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Obstructions to Embedding and Isotopy in the Metastable Range

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0. Introduction

Let $f:(M,\partial M)\to(Q,\partial Q)$ be a piecewise linear map from a manifold of dimension m to one of dimension q. Assuming that $3(m+1)<2q$, and that certain connectivity conditions are satisfied, this paper describes an obstruction to homotoping $f\operatorname{rel}\partial M$ to an embedding.

If $f,g:(M,\partial M)\to(Q,\partial Q)$ are embeddings which agree on ∂M and are homotopic $\operatorname{rel}\partial M$ by a homotopy F, then under similar conditions to those mentioned above, the theory yields an obstruction $d(f,g;F)$ to isotoping f to $g\operatorname{rel}\partial M$. The drawback to this obstruction is its dependence on the homotopy F; examples are given where this can be overcome.

Finally, smoothing theory is used to prove the main results in the smooth category.

1. Preliminaries

The stable r-stem of the sphere will be denoted by π^r; thus $\pi^r\cong\pi_{n+r}(S^n)$ for sufficiently large values of n. Let π be any group, and for $\alpha\in\pi$ let A^α be a $k\times k$ matrix with entries in π^r. Then a (π,π^r)-matrix of order k is a formal sum $A=\sum_{\alpha\in\pi}\alpha A^\alpha$, with $A^\alpha=0$ for all but finitely many α. Thus the ijth entry of A is $A_{ij}=\sum_{\alpha\in\pi}\alpha A^\alpha_{ij}$.

Two such matrices A, B, will be called *equivalent* ($A\sim B$) if there exist $\gamma_1,\dots,\gamma_k\in\pi$ such that

$$A^\alpha_{ij}=B^{\gamma_i^{-1}\alpha\gamma_j}_{ij},\qquad \forall\alpha\in\pi,\qquad 1\leqq i\leqq k,\quad 1\leqq j\leqq k.$$

If A is a (π,π^r)-matrix, then the conjugate transpose $\bar{A}'$ of A is defined by

$$\bar{A}'=\sum_{\alpha\in\pi}\alpha^{-1}(A^\alpha)'$$

where $(A^\alpha)'$ is the transpose of A^α.

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We shall work in the PL category throughout. A map $f: M \to Q$ of one manifold to another is *proper* if it is a map of triads $f:(M, \partial M, \operatorname{int} M) \to (Q, \partial Q, \operatorname{int} Q)$. If f is a proper map, then a homotopy of f rel ∂M is a homotopy through proper maps which is fixed on ∂M. A homotopy of $f \bmod \partial M$ is a homotopy through proper maps, but allows movement on the boundary. A homotopy of f Rel ∂M is a homotopy of $f \bmod \partial M$ which is an isotopy on the boundary.

If π is the fundamental group of Q, and $\tilde{Q}$ the universal cover of Q, then π acts on $\tilde{Q}$ as the group of covering transformations. The group action will be written on the left; thus if $\alpha, \beta \in \pi$ and $x \in \tilde{Q}$, then $\alpha x \in \tilde{Q}$ and $\alpha(\beta x) = (\alpha\beta)x$. If $f: M \to Q$ is a map, and $f^e: M \to \tilde{Q}$ is a lift of f, then $f^\alpha: M \to \tilde{Q}$ is defined by

$$f^\alpha(x) = \alpha(f^e(x)) \qquad \forall x \in M.$$

Here e denotes the identity element of π.

All manifolds will be oriented.

Suppose that for $i = 1, 2$ we have proper embeddings $f_i: B_i^m \to B^q$, whose images are disjoint on the boundary, where B^n denotes the n-dimensional ball. Assume that the three balls are oriented, and that the following inequalities hold: $m \geqq 3$, $q - m \geqq 3$, $3m - 2q + 2 < 0$.

Then $\partial B^q - f_2(\partial B_2^m)$ has the homotopy type of S^{q-m-1}, since spheres unknot in codimension 3 or more. Moreover, S^{q-m-1} inherits an orientation from ∂B^q and ∂B_2^m, via the duality isomorphisms of Alexander and Poincaré. Thus $f_1 | \partial B_1^m$ induces an element of $\pi_{m-1}(S^{q-m-1}) \cong \pi^{2m-q}$ since $m - 1 < 2(q - m - 1) - 1$.

Define $L(f_1, f_2)$ to be this element of π^{2m-q}. Then the following result is a consequence of [4], Lemma 5.1.

Lemma 1.1. $L(f_1, f_2) = (-1)^{m+q} L(f_2, f_1)$.

2. Embedding rel the Boundary

Theorem 2.1. *Let M be a compact m-dimensional manifold with k components, each of which is s-connected; and let Q be a compact connected q-dimensional manifold with fundamental group π and $\pi_i(Q) = 0$ for $1 < i \leqq t$. Let $f: M \to Q$ be a proper map which is an embedding on the boundary. Assume that the following inequalities are satisfied: $s \geqq 1$, $m \geqq 3$, $q - m \geqq 3$, $2m - q < s$, $2m - q + 1 < t$, $3m - 2q + 3 < 0$.*

Then f determines, up to equivalence, a (π, π^{2m-q})-matrix A of order k, satisfying $A = (-1)^{m+q} \bar{A}'$, and with $A_{ii}^e = 0$ for all i.

The equivalence class of A depends only on the homotopy class of f rel ∂M, and $A = 0$ iff f can be homotoped rel ∂M to an embedding.

The proof of this theorem will occupy the rest of the section.

Let M_i denote the ith component of M, and set $f_i = f | M_i$. Since $s \geqq 1$, M_i is simply-connected and f lifts to $\tilde{Q}$; let f^e denote some fixed lift of f. Then as in the preceding section, f^α is a lift of f, for each $\alpha \in \pi$. Set $f_i^\alpha = f^\alpha | M_i$. Note that $f^\alpha | \partial M$ is an embedding, and that $f^\alpha(\partial M) \cap f^\beta(\partial M) = \emptyset$ unless $\alpha = \beta$.

