COBORDISM OF KNOTS AND BLANCHFIELD DUALITY

C. KEARTON

An *n-knot* is a smooth submanifold k of the $(n+2)$-sphere S^{n+2}, such that k is homeomorphic to S^n. The knot k is *null-cobordant* if (S^{n+2}, k) is the boundary of a smooth ball pair (B^{n+3}, B^{n+1}). It is *doubly-null-cobordant* if (S^{n+2}, k) is an equatorial pair of the unknotted pair (S^{n+3}, S^{n+1}). M. A. Kervaire [6] has studied the case n even, and for n odd a necessary and sufficient condition for k to be null-cobordant has been given by J. Levine [7]. Conditions for k to be doubly-null-cobordant when n is odd have been given by D. W. Sumners [11] and C. Kearton [5]. These have all been in terms of the associated Seifert matrices, and it is the purpose of this paper to translate these conditions into an intrinsic form, in terms of the Blanchfield duality pairing. For details of this pairing the reader is referred to [1, 2, 3, 13].

1. If k is an n-knot, we denote by K the complement in S^{n+2} of some open tubular neighbourhood of k. $\tilde{K}$ denotes the (infinite cyclic) cover of K corresponding to the kernel of the Hurewicz map $\pi_1(K) \to H_1(K)$. By Alexander duality, $H_1(K) = (t :)$ the free group on one generator, and the orientations of S^{n+2} and k yield a preferred generator t. Let R denote the integral group ring of $(t :)$, written as the ring of Laurent polynomials in t with integer coefficients, and let R_0 be the field of fractions of R. Then $H_*(\tilde{K})$ is an R-module in each dimension. If $T_*(\tilde{K})$ is the Z-torsion submodule of $H_*(\tilde{K})$, then $M_*(\tilde{K}) = H_*(\tilde{K})/T_*(\tilde{K})$ is also an R-module.

For the case $n = 2q - 1$, we recall the following facts (compare [1, 2, 3, 10]).

(i) $M_q(\tilde{K})$ is a finitely-generated R-torsion-module.

(ii) $J : M_q(\tilde{K}) \to M_q(\tilde{K})$ is an isomorphism, where $Jx = (1-t)x$.

(iii) There is a non-singular $(-1)^{q+1}$-Hermitian pairing,

$$[,] : M_q(\tilde{K}) \times M_q(\tilde{K}) \to R_0/R.$$

The pairing in (iii) is due to Blanchfield [1].

If M is an R-torsion-module, define the *rank* of M to be the dimension of $M \otimes_Z Q$, regarded as a rational vector space. If M satisfies (i)-(iii), we shall say that $(M, [,])$ is *null-cobordant* if M has a sub-module N which is self-annihilating under $[,]$, such that rank $N = \frac{1}{2}$ rank M. $(M, [,])$ is *doubly-null-cobordant* if M is the direct sum of two such submodules.

Suppose that k is a $(2q-1)$-knot, and that V is a Seifert matrix of k, of order $2n$. Then, by [2], $M_q(\tilde{K})$ is presented as an R-module by $tV + (-1)^q V'$; i.e. there is an exact sequence

$$G \xrightarrow{d} F \xrightarrow{\phi} M_q(\tilde{K}) \to 0$$

where G and F are free R-modules of rank $2n$, and the map d is represented by $tV + (-1)^q V'$ with respect to some pair of bases of G and F. Either of these bases determines the other by a completely dual pairing; see [9] for details. If x and y

are column vectors representing elements of F with respect to the chosen basis, then the Blanchfield pairing is described by

$$[\phi x, \phi y] = x'(1-t)\left(tV + (-1)^q V'\right)^{-1} y \quad (\text{mod } R). \tag{1}$$

See [4] or [13] for details.

Following Trotter [13], we shall write $(M_V, [\ ,\ ]_V)$ for the R-module and pairing associated with a Seifert matrix V; thus $M_V = M_q(\tilde{K})$ and $[\ ,\ ]_V = [\ ,\ ]$ if V is a Seifert matrix of k.

Finally, recall that a Seifert matrix is *null-cobordant* (resp. *doubly-null-cobordant*) if it is congruent by an integer unimodular matrix to one of the form

$$\begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \left(\text{resp. } \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\right),$$

all the blocks being square.

