}$, then $Q(\mathcal{A}^{op})$ is isomorphic to $Q\mathcal{A}$ by Ex 4.3, so we have $K_n(\mathcal{A}^{op}) = K_n(\mathcal{A})$. For example, if R is a ring then $\mathbf{P}(R^{op}) \cong \mathbf{P}(R)^{op}$ by $P \mapsto \mathrm{Hom}_R(P, R)$, so we have $K_n(R) \cong K_n(R^{op})$.

The product or direct sum $\mathcal{A} \oplus \mathcal{A}'$ of two exact categories is exact by Example II.7.1.6, and $Q(\mathcal{A} \oplus \mathcal{A}') = Q\mathcal{A} \times Q\mathcal{A}'$. Since the geometric realization preserves products by 2.1(4), we have $BQ(\mathcal{A} \oplus \mathcal{A}') = BQ\mathcal{A} \times BQ\mathcal{A}'$ and hence $K_n(\mathcal{A} \oplus \mathcal{A}') \cong K_n(\mathcal{A}) \oplus K_n(\mathcal{A}')$. For example, if R_1 and R_2 are rings then $\mathbf{P}(R_1 \times R_2) \cong \mathbf{P}(R_1) \oplus \mathbf{P}(R_2)$ and we have $K_n(R_1 \times R_2) \cong K_n(R_1) \oplus K_n(R_2)$.

Finally, suppose that $i \mapsto \mathcal{A}_i$ is a functor from some small filtering category I to exact categories and exact functors. Then the filtered colimit $\mathcal{A} = \varinjlim \mathcal{A}_i$ is an exact category (Ex. II.7.9), and $Q\mathcal{A} = \varinjlim Q\mathcal{A}_i$. Since geometric realization preserves filtered colimits by 2.1(3), we have $BQ\mathcal{A} = \varinjlim BQ\mathcal{A}_i$ and hence $K_n(\mathcal{A}) = \varinjlim K_n(\mathcal{A}_i)$. The K_0 version of this result was given in chapter II, as 6.2.7 and 7.1.7.

For example, if a ring R is the filtered union of subrings R_i we have $K_n(R) \cong \varinjlim K_n(R_i)$. However, $i \mapsto \mathbf{P}(R_i)$ is not a functor. To fix this, we need to replace the category $\mathbf{P}(R_i)$ by the equivalent category $\mathbf{P}'(R_i)$ whose objects are idempotent matrices over R_i; $\mathbf{P}(R)$ is equivalent to the category $\mathbf{P}'(R) = \varinjlim \mathbf{P}_i$. Alternatively one could use the Kleisli rectification, which is described in Ex. 4.5.

EXERCISES

4.1 *Admissible subquotients.* Let B be an object in an exact category $\mathcal{A}$. An *admissible layer* in B is a pair of subobjects represented by a sequence $B_1 \rightarrowtail B_2 \rightarrowtail B$ of admissible monics, and we call the quotient B_2/B_1 an *admissible subquotient* of B. Show that a morphism $A \to B$ in $Q\mathcal{A}$ may be identified with an isomorphism $j\colon B_2/B_1 \cong A$ of A with an admissible subquotient of B, and that composition in $Q\mathcal{A}$ arises from the fact that a subquotient of a subquotient is a subquotient.

4.2 If two exact categories $\mathcal{A}$ and $\mathcal{A}'$ are equivalent (and the equivalence respects exactness), show that $Q\mathcal{A}$ and $Q\mathcal{A}'$ are equivalent. If both are small categories, conclude that $K_n(\mathcal{A}) \cong K_n(\mathcal{A}')$ for all n.

4.3 If $\mathcal{A}$ is an exact category, so is its opposite category $\mathcal{A}^{op}$ (see Example II.7.1.5). Show that $Q(\mathcal{A}^{op})$ is isomorphic to $Q\mathcal{A}$.

4.4 Let B be an object in an exact category $\mathcal{A}$. Show that the comma category $Q\mathcal{A}/B$ is equivalent to the poset of admissible layers of B in the sense of Ex. 4.1. If $\mathcal{P}$ is an exact subcategory of $\mathcal{A}$ and i denotes the inclusion $Q\mathcal{P} \subset Q\mathcal{A}$, show that i/B is equivalent to the poset of admissible layers of B with $B_2/B_1 \in \mathcal{P}$.

4.5 *Kleisli rectification.* Let I be a filtering category, and let $I \to CAT$ be a lax functor in the sense of Ex. 2.8. Although the family of exact categories $Q\mathcal{A}(i)$ is not filtering, the family of homotopy groups $K_n\mathcal{A}(i)$ is filtering. The following trick allows us make K-theoretic sense out of the phantom category $\mathcal{A} = \varinjlim \mathcal{A}(i)$.

Let $\mathcal{A}_i$ be the category whose objects are pairs $(A_j, j \xrightarrow{f} i)$ with A_j in $\mathcal{A}(j)$ and f a morphism in I. A morphism from $(A_j, j \xrightarrow{f} i)$ to $(A_k, k \xrightarrow{g} i)$ is a pair $(j \xrightarrow{h} k, \theta_j)$ where $f = gh$ in I and θ_j is an isomorphism $h_*(A_j) \cong A_k$ in $\mathcal{A}(k)$. Clearly $\mathcal{A}_i$ is equivalent to $\mathcal{A}(i)$, and $i \mapsto \mathcal{A}_i$ is a functor. Thus if $\mathcal{A}$ denotes $\varinjlim \mathcal{A}_i$ we have $K_n\mathcal{A} = \varinjlim K_n\mathcal{A}(i)$.

4.6 (Gersten) Suppose given a surjective homomorphism $\phi\colon K_0\mathcal{A} \to G$.

(a) Show that there is a functor $\psi\colon Q\mathcal{A} \to G$ sending the morphism (4.1.1) of $Q\mathcal{A}$ to $\phi[\ker(j)]$, where G is regarded as a category with one object $*$. Using 4.2, show that the map $\pi_1(Q\mathcal{A}) \to \pi_1(G)$ is just ϕ.

(b) Show that the hypotheses of Theorem B are satisfied by ψ, so that $B(\psi/*)$ is the homotopy fiber of $BQ\mathcal{A} \to BG$.

(c) Let $\mathcal{B}$ denote the full subcategory of all B in $\mathcal{A}$ with $\phi[B] = 0$ in G. Use Theorem A to show that $Q\mathcal{B} \to \psi^{-1}(*)$ is a homotopy equivalence.

(d) Suppose in addition that for every A in $\mathcal{A}$ there is an A' such that $\phi[A'] = -\phi[A]$, so that K_0B is the subgroup $\ker(\phi)$ of $K_0\mathcal{A}$ by II.7.2. Use Theorem A to show that $\psi^{-1}(*) \simeq \psi/*$. This proves that $BQ\mathcal{B}$ is the homotopy fiber of $BQ\mathcal{A} \to BG$. Conclude that $K_n\mathcal{B} \cong K_n\mathcal{A}$ for all $n \geq 1$.

§5. The "$+ = Q$" Theorem

Suppose that $\mathcal{A}$ is an additive category. One way to define the K-theory of $\mathcal{A}$ is to consider the symmetric monoidal category $S = \operatorname{iso}\mathcal{A}$ (where $\Box = \oplus$) and use the $S^{-1}S$ construction: $K_n^{\oplus}\mathcal{A} = \pi_n B(S^{-1}S)$ and $K^{\oplus}\mathcal{A} = K(S) = B(S^{-1}S)$.

Another way is to suppose that $\mathcal{A}$ has the structure of an exact category and form the Q-construction on $\mathcal{A}$ with the $S^{-1}S$ construction on S. Comparing the definitions of $K_0^{\oplus}\mathcal{A}$ and $K_0\mathcal{A}$ in II.5.1.2 and II.7.1, we see that the K_0 groups are not isomorphic in general, unless perhaps every exact sequence splits in $\mathcal{A}$, *i.e.,* unless $\mathcal{A}$ is a split exact category in the sense of II.7.1.2.

Here is the main theorem of this section.

THEOREM 5.1 (QUILLEN). *If $\mathcal{A}$ is a split exact category and $S = \operatorname{iso}\mathcal{A}$, then $\Omega BQ\mathcal{A} \simeq B(S^{-1}S)$. Hence $K_n(\mathcal{A}) \cong K_n(S)$ for all $n \geq 0$.*

In fact, $B(S^{-1}S)$ is the group completion of BS by Ex. 5.1. In some circumstances (see 3.7, 3.7.3 and 3.8.1), the $S^{-1}S$ construction is a $+$–construction. In these cases, theorem 5.1 shows that the Q–construction is also a $+$–construction. For $\mathcal{A} = \mathbf{P}(R)$, this yields the "$+ = Q$" theorem:

COROLLARY 5.2 ($+ = Q$). *For every ring R,*

$$\Omega BQ\mathbf{P}(R) \simeq K_0(R) \times BGL(R)^+.$$

Hence $K_n(R) \cong K_n\mathbf{P}(R)$ for all $n \geq 0$.

DEFINITION 5.3. Given an exact category $\mathcal{A}$, we define the category $\mathcal{E}\mathcal{A}$ as follows. The objects of $\mathcal{E}\mathcal{A}$ are admissible exact sequences in $\mathcal{A}$. A morphism from $E' : (A' \rightarrowtail B' \twoheadrightarrow C')$ to $E : (A \rightarrowtail B \twoheadrightarrow C)$ is an equivalence class of diagrams of the following form, where the rows are exact sequences in $\mathcal{A}$:

$$\begin{array}{cccccccc} E': & & A' & \rightarrowtail & B' & \twoheadrightarrow & C' \\ & & \alpha\uparrow & & \| & & \uparrow \\ (5.3.1) & \downarrow & A & \rightarrowtail & B' & \twoheadrightarrow & C'' \\ & & \| & & \downarrow\beta & & \downarrow \\ E: & & A & \rightarrowtail & B & \twoheadrightarrow & C. \end{array}$$

Two such diagrams are equivalent if there is an isomorphism between them which is the identity at all vertices except for the C'' vertex.

Notice that the right column in (5.3.1) is just a morphism φ in $Q\mathcal{A}$ from C' to C, so the target $C = t(A \rightarrowtail B \twoheadrightarrow C)$ is a functor $t\colon \mathcal{E}\mathcal{A} \to Q\mathcal{A}$.

In order to improve legibility, it is useful to write $\mathcal{E}_C$ for the fiber category $t^{-1}(C)$.

FIBER CATEGORIES 5.4. If we fix φ as the identity map of $C = C'$, we see that the fiber category $\mathcal{E}_C = t^{-1}(C)$ of exact sequences with target C has for its morphisms all pairs (α, β) of isomorphisms fitting into a commutative diagram:

$$\begin{array}{ccccc} A' & \rightarrowtail & B' & \twoheadrightarrow & C \\ \alpha\uparrow\cong & & \cong\downarrow\beta & & \| \\ A & \rightarrowtail & B & \twoheadrightarrow & C. \end{array}$$

In particular, every morphism in $\mathcal{E}_C$ is an isomorphism.

EXAMPLE 5.4.1. The fiber category $\mathcal{E}_0 = t^{-1}(0)$ is equivalent to $S = \operatorname{iso}\mathcal{A}$. To see this, consider the functor from $\operatorname{iso}\mathcal{A}$ to $\mathcal{E}_0$ sending A to the trivial sequence $A \overset{\mathrm{id}}{\rightarrowtail} A \twoheadrightarrow 0$. This functor is a full embedding. Moreover, every object of $\mathcal{E}_0$ is naturally isomorphic to such a trivial sequence, whence the claim.