Homotop f, rel ∂M, to a proper map in general position. Then the singular set, $S(f)$, of f has dimension at most $2m - q$. There are no triple points, because $3m - 2q < 0$.

Lemma 2.2. *f can be homotoped*, $\operatorname{rel}\partial M$, *so that f_i^α is an embedding for each i and each α.*

Proof. For each i, consider the map $f_i^e: M_i \to \tilde{Q}$. By Irwin's theorem, there is a homotopy, $\operatorname{rel}\partial M_i$, taking f_i^e to an embedding. Composing with the projection $p:\tilde{Q}\to Q$ gives a homotopy of f_i for each i, and hence of f. □

For each $(i,\alpha)\neq(j,\beta)$, define $T_{i,j}^{\alpha,\beta}\subset \operatorname{int} M_i$ by

$$T_{i,j}^{\alpha,\beta}=(f_i^\alpha)^{-1}[\operatorname{Im} f_i^\alpha \cap \operatorname{Im} f_j^\beta].$$

Of course, $T_{i,j}^{\alpha,\beta}$ is a subset of $S(f)$, and so has dimension at most $2m-q$.

Lemma 2.3. $T_{i,j}^{\alpha,\beta}\cap T_{i',j'}^{\alpha',\beta'}\neq\emptyset$

$$\Rightarrow\ (i,\gamma\alpha)=(i',\alpha') \quad \textit{for some} \quad \gamma\in\pi$$

$$(j,\gamma\beta)=(j',\beta')$$

$$\Rightarrow\ T_{i,j}^{\alpha,\beta}=T_{i',j'}^{\alpha',\beta'}.$$

Proof. For the intersection to be non-empty, we must have $i=i'$, for otherwise $M_i\cap M_{i'}=\emptyset$. If $j=i$, then $\alpha\neq\beta$ by definition, and since f has no triple points, $j'=j=i$. If $j\neq i$, then $j'=j$ again because f has no triple points. Thus $i=i'$ and $j=j'$.

Define $\gamma\in\pi$ by $\gamma\alpha=\alpha'$. If $x\in T_{i,j}^{\alpha,\beta}\cap T_{i,j}^{\gamma\alpha,\beta'}$, then

$$f_i^\alpha(x)=f_j^\beta(y) \quad \text{for some} \quad y\in M_j$$

$$f_i^{\gamma\alpha}(x)=f_j^{\beta'}(y') \quad \text{for some} \quad y'\in M_j.$$

Therefore $f_j^{\gamma\beta}(y)=\gamma f_j^\beta(y)=\gamma f_i^\alpha(x)=f_i^{\gamma\alpha}(x)=f_j^{\beta'}(y')$. Now $x=y$ implies that $i=j$, and $f_i^\alpha(x)=f_i^\beta(x)$, which is impossible as $\alpha\neq\beta$ in this case. Similarly $x=y'$ implies that $i=j$, $f_i^{\gamma\alpha}(x)=f_i^{\beta'}(x)$, $\beta'=\gamma\alpha=\alpha'$ which is impossible. Thus $y=y'$, as f has no triple points, and so $f_j^{\beta'}(y)=f_j^{\gamma\beta}(y)$ and hence $\beta'=\gamma\beta$.

This proves the first implication; the second implication is trivial. □

Lemma 2.4. $S(f_i)=\bigcup_{\beta\in\pi-\{e\}} T_{i,i}^{e,\beta}$

$$S(f)=\bigcup_{(i,e)\neq(j,\beta)} T_{i,j}^{e,\beta}$$

the unions being disjoint.

Proof. Immediate from Lemma 2.3. □

Lemma 2.5. *Only finitely many $T_{i,j}^{e,\beta}$ are non-empty.*

Proof. In any given triangulation, $S(f)$ is a finite polyhedron. □

Lemma 2.6. *For each (i,α), (j,β) with $(i,\alpha)\neq(j,\beta)$ and $T_{i,j}^{\alpha,\beta}\neq\emptyset$, there exists an m-ball $B_{i,j}^{\alpha,\beta}\subset\operatorname{int}M_i$ and a q-ball $A_{i,j}^{\alpha,\beta}\subset\operatorname{int}Q$ with the following properties.*

$$B_{i,j}^{\alpha,\beta}=B_{i,j}^{\gamma\alpha,\gamma\beta} \quad \textit{for all} \quad \gamma\in\pi$$

$$A_{i,j}^{\alpha,\beta}=A_{i,j}^{\gamma\alpha,\gamma\beta}$$

$$A_{i,j}^{\alpha,\beta}=A_{j,i}^{\beta,\alpha}.$$

The $B_{i,j}^{\alpha,\beta}$ are identical or disjoint.
The $A_{i,j}^{\alpha,\beta}$ are identical or disjoint.
There are only finitely many $B_{i,j}^{e,\beta}$ and $A_{i,j}^{e,\beta}$.

$$f^{-1}(A_{i,j}^{\alpha,\beta}) = B_{i,j}^{\alpha,\beta} \cup B_{j,i}^{\beta,\alpha}$$

$$S(f|B_{i,j}^{\alpha,\beta} \cup B_{j,i}^{\beta,\alpha}) = T_{i,j}^{\alpha,\beta} \cup T_{j,i}^{\beta,\alpha}$$

$$f|B_{i,j}^{\alpha,\beta} \cup B_{j,i}^{\beta,\alpha} : B_{i,j}^{\alpha,\beta} \cup B_{j,i}^{\beta,\alpha} \to A_{i,j}^{\alpha,\beta}$$

is a proper map which is an embedding on the boundary.