2. THEOREM. *Let V be a Seifert matrix of a $(2q-1)$-knot. Then V is null-cobordant (S-equivalent to a doubly null-cobordant matrix) if and only if $(M_V, [\ ,\ ]_V)$ is null-cobordant (doubly-null-cobordant).*

Remark. For the definition of S-equivalence see [8] or [13].

Proof. By [12; p. 484], V is S-equivalent to a non-singular matrix U, and by [13] M_U is isomorphic to M_V by a map which preserves the Blanchfield pairing. If $(M_V, [\ ,\ ]_V)$ is null-cobordant, then consider the exact sequence

$$G \xrightarrow{d} F \xrightarrow{\phi} M_U \to 0$$

associated with U. As we remarked above, d is represented by the matrix $tU + (-1)^q U'$ with respect to some bases of G and F; let F' be the free abelian group generated by the latter basis. Then $\phi(F')$ spans $M_U \otimes Q$ as a vector space (*cf.* [9]). If N is the self-annihilating submodule of M_U, then $\phi^{-1}(N) \cap F'$ is a subgroup of F' of rank n, where $2n = \text{rank } M_U$. Let $\alpha_1, \ldots, \alpha_{2n}$ be a basis of F' such that $m_1 \alpha_{n+1}, \ldots, m_n \alpha_{2n}$ is a basis of $\phi^{-1}(N) \cap F'$, where each m_i is a non-zero integer.

Since

$$[\phi(m_i \alpha_{n+i}), \phi(m_j \alpha_{n+j})] \equiv 0 \quad (\text{mod } R)$$

for $1 \leq i, j \leq n$, we have $[\phi\alpha_{n+i}, \phi\alpha_{n+j}] \equiv \beta_{ij}/m_i m_j$ where $\beta_{ij} \in R$. If W is the Seifert matrix with respect to $\alpha_1, \ldots, \alpha_{2n}$ then from (1)

$$[\phi x, \phi y] \equiv x'(1-t) \frac{\text{adj}\left(tW + (-1)^q W'\right)}{\det\left(tW + (-1)^q W'\right)} y$$

and since $\det\left(W + (-1)^q W'\right) = \pm 1$ it follows that $m_i m_j \mid \beta_{ij}$ and so

$$[\phi\alpha_{n+i}, \phi\alpha_{n+j}] \equiv 0 \quad (\text{mod } R) \quad 1 \leq i, j \leq n. \tag{2}$$

Now the degree of $\det\left(tW + (-1)^q W'\right)$ is $2n$ and the degree of any element of $\text{adj}\left(tW + (-1)^q W'\right)$ is at most $2n-1$. Thus (2) and property (ii) imply that $\left(tW + (-1)^q W'\right)^{-1}$ has the form

$$\begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \text{ over } R_0,$$

all the blocks being square. Therefore $tW + (-1)^q W'$, and hence W, has the form

$$\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}.$$

But W is congruent to U, and so S-equivalent to V, and thus V is null-cobordant.

If $(M_V, [,]_V)$ is doubly-null-cobordant, a similar argument applies, except that V being S-equivalent to a doubly-null-cobordant matrix does not imply that V is doubly-null-cobordant.

The converse results are obtained by a straightforward computation.

COROLLARY 1. *Let k be a null-cobordant $(2q-1)$-knot. Then $\left(M_q(\tilde{K}), [,]\right)$ is null-cobordant.*

Proof. See [7; Lemma 2].

COROLLARY 2. *Let k be a $(2q-1)$-knot, $q \geqslant 2$. If $\left(M_q(\tilde{K}), [,]\right)$ is null-cobordant, then so is k.*

Proof. See [7; Lemmas 4, 5].

COROLLARY 3. *Let k be a doubly-null-cobordant $(2q-1)$-knot. Then $\left(M_q(\tilde{K}), [,]\right)$ is doubly-null-cobordant.*

Proof. See [11; Theorem 2.3].

COROLLARY 4. *Let k be a simple $(2q-1)$-knot, $q \geqslant 2$. If $\left(M_q(\tilde{K}), [,]\right)$ is doubly-null-cobordant, then so is k.*

Proof. See [5, 11].

Added in proof: Part of the theorem above can also be deduced from results in the paper by M. A. Kervaire in *Manifolds* (Amsterdam, 1970).

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Corpus Christi College,
    Cambridge