SIGNATURES OF KNOTS AND THE FREE DIFFERENTIAL CALCULUS

By C. KEARTON

[Received 13 April 1978; in revised form 27 September 1978]

0. Introduction

THE free differential calculus was invented by Fox in order to calculate the Alexander polynomials and ideals of a classical knot, starting from the knot diagram. We show here that with a little extra effort the method can be adapted to write down the Blanchfield duality pairing of the knot. From this it is possible to define a signature for each irreducible factor of the Alexander polynomial (regarded as a polynomial with real coefficients), and we show that these signatures coincide with certain signatures defined by Milnor in [**6**].

1. Notation

Given a presentation of a knot k, $S^1 \subset S^3$, divide the knot into overpasses $a_0, a_1, \ldots, a_n$ and underpasses $b_0, b_1, \ldots, b_n$ so that in progressing round S^1 in the positive direction they lie in the order $a_0, b_0, a_1, b_1, \ldots, a_n, b_n$. Around each b_i draw a rectangular neighbourhood B_i, oriented as shown in Fig. 1, and let e_i be the corner of B_i indicated.

Let $*$ be a point of S^2 not lying on the projection of the knot, or on any of the B_i. For each i, join $*$ to e_i by an arc $\beta_i \subset S^2$ which crosses the knot only on the overpasses, and meets $\bigcup_{j=0}^{n} B_j$ only in e_i. A small circle centred at $*$ should cross the β_i in order of increasing i when traversed anti-clockwise.

Define a homomorphism ε from the free group on generators $x_0, x_1, \ldots, x_n$ to the free group on one generator t by $\varepsilon(x_i) = t$, $0 \leqslant i \leqslant n$.

In order to read an arc γ which crosses the knot transversely, on its overpasses, we write x_i when γ crosses a_i from left to right, and x_i^{-1} when γ crosses a_i from right to left. In Fig. 2, γ is read as $x_1x_3^{-1}x_2^{-1}$.

For orientation, we adopt a left-hand convention. Thus if $T(A, x)$ denotes the intersection number of a 2-cell A and a 1-cell x, and $L(y, x)$ the linking number of two 1-cycles, Fig. 3 illustrates the convention.

Let $\Lambda = \mathbf{Z}[t, t^{-1}]$, the integral group ring of $(t\ :\ )$, regarded as the ring of Laurent polynomials in t with integer coefficients.

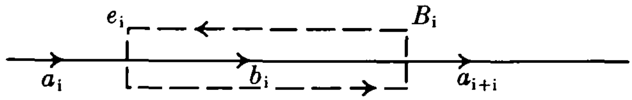


FIG. 1

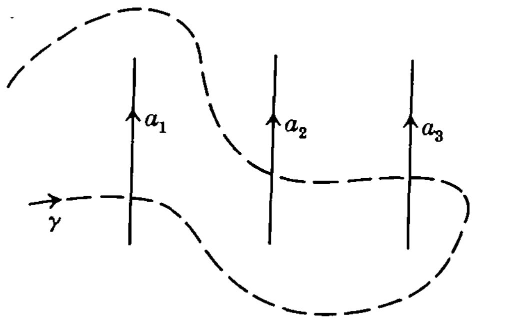


FIG. 2

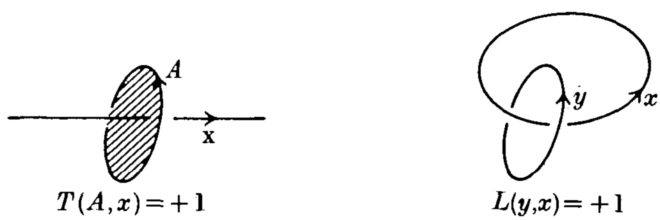


FIG. 3

2. The Blanchfield duality pairing

If N is a regular neighbourhood of k, set $K = cl[S^3 - N]$. Let $\tilde{K}$ be the covering of K corresponding to the kernel of the Hurewicz homomorphism $\pi_1(K) \to H_1(K) = (t :)$. Then $H_1(\tilde{K}, \partial\tilde{K})$, $H_1(\tilde{K})$ are finitely-generated modules over Λ. Blanchfield [1] shows that there is an Hermitian pairing

$$\langle\ ,\ \rangle \colon H_1(\tilde{K}, \partial\tilde{K}) \times H_1(\tilde{K}) \to \Lambda_0/\Lambda$$

where Λ_0 denotes the field of fractions of Λ, and conjugation in Λ is the linear extensions of $t \mapsto t^{-1}$, denoted by $^{-}$. This pairing is non-singular in the strong sense, that the induced map $H_1(\tilde{K}, \partial\tilde{K}) \to \overline{\text{Hom}}\,(H_1(\tilde{K}), \Lambda_0/\Lambda)$ is an isomorphism; $\overline{\text{Hom}}\,(\ ,\ )$ denotes the module of conjugate linear maps.

Since the usual map $H_1(\tilde{K}) \to H_1(\tilde{K}, \partial\tilde{K})$ is an isomorphism, the Blanchfield pairing induces another pairing

$$\langle\ ,\ \rangle\colon\ H_1(\tilde{K})\times H_1(\tilde{K})\to\Lambda_0/\Lambda,$$

and it is this one that we shall study.

Further details of the Blanchfield pairing may be found in [5].

3. Overpasses, underpasses and handles

A presentation of the knot may be regarded as a handle decomposition of S^1, the underpasses being 0-handles and the overpasses being 1-handles. By a trivial isotopy we may regard S^1 as embedded in $S^2\times I$, with each 0-handle in $S^2\times\frac{3}{4}$, and vertical collars in $S^2\times[\frac{1}{4},\frac{3}{4}]$. If N is a regular neighbourhood of S^1, then $S^2\times I-\operatorname{int}N$ has a handle decomposition on $S^2\times 0$ with one 1-handle h_i^1 for each 0-handle of S^1, and one 2-handle h_i^2 for each 1-handle of S^1. Figure 4 illustrates this: for a fuller discussion see [3], [4].

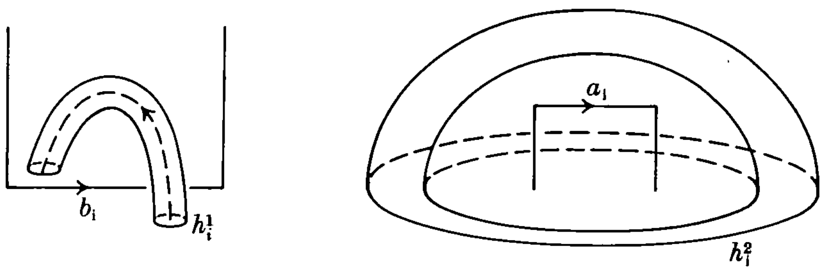


FIG. 4

The knot has a dual presentation obtained by turning it over, so that the overpasses become underpasses and vice-versa. The dual presentation corresponds to the dual handle decomposition of S^1, in which 1-handles dualise to 0-handles, 0-handles to 1-handles, yielding a handle decomposition of $S^2\times I-\operatorname{int}N$ on $S^2\times 1$. Figure 5 illustrates the relationship

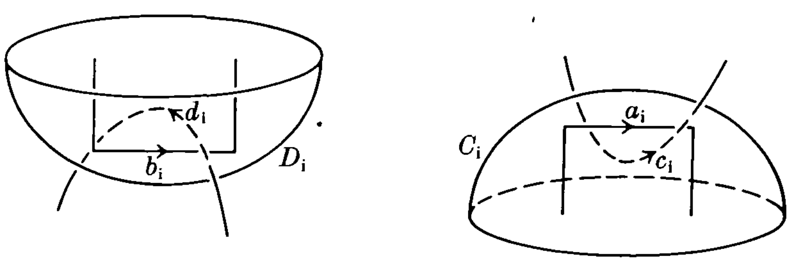


FIG. 5

between the two handle decompositions of $S^2 \times I - \text{int } N$, only the cores of the handles being shown.

For an underpass b_i, let d_i denote the core of the corresponding 1-handle, and D_i the core of the corresponding 2-handle in the dual presentation (where b_i is regarded as an overpass). For an overpass a_i, let C_i denote the core of the corresponding 2-handle, and c_i the core of the corresponding 1-handle in the dual presentation. We take $\partial D_i = \partial B_i$, and B_i is the projection of D_i onto $S^2 = S^2 \times \frac{1}{2}$.

When dealing with intersection theory on a PL manifold M^m, one normally considers a triangulation of M and a dual triangulation of $(M, \partial M)$. In terms of handle-body theory, this is dealt with by taking a handle decomposition of M, and considering the dual handle decomposition of M on ∂M. The cores of the r-handles play the role of r-chains, and the cocores of the r-handles play the role of $(m-r)$-chains in the dual triangulation. Here the c_i will play the role of 1-chains, the D_i the role of 2-chains in one decomposition; and the d_i will be the 1-chains, the C_i the 2-chains in the "dual" decomposition.

Let x_i denote the following composite arc: from $* \times 1 \in S^2 \times 1$ to the point of $S^2 \times 1$ vertically above the tail end of c_i, from this point straight down to c_i, along c_i in the positive direction, straight up again to $S^2 \times 1$, and back to $* \times 1$. Clearly the x_i, $0 \leq i \leq n$, generate $\pi_1(K, * \times 1)$.

Let p_i' denote the following arc: from $* \times 1$ straight down to $* = * \times \frac{1}{2}$; from $*$ along β_i, around ∂B_i in the positive direction, back along β_i, and from $*$ up to $* \times 1$. The arc p_i starts from $* \times 1$, and follows the projection of p_i' into $S^2 \times 1$ except when p_i' crosses the knot. If p_i' crosses the knot at a_j say, then p_i leaves $S^2 \times 1$ to pass along c_j in the appropriate direction, and then back to $S^2 \times 1$.

Let y_i denote the following arc: from $* \times 0$ follow the projection of β_i in $S^2 \times 0$ to $e_i \times 0$, from there straight to the tail end of d_i, along d_i, straight back to $e_i \times 0$, and back along β_i to $* \times 0$.

Let δ_i denote an arc in $S^2 \times 1$ from $* \times 1$ to the point vertically above e_i, and straight down from this point to e_i.