LEMMA 5.5. *For any C in $\mathcal{A}$, $\mathcal{E}_C$ is a symmetric monoidal category, and there is a faithful monoidal functor $S \to \mathcal{E}_C$ sending A to $A \rightarrowtail A \oplus C \twoheadrightarrow C$.*

PROOF. Given $E_i = (A_i \rightarrowtail B_i \twoheadrightarrow C)$ in $\mathcal{E}_C$, set $E_1 * E_2$ equal to

$$A_1 \oplus A_2 \rightarrowtail (B_1 \times_C B_2) \twoheadrightarrow C. \tag{5.5.1}$$

This defines a symmetric product on $\mathcal{E}_C$ with identity $e\colon 0 \rightarrowtail C \twoheadrightarrow C$. It is now straightforward to check that $S \to \mathcal{E}_C$ is a monoidal functor, and that it is faithful.

REMARK 5.5.2. If $\mathcal{A}$ is split exact then every object of $\mathcal{E}_C$ is isomorphic to one coming from S. In particular, the category $\langle S, \mathcal{E}_C\rangle$ is connected. This fails if $\mathcal{A}$ has a non-split exact sequence.

PROPOSITION 5.6. *If $\mathcal{A}$ is split exact, each $S^{-1}S \to S^{-1}\mathcal{E}_C$ is a homotopy equivalence.*

PROOF. By Ex. 3.6 and 5.1, $S^{-1}S \to S^{-1}\mathcal{E}_C \to \langle S, \mathcal{E}_C\rangle$ is a fibration, so it suffices to prove that $L = \langle S, \mathcal{E}_C\rangle$ is contractible. First, observe that the monoidal product on $\mathcal{E}_C$ induces a monoidal product on L, so BL is an H-space (as in 3.1). We remarked in 5.5.2 that L is connected. By [Wh, X.2.2], BL is group-like, *i.e.,* has a homotopy inverse.

For every exact sequence E, there is a natural transformation $\delta_E\colon E \to E * E$ in L, given by the diagonal.

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \| \\ A \oplus A & \rightarrowtail & B \times_C B & \twoheadrightarrow & C \end{array}$$

Now δ induces a homotopy between the identity on BL and muliplication by 2. Using the homotopy inverse to subtract the identity, this gives a homotopy between zero and the identity of BL. Hence BL is contractible.

We also need a description of how $\mathcal{E}_C$ varies with C.

LEMMA 5.7. *For each morphism $\varphi\colon C' \to C$ in $Q\mathcal{A}$, there is a canonical functor $\varphi^*\colon \mathcal{E}_C \to \mathcal{E}_{C'}$ and a natural transformation $\eta_E\colon \varphi^*(E) \to E$ from φ^* to the inclusion of $\mathcal{E}_C$ in $\mathcal{E}\mathcal{A}$.*

In fact, t is a fibered functor with base-change φ^* (Ex. 5.2). It follows (from 2.6.4) that $C \mapsto \mathcal{E}_C$ is a contravariant functor from $Q\mathcal{A}$ to CAT.

PROOF. Choose a representative $C' \twoheadleftarrow C'' \rightarrowtail C$ for φ and choose a pullback B' of B and C'' along C. This yields an exact sequence $A \rightarrowtail B' \twoheadrightarrow C''$ in $\mathcal{A}$. (Why?) The composite $B' \twoheadrightarrow C'' \twoheadrightarrow C'$ is admissible; if A' is its kernel then set

$$\varphi^*(A \rightarrowtail B \twoheadrightarrow C) = (A' \rightarrowtail B' \twoheadrightarrow C').$$

Since every morphism in $\mathcal{E}_C$ is an isomorphism, it is easy to see that φ^* is a functor, independent (up to isomorphism) of the choices made. Moreover, the construction yields a diagram (5.3.1), natural in E; the map β is an admissible monic because $A \rightarrowtail B' \xrightarrow{\beta} B'$ is. Hence (5.3.1) constitutes the natural map η_E from E to $\varphi^*(E)$.

Now the direct sum of sequences defines an operation $\oplus$ on $\mathcal{EA}$, and S acts on $\mathcal{EA}$ via the inclusion of S in $\mathcal{EA}$ given by 5.4.1. That is, $A' \Box (A \rightarrowtail B \twoheadrightarrow C)$ is the sequence $A' \oplus A \rightarrowtail A' \oplus B \twoheadrightarrow C$. Since $t(A' \Box E) = t(E)$ we have an induced map $T = S^{-1}t \colon S^{-1}\mathcal{EA} \to Q\mathcal{A}$. This is also a fibered functor (Ex. 5.2).

THEOREM 5.8. *If $\mathcal{A}$ is a split exact category and $S = \operatorname{iso}\mathcal{A}$, then the sequence $S^{-1}S \to S^{-1}\mathcal{EA} \xrightarrow{T} Q\mathcal{A}$ is a homotopy fibration.*

PROOF. We have to show that Quillen's Theorem B applies, *i.e.,* that the base-changes φ^* of 5.7 are homotopy equivalences. It suffices to consider φ of the form $0 \rightarrowtail C$ and $0 \twoheadleftarrow C$. If φ is $0 \rightarrowtail C$, the composition of the equivalence $S^{-1}S \to S^{-1}\mathcal{E}_C$ of 5.6 with φ^* is the identity by Ex. 5.5, so φ^* is a homotopy equivalence.

Now suppose that φ is $0 \twoheadleftarrow C$. The composition of the equivalence $S^{-1}S \to S^{-1}\mathcal{E}_C$ of 5.6 with φ^* sends A to $A \oplus C$ by Ex. 5.5. Since there is a natural transformation $A \to A \oplus C$ in $S^{-1}S$, this composition is a homotopy equivalence. Hence φ^* is a homotopy equivalence.

PROOF OF THEOREM 5.1. This will follow from theorem 5.8, once we show that $S^{-1}\mathcal{EA}$ is contractible. By Ex. 5.3, $\mathcal{EA}$ is contractible. Any action of S on a contractible category must be invertible (3.5.1). By Ex. 3.5 and Ex. 5.1, $\mathcal{EA} \to S^{-1}\mathcal{EA}$ is a homotopy equivalence, and therefore $S^{-1}\mathcal{EA}$ is contractible.

EXERCISES

5.1 If $\mathcal{A}$ is an additive category, $S = \operatorname{iso}\mathcal{A}$ is equivalent to the disjoint union of one-object categories $\operatorname{Aut}(A)$, one for every isomorphism class in $\mathcal{A}$. Show that the translations $\operatorname{Aut}(A) \to \operatorname{Aut}(A \oplus B)$ are injections. Then conclude using theorem 3.6 that $B(S^{-1}S)$ is the group completion of the H-space $BS = \coprod \operatorname{Aut}(A)$.

5.2 Show that the target functor $t \colon \mathcal{EA} \to Q\mathcal{A}$ is a fibered functor in the sense of definition 2.6.2, with base-change φ^* given by 5.7. Then show that the action of S on $\mathcal{EA}$ is cartesian (Ex. 3.11), so that the induced functor $S^{-1}\mathcal{EA} \to Q\mathcal{A}$ is also fibered, with fiber $S^{-1}S$ over 0.

5.3 Let $iQ\mathcal{A}$ denote the subcategory of $Q\mathcal{A}$ whose objects are those of $\mathcal{A}$ but whose morphisms are admissible monomorphisms. Show that the category $\mathcal{EA}$ of 5.3 is equivalent to the subdivision category $Sub(iQ\mathcal{A})$ of Ex. 2.9. Conclude that the category $\mathcal{EA}$ is contractible.

5.4 Show that Quillen's Theorem B *can not* apply to $\mathcal{EA} \to Q\mathcal{A}$ unless $\mathcal{A} \cong 0$. *Hint:* Compare $\pi_0 S$ to $K_0\mathcal{A}$.

5.5 If φ is the map $0 \rightarrowtail C$, resp. $0 \twoheadleftarrow C$, show that $\varphi^* \colon \mathcal{E}_C \to \mathcal{E}_0 \cong S$ sends $A \rightarrowtail B \twoheadrightarrow C$ to A, resp. to B.

5.6 Describe $\mathcal{E}'\mathcal{A} = (\mathcal{EA})^{op}$, which by Ex. 2.6 and 5.2 is cofibered over $(Q\mathcal{A})^{op}$. Use $\mathcal{E}'\mathcal{A}$ to prove the $+ = Q$ Theorem 5.1. *Hint:* There is a new action of S. Use pushout instead of pullback in (5.5.1) to prove the analogue of proposition 5.6.

5.7 *Finite Sets.* Let $\mathbf{Sets}_f$ denote the category of pointed finite sets, as in 3.1.1(a). We say that a sequence $A \rightarrowtail B \twoheadrightarrow C$ is exact just in case the first map is an injection and the second map identifies C with B/A.
(a) Copy the Q-construction 4.1 to form a category $Q\mathbf{Sets}_f$.
(b) Show that there is an extension category $\mathcal{E}'\mathbf{Sets}_f$, defined as in Ex. 5.6, which is cofibered over $(Q\mathbf{Sets}_f)^{op}$ with $S = \operatorname{iso}\mathbf{Sets}_f$ as the fiber over the basepoint.
(c) Modify the proof of the $+ = Q$ theorem to prove that $\Omega BQ\mathbf{Sets}_f \simeq S^{-1}S$.

§6. Waldhausen's $wS.$ construction

Our last construction of K-theory applies to Waldhausen categories, *i.e.*, "categories with cofibrations and weak equivalences." Unfortunately, this will occur only after a lengthy list of definitions, and we ask the reader to be forgiving.

Recall from chapter II, section 9 that a *category with cofibrations* is a category $\mathcal{C}$ with a distinguished zero object '0' and a subcategory $co(\mathcal{C})$ of morphisms in $\mathcal{C}$ called "cofibrations" (indicated with feathered arrows $\rightarrowtail$). Every isomorphism in $\mathcal{C}$ is to be a cofibration, and so are the unique arrows $0 \rightarrowtail A$ for every object A in $\mathcal{C}$. In addition, the pushout $C \rightarrowtail B \cup_A C$ of any cofibration $A \rightarrowtail B$ is a cofibration. (See Definition II.9.1 more precise statements.) These axioms imply that two constructions make sense: the coproduct $B \amalg C = B \cup_0 C$ of any two objects, and every cofibration $A \rightarrowtail B$ fits into a *cofibration sequence* $A \rightarrowtail B \twoheadrightarrow B/A$, where B/A is the cokernel of $A \rightarrowtail B$. The following is a restatement of Definition II.9.1.1:

Definition 6.1. A *Waldhausen category* $\mathcal{C}$ is a category with cofibrations, together with a family $w(\mathcal{C})$ of morphisms in $\mathcal{C}$ called "weak equivalences" (indicated with decorated arrows $\xrightarrow{\sim}$). Every isomorphism in $\mathcal{C}$ is to be a weak equivalence, and weak equivalences are to be closed under composition (so we may regard $w(\mathcal{C})$ as a subcategory of $\mathcal{C}$). In addition, a "Glueing axiom" (W3) must be satisfied, saying that the pushout of weak equivalences is a weak equivalence.

A functor $F : \mathcal{A} \to \mathcal{C}$ between two Waldhausen categories is called an *exact functor* if it preserves all the relevant structure: zero, cofibrations, weak equivalences and the pushouts along a cofibration.