Proof. Consider those $T_{i,j}^{e,\beta}$ which are non-empty. By [2], Theorem 7.7, there exists for each $T_{i,j}^{e,\beta}$ a compact PL subspace $C_{i,j}^{e,\beta}$ such that

$$C_{i,j}^{e,\beta} \subset (\operatorname{int} M - S(f)) \cup T_{i,j}^{e,\beta}, \qquad C_{i,j}^{e,\beta} \searrow \text{point}$$

$$C_{i,j}^{e,\beta} \supset T_{i,j}^{e,\beta}, \qquad \dim C_{i,j}^{e,\beta} \leqq 2m - q + 1 .$$

By general position, we can ambient isotop the $C_{i,j}^{e,\beta}$ rel $T_{i,j}^{e,\beta}$ so that they are pairwise disjoint: this requires the inequality $2(2m-q+1)-q<0$, which is implied by the hypotheses of the theorem.

Similarly, there exist $D_{i,j}^{e,\beta}$ such that

$$D_{i,j}^{e,\beta} \subset \operatorname{int} \tilde{Q}, \qquad D_{i,j}^{e,\beta} \searrow \text{point}$$

$$D_{i,j}^{e,\beta} \supset f_i^e(C_{i,j}^{e,\beta}) \cup f_j^\beta(C_{j,i}^{e,\beta^{-1}}), \qquad \dim D_{i,j}^{e,\beta} \leqq 2m - q + 2 .$$

Note that

$$\beta D_{j,i}^{e,\beta^{-1}} \supset \beta f_j^e(C_{j,i}^{e,\beta^{-1}}) \cup \beta f_i^{\beta^{-1}}(C_{i,j}^{e,\beta}) = f_j^\beta(C_{j,i}^{e,\beta^{-1}}) \cup f_i^e(C_{i,j}^{e,\beta}),$$

so that we may take $D_{i,j}^{e,\beta} = \beta D_{j,i}^{e,\beta^{-1}}$.

By general position, we may arrange that the $D_{i,j}^{e,\beta}$ are pairwise disjoint (or identical), and that

$$\dim\left[(D_{i,j}^{e,\beta} - f_i^e(C_{i,j}^{e,\beta}) \cup f_j^\beta(C_{j,i}^{e,\beta^{-1}})) \cap \left(\bigcup_{\gamma\in\pi} \operatorname{Im} f^\gamma\right)\right]$$

$$\leqq 2m - q + 2 + m - q$$

$$= 3m - 2q + 2 < 0 .$$

Thus $C_{i,j}^{e,\beta} = (f_i^e)^{-1} D_{i,j}^{e,\beta}$ and $C_{j,i}^{e,\beta^{-1}} = (f_j^\beta)^{-1} D_{i,j}^{e,\beta}$.

For fixed i, j, β, set $D = D_{i,j}^{e,\beta}$ and $C = f_i^e(C_{i,j}^{e,\beta}) \cup f_j^\beta(C_{j,i}^{e,\beta^{-1}})$. Let $v: D_0 \to D$ be a homeomorphism, and $C_0 = v^{-1}C$. Recall that $p: \tilde{Q} \to Q$ is the projection, and note $S(pv) \cap C_0 = \emptyset$ since f has no triple points. Regarding v as $v: D_0 \to \tilde{Q}$, we may by general position homotop v, rel C_0, so that $\dim S(pv) \leqq 2(2m-q+2)-q<0$ by the hypotheses of the theorem. Thus $pv: D_0 \to Q$ is an embedding, and so $p|D$ is an embedding.

Let $B_{i,j}^{e,\beta}$ be a second derived neighbourhood of $C_{i,j}^{e,\beta}$ in some triangulation of M, and $A_{i,j}^{e,\beta}$ a second derived neighbourhood of $pD_{i,j}^{e,\beta}$ in a corresponding triangulation of Q; that is, a triangulation compatible with f.

For each $(i,\alpha) \neq (j,\beta)$, set $B_{i,j}^{\alpha,\beta} = B_{i,j}^{e,\alpha^{-1}\beta}$, and $A_{i,j}^{\alpha,\beta} = A_{i,j}^{e,\alpha^{-1}\beta}$. These are the required balls, and their properties follow at once from those of $C_{i,j}^{e,\beta}$ and $D_{i,j}^{e,\beta}$. □

We are now in a position to define the matrix A. Set

$$A_{i,j}^{\beta}=L(f|B_{i,j}^{e,\beta}, f|B_{j,i}^{\beta,e})$$

where each map is a proper embedding into $A_{i,j}^{e,\beta}$. When $T_{i,j}^{e,\beta}=\emptyset$, and therefore $T_{j,i}^{\beta,e}=\emptyset$, we define $A_{i,j}^{\beta}$ to be $0\in\pi^{2m-q}$.

Lemma 2.7. $\bar{A}'=(-1)^{m+q}A$.

Proof. $A_{j,i}^{\beta}$ is the homotopy class of

$$f|\partial B_{j,i}^{e,\beta}:\partial B_{j,i}^{e,\beta}\to\partial A_{j,i}^{e,\beta}-f(\partial B_{i,j}^{\beta,e})$$

which is the same as that of

$$f|\partial B_{j,i}^{\beta^{-1},e}:\partial B_{j,i}^{\beta^{-1},e}\to\partial A_{j,i}^{\beta^{-1},e}-f(\partial B_{i,j}^{e,\beta^{-1}})$$

by Lemma 2.6. By Lemma 1.1 this is $(-1)^{m+q}$ times the homotopy class of

$$f|\partial B_{i,j}^{e,\beta^{-1}}:\partial B_{i,j}^{e,\beta^{-1}}\to\partial A_{j,i}^{\beta^{-1},e}-f(\partial B_{j,i}^{\beta^{-1},e}).$$

Since $A_{j,i}^{\beta^{-1},e}=A_{i,j}^{e,\beta^{-1}}$, this is $A_{i,j}^{\beta^{-1}}$, and so $A_{j,i}^{\beta}=(-1)^{m+q}A_{i,j}^{\beta^{-1}}$. □

Recall that we made an arbitrary choice of the lifts f_i^e.