4. Presentation of the knot group

Corresponding to each overpass a_i, we have a loop x_i based at $* \times 1$. For each underpass b_j, read the word in the x_i defined by the arc p_j' as in §1; call this r_j. Then $(x_0, x_1, \ldots, x_n :)$, the free group on $n+1$ generators, is $\pi_1((S^2 \times [\frac{3}{4}, 1]) \cup \bigcup_{i=0}^{n} c_i, * \times 1)$, and the word r_j represents the element killed off by the 2-cell D_j. From the theory of handle-bodies, or CW-complexes, $\pi_1(K_1 * \times 1) \cong \pi_1(S^2 \times I - \text{int } N, * \times 1) \cong (x_0, x_1, \ldots, x_n : r_0, r_1, \ldots, r_n)$.

Note that the β_j have been so chosen that $\prod_{j=0}^{n} r_j = 1$. This is the over-presentation of $\pi_1(K)$.

The under-presentation is obtained in a similar way from the dual presentation of the knot, using d_i in place of c_i, C_j in place of D_j.

5. Presentation of the knot module

Let $\tilde{*}$ be a lift of $*$ to $\tilde{K}$, and let $\tilde{x}_i$ be the lift of x_i which starts at $\tilde{*}\times 1$. Thus $\tilde{x}_0, \ldots, \tilde{x}_n$ forms a basis for the free Λ-module of 1-chains $C_1(\tilde{K})$. Let $\tilde{\delta}_i$ be the lift of δ_i which starts at $\tilde{*}\times 1$ and ends at $\tilde{e}_i$: this defines the lift $\tilde{D}_i$ of D_i.

Using the free differential calculus of Fox [2], we can lift the loop p_i to $\tilde{K}$; the lift is represented by $\sum_{j=0}^{n} \varepsilon\left(\frac{\partial r_i}{\partial x_j}\right)\tilde{x}_j$. Setting $A_{ij} = \varepsilon\left(\frac{\partial r_i}{\partial x_j}\right)$, we obtain an Alexander matrix A; this matrix presents the Λ-module $H_1(\tilde{K})\oplus\Lambda$, with generators $\tilde{x}_0, \ldots, \tilde{x}_n$.

Because $\prod_{j=0}^{n} r_j = 1$, the sum of the rows of A is zero, and so is the sum of the columns. So we can delete the 0th row of A and still have a presentation matrix for $H_1(\tilde{K})\oplus\Lambda$. If we change basis in $C_1(\tilde{K})$ to $\tilde{x}_0$, $\tilde{x}_1-\tilde{x}_0, \ldots, \tilde{x}_n-\tilde{x}_0$, we obtain a matrix with 0th column zero but whose other columns are the same as those of A. Thus the Alexander matrix A with its 0th row and column deleted gives a presentation matrix B for $H_1(\tilde{K})$ as a Λ-module.

Let s_i be the word in the x_j read from the arc β_i. Then the ith row of B represents the boundary of $\varepsilon(s_i)\tilde{D}_i$ with respect to the chains $\tilde{x}_1-\tilde{x}_0, \ldots, \tilde{x}_n-\tilde{x}_0$. The chains $\tilde{x}_1-\tilde{x}_0, \ldots, \tilde{x}_n-\tilde{x}_0$ form a basis for $Z_1(\tilde{K})$, the module of 1-cycles.

Let $\tilde{y}_i$ be the lift of y_i which starts at $\tilde{*}\times 0$; then $\tilde{y}_0, \ldots, \tilde{y}_n$ forms a basis for the dual module of 1-chains $\bar{C}_1(\tilde{K})$.

After Blanchfield [1], define

$$S(\tilde{D}_i, \tilde{y}_j) = \sum_{-\infty<k<\infty} T(\tilde{D}_i, t^k\tilde{y}_j)t^k$$

where T denotes the ordinary intersection of chains. Clearly $T(\tilde{D}_i, \varepsilon(s_j)^{-1}\tilde{y}_j) = -\delta_{ij}$, $T(\tilde{D}_i, \varepsilon(s_j)^{-1}t^{-1}\tilde{y}_j) = \delta_{ij}$, and zero otherwise. Thus

$$S(\tilde{D}_i, \tilde{y}_j) = \varepsilon(s_j)^{-1}(t^{-1}-1)\,\delta_{ij}.$$

Define the matrix S by $S_{ij} = S(\tilde{D}_i, \tilde{y}_j - \tilde{y}_0) = S(\tilde{D}_i, \tilde{y}_j)$ for $1 \leq i \leq n$, $1 \leq j \leq n$.

If x is a 1-cycle in $\tilde{K}$, y a dual 1-cycle, and $\alpha x = \partial u$ for some non-zero

$\alpha \in \Lambda$, and 2-cycle u, then Blanchfield [1] defines V by

$$V(x, y) = \frac{1}{\alpha} S(u, y).$$

Setting $V_{ij} = V(\tilde{x}_i - \tilde{x}_0, \tilde{y}_j - \tilde{y}_0)$ for $1 \leq i \leq n$, $1 \leq j \leq n$, we obtain

$$\begin{aligned}\sum_{j=1}^{n} B_{ij} V_{jk} &= \sum_{j=1}^{n} B_{ij} V(\tilde{x}_j - \tilde{x}_0, \tilde{y}_k - \tilde{y}_0) \\ &= V\Big(\sum_{j=1}^{n} B_{ij}(\tilde{x}_j - \tilde{x}_0), \tilde{y}_k - \tilde{y}_0\Big) \\ &= S(\varepsilon(s_i)\tilde{D}_i, \tilde{y}_k - \tilde{y}_0) \\ &= \varepsilon(s_i) S_{ik}.\end{aligned}$$

Now S is a diagonal matrix, and the ith diagonal entry is $\varepsilon(s_i)^{-1}(t^{-1} - 1)$. Thus

$$BV = (t^{-1} - 1)I.$$

To express $\tilde{y}_i$ in terms of the $\tilde{x}_j$, we use the free differential calculus again. First, let γ_i denote the loop illustrated in Fig. 6. By changing base point from $* \times 0$ to $* \times 1$, we obtain from y_i a loop based at $* \times 1$. Reading the arc $\beta_i \gamma_i \beta_i^{-1}$ gives the word $s_i x_i s_i^{-1}$ corresponding to this loop. The free differential calculus applied to $s_i x_i s_i^{-1}$ gives $\tilde{y}_i$ in terms of the $\tilde{x}_j$. Strictly speaking we should distinguish between the chain $\tilde{y}_i$ based at $* \times 0$, and that based $* \times 1$; but there will be no difference when we pass to $H_1(\tilde{K})$, and so we avoid introducing the extra notation.

Defining $C_{ij} = \varepsilon \dfrac{\partial}{\partial x_j}(s_i x_i s_i^{-1})$, we obtain

$$\tilde{y}_i = \sum_{j=0}^{n} C_{ij} \tilde{x}_j \qquad 0 \leq i \leq n.$$

$$\tilde{y}_i = \sum_{j=0}^{n} C_{ij} \tilde{x}_0 + \sum_{j=1}^{n} C_{ij}(\tilde{x}_j - \tilde{x}_0) \qquad 0 \leq i \leq n.$$

$$\tilde{y}_i - \tilde{y}_0 = \sum_{j=0}^{n} (C_{ij} - C_{0j})\tilde{x}_0 + \sum_{j=1}^{n} (C_{ij} - C_{0j})(\tilde{x}_j - \tilde{x}_0) \qquad 1 \leq i \leq n.$$

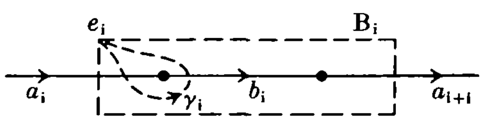


FIG. 6

But for each i, $\sum_{j=1}^{n} C_{ij}=1$, and so defining $D_{ij}=C_{ij}-C_{0j}$ we obtain

$$\tilde{y}_i-\tilde{y}_0=\sum_{j=1}^{n} D_{ij}(\tilde{x}_j-\tilde{x}_0). \qquad 1\leq i\leq n.$$

Thus D is obtained from C by subtracting the 0th row from every other row, and then deleting the 0th row and column.

Let U be the linking matrix with respect to the basis $\tilde{x}_1-\tilde{x}_0,\ldots,\tilde{x}_n-\tilde{x}_0$; so $U_{ij}=V(\tilde{x}_i-\tilde{x}_0,\tilde{x}_j-\tilde{x}_0)$. Then

$$\begin{aligned}\sum_{j=1}^{n} U_{ij}\bar{D}_{kj}&=\sum_{j=1}^{n} V(\tilde{x}_i-\tilde{x}_0,\tilde{x}_j-x_0)\bar{D}_{kj}\\&=V\left(\tilde{x}_i-\tilde{x}_0,\sum_{j=1}^{n} D_{ij}(\tilde{x}_j-\tilde{x}_0)\right)\\&=V(\tilde{x}_i-\tilde{x}_0,\tilde{y}_k-\tilde{y}_0)\\&=V_{ik}.\end{aligned}$$

So $UD^*=V$, where $D^*=\bar{D}'$, and hence

$$BUD^*=BV=(t^{-1}-1)I.$$
$$UD^*=(t^{-1}-1)B^{-1}.$$

Now let φ: $\langle\tilde{x}_1-\tilde{x}_0,\ldots,\tilde{x}_n-\tilde{x}_0:\rangle\to H_1(\tilde{K})$, ψ: $\langle\tilde{y}_1-\tilde{y}_0,\ldots,\tilde{y}_n-\tilde{y}_0:\rangle\to H_1(\tilde{K})$ be the quotient maps, and define the matrix W over Λ_0/Λ by

$$W_{ij}\equiv\langle\varphi(\tilde{x}_i-\tilde{x}_0),\qquad \varphi(\tilde{x}_j-\tilde{x}_0)\rangle.$$

Of course, W is just the image of the matrix U under the map $\Lambda_0\to\Lambda_0/\Lambda$.

For each i, $1\leq i\leq n$, we have

$$\psi(\tilde{y}_i-\tilde{y}_0)=\sum_{j=1}^{n} D_{ij}\varphi(\tilde{x}_j-\tilde{x}_0).$$

But since

$$\varphi\left(\sum_{j=1}^{n} B_{kj}(\tilde{x}_j-\tilde{x}_0)\right)=0$$

for each k, $1\leq k\leq n$, we deduce that

$$\psi(\tilde{y}_i-\tilde{y}_0)=\sum_{j=1}^{n}(\alpha B_{kj}+D_{ij})\varphi(\tilde{x}_j-\tilde{x}_0)$$

for any $\alpha\in\Lambda$. Thus we may add a multiple of any row of B to any row of D without affecting the matrix W.