A *Waldhausen subcategory* $\mathcal{A}$ of a Waldhausen category $\mathcal{C}$ is a subcategory which is also a Waldhausen category in such a way that: (i) the inclusion $\mathcal{A} \subseteq \mathcal{C}$ is an exact functor, (ii) the cofibrations in $\mathcal{A}$ are the maps in $\mathcal{A}$ which are cofibrations in $\mathcal{C}$ and whose cokernel lies in $\mathcal{A}$, and (iii) the weak equivalences in A are the weak equivalences of $\mathcal{C}$ which lie in $\mathcal{A}$.

In order to describe Waldhausen's $wS.$ construction for K-theory, we need a sequence of Waldhausen categories $S_n(\mathcal{C})$. $S_0(\mathcal{C})$ is the zero category, and $S_1(\mathcal{C})$ is the category $\mathcal{C}$, but whose objects A are thought of as the cofibrations $0 \rightarrowtail A$. The category $S_2(\mathcal{C})$ is the extension category $\mathcal{E}$ of II.9.3. For convenience, we repeat its definition here.

Extension Categories 6.2. The objects of the extension category $S_2(\mathcal{C})$ are the cofibration sequences $A_1 \rightarrowtail A_2 \twoheadrightarrow A_{12}$ in $\mathcal{C}$. A morphism $E \to E'$ in $S_2(\mathcal{C})$ is a commutative diagram:

$$\begin{array}{ccccccc} E: & & A_1 & \rightarrowtail & A_2 & \twoheadrightarrow & A_{12} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E': & & A_1' & \rightarrowtail & A_2' & \twoheadrightarrow & A_{12}' \end{array}$$

We make $S_2(\mathcal{C})$ in to a Waldhausen category as follows. A morphism $E \to E'$ in $S_2(\mathcal{C})$ is a cofibration if $A_1 \to A_1'$, $A_{12} \to A_{12}'$ and $A_1' \cup_{A_1} A_2 \to A_2'$ are cofibrations in $\mathcal{C}$. A morphism in $S_2(\mathcal{C})$ is a weak equivalence if its component maps $A_i \to A_i'$ $(i = 1, 2, 12)$ are weak equivalences in $\mathcal{C}$.

The following technically convenient axiom is often imposed on $\mathcal{C}$. It says that weak equivalences are closed under extensions.

EXTENSION AXIOM 6.2.1. Suppose that $f\colon E \to E'$ is a map between cofibration sequences, as in 6.2. If the source and quotient maps of f ($A \to A'$ and $C \to C'$) are weak equivalences, so is the total map of f ($B \to B'$).

DEFINITION 6.3. ($S.\mathcal{C}$) If $\mathcal{C}$ is a Waldhausen category, let $S_n\mathcal{C}$ be the category whose objects $A.$ are sequences of n cofibrations in $\mathcal{C}$:

$$A.: \quad 0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$$

together with a choice of every subquotient $A_{ij} = A_j/A_i$ ($0 < i < j \le n$). These choices are to be compatible in the sense that there is a commutative diagram:

(6.3.0)
$$\begin{array}{ccccccc}
 & & & & & & A_{n-1,n} \\
 & & & & & & \uparrow\!\!\uparrow \\
 & & & & & & \cdots \\
 & & & & A_{23} & \rightarrowtail \cdots \rightarrowtail & A_{2n} \\
 & & & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\
 & & A_{12} & \rightarrowtail & A_{13} & \rightarrowtail \cdots \rightarrowtail & A_{1n} \\
 & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\
A_1 & \rightarrowtail & A_2 & \rightarrowtail & A_3 & \rightarrowtail \cdots \rightarrowtail & A_n
\end{array}$$

The conventions $A_{0j} = A_j$ and $A_{jj} = 0$ will be convenient at times. A morphism $A. \to B.$ in $S_n\mathcal{C}$ is a natural transformation of sequences.

If we forget the choices of the subquotients A_{ij} we obtain the higher extension category $\mathcal{E}_n(\mathcal{C})$ constructed in II.9.3.2. Since we can always make such choices, it follows that the categories $S_n\mathcal{C}$ and $\mathcal{E}_n(\mathcal{C})$ are equivalent. By Ex. II.9.4, $\mathcal{E}_n(\mathcal{C})$ is a Waldhausen category, so $S_n\mathcal{C}$ is also a Waldhausen category. Here are the relevant definitions for S_n, translated from the definitions II.9.3.2 for $\mathcal{E}$.

A weak equivalence in $S_n\mathcal{C}$ is a map $A. \to B.$ such that each $A_i \to B_i$ (hence, each $A_{ij} \to B_{ij}$) is a weak equivalence in $\mathcal{C}$. A map $A. \to B.$ is a cofibration when for every $0 \le i < j < k \le n$ the map of cofibration sequences

$$\begin{array}{ccccc}
A_{ij} & \rightarrowtail & A_{ik} & \twoheadrightarrow & A_{jk} \\
\downarrow & & \downarrow & & \downarrow \\
B_{ij} & \rightarrowtail & B_{ik} & \twoheadrightarrow & B_{jk}
\end{array}$$

is a cofibration in $S_2(\mathcal{C})(\mathcal{C})$.

The reason for including choices in the definition of the categories $S_n(\mathcal{C})$ is that we can form a simplicial Waldhausen category.

DEFINITION 6.3.1. For each $n \ge 0$, the exact functor $\partial_0\colon S_n(\mathcal{C}) \to S_{n-1}(\mathcal{C})$ is defined deletion of the bottom row of (6.3.0). That is, *i.e.*, ∂_0 is defined by the formula

$$\partial_0(A.): \quad 0 = A_{11} \rightarrowtail A_{12} \rightarrowtail A_{13} \rightarrowtail \cdots \rightarrowtail A_{1n}$$

together with the choices $\partial_0(A.)_{ij} = A_{i+1,j+1}$. By Ex. 6.1, $\partial_0(A.)$ is in $S_{n-1}(\mathcal{C})$.

For $0 < i \le n$ we define the exact functors $\partial_i\colon S_n(\mathcal{C}) \to S_{n-1}(\mathcal{C})$ by omitting the row A_{i*} and the column containing A_i in (6.3.0), and reindexing the A_{jk} as needed. Similarly, we define the exact functors $s_i\colon S_n(\mathcal{C}) \to S_{n+1}(\mathcal{C})$ by duplicating A_i, and reindexing with the normalization $A_{i,i+1} = 0$.

By Ex. 6.2, the $S_n\mathcal{C}$ fit together to form a simplicial Waldhausen category $S.\mathcal{C}$, and the subcategories $wS_n\mathcal{C}$ of weak equivalences fit together to form a simplicial category $wS.\mathcal{C}$. Hence their geometric realizations $B(wS_n\mathcal{C})$ fit together to form a simplicial topological space $BwS.\mathcal{C}$, and we write $|wS.\mathcal{C}|$ for the realization of $BwS.\mathcal{C}$. Since $S_0(\mathcal{C})$ is trivial, $|wS.\mathcal{C}|$ is a connected space.

Recall from chapter II, 9.1.2, that $K_0(\mathcal{C})$ is defined as the group generated by the set of weak equivalence classes [A] of objects of $\mathcal{C}$ with the relations that $[B] = [A] + [B/A]$ for every cofibration sequence

$$A \rightarrowtail B \twoheadrightarrow B/A.$$

Proposition 6.4. *If $\mathcal{C}$ is a Waldhausen category then $\pi_1|wS.\mathcal{C}| \cong K_0(\mathcal{C})$.*

Proof. If $X.$ is any simplicial space with X_0 a point, then $|X.|$ is connected and $\pi_1|X.|$ is the free group on $\pi_0(X_1)$ modulo the relations $d_1(x) = d_2(x)d_0(x)$ for every $x \in \pi_0(X_2)$. For $X. = BwS.\mathcal{C}$, $\pi_0(BwS_1\mathcal{C})$ is the set of weak equivalence classes of objects in $\mathcal{C}$, $\pi_0(BwS_2\mathcal{C})$ is the set of equivalence classes of cofibration sequences, and the maps $\partial_i\colon S_2(\mathcal{C}) \to S_1(\mathcal{C})$ of 6.3.1 send $A \rightarrowtail B \twoheadrightarrow B/A$ to B/A, B and A, respectively.

Definition 6.5. If $\mathcal{C}$ is a small Waldhausen category, its *algebraic K-theory space* $K(\mathcal{C}) = K(\mathcal{C}, w)$ is the loop space

$$K(\mathcal{C}) = \Omega|wS.\mathcal{C}|.$$

The K-groups of $\mathcal{C}$ are defined to be its homotopy groups:

$$K_i(\mathcal{C}) = \pi_i K(\mathcal{C}) = \pi_{i+1}|wS.\mathcal{C}| \qquad \text{if } i \geq 0.$$

Remark 6.5.1. Since the subcategory $w\mathcal{C}$ is closed under coproducts in $\mathcal{C}$ by axiom (W3), the coproduct gives an H-space structure to $|wS.\mathcal{C}|$ via the map

$$|wS.\mathcal{C}| \times |wS.\mathcal{C}| \cong |wS.\mathcal{C} \times wS.\mathcal{C}| \xrightarrow{\amalg} |wS.\mathcal{C}|.$$

Simplicial Model 6.5.2. Suppose that $\mathcal{C}$ is a small Waldhausen category in which the isomorphisms $i\mathcal{C}$ are the weak equivalences. Let $s_n\mathcal{C}$ denote the set of objects of $S_n\mathcal{C}$; as n varies, we have a simplicial set $s.\mathcal{C}$. Waldhausen proved in [W1126, 1.4] that the inclusion $|s.\mathcal{C}| \to |iS.\mathcal{C}|$ is a homotopy equivalence. Therefore $\Omega|s.\mathcal{C}|$ is a simplicial model for the space $K(\mathcal{C})$.

Remark 6.5.3. In fact $K(\mathcal{C})$ is an infinite loop space. To see this, note that we can apply the $S.$ construction to each $S_n\mathcal{C}$, obtaining a bisimplicial Waldhausen category $S.S.\mathcal{C}$. Iterating this construction, we can form the multisimplicial Waldhausen categories $S.^n\mathcal{C} = S.S.\cdots S.\mathcal{C}$ and the multisimplicial categories $wS.^n\mathcal{C}$ of weak equivalences. Waldhausen points out on p. 330 of [W1126] that $|wS.^n\mathcal{C}|$ is the loop space of $|wS.^{n+1}\mathcal{C}|$, and that the sequence of spaces

$$\Omega|wS.\mathcal{C}|,\ \Omega|wS.S.\mathcal{C}|,\ \dots,\ \Omega|wS.^n\mathcal{C}|,\ \dots$$

forms a connective Ω-spectrum $\mathbf{K}\mathcal{C}$, called the *K-theory spectrum of $\mathcal{C}$*. Many authors think of the K-theory of $\mathcal{C}$ in terms of this spectrum. This does not affect the K-groups, because:

$$\pi_i(\mathbf{K}\mathcal{C}) = \pi_i K(\mathcal{C}) = K_i(\mathcal{C}), \qquad i \geq 0.$$

An exact functor F induces a map $F_*\colon K(\mathcal{B}) \to K(\mathcal{C})$ of spaces, and spectra, and of their homotopy groups $K_i(\mathcal{B}) \to K_i(\mathcal{C})$.