Lemma 2.8. *Suppose that using another choice of lifts, $f_i'^e=f_i^{\gamma_i}$, the matrix B is obtained in place of A. Then $B_{i,j}^{\beta}=A_{i,j}^{\gamma_i^{-1}\beta\gamma_j}$ for all i,j,β.*

Proof. Let $'$ distinguish the sets corresponding to the new choice of lifts $f_i'^e$. Then

$$\begin{aligned}T_{i,j}'^{\alpha,\beta}&=(f_i'^{\alpha})^{-1}[\operatorname{Im}f_i'^{\alpha}\cap\operatorname{Im}f_j'^{\beta}]\\&=(f_i^{\alpha\gamma_i})^{-1}[\operatorname{Im}f_i^{\alpha\gamma_i}\cap\operatorname{Im}f_j^{\beta\gamma_j}]\\&=T_{i,j}^{\alpha\gamma_i,\beta\gamma_j},\end{aligned}$$

and so $B_{i,j}'^{\alpha,\beta}=B_{i,j}^{\alpha\gamma_i,\beta\gamma_j}$.

$B_{i,j}^{\beta}$ is the homotopy class of $f|\partial B_{i,j}'^{e,\beta}:\partial B_{i,j}'^{e,\beta}\to\partial A_{i,j}'^{e,\beta}-f(\partial B_{j,i}'^{\beta,e})$, which is the same as that of

$$f|\partial B_{i,j}^{\gamma_i,\beta\gamma_j}:\partial B_{i,j}^{\gamma_i,\beta\gamma_j}\to\partial A_{i,j}^{\gamma_i,\beta\gamma_j}-f(\partial B_{j,i}^{\beta\gamma_j,\gamma_i}),$$

$$f|\partial B_{i,j}^{e,\gamma_i^{-1}\beta\gamma_j}:\partial B_{i,j}^{e,\gamma_i^{-1}\beta\gamma_j}\to\partial A_{i,j}^{e,\gamma_i^{-1}\beta\gamma_j}-f(\partial B_{j,i}^{\gamma_i^{-1}\beta\gamma_j,e}).$$

Whence the result. □

In other words, f determines A up to equivalence, modulo our choice of $C_{i,j}^{e,\beta}$, etc.

Now suppose that $F:M\times I\to Q\times I$ is a homotopy of f rel ∂M, with ${}_0F=f$ and ${}_1F=g$. Choose a lift $F^e:M\times I\to\tilde{Q}\times I$ so that ${}_0F^e=f^e$, and let $g^e={}_1F^e$. Note that ${}_tF=F|M\times t$.

First note that F can be homotoped rel $\partial(M\times I)$ so that F_i^{α} is an embedding for each (i,α). This follows from Lemma 2.2, replacing f by F, M by $M\times I$, and Q by $Q\times I$. The necessary inequalities are implied by the hypotheses of the theorem.

Lemma 2.9. *If A is the matrix given by f, and B that given by g, then $A=B$.*

Proof. Let ${}_\varepsilon C_{i,j}^{\alpha,\beta}$, ${}_\varepsilon D_{i,j}^{\alpha,\beta}$ be the sets arising in the proof of Lemma 2.6 for ${}_\varepsilon F$, $\varepsilon=0,1$.

If ${}_\varepsilon C_{i,j}^{e,\beta}\neq\emptyset$ for $\varepsilon=0,1$, join ${}_0C_{i,j}^{e,\beta}$ to a point $(x,0)\in\partial M\times 0$, ${}_1C_{i,j}^{e,\beta}$ to $(x,1)\in\partial M\times 1$, and let $C_{i,j}^{e,\beta}$ be the union of these two sets with $x\times I\subset\partial M\times I$. We may assume that the arcs are embedded and disjoint for different i, j, β, and do not meet $S(F)$. Let $D^{e,\beta}_{i,j}$ be similarly defined.

For $(i,\alpha) \neq (j,\beta)$, define

$$S^{\alpha,\beta}_{i,j} = (F^\alpha_i)^{-1}(\operatorname{Im} F^\alpha_i \cap \operatorname{Im} F^\beta_j) \subset (\operatorname{int} M_i) \times I .$$

By general position we may assume that $\dim S^{\alpha,\beta}_{i,j} \leqq 2m - q + 1$, and that F has no triple points as $3(m+1) - 2(q+1) = 3m - 2q + 1 < 0$.

As in Lemma 2.6, using [2], Lemma 7.8, we can find sets $E^{e,\beta}_{i,j}$, $G^{e,\beta}_{i,j}$ satisfying

$$E^{e,\beta}_{i,j} \subset (M \times I - S(F)) \cup S^{e,\beta}_{i,j}, \qquad E^{e,\beta}_{i,j} \searrow E^{e,\beta}_{i,j} \cap \partial(M \times I) = C^{e,\beta}_{i,j},$$

$$E^{e,\beta}_{i,j} \supset C^{e,\beta}_{i,j} \cup S^{e,\beta}_{i,j}, \qquad \dim E^{e,\beta}_{i,j} \leqq 2m - q + 2,$$

$$G^{e,\beta}_{i,j} \subset \tilde{Q} \times I, \qquad G^{e,\beta}_{i,j} \searrow G^{e,\beta}_{i,j} \cap \partial(\tilde{Q} \times I) = D^{e,\beta}_{i,j},$$

$$G^{e,\beta}_{i,j} = \beta G^{e,\beta^{-1}}_{j,i}, \qquad \dim G^{e,\beta}_{i,j} \leqq 2m - q + 3,$$

$$G^{e,\beta}_{i,j} \supset F^e_i(E^{e,\beta}_{i,j}) \cup F^\beta_j(E^{e,\beta^{-1}}_{j,i}).$$

This requires the inequality

$$\begin{aligned} 2m - q + 1 &\leqq \min(s, m + 1 - (2(m+1) - q - 1) - 2) \\ &= \min(s, q - m - 2), \end{aligned}$$

which is satisfied under the hypotheses of the theorem.