6. Matrix moves

Let E_{ij}^{α} denote the identity matrix added to the matrix whose only non-zero entry is α in the ijth place, $i \neq j$. Then $E_{ij}^{\alpha}B$ is obtained by adding α times the jth row of B to the ith row, and BE_{ij}^{α} is obtained by adding α times the ith column of B to the jth column. Furthermore, note that $(E_{ij}^{\alpha})^{-1} = E_{ij}^{-\alpha}$.

Recall that the matrix B is a presentation matrix for the Λ-module $H_1(\tilde{K})$. As such, it may be altered by certain row and column operations, and we now analyse the effect of such operations on the matrices B, D, and U.

Premultiplying B by E_{ij}^{α} corresponds to a change in basis from $\tilde{D}_1, \ldots, \tilde{D}_i, \ldots, \tilde{D}_n$ to $\tilde{D}_1, \ldots, \tilde{D}_i + \alpha\tilde{D}_j, \ldots, \tilde{D}_n$. U is unchanged, and D becomes PD where $(E_{ij}^{\alpha})^{-1}P^{*-1} = I$, i.e. $P = E_{ji}^{-\bar{\alpha}}$.

Postmultiplying B by E_{ij} corresponds to a change in basis from $\tilde{x}_1 - \tilde{x}_0, \ldots, \tilde{x}_i - \tilde{x}_0, \ldots, \tilde{x}_n - \tilde{x}_0$ to $\tilde{x}_1 - \tilde{x}_0, \ldots, \tilde{x}_i - \tilde{x}_0 - \alpha(\tilde{x}_j - \tilde{x}_0), \ldots, \tilde{x}_n - \tilde{x}_0$. This alters U to $E_{ij}^{-\alpha}UE_{ji}^{-\bar{\alpha}}$, and so D is changed to DE_{ij}^{α}.

The third move we shall consider is enlargement of B by an extra row and column to $\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$; this requires an enlargement of D to $\begin{bmatrix} D & 0 \\ 0 & t-1 \end{bmatrix}$, for then U is enlarged to $\begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}$. In view of the final remark in the last section, we can enlarge D to $\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, or indeed to $\begin{bmatrix} D & * \\ 0 & * \end{bmatrix}$.

Let E_{ij} denote the matrix with 1 in the ijth and jith places, 1 on the diagonal except for the ith and jth places, and zeros elsewhere. Premultiplication by E_{ij} corresponds to swopping the ith and jth rows, postmultiplication corresponds to swopping the ith and jth columns.

So if B is replaced by $E_{ij}B$, D must be replaced by PD where $E_{ij}^{-1}P^{*-1} = I$; i.e., $P = E_{ij}^{-1*} = E_{ij}^{*} = E_{ij}$. U is unchanged.

If B is replaced by BE_{ij}, corresponding to a swop of basis elements $\tilde{x}_i - \tilde{x}_0$ and $\tilde{x}_j - \tilde{x}_0$, then U is altered to $E_{ij}UE_{ij}$, and so D is altered to DE_{ij}.

Let E_i^{α} denote the identity matrix with the ith diagonal entry replaced by the unit $\alpha \in \Lambda$. Premultiplying by E_i^{α} multiplies the ith row by α, postmultiplying by E_i^{α} multiplies the ith column by α. Premultiplying B by E_i^{α} corresponds to a change in basis of the $\tilde{D}_j$, so does not affect U. Thus D is replaced by PD where $(E_i^{\alpha})^{-1}P^{*-1} = I$, and hence $P = E^{\bar{\alpha}^{-1}} = E_i^{\alpha}$. Postmultiplying B by E_i^{α} corresponds to changing base in the $\tilde{x}_j - \tilde{x}_0$, replacing $\tilde{x}_i - \tilde{x}_0$ by $\alpha^{-1}(\tilde{x}_i - \tilde{x}_0)$. U is altered to $E_i^{\alpha^{-1}}UE^{\bar{\alpha}^{-1}}$, and D to DE_i^{α}.

To sum up, the moves we allow are as follows.

(i) Add α times the jth row of B to the ith.
Subtract α times the ith row of D from the jth.

(ii) Add α times the ith column of B to the jth.
Add α times the ith column of D to the jth.

(iii) Enlarge B to $\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$; enlarge D to $\begin{bmatrix} D & * \\ 0 & * \end{bmatrix}$.

(iv) The inverse of (iii).

(v) Swap the ith and jth rows of B.
Swap the ith and jth rows of D.

(vi) Swap the ith and jth columns of B.
Swap the ith and jth columns of D.

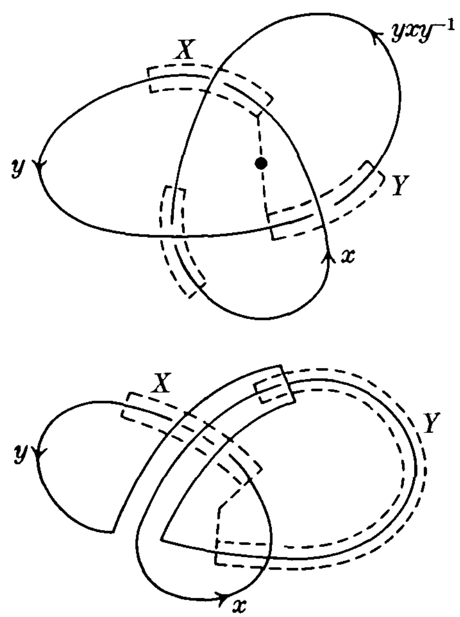
(vii) Multiply the ith row of B by a unit $\alpha \in \Lambda$.
Multiply the ith row of D by D by α.

(viii) Multiply the ith column of B by a unit $\alpha \in \Lambda$.
Multiply the ith column of D by α.

(ix) Add a multiple of any row of B to any row of D.

7. Economy

So far we have given each overpass and each underpass its own label. In practice, it is often convenient to label some of the overpasses in terms of the others, using the relations given by the underpasses. This is justified by passing to a new knot diagram; the example in Fig. 7 illustrates the principle involved.



$$r_X = xyxy^{-1}y^{-1}(xyx^{-1})^{-1}$$

$$r_Y = yx(yxy^{-1})^{-1}x^{-1}$$

$$r_X = xyx\,y^{-1}y^{-1}yx^{-1}y^{-1}$$

$$r_Y = yxy\,x^{-1}y^{-1}x^{-1}$$

FIG. 7

8. Examples

The first example is the trefoil knot, illustrated in Fig. 7. As subscripts are rather tedious in practice, the overpasses are labelled x, y, yxy^{-1}, and the B_i are labelled X, Y. The third underpass has already been used to label yxy^{-1}, and plays no further part in the computation.

The relations r_X, r_Y are given by

$$\begin{aligned} r_X &= xyxy^{-1}y^{-1}(yxy^{-1})^{-1} \\ &= xyxy^{-1}y^{-1}yx^{-1}y^{-1} \\ &= xyxy^{-1}x^{-1}y^{-1} \\ r_Y &= yx(yxy^{-1})^{-1}x^{-1} \\ &= yxyx^{-1}y^{-1}x^{-1} \end{aligned}$$

Choosing x to play the role of x_0, the matrix B is obtained by calculating $\varepsilon\left(\frac{\partial r_Y}{\partial y}\right) = 1+t^2-t$. Thus $B = (1-t+t^2)$.

In this case $s_0x_0s_0^{-1} = x$ and $s_1x_1s_1^{-1} = y$, so that C is given by

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and so $D = (1)$. It follows that $W = \left(\frac{t^{-1}-1}{1-t+t^2}\right)$.

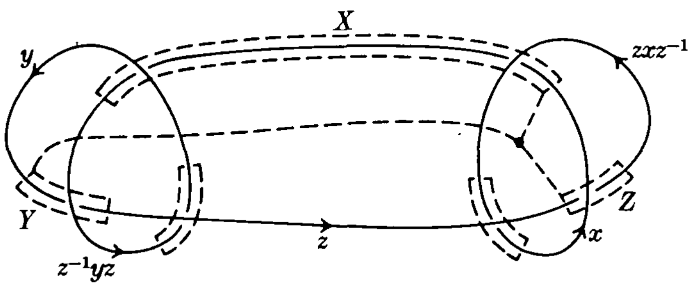


FIG. 8

In the second example, the reef knot shown in Fig. 8, three labels x, y, z are necessary. Note that r_X is not used, and so it is not necessary to compute it.

$$\begin{aligned} r_Y &= zxz^{-1}y^{-1}z^{-1}yzy(z^{-1}yz)^{-1}z^{-1}z^{-1}yz(z^{-1}yz)^{-1}y(zxz^{-1})^{-1} \\ &= zxz^{-1}y^{-1}z^{-1}yzyz^{-1}y^{-1}zz^{-1}yzx^{-1}z^{-1} \\ &= zxz^{-1}y^{-1}z^{-1}yzyx^{-1}z^{-1} \\ r_Z &= zx(zxz^{-1})^{-1}x^{-1} \\ &= zxzx^{-1}z^{-1}x^{-1} \end{aligned}$$

$$B=\begin{bmatrix}\varepsilon\dfrac{\partial r_Y}{\partial_Y} & \varepsilon\dfrac{\partial r_Y}{\partial_Z}\\ \\ \varepsilon\dfrac{\partial r_Z}{\partial y} & \varepsilon\dfrac{\partial r_Z}{\partial z}\end{bmatrix}$$

$$=\begin{bmatrix}-1+t^{-1}+t & 1-t-t^{-1}+1-1\\ 0 & 1+t^2-t\end{bmatrix}$$

$$=\begin{bmatrix}t^{-1}-1+t & -t^{-1}+1-t\\ 0 & 1-t+t^2\end{bmatrix}$$

$$s_0x_0s_0^{-1}=x$$

$$s_1x_1s_1^{-1}=zxz^{-1}y^{-1}z^{-1}yzy(z^{-1}yz)^{-1}y(zxz^{-1})^{-1}$$

$$=zxz^{-1}y^{-1}z^{-1}yzyz^{-1}y^{-1}zyzx^{-1}z^{-1}$$

$$s_2x_2s_2^{-1}=z$$

$$C=\begin{bmatrix}1 & 0 & 0\\ t-t^2 & -1+t^{-1}+t-1+t & 1-t-t^{-1}+1-t+1+t^2-t\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ t-t^2 & t^{-1}-2+2t & -t^{-1}+3-3t+t^2\\ 0 & 0 & 1\end{bmatrix}$$

$$D=\begin{bmatrix}t^{-1}-2+2t & -t^{-1}+3-3t+t^2\\ 0 & 1\end{bmatrix}$$

Multiplying the first row of B by t gives

$$B=\begin{bmatrix}1-t+t^2 & -1+t-t^2\\ 0 & 1-t+t^2\end{bmatrix}$$

$$D=\begin{bmatrix}1-2t+2t^2 & -1+3t-3t^2+t^3\\ 0 & 1\end{bmatrix}$$

Adding the second row of B to the first gives

$$B=\begin{bmatrix}1-t+t^2 & 0\\ 0 & 1-t+t^2\end{bmatrix}$$

$$D=\begin{bmatrix}1-2t+2t^2 & -1+3t-3t^2+t^3\\ -1+2t-2t^2 & 2-3t+3t^2-t^3\end{bmatrix}$$