Exact Categories 6.6. We saw in II.9.1.3 that any exact category $\mathcal{A}$ becomes a Waldhausen category in which the cofibration sequences are just the admissible exact sequences, and the weak equivalences are just the isomorphisms. We write $i(\mathcal{A})$ for the family of isomorphisms, so that we can form the K-theory space $K(\mathcal{A}) = \Omega|iS.\mathcal{A}|$. Waldhausen proved in [W1126, 1.9] that there is a homotopy equivalence between $|iS.\mathcal{A}|$ and $BQ\mathcal{A}$, so that this definition is consistent with the definition of $K(\mathcal{A})$ in definition 4.3. His proof is given in exercises 6.5 and 6.6 below.

Another important example of a Waldhausen category is $\mathcal{R}_f(X)$, introduced in II.9.1 and Ex. II.9.1. The so-called *K-theory of spaces* refers to the corresponding K-theory spaces $A(X)$.

Example 6.7 $\big(A(*)\big)$. Recall from II.9.1.4 that the category $\mathcal{R}_f = \mathcal{R}_f(*)$ of finite based CW complexes is a Waldhausen category in which the family $h\mathcal{R}_f$ of weak equivalences is the family of weak homotopy equivalences. This category is saturated (II.9.7.1) and satisfies the extension axiom 6.2.1. Following Waldhausen [W1126], we write $A(*)$ for the space $K(\mathcal{R}_f) = \Omega|hS.\mathcal{R}_f|$. In particular, $A_0(*) = K_0\mathcal{R}_f = \mathbb{Z}$ by II.9.1.5.

Example 6.7.1 $\big(A(X)\big)$. More generally, let X be a CW complex. The category $\mathcal{R}(X)$ of CW complexes Y obtained from X by attaching cells, and having X as a retract, is a Waldhausen category in which cofibrations are cellular inclusions (fixing X) and weak equivalences are homotopy equivalences (see Ex. II.9.1). Consider the Waldhausen subcategory $\mathcal{R}_f(X)$ of those Y obtained by attaching only finitely many cells. Following Waldhausen [W1126], we write $A(X)$ for the space $K(\mathcal{R}_f(X)) = \Omega|hS.\mathcal{R}_f(X)|$. Thus $A_0(X) = K_0\mathcal{R}_f(X)$ is $\mathbb{Z}$ by Ex. II.9.1.

Similarly, we can form the Waldhausen subcategory $\mathcal{R}_{fd}(X)$ of those Y which are finitely dominated. We write $A^{fd}(X)$ for $K(\mathcal{R}_{fd}(X)) = \Omega|hS.\mathcal{R}_{fd}(X)|$. Note that $A_0^{fd}(X) = K_0\mathcal{R}_{fd}(X)$ is $\mathbb{Z}[\pi_1(X)]$ by Ex. II.9.1.

Cylinder Functors

When working with Waldhausen categories, it is often technically convenient to have mapping cylinders. Recall from Ex. 2.10 that the category $\mathcal{C}/\mathcal{C}$ of arrows in $\mathcal{C}$ has the morphisms of $\mathcal{C}$ as its objects, and a map $(a,b)\colon f \to f'$ in $Ar(\mathcal{C})$ is a commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array} \qquad (6.8.0)$$

The source $s(f) = A$ and target $t(f) = B$ of f define functors $s, t\colon \mathcal{C}/\mathcal{C} \to \mathcal{C}$.

Definition 6.8 (Cylinders). Let $\mathcal{C}$ be a Waldhausen category. A (mapping) *cylinder functor* on $\mathcal{C}$ is a functor T from the category $\mathcal{C}/\mathcal{C}$ of arrows in $\mathcal{C}$ to the category $\mathcal{C}$, together with natural transformations $j_1\colon s \Rightarrow T$, $j_2\colon t \Rightarrow T$ and $p\colon T \Rightarrow t$ so that for every $f\colon A \to B$ the diagram

$$\begin{array}{ccccc} A & \xrightarrow{j_1} & T(f) & \xleftarrow{j_2} & B \\ & f\searrow & \downarrow p & \swarrow = & \\ & & B & & \end{array}$$

commutes in $\mathcal{C}$. The following conditions must also hold:

(i) $T(0 \rightarrowtail A) = A$, with p and j_2 the identity map, for all $A \in \mathcal{C}$.

(ii) $j_1 \amalg j_2 \colon A \amalg B \rightarrowtail T(f)$ is a cofibration for all $f \colon A \to B$.

(iii) Given a map $(a,b)\colon f \to f'$ in $Ar(\mathcal{C})$, *i.e.,* a commutative square (6.8.0), if a and b are weak equivalences in $\mathcal{C}$ then so is $T(f) \to T(f')$.

(iv) Given a map $(a,b)\colon f \to f'$ in $Ar(\mathcal{C})$, if a and b are cofibrations in $\mathcal{C}$, then so is $T(f) \to T(f')$, and the following map, induced by condition (ii), is also a cofibration in $\mathcal{C}$.

$$A' \amalg_A T(f) \amalg_B B' \to T(f')$$

We often impose the following extra axiom on the weak equivalences of $\mathcal{C}$.

Cylinder Axiom 6.8.1. All maps $p : T(f) \to B$ are weak equivalences in $\mathcal{C}$.

Suppose $\mathcal{C}$ has a cylinder functor T. The *cone* of an object A is $\mathrm{cone}(A) = T(A \twoheadrightarrow 0)$, and the *suspension* of A is $\Sigma A = \mathrm{cone}(A)/A$. The cylinder axiom implies that $\mathrm{cone}(A) \xrightarrow{\sim} 0$ is a weak equivalence. Since $A \rightarrowtail \mathrm{cone}(A) \twoheadrightarrow \Sigma A$ is a cofibration sequence it follows from the description of $K_0(\mathcal{C})$ in II.9.1.2 that $[\Sigma A] = -[A]$ in $K_0(\mathcal{C})$. (Cf. Lemma II.9.2.1.) In fact, the Additivity Theorem (see Chapter V below) implies that the map $\Sigma\colon K(\mathcal{C}) \to K(\mathcal{C})$ is a homotopy inverse with respect to the H-space structure on $K(\mathcal{C})$, because $\Sigma_* + 1 = \mathrm{cone}_* = 0$.

The name 'cylinder functor' comes from the following two paradigms.

Example 6.8.2. The Waldhausen categories $\mathcal{R}_f(*)$ and $\mathcal{R}_f(X)$ of examples 6.7 and 6.7.1 have a cylinder functor: $T(f)$ is the usual (based) mapping cylinder of f. By construction, the mapping cylinder satisfies the cylinder axiom 6.8.1. Because of this paradigm, j_1 and j_2 are sometimes called the *front* and *back* inclusions.

Example 6.8.3. Let **Ch** be the Waldhausen category of chain complexes and quasi-isomorphisms constructed from an abelian (or exact) category $\mathcal{C}$; see II.9.2. The mapping cylinder of $f\colon A. \to B.$ is the usual mapping cylinder chain complex, with

$$T(f)_n = A_n \oplus A_{n-1} \oplus B_n.$$

The suspension functor $\Sigma(A.) = A.[-1]$ here is the shift operator: $\Sigma(A.)_n = A_{n-1}$.

Example 6.8.4. Exact categories usually do not have cylinder functors. This is reflected by the fact that for some $A \in \mathcal{A}$ there may be no B such that $[A \oplus B] = 0$ in $K_0(\mathcal{A})$. However, the Waldhausen category $\mathbf{Ch}^b(\mathcal{A})$ of bounded chain complexes does have a cylinder functor, and we used it to prove that $K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A})$ in II.9.2.2. In fact, $K(\mathcal{A}) \simeq K(\mathbf{Ch}^b(\mathcal{A})$ by the Gillet-Waldhausen theorem presented in chapter V. Thus many results requiring mapping cylinders in Waldhausen K-theory can be translated into results for Quillen K-theory.

Cofinality

Let $\mathcal{A}$ be a Waldhausen subcategory of $\mathcal{B}$. We say that $\mathcal{A}$ is *strictly cofinal* in $\mathcal{B}$ if for every B in $\mathcal{B}$ there is an A in $\mathcal{A}$ so that $A \wedge B$ is in $\mathcal{A}$. The following result was proven by Waldhausen in [W1126, 1.5.9]. We will prove a different cofinality result in chapter VI.

Strict Cofinality Theorem 6.9. *If $\mathcal{A}$ is a strictly cofinal Walhausen subcategory of $\mathcal{B}$, then $K(\mathcal{A}) \to K(\mathcal{B})$ is a homotopy equivalence. In particular, $K_n(\mathcal{A}) \cong K_n(\mathcal{B})$ for all n.*

EXERCISES

6.1 Show that for every $0 \le i < j < k \le n$ the diagram

$$A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk}$$

is a cofibration sequence, and that this gives an exact functor from $S_n\mathcal{C}$ to $S_2\mathcal{C}$.

6.2 Show that each functor $\partial_i\colon S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ is exact. Then show that $S.\mathcal{C}$ is a simplicial category.

6.3 Let $F, F'\colon \mathcal{A} \to \mathcal{B}$ be a pair of exact functors. A natural transformation $\eta\colon F \to F'$ is called a *weak equivalence* from F to F' if each $F(A) \xrightarrow{\sim} F'(A)$ is a weak equivalence in $\mathcal{B}$. Show that a weak equivalence induces a homotopy between the two maps $K(\mathcal{A}) \to K(\mathcal{B})$. *Hint:* Show that the maps $wS_n\mathcal{A} \to wS_n\mathcal{B}$ are homotopic in a compatible way.

6.4 If $X.$ is any simplicial space with X_0 a point, the formula for $|X.|$ gives a map from the reduced suspension ΣX_1 to $|X.|$. Using the Freudenthal map $X_1 \to \Omega\Sigma X_1$ this gives a map $X_1 \to \Omega|X.|$ and hence maps $\pi_i(X_1) \to \pi_{i+1}|X.|$. When $X. = BwS.\mathcal{C}$, this yields a map from $X_1 = Bw(\mathcal{C})$ to $K(\mathcal{C})$.
(a) Show that every weak self-equivalence $\alpha\colon A \xrightarrow{\sim} A$ determines an element $[\alpha]$ of $K_1(\mathcal{C})$. If β is another weak self-equivalence, show that $[\alpha][\beta] = [\alpha \wedge \beta]$.
(b) If $\mathcal{A}$ is an exact category, considered as a Waldhausen category, show that the resulting map $B(\text{iso}\,\mathcal{A}) \to K(\mathcal{A})$ induces a map from the group $K_1^{\oplus}\mathcal{A}$ of 3.6.1 to $K_1(\mathcal{A})$.

6.5 (Waldhausen) Let $\mathcal{A}$ be a small exact category. In this exercise we produce a map from $|iS.\mathcal{A}| \simeq |s.\mathcal{A}|$ to $BQ\mathcal{A}$, where $s.\mathcal{A}$ is defined in 6.5.2.
(a) Show that an object $A.$ of $iS_3\mathcal{A}$ determines a morphism in $Q\mathcal{A}$ from A_{12} to A_3.
(b) Show that an object $A.$ of $iS_5\mathcal{A}$ determines a sequence $A_{23} \to A_{14} \to A_5$ of row morphisms in $Q\mathcal{A}$.
(c) Recall from Ex. 2.10 that the Segal subdivision $Sub(s.\mathcal{A})$ is homotopy equivalent to $s.\mathcal{A}$. Show that (a) and (b) determine a simplicial map $Sub(s.\mathcal{A}) \to Q\mathcal{A}$. Composing with $|iS.\mathcal{A}| \simeq Sub(s.\mathcal{A})$, this yields a map from $|iS.\mathcal{A}|$ to $BQ\mathcal{A}$.