By general position we may move $G^{e,\beta}_{i,j}$ keeping $X^\beta_{i,j} = F^e_i(E^{e,\beta}_{i,j}) \cup F_j(E^{e,\beta^{-1}}_{j,i})$ fixed, so that

$$\begin{aligned} \dim\left[(G^{e,\beta}_{i,j} - X^\beta_{i,j}) \cap \left(\bigcup_{\gamma \in \pi} \operatorname{Im} F^\gamma\right)\right] &\leqq (2m - q + 3) + (m+1) - (q+1) \\ &= 3m - 2q + 3 < 0 . \end{aligned}$$

Thus $E^{e,\beta}_{i,j} = (F^e_i)^{-1} G^{e,\beta}_{i,j}$, $E^{e,\beta^{-1}}_{j,i} = (F^\beta_j)^{-1} G^{e,\beta}_{i,j}$. As in Lemma 2.2, we can arrange for $p|G^{e,\beta}_{i,j}$ to be an embedding.

By the nature of $C^{e,\beta}_{i,j}$ we can arrange that near $\partial M \times I$, $E^{e,\beta}_{i,j}$ has the form $I \times I$, with $(I \times I) \cap (\partial M \times I) = 0 \times I$ and $(I \times I) \cap (M \times \partial I) = I \times \partial I$. A similar arrangement can be made for $pG^{e,\beta}_{i,j}$. Excising $[0,1) \times I$ in each case, and taking second derived neighbourhoods in some triangulation, we obtain $(m+1)$-balls $K^{e,\beta}_{i,j} \subset M \times I$ and $(q+1)$-balls $L^{e,\beta}_{i,j} \subset Q \times I$. For each $(i,\alpha) \neq (j,\beta)$, set $K^{\alpha,\beta}_{i,j} = K^{e,\alpha^{-1}\beta}_{i,j}$ and $L^{\alpha,\beta}_{i,j} = L^{e,\alpha^{-1}\beta}_{i,j}$. These balls have the following properties.

$$K^{\alpha,\beta}_{i,j} = K^{\gamma\alpha,\gamma\beta}_{i,j}, \qquad L^{\alpha,\beta}_{i,j} = L^{\gamma\alpha,\gamma\beta}_{i,j} = L^{\beta,\alpha}_{j,i},$$

$$F^{-1}(L^{\alpha,\beta}_{i,j}) = K^{\alpha,\beta}_{i,j} \cup K^{\beta,\alpha}_{j,i},$$

$$K^{\alpha,\beta}_{i,j} \cap \partial(M \times I) = {}_0B^{\alpha,\beta}_{i,j} \cup {}_1B^{\alpha,\beta}_{i,j},$$

$$L^{\alpha,\beta}_{i,j} \cap \partial(Q \times I) = {}_0A^{\alpha,\beta}_{i,j} \cup {}_1A^{\alpha,\beta}_{i,j},$$

$$S(F|K^{\alpha,\beta}_{i,j} \cup K^{\beta,\alpha}_{j,i}) = S^{\alpha,\beta}_{i,j} \cup S^{\beta,\alpha}_{j,i} \subset (\operatorname{int} M) \times I,$$

$$F|K^{\alpha,\beta}_{i,j} \cup K^{\beta,\alpha}_{j,i} : K^{\alpha,\beta}_{i,j} \cup K^{\beta,\alpha}_{j,i} \to L^{\alpha,\beta}_{i,j}$$

is a proper map.

Now observe that F restricts to a map

$$cl[\partial K_{i,j}^{e,\beta}-({}_0B_{i,j}^{e,\beta}\cup{}_1B_{i,j}^{e,\beta})]\to cl[\partial L_{i,j}^{e,\beta}-({}_0A_{i,j}^{e,\beta}\cup{}_1A_{i,j}^{e,\beta})]-F(K_{j,i}^{\beta,e})\,;$$

this is a homotopy which implies that $A_{i,j}^{\beta}=B_{i,j}^{\beta}$.

A modification of the argument above works when ${}_\varepsilon C_{i,j}^{e,\beta}=\emptyset$ for either $\varepsilon=0$ or 1. □

The proof of the theorem is now almost complete. Clearly if f is homotopic $\mathrm{rel}\,\partial M$ to an embedding, then $A=0$. Conversely, if $A=0$, then for each $(i,e)\neq(j,\beta)$ we can homotop $f|B_{i,j}^{e,\beta}\cup B_{j,i}^{\beta,e}$ to an embedding $\mathrm{rel}\,\partial(B_{i,j}^{e,\beta}\cup B_{j,i}^{\beta,e})$, and hence homotop $f\,\mathrm{rel}\,\partial M$ to an embedding.

The theorem can be strengthened as follows.

Proposition 2.10. *The matrix A is determined up to equivalence by the homotopy class of f $\mathrm{Rel}\,\partial M$.*

Proof. If F is a homotopy from f to g $\mathrm{Rel}\,\partial$, then F may be written as a product of homotopies G, H, where G is an isotopy on a collar neighbourhood of ∂M and the identity elsewhere, and H is fixed on the boundary. Then G preserves A, as the singular set is unaffected, and H preserves A by the theorem. □

3. Connected Sums

Assume that M, Q, and f satisfy the hypotheses of Theorem 2.1, and that M has an extra component M_0. For $2\leqq i\leqq k$, set $N_i=M_i$ and $g_i=f_i$. Let $N'=M_0\cup(I\times B^m)\cup M_1$, with $M_i\cap(I\times B^m)=i\times B^m\subset\mathrm{int}\,M_i$, for $i=0,1$. Clearly

$$N_1=(M_0-0\times B^m)\cup(I\times\partial B^m)\cup(M_1-1\times B^m)\cong M_0\# M_1\,,$$

the interior connected sum. Define $h:N'\to Q$ so that $(h|M_0)^e=f_0^e$ and $(h|M_1)^e=f_1^e$, and let $g_1:N_1\to Q$ be the restriction of h to N_1. I claim that g_1 is determined up to homotopy $\mathrm{rel}\,\partial N_1$. For let $r:N'\to M_0\cup I\times 0\cup M_1$ be a retraction. Then $h\simeq h_0r\,\mathrm{rel}\,M_0\cup M_1$, and h_0r is determined up to homotopy $\mathrm{rel}\,M_0\cup M_1$ by $h|I\times 0$, which is determined up to homotopy $\mathrm{rel}\,\partial I\times 0$ by $f_0^e(0\times 0)$ and $f_1^e(1\times 0)$. This establishes the claim, and so g is determined up to homotopy $\mathrm{rel}\,\partial N$.