The form of B allows us to add any multiple of $1-t+t^2$ to any entry of D: thus we easily obtain

$$D=\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}$$

From the equation $WD^* = (t^{-1}-1)B^{-1}$ it follows that

$$W = \frac{(t^{-1}-1)}{(1-t+t^2)^2}\begin{bmatrix}1-t+t^2 & 0\\ 0 & 1-t+t^2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$$

$$= \frac{t^{-1}-1}{1-t+t^2}\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$$

$$= \frac{t^{-2}-t^{-1}}{t^{-1}-1+t}\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$$

$$= \frac{-1}{t^{-1}-1+t}\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}.$$

9. Signatures

Let $\Gamma = \mathbb{R}[t, t^{-1}]$; then Γ is a *PID*, and so $H_1(\tilde{K}; \mathbb{R})$ is a direct sum of cyclic Γ-modules of the form $\Gamma/(\alpha)$, $\alpha \neq 0$ (since $\dim_{\mathbb{R}} H_1(\tilde{K}; \mathbb{R}) < \infty$). The Blanchfield pairing yields a non-singular Hermitian pairing $\langle\ ,\ \rangle$: $H_1(\tilde{K}: \mathbb{R}) \times H_1(\tilde{K}; \mathbb{R}) \to \Gamma_0/\Gamma$.

Let Δ be the Alexander polynomial of the knot, so that Δ annihilates $H_1(\tilde{K}; \mathbb{R})$. Let π be a prime in Γ dividing Δ, and let V_π be the π-primary summand of $H_1(\tilde{K}; \mathbb{R})$. As in [7], V_π is orthogonal to V_σ unless $(\pi(t)) = (\sigma(t^{-1}))$. Arguing as in [7]; pp. 95–96, the following facts can be established. Assuming $\pi(t) = \pi(t^{-1})$, V_π splits as an orthogonal direct sum $V_\pi^1 \cdots V_\pi^m$ for some integer m, where V_π^i is free over $\Gamma/(\pi^i)$. Setting $H_\pi^i = V_\pi^i/\pi V_\pi^i$, let (x) denote the image in H_π^i of $x \in V_\pi^i$. Setting $\frac{1}{\pi}\langle (x), (y)\rangle_\pi^i = \langle \pi^{i-1}x, y\rangle$ for x, $y \in V_\pi^i$ defines a non-singular Hermitian pairing on H_π^i. In fact, H_π^i is a finite dimensional vector space over the complex numbers $\mathbb{C} = \mathbb{R}(\tau) \cong \Gamma/(\pi)$, where τ is a root of π, and conjugation in $\mathbb{R}(\tau)$ is given by $\tau \mapsto \tau^{-1}$. This of course is just complex conjugation since π is quadratic, and $\tau^{-1} = \bar{\tau}$ lies on the complex unit circle. The pairs $(H_\pi^i, \langle\ ,\ \rangle_\pi^i)$ are independent of the choices made.

Furthermore, the pairs $(H_\pi^i, \langle\ ,\ \rangle_\pi^i)$ determine $(V_\pi, \langle\ ,\ \rangle \mid V_\pi)$. To see this, consider an orthogonal basis $(v_1), \ldots, (v_r)$ of H_π^m. Then if $\langle (v_1), (v_1)\rangle_\pi^m = b \in R$, we have

$$\langle \pi^{m-1}v_1, v_1\rangle \equiv b/\pi \qquad (\mathrm{mod}\ \Gamma).$$

Write $v_1' = v_1 + a(t)\pi v_1$, where $a(t) \in \Gamma$ will be defined shortly. As in [7]; p. 96,

$$\langle \pi^{m-2}v_1', v_1'\rangle \equiv \langle \pi^{m-2}v_1, v_1\rangle + (a(t) + a(t^{-1}))\langle \pi^{m-1}v_1, v_1\rangle.$$

Since $\langle \pi^{m-2}v_1, v_1\rangle \equiv b/\pi^2 + c/\pi$ (mod Γ), we can make $\langle \pi^{m-2}v_1', v_1'\rangle \equiv$

$b/\pi^2 \pmod{\Gamma}$ if we can solve the equation

$$0 = c/\pi + (a(t) + a(t^{-1}))b/\pi \qquad \pmod{\Gamma}.$$

But $c \in \mathbb{R}$, and so we only need to set $a(t) = -c/2b$.

Continuing in this way, we arrive at an element $x_1 \in V_\pi^m$ with $(x_1) = (v_1)$ and $\langle x_1, x_1\rangle \equiv b/\pi^m \pmod{\Gamma}$.

Now V_π^m splits as the direct sum of the submodule $\langle x_1\rangle$ and its orthogonal complement. Choosing a representative for (v_2) in this orthogonal complement, an inductive argument shows that $(H_\pi^m, \langle\ ,\ \rangle_\pi^m)$ determines $(V_\pi^m, \langle\ ,\ \rangle \mid V_\pi^m)$. A similar argument applies to V_π^{m-1}, and so on.

The trace function $\mathbb{R}(\tau) \to \mathbb{R}$ makes H_π^i into a quadratic space over $\mathbb{R}$, with signature σ_π^i. Define σ_π to be the sum over odd i of the σ_π^i.

Writing $\pi(t) = t^{-1} - 2\cos\theta + t$, $0 < \theta < \pi$, Milnor **[6]** has defined signatures σ_θ for a certain quadratic form on $H_1(\tilde{K}; R)$.

THEOREM. $\sigma_\pi = \sigma_\theta$.

The proof will be given in the following sections.

In the discussion above, it was assumed that $(\pi(t)) = (\pi(t^{-1}))$. The other cases are not of interest here, as the quadratic spaces obtained are hyperbolic and so make no contribution to the signature. Compare **[7]**; pp. 93, 94.

10. The Milnor signatures

Cut K open along a connected Seifert surface V, and let X be a lift of the resulting manifold to $\tilde{K}$, so that $\tilde{K}$ is covered by the sets $\ldots, t^{-1}X, X, tX, t^2X, \ldots$. Note that X is connected and that $X \cap tX \cong V$. Furthermore, $N_p = \bigcup_{i=p}^{\infty} t^iX$ is a neighbourhood of one end ω of $\tilde{K}$, and the sets $N_0 \supset N_1 \supset N_2 \supset \cdots$ form a cofinal sequence of neighbourhoods of ω. Similarly the sets $N'_q = \bigcup_{i=-q}^{-\infty} t^iX$ form a cofinal sequence of neighbourhoods of the other end ω' of $\tilde{K}$.

For $p \geq p'$, there is a homomorphism

$$i'_{p,p'}: H_*(\tilde{K}, N_p; \mathbb{R}) \to H_*(\tilde{K}, N_{p'}; \mathbb{R})$$

induced by inclusion. Define $H_*(\tilde{K}, \omega; \mathbb{R}) = \varprojlim H_*(\tilde{K}, N_p; \mathbb{R})$.

By a result of Milnor ([6]; p. 125), there exists an integer $s > 0$ such that for all p, $H_*(\tilde{K}, N_{p+s}; \mathbb{R}) \to H_*(\tilde{K}, N_p; \mathbb{R})$ is the zero map. It follows that $H_*(\tilde{K}, \omega; \mathbb{R}) = 0$, and similarly for ω'.

For $p \geq p'$, $q \geq q'$, let $i_{p,q}^{p',q'}$: $H_*(\tilde{K}, N_p \cup N'_q; \mathbb{R}) \to H_*(\tilde{K}, N_{p'} \cup N'_{q'}; \mathbb{R})$ be induced by inclusion. Define $H_*(\tilde{K}, \omega \cup \omega'; \mathbb{R}) = \varprojlim H_*(\tilde{K}, N_p \cup N'_q; \mathbb{R})$. The Mayer-Vietoris sequence yields

$$\cdots \to H_2(\tilde{K}, N_p \cup N'_q; \mathbb{R}) \xrightarrow{\partial_*} H_1(\tilde{K}; \mathbb{R}) \to H_1(\tilde{K}, N_p; \mathbb{R}) \oplus H_1(\tilde{K}, N'_q; \mathbb{R}) \to$$

for large p, q. Since we are working with real coefficients, these are vector spaces over $\mathbb{R}$, and taking inverse limits yields an exact sequence

$$0 \to H_2(\tilde{K}, \omega \cup \omega'; \mathbb{R}) \xrightarrow{\partial_*} H_1(\tilde{K}; \mathbb{R}) \to 0 \oplus 0.$$

Let $F_{p,q} = \bigcup_{i=-q+1}^{p-1} t^i X$, and $\partial_0 F_{p,q} = \partial \tilde{K} \cap F_{p,q}$. Then for $p \geq p'$, $q \geq q'$, there is a homomorphism

$$j_{p,q}^{p',q'}\colon\ H_*(F_{p',q'}, \partial_0 F_{p',q'}; \mathbb{R}) \to H_*(F_{p,q}, \partial_0 F_{p,q}; \mathbb{R})$$

induced by inclusion. Taking the direct limit,

$$\varinjlim H_*(F_{p,q}, \partial_0 F_{p,q}; \mathbb{R}) \cong H_*(\tilde{K}, \partial \tilde{K}; \mathbb{R}).$$

Here is a technical result we need.