6.6 We now show that the map $|iS.\mathcal{A}| \to BQ\mathcal{A}$ constructed in the previous exercise is a homotopy equivalence. Let $iQ_n\mathcal{A}$ denote the category whose objects are the degree n elements of the nerve of $Q\mathcal{A}$, *i.e.*, sequences $A_0 \to \cdots \to A_n$ in $Q\mathcal{A}$, and whose morphisms are isomorphisms.
(a) Show that $iQ.\mathcal{A}$ is a simplicial category, and that the nerve of $Q\mathcal{A}$ is the simplicial set of objects. Waldhausen proved in [W1126, 1.6.5] that $BQ\mathcal{A} \to |iQ.\mathcal{A}|$ is a homotopy equivalence.
(b) Show that for each n there is an equivalence of categories $Sub(iS_n\mathcal{A}) \xrightarrow{\sim} iQ_n\mathcal{A}$, where $Sub(iS_n\mathcal{A})$ is the Segal subdivision category of Ex. 2.9. Then show that the equivalences form a map of simplicial categories $Sub(iS.\mathcal{A}) \to iQ.\mathcal{A}$. This map must be a homotopy equivalence, because it is a homotopy equivalence in each degree.

(c) Show that the map of the previous exercise fits into a diagram

$$\begin{tikzcd} |s.\mathcal{A}| & |Sub(s.\mathcal{A})| \arrow[l, "\simeq"'] \arrow[r] \arrow[d, "\simeq"'] & BQ\mathcal{A} \arrow[d, "\simeq"'] \\ |iS.\mathcal{A}| \arrow[from=u, "\simeq"'] & |Sub(iS.\mathcal{A})| \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & |iQ.\mathcal{A}|. \end{tikzcd}$$

Conclude that the map $|iS.\mathcal{A}| \to BQ\mathcal{A}$ of Ex. 6.5 is a homotopy equivalence.

6.7 Let $\mathbf{Ch}(\mathcal{C})$ be the Waldhausen category of chain complexes in an exact category $\mathcal{C}$, as in 6.8.3. Show that $\mathbf{Ch}(\mathcal{C})$ and $\mathbf{Ch}^b(\mathcal{C})$ satisfy the Saturation axiom II.9.7.1, the Extension axiom 6.2.1, and the Cylinder Axiom 6.8.1.

6.8 *Finite Sets.* Let $\mathbf{Sets}_f$ be the Waldhausen category of finite pointed sets, where the cofibrations are the injections and the weak equivalences are the isomorphisms. Show that the argument of Ex. 6.5 and 6.6 may be used, together with theorem 3.7.2 and Ex. 5.7, to prove that the Waldhausen K-theory space $K(\mathbf{Sets}_f)$ is $\mathbb{Z} \times \Sigma^+ \simeq \Omega^\infty\Sigma^\infty$. Thus $K_n(\mathbf{Sets}_f) \cong \pi_n^s$ for all n.

§7. The Gillet-Grayson construction

Let $\mathcal{A}$ be an exact category. Following Grayson and Gillet [GG], we define a simplicial set $G. = G.\mathcal{A}$ as follows.

Definition 7.1. If $\mathcal{A}$ is a small exact category, $G.$ is the simplicial set defined as follows. The set G_0 of vertices consists of all pairs of objects (A, B) in $\mathcal{A}$. The set G_1 of edges consists of all pairs of short exact sequences with the same cokernel:

$$A_0 \rightarrowtail A_1 \twoheadrightarrow A_{01}, \quad B_0 \rightarrowtail B_1 \twoheadrightarrow A_{01}. \tag{7.1.0}$$

The degeneracy maps $G_1 \to G_0$ send (7.1.0) to (A_1, B_1) and (A_0, B_0), respectively.
The set G_n consists of all pairs of triangular commutative diagrams in $\mathcal{A}$ of the form

$$\begin{tikzcd}
 & & & & A_{n-1,n} & & & & & A_{n-1,n} \\
 & & & & \cdots \arrow[u, two heads] & & & & & \cdots \arrow[u, two heads] \\
 & & A_{12} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{1n} & & & A_{12} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{1n} \\
 & A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A_{0n} \arrow[u, two heads] & & A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A_{0n} \arrow[u, two heads] \\
A_0 \arrow[r, tail] & A_1 \arrow[r, tail] \arrow[u, two heads] & A_2 \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A_n \arrow[u, two heads] & B_0 \arrow[r, tail] & B_1 \arrow[r, tail] \arrow[u, two heads] & B_2 \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & B_n \arrow[u, two heads]
\end{tikzcd} \tag{7.1.1}$$

so that each sequence $A_i \rightarrowtail A_j \twoheadrightarrow A_{ij}$ and $B_i \rightarrowtail B_j \twoheadrightarrow A_{ij}$ is exact. As in the definition of $S.\mathcal{A}$ (6.3.1), the face maps $\partial_i \colon G_n \to G_{n-1}$ are obtained by deleting the row A_{i*} and the columns containing A_i and B_i, while the degeneracy maps $\sigma_i \colon G_n \to G_{n+1}$ are obtained by duplicating A_i and B_i, and reindexing.

Omitting the choices A_{ij} for the cokernels, we can abbreviate (7.1.1) as:

$$\frac{A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n}{B_0 \rightarrowtail B_1 \rightarrowtail B_2 \rightarrowtail \cdots \rightarrowtail B_n}\,. \tag{7.1.2}$$

Remark 7.1.3. $|G.|$ is a homotopy commutative and associative H-space. Its product $|G.| \times |G.| \to |G.|$ arises from the simplicial map $G. \times G. \to G.$ whose components $G_n \times G_n \to G_n$ are termwise $\oplus$.
Note that for each isomorphism $A \cong A'$ in $\mathcal{A}$ there is an edge in G_1 from $(0, 0)$ to (A, A'), represented by $(0 \rightarrowtail A \twoheadrightarrow A, 0 \rightarrowtail A' \twoheadrightarrow A)$. Hence (A, A') represents zero in the group $\pi_0|G.|$.

Lemma 7.2. *There is a group isomorphism* $\pi_0|G.| \cong K_0(\mathcal{A})$.

Proof. As in 2.3, $\pi_0|G.|$ is presented as the set of elements (A, B) of G_0, modulo the equivalence relation that for each edge (7.1.0) we have

$$(A_1, B_1) = (A_0, B_0).$$

It is an abelian group by 7.1.3, with operation $(A, B) \oplus (A', B') = (A \oplus A', B \oplus B')$. Since $(A \oplus B, B \oplus A)$ represents zero in $\pi_0|G.|$, it follows that (B, A) is the inverse of (A, B). From this presentation, we see that there is a map $K_0(\mathcal{A}) \to \pi_0|G.|$ sending $[A]$ to $(A, 0)$, and a map $\pi_0|G.| \to K_0(\mathcal{A})$, sending (A, B) to $[A] - [B]$. These maps are inverses to each other.

(7.3) We now compare $G.$ with the loop space of the simplicial set $s.\mathcal{A}$ of 6.5.2. If we forget the bottom row of either of the two triangular diagrams in (7.1.1), we get a triangular commutative diagram of the form (6.3.0), *i.e.,* an element of $s_n\mathcal{A}$. The resulting set maps $G_n \to s_n\mathcal{A}$ fit together to form a simplicial map $\partial_0 \colon G. \to s.\mathcal{A}$.

PATH SPACES 7.3.1. Recall from [WHomo, 8.3.14] that the *path space* $PX.$ of a simplicial set $X.$ has $PX_n = X_{n+1}$, its ith face operator is the ∂_{i+1} of $X.$, and its ith degeneracy operator is the σ_{i+1} of $X.$. The forgotten face maps $\partial_0 \colon X_{n+1} \to X_n$ form a simplicial map $PX. \to X.$, and $\pi_0(PX.) \cong X_0$. In fact, σ_0 induces a canonical simplicial homotopy equivalence from $PX.$ to the constant simplicial set X_0; see [WHomo, Ex. 8.3.7]. Thus $PX.$ is contractible exactly when X_0 is a point.

Now there are two maps $G_n \to s_{n+1}\mathcal{A}$, obtained by forgetting one of the two triangular diagrams (7.1.1) giving an element of G_n. The face and degeneracy maps of $G.$ are defined so that these yield two simplicial maps from $G.$ to the path space $P. = P(s.\mathcal{A})$. Clearly, either composition with the canonical map $P. \to s.\mathcal{A}$ yields the map $\partial_0 \colon G. \to s.\mathcal{A}$. Thus we have a commutative diagram

(7.3.2)
$$\begin{array}{ccc} G. & \longrightarrow & P. \\ \big\downarrow & & \big\downarrow \\ P. & \longrightarrow & s.\mathcal{A}. \end{array}$$

Since $s_0\mathcal{A}$ is a point, the path space $|P.|$ is canonically contractible. Therefore this diagram yields a canonical map $|G.| \to \Omega|s.\mathcal{A}|$. On the other hand, we saw in 6.6 that $|s.\mathcal{A}| \simeq BQ\mathcal{A}$, so $\Omega|s.\mathcal{A}| \simeq \Omega BQ\mathcal{A} = K(\mathcal{A})$.

We cite the following result from [GG, 3.1]. Its proof uses simplicial analogues of Quillen's theorems A and B.

THEOREM 7.4. *(Gillet-Grayson) Let $\mathcal{A}$ be a small exact category. Then the map of (7.3) is a homotopy equivalence:*

$$|G.| \simeq \Omega|s.A| \simeq K(\mathcal{A}).$$

Hence $\pi_i|G.| = K_i(\mathcal{A})$ for all $i \geq 0$.

EXAMPLE 7.5. A *double s.e.s.* in $\mathcal{A}$ is a pair ℓ of short exact sequences in $\mathcal{A}$ on the same objects:

$$\ell\colon \qquad A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C, \quad A \overset{f'}{\rightarrowtail} B \overset{g'}{\twoheadrightarrow} C.$$

Thus ℓ is an edge (in G_1) from (A,A) to (B,B). To ℓ we attach the element $[\ell]$ of $K_1(\mathcal{A}) = \pi_1|G.|$ given by the following 3-edged loop.

$$\begin{array}{ccc} (A,A) & \xrightarrow{\ell} & (B,B) \\ {\scriptstyle e_A}\nwarrow & & \nearrow{\scriptstyle e_B} \\ & (0,0) & \end{array}$$

where e_A denotes the canonical double s.e.s. $(0 \rightarrowtail A \twoheadrightarrow A,\ 0 \rightarrowtail A \twoheadrightarrow A)$.

The following theorem was proven by A. Nenashev in [Nen].