We can choose g_1^e to agree with f_0^e and f_1^e on $N_1\cap M_0$ and $N_1\cap M_1$ respectively, and we choose $g_i^e=f_i^e$ for $2\leqq i\leqq k$.

Proposition 3.1. *With the choice of lifts above, let A be the obstruction to homotoping $f\,\mathrm{rel}\,\partial M$ to an embedding, and B the obstruction to homotoping $g\,\mathrm{rel}\,\partial N$ to an embedding. Then for each $\beta\in\pi$,*

$$B_{i,j}^{\beta}=A_{i,j}^{\beta}\,,\quad 2\leqq i\leqq k\,,\quad 2\leqq j\leqq k\,,$$

$$B_{1,j}^{\beta}=A_{1,j}^{\beta}+A_{0,j}\,,\quad 2\leqq j\leqq k\,,$$

$$B_{1,1}^{\beta}=A_{0,0}^{\beta}+A_{0,1}^{\beta}+A_{1,0}^{\beta}+A_{1,1}^{\beta}\,,\quad \beta\neq e\,.$$

Proof. Note that $S(g)\subset(N_1\cap M_0)\cup(N_1\cap M_1)\cup\bigcup_{i=2}^{k}N_i$, so that $S(g)$ is the union of the $T_{i,j}^{e,\beta}$ determined by f. For $i\geqq 2$ and $j\geqq 2$, these are the same as those determined by g, and so the first assertion follows.

For $j \geqq 2$, $T_{0,j}^{e,\beta}$ and $T_{1,j}^{e,\beta}$ lie in N_1, and if they are non-empty then we may join them by an embedded arc x_j^β which apart from its endpoints misses all the $C_{i,k}^{e,\alpha}$ of Lemma 2.6. Similarly, $T_{j,0}^{e,\beta}$ and $T_{j,1}^{e,\beta}$ lie in N_j, and we may join them by an embedded arc y_j^β which apart from its endpoints misses all the $C_{i,k}^{e,\alpha}$. Moreover, we can by general position arrange that the arcs are mutually disjoint, and we can also choose the endpoints of x_j^β so that their images under g coincide with those of $y_j^{\beta^{-1}}$.

Now $g_1^e(x_j^\beta) \cup g_j(y_j^{\beta^{-1}})$ is an embedded loop in $\tilde{Q}$, and hence spans a 2-disc H_j^β; by general position we may assume that H_j^β is embedded and misses all the $D_{i,k}^{\alpha,\gamma}$ of Lemma 2.6 except for the endpoints of the two arcs which make up ∂H_j^β.

Then $T_{0,j}^{e,\beta} \cup x_j^\beta \cup T_{1,j}^{e,\beta}$, $T_{j,0}^{e,\beta^{-1}} \cup y_j^{\beta^{-1}} \cup T_{j,1}^{e,\beta^{-1}}$, and $D_{0,j}^{e,\beta} \cup H_j^\beta \cup D_{1,j}^{e,\beta}$ are all collapsible. Taking regular neighbourhoods as in Lemma 2.6 we obtain m-balls and q-balls for g, and the second assertion follows by homotopy addition.

Finally, for $\beta \neq e$, note that

$$(g_1^e)^{-1}[\operatorname{Im} g_1^e \cap \operatorname{Im} g_1^\beta] = T_{0,0}^{e,\beta} \cup T_{0,1}^{e,\beta} \cup T_{1,0}^{e,\beta} \cup T_{1,1}^{e,\beta},$$

with a similar result for $(g_1^\beta)^{-1}[\operatorname{Im} g_1^e \cap \operatorname{Im} g_1^\beta]$. An adaptation of the argument above yields the final assertion. □

Proposition 3.2. *Let M, Q, and f satisfy the hypotheses of Theorem 2.1, with obstruction matrix A with respect to some choice of lifts. Let $h: S^{q-m} \to Q$ be an embedding such that $\operatorname{Im} h \cap \operatorname{Im} f = \operatorname{Im} h \cap \operatorname{Im} f_n$ is a single point, and the intersection is transverse. Let the following be given: $a \in \pi^{2m-q}$, $\alpha \in \pi$, such that $(n, \alpha) \neq (1, e)$.*

Then there is a proper map $g: M \to Q$, agreeing with f on ∂M, with obstruction matrix B such that $B = A$ except for $B_{1,n}^\alpha$ and $B_{n,1}^{\alpha^{-1}}$. Furthermore, $B_{1,n}^\alpha = A_{1,n}^\alpha + a$, which determines B.

Proof. Let $x = \operatorname{Im} h \cap \operatorname{Im} f$; by taking regular neighbourhoods in a suitable triangulation, there are balls $B^{q-m} \subset S^{q-m}$, $B^m \subset M_1$, $B^q \subset Q$ such that

$$h|B^{q-m}: B^{q-m} \to B^{q-m} \times B^m = B^q$$

and $f|B^m: B^m \to B^{q-m} \times B^m = B^q$ are the standard embeddings.