LEMMA. *Let M be a 3-manifold, A and B 2-manifolds with $\partial M = A \cup B$, $A \cap B = \partial A = \partial B$. Using real coefficients, the intersection pairing $H_1(M, A) \times H_2(M, B) \to \mathbb{R}$ is non-singular.*

Proof. There is a ladder of exact sequences and intersection pairings which is commutative up to sign.

$$\begin{array}{ccccccccc}
H_2(M, \partial M) & \to & H_1(\partial M, A) & \to & H_1(M, A) & \to & H_1(M, \partial M) & \to & H_0(\partial M, A) \\
\times & & \times & & \times & & \times & & \times \\
H_1(M) & \leftarrow & H_1(B) & \leftarrow & H_2(M, B) & \leftarrow & H_2(M) & \leftarrow & H_2(B)
\end{array}$$

The first and fourth pairings are non-singular by Lefschetz duality. By excision, $H_*(\partial M, A) \cong H_*(B, \partial B)$, and so the second and fifth pairings are also non-singular.

If U, V are finite-dimensional vector spaces over $\mathbb{R}$, then a bilinear pairing $U \times V \to \mathbb{R}$ defines a map $U \to V^*$, where V^* is the dual space of V. Hence the ladder above gives rise to a diagram, commutative up to sign:

$$\begin{array}{ccccccccc}
H_2(M, \partial M) & \to & H_1(\partial M, A) & \to & H_1(M, A) & \to & H_1(M, \partial M) & \to & H_0(\partial M, A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_1(M)^* & \to & H_1(B)^* & \to & H_2(M, B)^* & \to & H_2(M)^* & \to & H_2(B)^*
\end{array}$$

The bottom row is exact because we are using real coefficients, and the four outer vertical maps are isomorphisms because the corresponding pairings are non-singular. By the five lemma, so is the middle map.

Now apply the lemma with $M=F_{p,q}$, $A=\partial_0 F_{p,q}$, $B=F_{p,q}\cap(N_p\cup N'_q)$, to obtain (after excision) a non-singular intersection pairing

$$\{\ ,\ \}_{p,q}\colon\ H_1(F_{p,q},\partial_0 F_{p,q};\mathbb{R})\times H_2(\tilde{K},N_p\cup N'_q;\mathbb{R})\to\mathbb{R}.$$

Taking the direct and inverse limits yields a pairing

$$\{\ ,\ \}\colon\ H_1(\tilde{K},\partial\tilde{K};\mathbb{R})\times H_2(\tilde{K},\omega\cup\omega';\mathbb{R})\to\mathbb{R}$$

as follows. Let

$$i_{p,q}\colon\ H_2(K,\omega\cup\omega';\mathbb{R})\to H_2(K,N_p\cup N'_q;\mathbb{R}),$$

$j_{p,q}\colon\ H_1(F_{p,q},\partial_0 F_{p,q};\mathbb{R})\to H_1(\tilde{K},\partial\tilde{K};\mathbb{R})$ be the maps corresponding to the inverse and direct limits. Let $x\in H_1(\tilde{K},\partial\tilde{K};\mathbb{R})$, $y\in H_2(\tilde{K},\omega\cup\omega';\mathbb{R})$. For some p', $q'>0$, there is an element $x_{p',q'}\in H_1(F_{p',q'},\partial_0 F_{p',q'};\mathbb{R})$ such that $j_{p',q'}x_{p',q'}=x$. For all $p\geqslant p'$, $q\geqslant q'$, set $x_{p,q}=j^{p',q'}_{p,q}x_{p',q'}$; then for all $p\geqslant p'$, $q\geqslant q'$, we have $j_{p,q}x_{p,q}=x$. Furthermore, for $p\geqslant p'$, $q\geqslant q'$,

$$\{x_{p,q},i_{p,q}y\}_{p,q}=\{j^{p',q'}_{p,q}x_{p',q'},i_{p,q}y\}_{p,q}=\{x_{p',q'},i^{p',q'}_{p,q}i_{p,q}y\}_{p',q'}=\{x_{p',q'},i_{p',q'}y\}_{p',q'}.$$

So we can define $\{x,y\}=\{x_{p',q'},i_{p',q'}y\}_{p,q}$ for large p, q. This is independent of the choice of $x_{p',q'}$, for if $j_{p',q'}x_{p',q'}=j_{p',q'}x'_{p',q'}$, consider the direct system obtained by restricting $j_{p,q}$ to the subspace spanned by $x_{p,q}-x'_{p,q}$. The direct limit of this system is zero, and so $j^{p',q'}_{p,q}(x_{p',q'}-x'_{p',q'})$ must be zero for sufficiently large p,q. Whence $\{x,y\}$ is independent of the choice of $x_{p',q'}$.

Suppose that $\{x,y\}=0$ for all $x\in H_1(\tilde{K},\partial\tilde{K};\mathbb{R})$. Let $z\in H_1(F_{p,q},\partial_0 F_{p,q};\mathbb{R})$; then $\{z,i_{p,q}y\}_{p,q}=\{j_{p,q}z,y\}=0$, so since $\{\ ,\ \}_{p,q}$ is non-singular, $i_{p,q}y=0$ and hence $y=0$. Since $H_1(\tilde{K},\partial\tilde{K};\mathbb{R})$ has the same dimension as $H_1(\tilde{K};\mathbb{R})\cong H_2(\tilde{K},\omega\cup\omega';\mathbb{R})$, it follows that $\{\ ,\ \}$ is non-singular.

Let $j_*\colon\ H_1(\tilde{K};\mathbb{R})\to H_1(\tilde{K},\partial\tilde{K};\mathbb{R})$ be the usual isomorphism, and define the non-singular skew-symmetric bilinear pairing

$$[\ ,\ ]\colon\ H_1(\tilde{K};\mathbb{R})\times H_1(\tilde{K};\mathbb{R})\to\mathbb{R}$$

by $[x,y]=\{j_*x,\partial_*^{-1}y\}$.

Define a symmetric pairing

$$(\ ,\ )\colon\ H_1(\tilde{K};\mathbb{R})\times H_1(\tilde{K};\mathbb{R})\to\mathbb{R}$$

by $(x,y)=[tx,y]+[ty,x]$.

The action of t on $H_1(\tilde{K};\mathbb{R})$ is easily seen to be an isometry of $[\ ,\ ]$, and hence also of $(\ ,\ )$. As in [**6**]; Assertion 10, $(\ ,\ )$ is non-singular.

The characteristic polynomial of the isometry t is Δ, the Alexander polynomial of the knot. Let λ be a symmetric quadratic factor of Δ, irreducible over $\mathbb{R}$: we may take $\lambda(t)=t^2-2t\cos\theta+1$, $0<\theta<\pi$. In [6], Milnor defines σ_θ to be the signature of (,) restricted to the λ-primary component of $H_1(\tilde{K};\mathbb{R})$ regarded as a Γ-module. If λ is not symmetric, then the λ-primary component makes no contribution to the signature, as the corresponding quadratic form is hyperbolic.

Finally, note that since the cup product pairing is dual to the intersection pairing, our σ_θ is the same as Milnor's.

11. Equality of signatures

In this section we show how the quadratic form (,) can be obtained from the Blanchfield pairing $\langle\ ,\ \rangle$. Let λ be a factor of Δ, irreducible over $\mathbb{R}$, of the form $\lambda(t)=1-2at+t^2$; and let V_λ be the λ-primary component of $H_1(\tilde{K};\mathbb{R})$. As in §9, V_λ splits as an orthogonal direct sum $V_\lambda^1\oplus\cdots\oplus V_\lambda^m$, with V_λ^r free over $\Gamma/(\lambda^r)$: note that the direct sum is orthogonal with respect to $\langle\ ,\ \rangle$.

Let $u\in V_\lambda^r$, $v\in V_\lambda^s$, $r\neq s$, and let x be a cycle representing u, y a dual cycle representing v. There is a 2-chain c such that $\lambda^r x=\partial c$. Because $\langle u, v\rangle=0$, we know that $V(x, y)\equiv 0 \pmod{\Gamma}$, and so $V(x, y)=f\in\Gamma$. Thus $S(c, y)=f.\lambda^r$. Suppose that we choose a disc d which meets $t^q y$ just once, and meets no other translate of y; thus $S(d, y)=t^q$. Let $z=\partial d$, and set $x'=x+\alpha z$ for some real number α. Note that x' is a cycle representing u. Moreover, $\lambda^r x'=\lambda^r x+\lambda^r\alpha z=\partial(c+\alpha\lambda^r d)$, and $S(c+\alpha\lambda^r d, y)=f\lambda^r+\alpha\lambda^r t^q$. It follows that by choosing the representative cycle x carefully, we can arrange that $S(c, y)=0$. A similar argument holds for y in place of x.

Suppose now that x is a cycle representing $u\in V_\lambda^r$, y is a cycle representing $v\in V_\lambda^s$, $r\neq s$, that $\lambda^s y=\partial d$, and that $S(x, d)=0$. We can expand as formal power series

$$\frac{1}{\lambda^s}=\sum_{i=0}^{\infty} a_i t^i$$

$$\frac{1}{(\bar{\lambda})^s}=\sum_{i=0}^{\infty} -a_{-2s-i}t^{-i}$$

$$y=\partial\left(\frac{d}{\lambda^s}\right)=\partial\left(\sum_{i=0}^{\infty} a_i t^i\right)d$$

$$y=\frac{t^{-2s}}{(\bar{\lambda})^s}d=\partial\left(\sum_{i=0}^{\infty} -a_{-2s-i}t^{-2s-i}\right)d.$$

Set $c=\sum_{-\infty<i<\infty} a_i t^i d$, where $a_i=0$ for $-2s<i<0$.

Then I claim that the homology class $w \in H_2(\tilde{K}, \omega \cup \omega'; \mathbb{R})$ represented by c is such that $\partial_* w = v$; this follows by considering the Mayer-Vietoris sequence and inverse limit of the preceding section. Furthermore, $\{u, w\} = 0$, and so $(u, v) = 0$. Thus V_λ^r is orthogonal to V_λ^s with respect to $(\ , \ )$.