NENASHEV'S THEOREM 7.6. *$K_1(\mathcal{A})$ may be described as follows.*
(a) Every element of $K_1(\mathcal{A})$ is represented by the loop $[\ell]$ of a double s.e.s.;
(b) $K_1(\mathcal{A})$ is presented as the abelian group with generators the double s.e.s. in $\mathcal{A}$, subject to two relations:

(i) If E is a short exact sequence, the loop of the double s.e.s. (E, E) is zero;
(ii) for any diagram of six double s.e.s. (7.6.1) such that the "first" diagram commutes, and the "second" diagram commutes, then

$$[r_0] - [r_1] + [r_2] = [c_0] - [c_1] + [c_2],$$

where r_i is the i^{th} row and c_i is the i^{th} column of (7.6.1).

$$\begin{array}{ccccc} A' & \rightrightarrows & A & \rightrightarrows & A'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ B' & \rightrightarrows & B & \rightrightarrows & B'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ C' & \rightrightarrows & C & \rightrightarrows & C''. \end{array} \tag{7.6.1}$$

EXAMPLE 7.6.2. If α is an automorphism of A, the class $[\alpha] \in K_1(\mathcal{A})$ is the class of the double s.e.s. $(0 \rightarrowtail A \overset{\alpha}{\twoheadrightarrow} A, 0 \rightarrowtail A \overset{=}{\twoheadrightarrow} A)$.

If β is another automorphism of A, the relation $[\alpha\beta] = [\alpha] + [\beta]$ comes from relation (ii) for

$$\begin{array}{ccccc} 0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ 0 & \rightrightarrows & A & \underset{1}{\overset{\alpha}{\rightrightarrows}} & A \\ \downdownarrows & & 1 \downdownarrows 1 & & \beta \downdownarrows 1 \\ 0 & \rightrightarrows & A & \underset{1}{\overset{\alpha\beta}{\rightrightarrows}} & A. \end{array}$$

EXERCISES

7.1 Verify that condition 7.6(i) holds in $\pi_1|G.|$.

7.2 Show that omitting the choice of quotients A_{ij} from the definition of $G.\mathcal{A}$ yields a homotopy equivalent simplicial set $G'.\mathcal{A}$. An element of $G'_n\mathcal{A}$ is a diagram (7.1.2) together with a compatible family of isomorphisms $A_j/A_i \cong B_j/B_i$.

7.3 Consider the involution on $G.$ which interchanges the two diagrams in (7.1.1). We saw in 7.2 that it induces multiplication by -1 on $K_0(\mathcal{A})$. Show that this involution is an additive inverse map for the H-space structure 7.1.3 on $|G.|$.

7.4 If $\alpha\colon A \cong A$ is an isomorphism, use relation (ii) in Nenashev's presentation 7.6 to show that $[\alpha^{-1}] \in K_1(\mathcal{A})$ is represented by the loop of the double s.e.s.:

$$\frac{A \overset{\alpha}{\to} A}{A \underset{=}{\to} A}$$

7.5 If $\mathcal{A}$ is a split exact category, use Nenashev's presentation 7.6 to show that $K_1(\mathcal{A})$ is generated by automorphisms (7.6.2).

§8. Karoubi-Villamayor K-theory

Following Gersten, we say that a functor F from rings (or rings without unit) to sets is *homotopy invariant* if $F(R) \cong F(R[t])$ for every R. Similarly, a functor F from rings to CW complexes (spaces) is called *homotopy invariant* if for every ring R the natural map $R \to R[t]$ induces a homotopy equivalence $F(R) \simeq F(R[t])$. Note that each homotopy group $\pi_n F(R)$ also forms a homotopy invariant functor.

Of course, this notion may be restricted to functors defined on any subcategory of rings which is closed under polynomial extensions and contains the evaluations as well as the inclusion $R \subset R[t]$. For example, we saw in II.6.5 and 7.8.3 that $G_0(R)$ is a homotopy invariant functor defined on noetherian rings (and schemes) and maps of finite Tor-dimension.

Conversely, recall from III.3.1 that R is called F-regular if $F(R) \cong F(R[t_1, ..., t_n])$ for all n. Clearly, any functor F from rings to sets becomes homotopy invariant when restricted to the subcategory of F-regular rings. For example, we see from II.7.7 that K_0 becomes homotopy invariant when restricted to regular rings. The Fundamental Theorem in chapter V implies that the functors K_n are also homotopy invariant when restricted to regular rings.

There is a canonical way to make F into a homotopy invariant functor.

Strict homotopization 8.1. Let F be a functor from rings to sets. Its *strict homotopization* $[F]$ is defined as the coequalizer of the evaluations at $t = 0, 1$: $F(R[t]) \rightrightarrows F(R)$. In fact, $[F]$ is a homotopy invariant functor and there is a universal transformation $F(R) \to [F](R)$; see Ex. 8.1. Moreover, if F takes values in groups then so does $[F]$; see Ex. 8.2.

Example 8.1.1. Recall that a matrix is called *unipotent* if it has the form $1 + \nu$ for some nilpotent matrix ν. Let $Uni(R)$ denote the subgroup of $GL(R)$ generated by the unipotent matrices. This is a normal subgroup of $GL(R)$, because the unipotent matrices are closed under conjugation. Since every elementary matrix $e_{ij}(r)$ is unipotent, this contains the commutator subgroup $E(R)$ of $GL(R)$.

We claim that $[E]R = [Uni]R = 1$ for every R. Indeed, if $1 + \nu$ is unipotent, $(1 + t\nu)$ is a matrix in $Uni(R[t])$ with $\partial_0(1 + t\nu) = 1$ and $\partial_1(1 + t\nu) = (1 + \nu)$. Since $Uni(R)$ is generated by these elements, $[Uni]R$ must be trivial. The same argument applies to the elementary group $E(R)$.

We now consider $GL(R)$ and its quotient $K_1(R)$. A priori, $[GL]R \to [K_1]R$ is a surjection. In fact, it is an isomorphism.

Lemma 8.2. *Both $[GL]R$ and $[K_1]R$ are isomorphic to $GL(R)/Uni(R)$.*

Definition 8.2.1. For each ring R, we define $KV_1(R)$ to be $GL(R)/Uni(R)$. Thus $KV_1(R)$ is the strict homotopization of $K_1(R) = GL(R)/E(R)$.

Proof. The composite $Uni(R) \to GL(R) \to [GL]R$ is trivial, because it factors through $[Uni]R = 1$. Hence $[GL]R$ (and $[K_1]R$) are quotients of $GL(R)/Uni(R)$. By Higman's trick III.3.4.1, if $g \in GL(R[t])$ is in the kernel of ∂_0 then $g \in Uni(R[t])$ and hence $\partial_1(g) \in Uni(R)$. Hence $\partial_1(NGL(R)) = Uni(R)$. Hence $GL(R)/Uni(R)$ is a strictly homotopy invariant functor; universality implies that the induced maps $[GL]R \to [K_1]R \to GL(R)/Uni(R)$ must be isomorphisms.

To define the higher Karoubi-Villamayor groups, we introduce the simplicial ring $R[\Delta^\cdot]$, and use it to define the notion of homotopization. The simplicial ring $R[\Delta^\cdot]$

will also play a critical role in the last chapter, when we construct higher Chow groups and Motivic cohomology.

Definition 8.3. For each ring R the coordinate rings of the standard simplices form a simplicial ring $R[\Delta^\cdot]$. It may described by the diagram

$$R \leftleftarrows R[t_1] \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} R[t_1,t_2] \overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}} \cdots R[t_1,\dots,t_n]\cdots$$

with $R[\Delta^n] = R[t_0,t_1,\cdots,t_n]/(\sum t_i = 1) \cong R[t_1,\cdots,t_n]$. The face maps are given by: $\partial_i(t_i) = 0$; $\partial_i(t_j)$ is t_j for $i < j$ and t_{j-1} for $i > j$. Degeneracies are given by: $\sigma_i(t_i) = t_i + t_{i+1}$; $\sigma_i(t_j)$ is t_j for $i < j$ and t_{j+1} for $i > j$.

Definition 8.4 (Homotopization). Let F be a functor from rings to CW complexes. Its *homotopization* $F^h(R)$ is the geometric realization of the simplicial space $F(R[\Delta^\cdot])$. Thus F^h is also a functor from rings to CW complexes, and there is a canonical map $F(R) \to F^h(R)$.

Lemma 8.4.1. *F^h is a homotopy invariant functor. Moreover, if F is homotopy invariant then $F(R) \simeq F^h(R)$ for all R.*

Proof. We claim that the inclusion $R[\Delta^\cdot] \subset R[x][\Delta^\cdot]$ is a simplicial homotopy equivalence, split by evaluation at $x = 0$. For this, we define ring maps $h_i\colon R[x][\Delta^n] \to R[x][\Delta^{n+1}]$ by: $h_i(f) = \sigma_i(f)$ if $f \in R[\Delta^n]$ and $h_i(x) = x(t_{i+1} + \cdots + t_n)$. These maps define a simplicial homotopy between the identity map of $R[x][\Delta^\cdot]$ and the composite

$$R[x][\Delta^\cdot] \xrightarrow{x=0} R[\Delta^\cdot] \subset R[x][\Delta^\cdot].$$

Applying F gives a simplicial homotopy equivalence between $F^h(R[\Delta^\cdot])$ and $F^h(R[x][\Delta^\cdot])$. Geometric realization converts this into a topological homotopy equivalence between $F^h(R)$ and $F^h(R[x])$.

Finally, if F is homotopy invariant then the map from the constant simplicial space $F(R)$ to $F(R[\Delta^\cdot])$ is a homotopy equivalence in each degree. It follows (see [Wa78]) that their realizations $F(R)$ and $F^h(R)$ are homotopy equivalent.

It is easy to see that $F \to F^h$ is universal (up to homotopy equivalence) for functors from F to homotopy invariant functors. A proof of this fact is left to Ex. 8.3.

Lemma 8.5. *Let F be a functor from rings to CW complexes. Then $\pi_0(F^h)$ is the strict homotopization $[F_0]$ of the functor $F_0(R) = \pi_0 F(R)$.*

Proof. For any simplicial space $X.$, the group $\pi_0(|X.|)$ is the coequalizer of $\partial_0, \partial_1\colon \pi_0(X_1) \rightrightarrows \pi_0(X_0)$. In this case $\pi_0(X_0) = \pi_0 F(R)$ and $\pi_0(X_1) = \pi_0 F(R[t])$.

Applying the functor GL gives us a simplicial group $GL. = GL(R[\Delta^\cdot])$.

Corollary 8.5.1. $KV_1(R) \cong \pi_0(GL.)$.

Definition 8.6. For $n \geq 1$, we set $KV_n(R) = \pi_{n-1}(GL.) = \pi_n(BGL.)$.

COROLLARY 8.6.1. *The abelian groups $KV_n(R)$ are homotopy invariant, i.e.,*

$$KV_n(R) \cong KV_n(R[t]) \quad \textit{for every } n \geq 1.$$

Gersten observed that $BGL. \simeq BGL(R[\Delta^\cdot])^+$. In this case, the standard spectral sequence for a simplicial space $E^1_{pq} = \pi_p(X_q) \Rightarrow \pi_{p+q}|X.|$ becomes the first quadrant spectral sequence (for $p \geq 1, q \geq 0$):

$$E^1_{pq} = K_p(R[\Delta^q]) \Rightarrow KV_{p+q}(R). \tag{8.6.2}$$

If R is regular, then each simplicial group $K_p(R[\Delta^\cdot])$ is constant (by the Fundamental Theorem in chapter V). Thus the spectral sequence degenerates at E^2:

THEOREM 8.7. *(Gersten) If R is regular, then $K_p(R) \cong KV_p(R)$ for all $p \geq 1$. For general R, there is a natural edge map $K_p(R) \to KV_p(R)$.*

We now quickly develop the key points in KV-theory.

DEFINITION 8.8. We say that a ring map $f\colon A \to B$ is a *GL-fibration* if

$$GL(A[t_1, ..., t_n]) \times GL(B) \to GL(B[t_1, ..., t_n])$$

is onto for every n. Note that we do not require A and B to have a unit.