Let $c: S^{m-1} \to S^{q-m-1}$ represent $a \in \pi^{2m-q} \cong \pi_{m-1}(S^{q-m-1})$, and define $f_0: S^m \to Q$ by $f_0 = h_0 Sc$ where Sc is the suspension of c. Note that $\pi^{2m-q} \cong \pi_{m-1}(S^{q-m-1})$ because $m-1 < 2(q-m-1)-1$ is guaranteed by the hypotheses of Theorem 2.1. Now $h_0 c: S^{m-1} \to B^q - \operatorname{Im} f \cong S^{q-m-1} \times B^m$ can be homotoped to an embedding by Irwin's theorem [3], so by the Alexander trick f_0 can be homotoped to an embedding. Setting $M_0 = S^m$, we have a proper map $f_0 \cup f: M_0 \cup M \to Q$ which satisfies the hypotheses of f in Proposition 3.1. By choosing an appropriate lift for f_0^e, we can arrange that $A_{0,n}^\alpha = a$ and $A_{0,j}^\beta = 0$ otherwise. Now just apply Proposition 3.1 to construct g. □

4. Obstructions to Isotopy rel ∂

Let M be a compact m-dimensional manifold with k components, each of which is s-connected; and let Q be a compact connected q-dimensional manifold with fundamental group π and $\pi_i(Q) = 0$ for $1 < i \leqq t$. Let $f, g: M \to Q$ be two proper embeddings which agree on ∂M, and let $F: M \times I \to Q \times I$ be a homotopy rel ∂M from f to g. Assuming that $s \geqq 1$, $m \geqq 2$, $q - m \geqq 3$, $2m - q + 1 < s$, $2m - q + 2 < t$, and

$3m-2q+4<0$, we define $d(f,g;F)$ to be the obstruction A to homotoping $F\operatorname{rel}\partial(M\times I)$ to an embedding. Thus $d(f,g;F)$ is a (π,π^{2m-q+1})-matrix of order k, defined up to equivalence.

The following result is an easy corollary of Theorem 2.1.

Proposition 4.1. *Let f,g,F,M,Q be as above. If $d(f,g;F)=0$, then f is isotopic to $g\operatorname{rel}\partial M$. Conversely, if f is isotopic to $g\operatorname{rel}\partial M$, then for some homotopy $F\operatorname{rel}\partial M$, $d(f,g;F)=0$.*

Proof. If $d(f,g;F)=0$, just apply Theorem 2.1 to $F:M\times I\to Q\times I$; we can homotop $F\operatorname{rel}\partial(M\times I)$ to an embedding $G:M\times I\to Q\times I$, such that G and F agree on $\partial(M\times I)$. G is thus a concordance between f and g which is fixed on ∂M. Since the codimension is at least 3, concordance implies isotopy [2], Theorem 9.1, and so f is isotopic to $g\operatorname{rel}\partial M$.

Conversely, if F is an isotopy from f to $g\operatorname{rel}\partial M$, then $d(f,g;F)=0$. □

In general, we may state the following results.

Theorem 4.2. *Let $r\geqq 0$ be such that π^r is the trivial group. Let M be a compact m-dimensional manifold, each of whose components is $(r+1)$-connected; and let Q be a compact connected $(2m-r)$-dimensional manifold such that $\pi_i(Q)=0$ for $1<i\leqq r+2$. Assume that $m\geqq 2r+2$; then every proper map $f:M\to Q$ which is an embedding on the boundary can be homotoped* rel ∂M *to an embedding.*

Theorem 4.3. *Let r, M, and Q be as above except that Q is $(2m-r+1)$-dimensional. Assume that $m\geqq 2r+3$; then any two proper embeddings $f,g:M\to Q$ which agree on the boundary and are homotopic* rel ∂M *are isotopic* rel ∂M.

Consider embeddings $f,g:S^m\to S^p\times S^1$, with $m\geqq 2$, $3m<2(p-1)$. Any two such embeddings are homotopic, as $\pi_m(S^p\times S^1)\cong\pi_m(S^p)\oplus\pi_m(S^1)=0$. Moreover, any two homotopies $F,G:S^m\times I\to S^p\times S^1\times I$ from f to g differ by an element of $\pi_{m+1}(S^p\times S^1\times I)=0$, and so F and G are homotopic $\operatorname{rel}\partial(S^m\times I)$. Thus the obstruction $d(f,g;F)$ does not depend upon the homotopy F in this case. Furthermore, there is a natural choice for a "null-embedding" f_0; let f_0 be the restriction to the boundary of an embedding $B^{m+1}\to S^p\times S^1$. The isotopy classes of embeddings $f:S^m\to S^p\times S^1$ are then classified by $d(f,f_0;F)$, which depends only on f. Compare the result of Hacon [8].

Alternatively, as every $f:S^m\to S^p\times S^1$ is null-homotopic, let A be the obstruction to embedding $F:B^{m+1}\to S^p\times S^1\times I$ where F is a null-homotopy of f. Since B^{m+1} has only one component, and $\pi=(t:)$ is abelian, A is independent of any choices, and is just a Laurent polynomial in t with coefficients in π^{2m-p}, which is $\pm$ symmetric with constant term zero. Thus only half the coefficients are needed to classify f, and these are the invariants used by Hacon.

5. An Example

The drawback of $d(f,g;F)$ is its dependence on the homotopy F. In general there seems to be little that one can do about this. However, there are circumstances which arise naturally in the study of high-dimensional knots in which progress can be made.

Assume that $2<2r\leqq n+1$, and consider a handle decomposition of S^n with one 0-handle h^0, $k(r-1)$-handles $h_i^{r-1}(1\leqq i\leqq k)$, k r-handles $h_i^r(1\leqq i\leqq k)$, and one n-handle. Suppose that h_i^r cancels h_i^{r-1}, and that the set of n-balls $h_i^{r-1}\cup h_i^r(1\leqq i\leqq k)$ are mutually disjoint.

Let $P=h^0\cup U_1^k h_i^{r-1}$, and $N=\partial P\cong \#_1^k(S^{r-1}\times S^{n-r})$. Then N has a regular neighbourhood in S^n of the form $B^1\times N$, and from the handle decomposition above we obtain a handle decomposition of S^n on $B^1\times N$ with k r-handles, $k(n-r+1)$-handles, and two n-handles.