Now let us restrict our attention to V_λ^r; in order to compute the two signatures of this subspace, it is enough to examine $H_\lambda^r = V_\lambda^r/\lambda V_\lambda^r$ (see [7]). Choose an orthogonal basis $(u_1), \ldots, (u_p)$ of H_λ^r with respect to $\langle \ , \ \rangle_\lambda^r$; as above, this is also an orthogonal basis of H_λ^r with respect to the quadratic form induced by $(\ , \ )$. To avoid subscripts, write $u = u_1$. If $\langle (u), (u) \rangle_\lambda^r = b$, then $\langle u, (t\bar{\lambda})^{r-1} u \rangle \equiv b/(t^{-1} - 2a + t) \equiv bt/\lambda \bmod \Gamma$. Arguing as above, there are cycles x, y representing u, and a 2-chain d such that $\lambda (t\bar{\lambda})^{r-1} y = \partial d$ and $S(x, d) = bt$.

We write

$$(t\bar{\lambda})^{r-1} y = \partial \frac{d}{\lambda} = \partial \{1 + t(2a - t) + t^2(2a - t)^2 + \cdots\} d$$

$$(t\bar{\lambda})^{r-1} y = \partial \frac{t^{-2}}{\bar{\lambda}} d = \partial \{t^{-2} 1 + t^{-1}(2a - t^{-1}) + t^{-2}(2a - t^{-1})^2 + \cdots\} d.$$

$$c = -\{1 + t(2a - t) + t^2(2a - t)^2 + \cdots\} d$$
$$+ t^{-2}\{1 + t^{-1}(2a - t^{-1}) + t^{-2}(2a - t^{-1})^2 + \cdots\} d.$$

Let $w \in H_2(\tilde{K}, \omega \cup \omega'; \mathbb{R})$ be the homology class represented by c. Then since $S(x, d) = bt$, we have $T(x, td) = b$ and $T(x, t^i d) = 0$ for $i \neq 1$. Thus

$$\begin{aligned}
\{u, w\} &= -2ab, & \{tu, w\} &= \{u, t^{-1} w\} = b; \\
[u, (t\bar{\lambda})^{r-1} u] &= -2ab, & [tu, (t\bar{\lambda})^{r-1} u] &= b; \\
(u, (t\bar{\lambda})^{r-1} u) &= 2b, \\
(tu, (t\bar{\lambda})^{r-1} u) &= [t^2 u, (t\bar{\lambda})^{r-1} u] + [t(t\bar{\lambda})^{r-1} u, tu] \\
&= [tu, (t + t^{-1})(t\bar{\lambda})^{r-1} u] \\
&= [tu, 2a(t\bar{\lambda})^{r-1} u] \\
&= 2ab.
\end{aligned}$$

Thus regarding H_λ^r as a vector space over $\mathbb{R}(\tau)$ the subspace spanned by (u) yields a real quadratic space with basic u, τu and matrix $\begin{bmatrix} 2b & 2ab \\ 2ab & 2b \end{bmatrix}$; this is the quadratic space determined by $(\ , \ )$.

But the pairing $\langle \ , \ \rangle^r$, restricted to the subspace spanned by (u) and regarded as an Hermitian pairing over $\mathbb{R}(\tau)$ has as its matrix b. Using the trace $\mathbb{R}(\tau) \to \mathbb{R}$, this yields a quadratic space whose matrix with respect to the real basis u, τu is $\begin{bmatrix} 2b & 2ab \\ 2ab & 2b \end{bmatrix}$.

In the case r even, V_λ^r contributes nothing to the Milnor signature [7]; p. 94, remark, and nothing to σ_λ by definition. If r is odd, then the contributions coincide, again by [7]; p. 94. This proves the theorem.

Note that the sign of σ_θ depends on the sign of ∂_* in the Mayer Vietoris sequence: by redefining the homomorphisms in the Mayer-Vietoris sequence we could change the sign of ∂_* and hence the sign of σ_θ.

12. Examples

Recall that for the trefoil knot we obtained

$$W = \left(\frac{t^{-1}-1}{1-t+t^2}\right) \equiv \left(\frac{t^{-2}-t^{-1}}{t^{-1}-1+t}\right) \equiv \left(\frac{-1}{t^{-1}-1+t}\right).$$

In this case $H_1(\tilde{K};\mathbb{R})$ is a 1-dimensional vector space over $\mathbb{R}(\tau)$, where $\tau^2-\tau+1=0$, and the Hermitian form has matrix (-1) with respect to some basis v say. Over $\mathbb{R}$, take v, τv as a basis; since $tr(-\tau)=-1$, the corresponding quadratic form has matrix $\begin{bmatrix}-2 & -1\\ -1 & -2\end{bmatrix}$, which is congruent to $\begin{bmatrix}-2 & 0\\ 0 & -\frac{3}{2}\end{bmatrix}$. Hence the signature is -2.

With the reef knot, there is a self-annihilating subspace with dimension equal to half the dimension of $H_1(\tilde{K};\mathbb{R})$; hence the signature is 0.

Fig. 9 illustrates the knot 6_2 in Reidemeister's table. The computation is as follows.

$$r_Y = yxy^{-1}x^{-1}yx^{-1}y^{-1}xy^{-1}x^{-1}yx^{-1}y^{-1}xyx^{-1}yxy^{-1}xyx^{-1}$$

$$s_0x_0s_0^{-1} = yxy^{-1}xyx^{-1}y^{-1}$$

$$s_1x_1s_1^{-1} = y$$

$$B = \left(\frac{\partial r_Y}{\partial y}\right) = (-t+3-3t^{-1}+3t^{-2}-t^{-3})$$

$$C = \begin{bmatrix} t & 1-2t+t^2\\ 0 & 1\end{bmatrix}$$

$$D = (2t-t^2)$$

$$B \underset{\text{(vii)}}{\longrightarrow} (t^2-3t+3-3t^{-1}+t^{-2})$$

$$D \underset{\text{(vii)}}{\longrightarrow} (t^3-2t^2)$$

$$t^2-3t+3-3t^{-1}+t^{-2} = (t-(3+\sqrt{5})/2+t^{-1})(t-(3-\sqrt{5})/2+t^{-1}).$$

Set $\lambda = t-(3+\sqrt{5})/2+t^{-1}$, $\mu = t-(3-\sqrt{5})/2+t^{-1}$; note that $\lambda=\bar{\lambda}$, $\mu=$

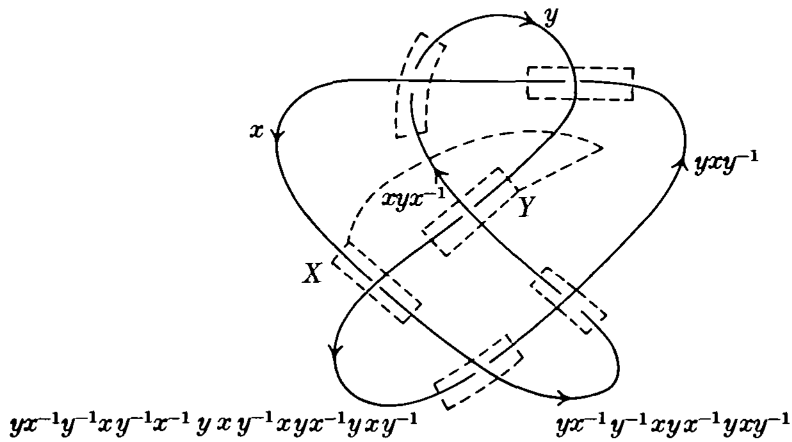


FIG. 9

$\bar{\mu}$, and that $\alpha\lambda+\beta\mu=1$ where $-\alpha=\beta=1/\sqrt{5}$.

$$B=(\lambda\mu)\xrightarrow[\text{(iii)}]{}\begin{bmatrix}\lambda\mu & 0\\ 0 & 1\end{bmatrix}\xrightarrow[\text{(i)}]{}\begin{bmatrix}\lambda\mu & \mu\\ 0 & 1\end{bmatrix}\xrightarrow[\text{(ii)}]{}\begin{bmatrix}0 & \mu\\ -\lambda & 1\end{bmatrix}\xrightarrow[\text{(v)}]{}\begin{bmatrix}-\lambda & 1\\ 0 & \mu\end{bmatrix}$$

$$\xrightarrow[\text{(ii)}]{}\begin{bmatrix}-\lambda & 1-\alpha\lambda\\ 0 & \mu\end{bmatrix}\xrightarrow[\text{(i)}]{}\begin{bmatrix}-\lambda & 1-\alpha\lambda-\beta\mu\\ 0 & \mu\end{bmatrix}=\begin{bmatrix}-\lambda & 0\\ 0 & \mu\end{bmatrix}.$$

Set $\nu=t^3-2t^2$:

$$D=(\nu)=\nu(1)\xrightarrow[\text{(iii)}]{}\nu\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\xrightarrow[\text{(i)}]{}\nu\begin{bmatrix}1 & 0\\ -\mu & 0\end{bmatrix}\xrightarrow[\text{(ii)}]{}\nu\begin{bmatrix}1 & 0\\ -\mu & 0\end{bmatrix}\xrightarrow[\text{(v)}]{}\nu\begin{bmatrix}-\mu & 0\\ 1 & 0\end{bmatrix}$$

$$\xrightarrow[\text{(ii)}]{}\nu\begin{bmatrix}-\mu & -\alpha\mu\\ 1 & \alpha\end{bmatrix}\xrightarrow[\text{(i)}]{}\nu\begin{bmatrix}-\mu & -\alpha\mu\\ 1-\beta\mu & \alpha-\alpha\beta\mu\end{bmatrix}=\nu\begin{bmatrix}-\mu & -\alpha\mu\\ -\beta\lambda & \alpha-\alpha\beta\mu\end{bmatrix}$$

$$\xrightarrow[\text{(ix)}]{}\nu\begin{bmatrix}-\mu & 0\\ 0 & \alpha\end{bmatrix}\xrightarrow[\text{(ix)}]{}\nu\begin{bmatrix}\lambda-\mu & 0\\ 0 & \alpha\end{bmatrix}=\nu\begin{bmatrix}-\sqrt{5} & 0\\ 0 & -1/\sqrt{5}\end{bmatrix}.$$