REMARK 8.8.1. Any GL-fibration must be onto. That is, $B \cong A/I$ for some ideal I of A. To see this, consider the $(1,2)$ entry α_{12} of a preimage of the elementary matrix $e_{12}(bt)$. Since $f(\alpha_{12}) = bt$, evaluation at $t = 1$ gives an element of A mapping to $b \in B$. However, not every surjection is a GL-fibration; see Ex. 8.5(b).

PROPOSITION 8.9. *If $A \to B$ is a GL-fibration with kernel I, there is a long exact sequence*

$$\begin{aligned} KV_{n+1}(B) \to KV_n(I) \to KV_n(A) \to KV_n(B) \to \cdots \\ KV_1(I) \to KV_1(A) \to KV_1(B) \to K_0(I) \to K_0(A) \to K_0(B). \end{aligned}$$

PROOF. Let $G_n \subset GL(B[\Delta^n])$ denote the image of $GL(A[\Delta^n])$. Then there is an exact sequence of simplicial groups

$$1 \to GL(I[\Delta^\cdot]) \to GL(A[\Delta^\cdot]) \to G. \to 1. \tag{8.9.1}$$

Now any short exact sequence of simplicial groups is a fibration sequence, meaning there is a long exact sequence of homotopy groups. Since the quotient $GL(B[\Delta^\cdot])/G.$ is a constant simplicial group, it is now a simple matter to splice the long exact sequences together to get the result. We have indicated the details in Ex. 8.3.

Here is an application of this result. Since $R[x] \to R$ has a section, it is a GL-fibration. By homotopy invariance, it follows that $KV_n(xR[x]) = 0$ for all $n \geq 1$. (Another proof is given in Ex. 8.4.)

DEFINITION 8.10. For any ring R (with or without unit), we define ΩR to be the ideal $(x^2 - x)R[x]$ of $R[x]$.

Since ΩR is the kernel of $x = 1\colon xR[x] \to R$, and this map is a GL-fibration by Ex. 8.7, we have the following corollary of 8.9.

COROLLARY 8.10.1. *For all R, $KV_1(R)$ is isomorphic to the kernel of the map $K_0(\Omega R) \to K_0(xR[x])$, and $KV_n(R) \cong KV_{n-1}(\Omega R)$ for all $n \geq 2$.*

This shows that we can define $KV_n(R)$ as $KV_1(\Omega^{n-1}R)$ for all $n \geq 2$.

The following definition is due to Karoubi and Villamayor.

DEFINITION 8.11. A *positive homotopy K-theory* consists of a sequence of functors K_n^h, $n \geq 1$, on the category of rings without unit, together with natural connecting maps $\delta_n\colon K_{n+1}^h(R/I) \to K_n^h(I)$ and $\delta_0\colon K_1^h(R/I) \to K_0(I)$, defined for every GL-fibration $R \to R/I$, satisfying the following axioms:

(1) The functors K_n^h are homotopy invariant;
(2) For every GL-fibration $R \to R/I$ the resulting sequence is exact:

$$K_{n+1}^h(R/I) \xrightarrow{\delta} K_n^h(I) \to K_n^h(R) \to K_n^h(R/I) \xrightarrow{\delta} K_{n-1}^h(I) \to \\ K_1^h(R) \to K_1^h(R/I) \xrightarrow{\delta} K_0(I) \to K_0(R) \to K_0(R/I).$$

THEOREM 8.11.1. *There is a unique positive homotopy K-theory (up to isomorphism), namely $K_n^h = KV_n$.*

PROOF. The fact that KV_n form a positive homotopy K-theory is given by 8.6.1 and 8.9. The axioms imply that any other positive homotopy K-theory must satisfy the conclusion of 8.10.1, and so must be isomorphic to KV-theory.

EXERCISES

8.1 Let F be a functor from rings to sets. Show that $[F]$ is a homotopy invariant functor, and that every natural transformation $F(R) \to G(R)$ to a homotopy invariant functor G factors uniquely through $F(R) \to [F](R)$.

8.2 If F is a functor from rings to groups, let $NF(R)$ denote the kernel of the map $t = 0\colon F(R[t]) \to F(R)$. Show that the image $F_0(R)$ of the induced map $t = 1\colon NF(R) \to F(R)$ is a normal subgroup of $F(R)$, and that $[F]R = F(R)/F_0(R)$. Thus $[F]R$ is a group.

8.3 Let F and G be functors from rings to CW complexes, and assume that G is homotopy invariant. Show that any natural transformation $F(R) \to G(R)$ factors through maps $F^h(R) \to G(R)$ such that for each ring map $R \to S$ the map $F^h(R) \to F^h(S) \to G(S)$ is homotopy equivalent to $F^h(R) \to F^h(S) \to G(S)$.

8.4 Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring. Show that for every homotopy invariant functor F on rings we have $F(R_0) \simeq F(R)$. In particular, if F is defined on rings without unit then $F(xA[x]) \simeq F(0)$ for every A. *Hint:* Copy the proof of III.3.3.2.

8.5 *GL-fibrations.* Let $f\colon A \to B$ be a GL-fibration with kernel I.
(a) Show that $tA[t] \to tB[t]$ and $\Omega A \to \Omega B$ are GL-fibrations.
(b) Show that $\mathbb{Z} \to \mathbb{Z}/4$ is not a GL-fibration, but $GL(\mathbb{Z}) \to GL(\mathbb{Z}/4)$ is onto.
(c) If B is a regular ring (with unit), show that every surjection $A \to B$ is a GL-fibration. *Hint:* $K_1(B) \cong K_1(B[t])$.

8.6 Let $f\colon A \to B$ is a GL-fibration with kernel I, and define $G_.$ as in the proof of Proposition 8.9. Show that $GL(B[\Delta^\cdot])/G.$ is a constant simplicial group. Use this to show that $\pi_i(G_.) = KV_{i+1}(B)$ for all $i > 0$, but that the cokernel of $\pi_0(G_.) \to \pi_0 GL(B[\Delta^\cdot])$ is the image of $K_1(B)$ in $K_0(I)$ under the map of III.2.3. Then combine this with the long exact sequence of homotopy groups for (8.9.1) to finish the proof of 8.9.

8.7 Show that the map $tA[t] \to A$, $f(t) \mapsto f(1)$, is a GL-fibration with kernel ΩA.

§9. Mod ℓ K-theory

In addition to the usual K-groups $K_i(\mathcal{C})$ of a category $\mathcal{C}$, it is often useful to study its "mod ℓ K-groups" $K_i(\mathcal{C};\mathbb{Z}/\ell)$, where ℓ is a positive integer. In this section we quickly recount the basic construction from mod ℓ homotopy theory. Basic properties of mod ℓ homotopy theory may be found in [N].

Recall [N] that if $m \geq 2$ the mod ℓ *Moore space* $P^m(\mathbb{Z}/\ell)$ is the space formed from the sphere S^{m-1} by attaching an m-cell via a degree ℓ map. The suspension of $P^m(\mathbb{Z}/\ell)$ is the Moore space $P^{m+1}(\mathbb{Z}/\ell)$, and as m varies this yields a suspension spectrum, called the *Moore spectrum* $P^\infty(\mathbb{Z}/\ell)$.

Definition 9.1. If $m \geq 2$, the mod ℓ homotopy "group" $\pi_m(X;\mathbb{Z}/\ell)$ of a based topological space X is defined to be the pointed set $[P^m(\mathbb{Z}/\ell), X]$ of based homotopy classes of maps from the Moore space $P^m(\mathbb{Z}/\ell)$ to X.

For a general space X, $\pi_2(X;\mathbb{Z}/\ell)$ isn't even a group, but the $\pi_m(X;\mathbb{Z}/\ell)$ are always groups for $m \geq 3$ and abelian groups for $m \geq 4$. However, if $X = \Omega Y$ then $\pi_2(X;\mathbb{Z}/\ell)$ is a group and we can define $\pi_1(X;\mathbb{Z}/\ell)$ as $\pi_2(Y;\mathbb{Z}/\ell)$; this is independent of the choice of U by Ex. 9.1. More generally, if $X = \Omega^k Y_k$ for $k >> 0$ and $P^m = P^m(\mathbb{Z}/\ell)$ then the formula

$$\pi_m(X;\mathbb{Z}/\ell) = [P^m, X] = [P^m, \Omega^k Y_k] \cong [P^{m+k}, Y_k] = \pi_{m+k}(Y_k;\mathbb{Z}/\ell)$$

shows that we can ignore these restrictions on m, and that $\pi_m(X;\mathbb{Z}/\ell)$ is an abelian group for all $m \geq 0$ (or even negative m, as long as $k > 2 + |m|$).

In particular, if X is an infinite loop space then abelian groups $\pi_m(X;\mathbb{Z}/\ell)$ are defined for all $m \in \mathbb{Z}$, using the explicit sequence of deloopings of X provided by the given structure on X.

If $m \geq 2$, the cofibration sequence $S^{m-1} \xrightarrow{\ell} S^{m-1} \longrightarrow P^m(\mathbb{Z}/\ell)$ defining $P^m(\mathbb{Z}/\ell)$ induces an exact sequence of homotopy groups

$$\pi_m(X) \xrightarrow{\ell} \pi_m(X) \to \pi_m(X;\mathbb{Z}/\ell) \xrightarrow{\partial} \pi_{m-1}(X) \xrightarrow{\ell} \pi_{m-1}(X).$$

It is convenient to adopt the notation that if A is an abelian group then ${}_\ell A$ denotes the subgroup of all elements a of A such that $\ell \cdot a = 0$. This allows us to restate the above exact sequence in a concise fashion.

Universal Coefficient Sequence 9.2. *For all $m \geq 3$ there is a natural short exact sequence*

$$0 \to (\pi_m X) \otimes \mathbb{Z}/\ell \to \pi_m(X;\mathbb{Z}/\ell) \xrightarrow{\partial} {}_\ell(\pi_{m-1}X) \to 0.$$

This sequence is split exact (but not naturally) when $\ell \not\equiv 2 \bmod 4$.

For π_2, the sequence 9.2 of pointed sets is also exact in a suitable sense; see [N, p. 3]. However this point is irrelevant for loop spaces, so we ignore it.

Example 9.2.2. When $\ell = 2$, the sequence need not split. For example, it is known that $\pi_{m+2}(S^m;\mathbb{Z}/2) = \mathbb{Z}/4$ for $m \geq 3$, and that $\pi_2(BO;\mathbb{Z}/2) = \mathbb{Z}/4$; see [AT65].

Now suppose that $\mathcal{C}$ is either a symmetric monoidal category, or an exact category, or a Waldhausen category, so that the K-theory space $K(\mathcal{C})$ is defined, and is an infinite loop space.

DEFINITION 9.3. The mod ℓ K-groups of $\mathcal{C}$ are defined to be the abelian group:

$$K_m(\mathcal{C};\mathbb{Z}/\ell) = \pi_m(K(\mathcal{C});\mathbb{Z}/\ell), \quad m \in \mathbb{Z}.$$

If $m \geq 2$ this definition states that $K_m(\mathcal{C};\mathbb{Z}/\ell) = [P^m(\mathbb{Z}/\ell), K(\mathcal{C})]$. Because $K(\mathcal{C}) \simeq Y$, we can define $K_1(\mathcal{C};\mathbb{Z}/\ell)$ in a way that is independent of the choice of Y. However, the groups $K_0(\mathcal{C};\mathbb{Z}/\ell)$ and $K_m(\mathcal{C};\mathbb{Z}/\ell)$ for $m < 0$ depend not only upon $K(\mathcal{C})$, but also upon the choice of the deloopings of $K(\mathcal{C})$ in the underlying K-theory spectrum $\mathbf{K}\mathcal{C}$. In fact, the literature is not consistent about $K_m(\mathcal{C};\mathbb{Z}/\ell)$ when $m < 2$, even for $K_1(R;Z/\ell)$.