Embed S^n in S^{n+1} as the equatorial n-sphere. Then S^n has a regular neighbourhood of the form $B^1\times S^n$, and so N has a regular neighbourhood of the form $B^1\times B^1\times N=B^2\times N$. Moreover, we obtain a handle decomposition of $B^1\times S^n$ on $B^2\times N$ with k r-handles, $k(n-r+1)$-handles, and two n-handles: each handle is of the form $B^1\times$(handle of S^n on $B^1\times N$).

If $Q=cl[S^{n+1}-B^2\times N]$, then by adding two $(n+1)$-handles to $B^1\times S^n$ we obtain a handle decomposition of S^{n+1} on $B^2\times N$, or of Q on a collar neighbourhood of ∂Q.

From the handle decomposition, it follows that $(Q,\partial Q)$ is $(r-1)$-connected and that $\pi_1(Q)=(t:)$, the infinite cyclic group.

Now assume the additional inequalities $r\geqq 3$, $3r>n+5$. Let $\tilde{Q}$ denote the universal cover of Q. From the handle decomposition we easily obtain the following result.

Lemma 5.1. *If* $2r<n+1$, *then* $H_r(\tilde{Q},\partial\tilde{Q})\cong\bigoplus_1^k \mathbb{Z}[t,t^{-1}]$ *with basis given by the cores of the* r-handles, and $H_{n-r+1}(\tilde{Q},\partial\tilde{Q})\cong\bigoplus_1^k \mathbb{Z}[t,t^{-1}]$ *with basis given by the cores of the* $(n-r+1)$*-handles.*

If $2r=n+1$, *then* $H_r(\tilde{Q},\partial\tilde{Q})\cong\bigoplus_1^{2k} \mathbb{Z}[t,t^{-1}]$ *with basis given by the cores of the r and* $(n-r+1)$*-handles.*

Now set $M=\bigcup_{i=1}^{l} B_i^m$, and let $f,g:M\to Q$ be two proper embeddings which agree on ∂M and are homotopic rel ∂M. If $m=r$ and $2r<n$, then f and g are isotopic rel ∂M: this is a consequence of [2], Theorem 10.1. In the cases $m=r$, $2r=n$ or $n+1$, there is an obstruction, as there is in the case $m=n-r+1$. For $2r=n+1$, note that $n-r+1=r$.

Let $f^e:M\to\tilde{Q}$ be a lift of f, and (f_i^e) the image of f_i^e under the Hurewicz map $\pi_m(\tilde{Q},\partial\tilde{Q})\to H_m(\tilde{Q},\partial\tilde{Q})$.

Proposition 5.1. *Assume that* $2r\leqq n+1$, $m=n-r+1$, $r\geqq 3$, $3r>n+5$. *If* $(f_1^e),\ldots,(f^e)$ *can be extended to a basis of* $H_{n-r+1}(\tilde{Q},\partial\tilde{Q})$, *regarded as a* $\mathbb{Z}[t,t^{-1}]$-module, then f is isotopic to g rel ∂M.

Proof. We may as well assume that $k=l$, and that $(f_1^e),\ldots,(f_k^e)$ is a basis of $H_{n-r+1}(\tilde{Q},\partial\tilde{Q})$. By the Hurewicz theorem, note that the classes of $H_r(\tilde{Q})$ are all spherical. According to [1] Theorem 2.6, therefore, there exist maps $h_i:S^r\to\operatorname{int}Q$ such that $(h_1^e),\ldots,(h_k^e)$ is a dual basis to $(f_1^e),\ldots,(f_k^e)$. Thus the algebraic intersection of (h_i^α) and (f_j^β) is 1 if $(i,\alpha)=(j,\beta)$ and 0 otherwise.

Let $F: M \times I \to Q \times I$ be a homotopy from f to g rel ∂M. Then $(F_1^e), \ldots, (F_k^e)$ is a $\mathbb{Z}[t, t^{-1}]$-basis for $H_{n-r+1}(\tilde{Q} \times I, \partial(\tilde{Q} \times I))$, and by homotoping h_i into $\operatorname{int}(Q \times I)$, $(h_1^e), \ldots, (h_k^e)$ is a dual basis to $(F_1^e), \ldots, (F_k^e)$.

In these dimensions algebraic and geometric intersection can be made to coincide, so we may assume that $\operatorname{Im} h_i^\alpha \cap \operatorname{Im} F_j^\beta$ is a single point if $(i, \alpha) = (j, \beta)$ and empty otherwise. This implies that $\operatorname{Im} h_i \cap \operatorname{Im} F_j$ is a single point if $i = j$, empty otherwise. By general position, as $2r < n + 2$, we can assume that h is an embedding.

Now by repeated use of Proposition 3.2, it is an easy matter to alter F until $d(f, g; F)$ is zero; whence the result. □

6. The Smooth Case

Finally we use an argument of Hudson [6] to show that Theorem 2.1 and Proposition 4.1 are true in the smooth category.

Theorem 6.1. *Let f, M, and Q be as in Theorem 2.1, except that everything is in the smooth category. Then the conclusion of Theorem 2.1 holds in this case.*

Proof. By [5] 10.6, there is a smooth triangulation of ∂M, and hence of $f(\partial M)$. By [5] 10.14, the latter extends to a smooth triangulation of ∂Q, and by [5] 10.6 the triangulations of ∂M, ∂Q extend to smooth triangulations of M, Q. Now f is a continuous map which is PL on ∂M, and by the simplicial approximation theorem is homotopic rel ∂M to a proper PL map. Since a homotopy between two PL maps can be approximated by a PL homotopy, and since by [5] 10.13 the triangulations are unique up to PL homeomorphism, the matrix A is well-defined up to equivalence by the homotopy class of f rel ∂M. Moreover, if $A = 0$, then f can be homotoped rel ∂M to a PL embedding. Now apply [6] Lemma 7 to homotop f rel ∂M to a smooth embedding.

The statement and proof of Proposition 4.1 in the smooth case are similar, except that one must appeal to [7] Theorem 2.3 for "concordance implies isotopy".

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