The polynomial λ is reducible over $\mathbb{R}$, and so contributes nothing to the signature. As an exercise, the reader may care to check directly that this is so. On the other hand, μ is irreducible, and we proceed as follows.

$$B^{-1}=\begin{bmatrix}-1/\lambda & 0\\ 0 & 1/\mu\end{bmatrix},\qquad D^*=\bar{\nu}\begin{bmatrix}-\sqrt{5} & 0\\ 0 & -1/\sqrt{5}\end{bmatrix}$$

$$\begin{bmatrix}w' & 0\\ 0 & w\end{bmatrix}\begin{bmatrix}-\sqrt{5}\bar{\nu} & 0\\ 0 & -\bar{\nu}/\sqrt{5}\end{bmatrix}=(t^{-1}-1)\begin{bmatrix}-1/\lambda & 0\\ 0 & 1/\mu\end{bmatrix}$$

$$\frac{w}{\sqrt{5}}(2t^{-2}-t^{-3})=\frac{t^{-1}-1}{\mu}$$

Let τ be a root of μ, so that $\tau-(3-\sqrt{5})/2+\tau^{-1}=0$. Passing to the Hermitian form over $\mathbf{R}(\tau)$, we obtain

$$\frac{w}{\sqrt{5}}(2\tau^{-2}-\tau^{-3})=\tau^{-1}-1$$

$$\frac{w}{\sqrt{5}}=\frac{\tau^{-1}-1}{2\tau^{-2}-\tau^{-3}}=\frac{\tau^2-\tau^3}{2\tau-1}=\frac{(\tau^2-\tau^3)(2\tau^{-1}-1)}{4-2(\tau+\tau^{-1})+1}$$

$$=\frac{2\tau-\tau^2-2\tau^2+\tau^3}{2+\sqrt{5}}$$

$$\frac{(2+\sqrt{5})}{\sqrt{5}}w=2\tau-3\tau^2+\tau^3$$

$$=2\tau-3\tau^2+\left(\frac{3-\sqrt{5}}{2}\right)\tau^2-\tau$$

$$=\tau-\frac{(3+\sqrt{5})}{2}\tau^2$$

$$=\tau-\left(\frac{3+\sqrt{5}}{2}\right)\left\{\left(\frac{3-\sqrt{5}}{2}\right)\tau-1\right\}$$

$$=\frac{3+\sqrt{5}}{2}$$

$$w=\frac{(3+\sqrt{5})\sqrt{5}}{2(2+\sqrt{5})}=\frac{5-\sqrt{5}}{2}>0.$$

Thus $\sigma_\mu=2$.

13. Torus knots

Finally we adapt the methods developed above to find the signatures of the (p,q)-torus knot. We regard S^3 as the union of $S^1\times B^2$ and $B^2\times S^1$ identified along their common boundary $S^1\times S^1$, oriented so that $L(S^1\times 0, 0\times S^1)=1$. For positive coprime integers p and q, the (p,q)-torus knot k is an embedded circle in $S^1\times S^1$ representing the element (p,q) in $H_1(S^1\times S^1)=\mathbf{Z}\oplus\mathbf{Z}$. Thus k winds p times around the first factor of $S^1\times S^1$ and q times around the second. Figure 7 illustrates the case $p=2$, $q=3$.

The Alexander polynomial of the (p,q)-torus knot is $\Delta(t)=(1-t^{pq})(1-t)/(1-t^p)(1-t^q)$, and so its roots are the pqth roots of unity, excluding the pth and qth roots of unity. These can be obtained in the form $\exp 2\pi i(a/p+b/q)$, where a runs over a complete set of non-zero residue classes mod p, and b runs over a complete set of non-zero residue classes mod q. In particular, we can take $0<a<p$, $0<b<q$. The complex

conjugate of $\exp 2\pi i(a/p+b/q)$ is $\exp 2\pi i((p-a)/p+(q-b)/q)$, and so we run through the real quadratic factors $1-2t\cos 2\pi(a/p+b/q)+t^2$ of $\Delta(t)$ by restricting attention to the set $X=\{(a,b)\colon 0<a<p,\ 0<b<q,\ a/p+b/q<1\}$. Denote the Milnor signature corresponding to $\exp 2\pi i(a/p+b/q)$ by $\sigma(a,b)$; then with the notation above we can state the following result.

THEOREM. $\sigma(a,b)=\ \ 2$ *if* $a/p+b/q<\frac{1}{2}$
$\qquad\qquad\qquad -2$ *if* $a/p+b/q>\frac{1}{2}$.

This result has previously been obtained by T. Matumoto [**8**].

I should like to thank D. Zagier for showing me how to handle the number theory involved in the proof.

Pick $*\in S^1\times S^1-k$, and let x be the loop which goes from $*$ along a radius of B^2 to $S^1\times 0\subset S^1\times B^2$, once around $S^1\times 0$ in the positive direction, and back along the radius to $*$. Similarly let y go from $*$ along a radius to $0\times S^1\subset B^2\times S^1$, once around $0\times S^1$ in the positive direction, and back to $*$. Because $L(x,k)=q$ and $L(y,k)=p$, we define $\varepsilon(x)=t^q$, $\varepsilon(y)=t^p$.

Let l be a path in $S^1\times S^1$, containing $*$, which is parallel to k: that is, represents (p,q) in $H_1(S^1\times S^1)$ and does not meet k. Join each point of l by the radius of $B^2\subset S^1\times B^2$ on which it lies to $S^1\times 0\subset S^1\times B^2$, and similarly join l to $0\times S^1\subset B^2\times S^1$. Call the resulting 2-complex A. Then ∂A consists of p copies of $S^1\times 0$ and q copies of $0\times S^1$. We can regard A as a singular 2-cell whose boundary gives a loop r_A based at $*$, such that $r_A=y^qx^{-p}$; then by Van Kampen's theorem the knot group has a presentation $(x,y\colon\ y^qx^{-p})$.

Choosing lifts $\tilde{x}$, $\tilde{y}$, $\tilde{A}$ based at $\tilde{*}$ in the infinite cyclic cover $\tilde{K}$, it is a standard exercise in the free differential calculus to establish that

$$\partial\tilde{A}=-\frac{(1-t^{pq})}{1-t^q}\,\tilde{x}+\frac{(1-t^{pq})}{1-t^p}\,\tilde{y}.$$

Setting $\Delta(t)=(1-t^{pq})(1-t)/(1-t^p)(1-t^q)$,

$$a(t)=(1-t^p)/(1-t),\ b(t)=(1-t^q)/(1-t),$$

this becomes

$$\partial\tilde{A}=\Delta(t)[-a(t)\tilde{x}+b(t)\tilde{y}].$$

Because p and q are coprime, there exist integers r and s such that $pr+qs=1$. Setting $f(t)=(1-t^{pr})/(1-t^p)$, $g(t)=(1-t^{qs})/(1-t^q)$, it is easily checked that $a(t)f(t)+b(t)g(t)=1$. Thus $H_1(K)$ is cyclic of order $\Delta(t)$, with $\tilde{z}=-a(t)\tilde{x}+b(t)\tilde{y}$ as a generator.

Now let $*'$ be a new base point, and x', y' be paths as indicated in Fig.

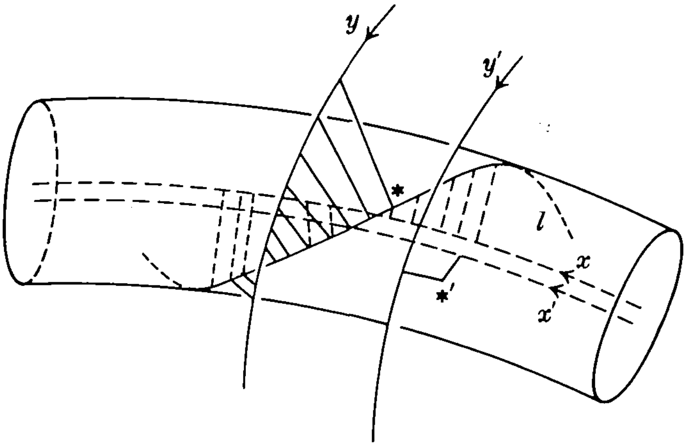


FIG. 10

10. Then x', y' play the role of a dual basis for $C_1(K)$. We must now compute $S(\tilde{A}, \tilde{x}')$. Recall that

$$\partial\tilde{A} = -(1+t^q+t^{2q}+\cdots+t^{(p-1)q})\tilde{x}+(1+t^p+\cdots+t^{(q-1)p})\tilde{y}.$$

The ordinary intersection of $\tilde{x}'$ with $\tilde{A}$ is obtained by starting at $*$ and tracing along l until one circuit of $S^1\times S^1$ in the x direction has been made. The number of whole revolutions in the y direction (plus one) is the number of times that $\tilde{A}$ meets $\tilde{x}'$, and this is the number of multiples of p in the interval $[0, q)$. Similarly the intersection of $\tilde{A}$ with $t^{rq}\tilde{x}'$ is equal to the number of multiples of p in the interval $[rq, (r+1)q)$, for $0\leq r<p$.

Define $\lambda(r)$ to be the number of multiples of p in the interval $[rq, (r+1)q)$, and $\mu(s)$ to be the number of multiples of q in $(sp, (s+1)p]$. Then arguing as above it can be seen that

$$S(\tilde{A}, \tilde{x}')=\sum_{r=0}^{p-1}\lambda(r)t^{rq}$$

$$S(\tilde{A}, \tilde{y}')=-\sum_{s=0}^{q-1}\mu(s)t^{sp}.$$

Setting $\tilde{z}'=-a(t)\tilde{x}'+b(t)\tilde{y}'$, we have

$$\begin{aligned}\Delta(t)V(\tilde{z}, \tilde{z}')&=S(\tilde{A}, \tilde{z}')\\&=-a(t^{-1})S(\tilde{A}, \tilde{x}')+b(t^{-1})S(\tilde{A}, \tilde{y}')\\&=-\frac{(1-t^{-p})}{(1-t^{-1})}\sum_{r=0}^{p-1}\lambda(r)t^{rq}-\frac{(1-t^{-q})}{(1-t^{-1})}\sum_{s=0}^{q-1}\mu(s)t^{sp}.\end{aligned}$$

Call the Laurent polynomial on the right $-F(t)$.