By Theorem 9.2, the mod ℓ K-groups are related to the usual K-groups.

UNIVERSAL COEFFICIENT THEOREM 9.4. *There is a short exact sequence*

$$0 \to K_m(\mathcal{C}) \otimes \mathbb{Z}/\ell \to K_m(\mathcal{C};\mathbb{Z}/\ell) \to {}_\ell K_{m-1}(\mathcal{C}) \to 0$$

for every $m \in \mathbb{Z}$, $\mathcal{C}$, and ℓ. It is split exact (not naturally) unless $\ell \equiv 2 \bmod 4$.

In particular, for $\mathcal{C} = \mathbf{P}(R)$ we have an exact sequence

$$0 \to K_m(R) \otimes \mathbb{Z}/\ell \to K_m(R;\mathbb{Z}/\ell) \to {}_\ell K_{m-1}(R) \to 0$$

Ex. 9.2 shows that the splitting in 9.4 is not natural in R.

EXAMPLE 9.4.1. ($\ell = 2$) Since the isomorphism $\Omega^\infty\Sigma^\infty \to \mathbb{Z} \times BO$ factors through $K(\mathbb{Z})$ and $K(\mathbb{R})$, the universal coefficient theorem and 9.2.2 show that

$$K_2(\mathbb{Z};\mathbb{Z}/2) \cong K_2(\mathbb{R};\mathbb{Z}/2) \cong \pi_2(BO;\mathbb{Z}/2) = \mathbb{Z}/4.$$

It turns out that for $\ell = 2$ the sequence for $K_m(R;\mathbb{Z}/2)$ is split whenever multiplication by $[-1] \in K_1(\mathbb{Z})$ is the zero map from $K_{m-1}(R)$ to $K_m(R)$. For example, this is the case for the finite fields $\mathbb{F}_q$, an observation made in [Br].

EXAMPLE 9.4.2 (BOTT ELEMENTS). Suppose that R contains a primitive ℓ^{th} root of unity ζ. The Universal Coefficient Theorem 9.4 provides an element $\beta \in K_2(R;\mathbb{Z}/\ell)$, mapping to $\zeta \in {}_\ell K_1(R)$. This element is called the *Bott element*, and it plays an important role in the product structure of the ring $K_*(R;\mathbb{Z}/\ell)$.

REMARK 9.4.3. A priori, β depends not only upon ζ but also upon the choice of the splitting in 9.4. One way to choose β is to observe that the inclusion of μ_ℓ in $GL_1(R)$ induces a map $B\mu_\ell \to BGL(R) \to BGL(R)^+$ and therefore a set function $\mu_\ell \to K_2(R;\mathbb{Z}/\ell)$. A posteriori, it turns out that this is a group homomorphism unless $\ell \equiv 2 \pmod 4$.

EXAMPLE 9.5. Let k be the algebraic closure of the field $\mathbb{F}_p$. Quillen's compution of $K_*(\mathbb{F}_q)$ in 1.9.1 shows that $K_n(k) = 0$ for m even ($m \geq 2$), and that $K_m(k) = \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ for m odd ($m \geq 1$). It follows that if ℓ is prime to p then:

$$K_m(k;\mathbb{Z}/\ell) = \begin{cases} \mathbb{Z}/\ell & \text{if } m \text{ is even, } m \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $K_*(k;\mathbb{Z}/\ell)$ is the polynomial ring $\mathbb{Z}/\ell[\beta]$ on the Bott element $\beta \in K_2(k;\mathbb{Z}/\ell)$, under the K-theory product we will discuss in the next chapter. See [Br] for more details.

Here is another way to define mod ℓ homotopy groups, and hence $K_*(\mathcal{C};\mathbb{Z}/\ell)$. Suppose that X is a loop space, and let F denote the homotopy fiber of the map $X \to X$ which is multiplication by ℓ.

PROPOSITION 9.6. *There are isomorphisms* $\pi_m(X;\mathbb{Z}/\ell) \cong \pi_{m-1}(F)$ *for all* $m \geq 2$.

PROOF. (Neisendorfer) Let $map(A,X)$ be the space of pointed maps. If $S = S^k$ is the k-sphere then the homotopy groups of $map(S^k,X)$ are the homotopy groups of X (reindexed by k), while if $P = P^k(\mathbb{Z}/\ell)$ is a mod ℓ Moore space, the homotopy groups of $map(P,X)$ are the mod ℓ homotopy groups of X (reindexed by k).

Now applying $map(-,X)$ to a cofibration sequence yields a fibration sequence, and applying $map(A,-)$ to a fibration sequence yields a fibration sequence; this may be formally deduced from the axioms (SM0) and (SM7) for any model structure, which hold for spaces. Applying $map(-,X)$ to $S^k \to S^k \to P^{k+1}(\mathbb{Z}/\ell)$ shows that $map(P,X)$ is the homotopy fiber of $map(S^k,X) \to map(S^k,X)$. Applying $map(S^k,-)$ to $F \to X \to X$ shows that $map(S^k,F)$ is also the homotopy fiber, and is therefore homotopy equivalent to $map(P,X)$. Taking the homotopy groups yields the result.

EXERCISES

9.1 Suppose that X is a loop space. Show that $\pi_1(X;\mathbb{Z}/\ell)$ is independent of the choice of Y such that $X \simeq \Omega Y$. This shows that $K_1(R;\mathbb{Z}/\ell)$ and even $K_1(\mathcal{C};\mathbb{Z}/\ell)$

9.2 Let R be a Dedekind domain with fraction field F. Show that the kernel of the map $K_1(R;\mathbb{Z}/\ell) \to K_1(F;\mathbb{Z}/\ell)$ is $SK_1(R)/\ell$. Hence it induces a natural map

$$_\ell\operatorname{Pic}(R) \xrightarrow{\rho} F^\times/F^{\times\ell}R^\times.$$

Note that $F^\times/R^\times$ is a free abelian group by I.3.6, so the target is a free $\mathbb{Z}/\ell$-module for every integer ℓ. Finally, use I.3.6 and I.3.8.1 to give an elementary description of ρ.

In particular, If R is the ring of integers in a number field F, the Bass-Milnor-Serre Theorem III.2.5 shows that the extension $K_1(R;\mathbb{Z}/\ell)$ of $_\ell\operatorname{Pic}(R)$ by $R^\times$ injects into $F^\times/F^{\times\ell}$.

INTRODUCTION AND REFERENCES

Algebraic K-theory has two components: the classical theory which centers around the Grothendieck group K_0 of a category and uses explicit algebraic presentations, and higher algebraic K-theory which requires topological or homological machinery to define.

There are three basic versions of the Grothendieck group K_0. One involves the group completion construction, and is used for projective modules over rings, vector bundles over compact spaces and other symmetric monoidal categories. Another adds relations for exact sequences, and is used for abelian categories as well as exact categories; this is the version first used in algebraic geometry. A third adds relations for weak equivalences, and is used for categories of chain complexes and other categories with cofibrations and weak equivalences ("Waldhausen categories").

Similarly, there are four basic constructions for higher algebraic K-theory: the +–construction (for rings), the group completion constructions (for symmetric monoidal categories), Quillen's Q-construction (for exact categories), and Waldhausen's wS. construction (for categories with cofibrations and weak equivalences). All these constructions give the same K-theory of a ring, but are useful in various distinct settings. These settings fit together like this:

$$\begin{tikzcd}
\boxed{\begin{array}{l}\text{number theory and}\\ \text{other classical topics}\end{array}} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \boxed{\text{algebraic geometry}} \arrow[d, leftrightarrow] \\
\boxed{\begin{array}{l}\text{Homological and}\\ +\text{-constructions}\\ K\text{-theory of rings}\end{array}} & \boxed{\begin{array}{l}Q\text{-construction: } K\text{-theory of}\\ \text{vector bundles on schemes,}\\ \text{exact categories, modules}\\ \text{and abelian categories}\end{array}} \arrow[l] \\
\boxed{\begin{array}{l}\text{Group Completions}\\ \text{relations to } L\text{-theory,}\\ \text{topological } K\text{-theory,}\\ \text{stable homotopy theory}\end{array}} \arrow[u] & \boxed{\begin{array}{l}\text{Waldhausen Construction:}\\ K\text{-theory of spaces,}\\ K\text{-theory of chain complexes}\\ \text{topological rings}\end{array}} \arrow[l] \arrow[u] \arrow[ul] \\
\boxed{\text{algebraic topology}} \arrow[u, leftrightarrow] \arrow[r, leftrightarrow] & \boxed{\text{geometric topology}} \arrow[u, leftrightarrow]
\end{tikzcd}$$

All the constructions have one feature in common: Some category C is concocted from the given setup, and one defines a K-theory space associated to the geometric realization BC of this category. The K-theory groups are then the homotopy groups of the K-theory space. In the first chapter, we introduce the basic cast of characters: projective modules and vector bundles (over a topological space, and over a scheme). Large segments of this chapter will be familiar to many readers, but which segments are familiar will depend upon the background and interests of the reader. The unfamiliar parts of this material may be skipped at first, and referred back to when relevant. We would like to warn the complacent reader that the material on the Picard group and Chern classes for topological vector bundles is in this first chapter.

In the second chapter, we define K_0 for all the settings in the above figure, and give the basic definitions appropriate to these settings: group completions for symmetric monoidal categories, K_0 for rings and topological spaces, λ-operations, abelian and exact categories, Waldhausen categories. All definitions and manipulations are in terms of generators and relations. Our philosophy is that this algebraic beginning is the most gentle way to become acquainted with the basic ideas of higher K-theory. The material on K-theory of schemes is isolated in a separate section, so it may be skipped by those not interested in algebraic geometry.

In the third chapter we give a brief overview of the classical K-theory for K_1 and K_2 of a ring. Via the Fundamental Theorem, this leads to Bass' "negative K-theory," meaning groups K_{-1}, K_{-2}, etc. We cite Matsumoto's presentation for K_2 of a field from [Milnor], and "Hilbert's Theorem 90 for K_2" (from chapter VI) in order to get to the main structure results. This chapter ends with a section on Milnor K-theory, including the transfer map, Izhboldin's theorem on the lack of p-torsion, the Galois symbol and the relation to the Witt ring of a field.

In the fourth chapter we shall describe the four constructions for higher K-theory. In the case of $\mathbf{P}(R)$, we show that all the constructions give the same K-groups, the groups $K_n(R)$. Very few theorems are present here, in order to keep this chapter short. We do not want to get involved in the technicalities lying just under the surface of each construction, so the key topological results we need are cited from the literature when needed.

The fundamental structural theorems for higher K-theory are presented in chapter 5?

In chapter 6 we apply the homological methods due to Suslin to describe the structure of the K-theory of fields. We also present the Merkurjev-Suslin material here.

In chapter 7 we intend to give several applications to Algebraic Geometry.

In chapter 8 we discuss Higher Chow Groups and Motivic Cohomology. This material is quite new, and I don't know how much I will say here.

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