Let c denote the integral part of the real number c, and define $((c))$ by

$$((c))=\begin{cases}c-[c]-\frac{1}{2} & \text{if} \quad c\notin \mathbf{Z}\\ 0 & \text{if} \quad c\in \mathbf{Z}.\end{cases}$$

Then we can write

$$F(t)=\frac{(1-t^{-p})}{(1-t^{-1})}\sum_{r=0}^{p-1}\left(\left[\frac{(r+1)q}{p}\right]-\left[\frac{rq}{p}\right]\right)t^{rq}$$

$$+\frac{(1-t^{-q})}{(1-t^{-1})}\sum_{s=0}^{q-1}\left(\left[\frac{(s+1)p}{q}\right]-\left[\frac{sp}{q}\right]\right)t^{sp}$$

$$+\frac{(1-t^{-p})}{(1-t^{-1})}(1-t^{(p-1)q})$$

Recall that $V(\tilde{z},\tilde{z}')$ is only defined mod Λ, and so we are only interested in $F(t)$ mod $\Delta(t)$.

$$\sum_{r=0}^{p-1}\left(\left[\frac{(r+1)q}{p}\right]-\left[\frac{rq}{p}\right]\right)t^{rq}$$

$$=\frac{q}{p}\sum_{r=0}^{p-1}t^{rq}-\sum_{r=0}^{p-1}\left\{\left(\left(\frac{(r+1)q}{p}\right)\right)-\left(\left(\frac{rq}{p}\right)\right)\right\}t^{rq}+\tfrac{1}{2}(t^{(p-1)q}-1)$$

$$\equiv\tfrac{1}{2}(t^{-q}-1)+\sum_{r=0}^{p-1}\left(\left(\frac{rq}{p}\right)\right)\{t^{rq}-t^{(r-1)q}\}.$$

To see the last step, set t equal to any root τ of $\Delta(t)$; thus $\tau^{pq}=1$ but $\tau^{p}\neq 1$ and $\tau^{q}\neq 1$. Then

$$\sum_{r=0}^{p-1}\tau^{rq}=\sum_{r=0}^{p-1}(\tau^{q})^{r}=0$$

and so $\Delta(t)$ divides $\sum_{r=0}^{p-1}t^{rq}$.

Noting that $(t^{rq}-t^{(r-1)q})=t^{rq}(1-t^{-q})$, we obtain

$$F(t)\equiv\frac{(1-t^{-p})(1-t^{-q})}{1-t^{-1}}\left\{\sum_{r=0}^{p-1}\left(\left(\frac{rq}{p}\right)\right)t^{rq}+\sum_{s=0}^{q-1}\left(\left(\frac{sp}{q}\right)\right)t^{sp}\right\}$$

Define p^*, q^* by $q^*q\equiv 1 \pmod p$ and $p^*p\equiv 1 \pmod q$; we shall be dealing with periodic functions and shall only be interested in the value of $p^* \pmod q$ and $q^* \pmod p$. Set $\zeta=\exp(2\pi imq^*/p)$, $\tau=\exp(2\pi im/pq)$, where τ is a root of $\Delta(t)$, and note that $\tau^q=\zeta^q$, $\zeta^p=1$, $\zeta\neq 1$. We have

$$\begin{aligned}\sum_{r=0}^{p-1}\left(\left(\frac{rq}{p}\right)\right)\tau^{rq} &= \sum_{r(\mathrm{mod}\,p)}\left(\left(\frac{rq}{p}\right)\right)\zeta^{rq}\\ &= \sum_{r(\mathrm{mod}\,p)}\left(\left(\frac{r}{p}\right)\right)\zeta^{r}\\ &= \sum_{r=1}^{p-1}\left(\frac{r}{p}-\frac{1}{2}\right)\zeta^{r}.\end{aligned}$$

But $(1-\zeta)\sum_{r=1}^{p-1} r\zeta^r = \zeta+2\zeta^2+\cdots+(p-1)\zeta^{p-1}-\zeta^2-2\zeta^3-\cdots-(p-1)^{-p}$

$$\begin{aligned}&=\zeta+\zeta^2+\cdots+\zeta^{p-1}+\zeta^p-p\zeta^p\\ &=-p.\end{aligned}$$

Whence

$$\sum_{r=0}^{p-1}\left(\left(\frac{rq}{p}\right)\right)\tau^{rp}=-\frac{1}{(1-\zeta)}+\frac{1}{2}=\frac{1}{2}\cdot\frac{(\zeta+1)}{(\zeta-1)}.$$

Setting $\xi=\exp(2\pi imp^*/q)$, we have

$$\begin{aligned}F(\tau)&=\frac{(1-\tau^{-p})(1-\tau^{-q})}{(1-\tau^{-1})}\left\{\frac{1}{2}\cdot\frac{(\zeta+1)}{(\zeta-1)}+\frac{1}{2}\cdot\frac{(\xi+1)}{(\xi-1)}\right\}\\ &=\frac{(\tau^{p/2}-\tau^{-p/2})(\tau^{q/2}-\tau^{-q/2})}{(\tau^{1/2}-\tau^{-1/2})}\cdot\frac{\tau^{(1-p-q)/2}}{2}\left\{\frac{(\zeta^{1/2}+\zeta^{-1/2})}{(\zeta^{1/2}-\zeta^{-1/2})}+\frac{(\xi^{1/2}+\xi^{-1/2})}{(\xi^{1/2}-\xi^{-1/2})}\right\}\end{aligned}$$

$$F\left(\exp\frac{2\pi im}{pq}\right)=\exp\pi i\left(\frac{m}{pq}-\frac{m}{p}-\frac{m}{q}\right)\cdot\frac{\sin(\pi m/q)\sin(\pi m/p)}{\sin(\pi m/pq)}\left\{\cot\frac{\pi mq^*}{p}+\cot\frac{\pi mp^*}{q}\right\}.$$

Working mod Λ, we have

$$\frac{F(t)}{\Delta(t)}=\sum_m\frac{F(\exp(2\pi im/pq))}{\Delta'(\exp(2\pi im/pq))}\cdot\frac{1}{t-\exp(2\pi im/pq)}$$

where the sum runs over a complete set of residues mod pq, excluding multiples of p and multiples of q. Here $\Delta'(t)$ denotes the derivative of $\Delta(t)$. An easy calculation gives

$$\Delta'(\exp(2\pi im/pq))=\frac{pq}{2i}\exp\pi i\left(-\frac{m}{q}-\frac{m}{p}-\frac{m}{pq}\right)\cdot\frac{\sin(\pi m/pq)}{\sin(\pi m/p)\sin(\pi m/q)},$$

and so

$$\frac{F(t)}{\Delta(t)}=\sum_m \frac{2i}{pq}\exp(2\pi im/pq)\cdot\frac{\sin^2(\pi m/p)\sin^2(\pi m/q)}{\sin^2(\pi m/pq)}$$
$$\cdot\left\{\cot\frac{\pi mq^*}{p}+\cot\frac{\pi mp^*}{q}\right\}\cdot\frac{1}{t-\exp(2\pi im/pq)}$$
$$=\frac{i}{pq}\sum_m\frac{\sin^2(\pi m/p)\sin^2(\pi m/q)}{\sin^2(\pi m/pq)}\left\{\cot\frac{\pi mq^*}{p}+\cot\frac{\pi mp^*}{q}\right\}\Delta_m(t)$$

where

$$\Delta_m(t)=\frac{\exp(2\pi im/pq)}{t-\exp(2\pi im/pq)}-\frac{\exp(-2\pi im/pq)}{t-\exp(-2\pi im/pq)}$$
$$=\frac{4it\sin(\pi m/pq)\cos(\pi m/pq)}{t^2-2t\cos(2\pi m/pq)+1}.$$

Thus

$$V(\tilde{z},\tilde{z}')=\frac{4}{pq}\sum_m\sin^2\frac{\pi m}{p}\sin^2\frac{\pi m}{q}\cot\frac{\pi m}{pq}\left\{\cot\frac{\pi mq^*}{p}+\cot\frac{\pi mp^*}{q}\right\}$$
$$\cdot\frac{t}{t^2-2t\cos(2\pi m/pq)+1}$$
$$=\sum_{\substack{m\,(\mathrm{mod}\,pq)\\ p\nmid m,\,q\nmid m}} c_m\frac{t}{t^2-2t\cos(2\pi m/pq)+1}$$

where

$$c_m=\frac{4}{pq}\cdot\sin^2\frac{\pi m}{p}\sin^2\frac{\pi m}{q}\cdot\cot\frac{\pi m}{pq}\left\{\cot\frac{\pi mq^*}{p}+\cot\frac{\pi mp^*}{q}\right\}.$$

If we set $m=aq+bp$, and allow a to run through a complete set of non-zero residue classes mod p, and b to run through a similar set mod q, then m runs through the required set of residue classes mod pq. So

$$c_m=c_{a,b}=\frac{4}{pq}\sin^2\frac{\pi aq}{p}\sin^2\frac{\pi bp}{q}\cot\pi\left(\frac{a}{p}+\frac{b}{q}\right)\left\{\cot\frac{\pi a}{p}+\cot\frac{\pi b}{q}\right\}.$$

Now $\cot x+\cot y=0\Leftrightarrow\cot x=\cot(-y)$

$$\Leftrightarrow x=n\pi-y$$
$$\Leftrightarrow x+y=n\pi.$$

But $0<\frac{a}{p}+\frac{b}{q}<1\Rightarrow 0<\pi\left(\frac{a}{p}+\frac{b}{q}\right)<\pi$, and so for a and b in this range the sign of $c_{a,b}$ is equal to that of $\cot\pi\left(\frac{a}{p}+\frac{b}{q}\right)$.

Recall the set X defined at the beginning of this section: we see that

$$V(\tilde{z}, \tilde{z}') = \sum_{(a,b)\in X} \frac{2c_{a,b}t}{t^2 - 2t\cos 2\pi\left(\frac{a}{p}+\frac{b}{q}\right)+1}.$$

The theorem will follow if we can show that the sign of $c_{a,b}$ is the same as that of the corresponding Milnor signature, since for $(a, b)\in X$,

$$\cot \pi\left(\frac{a}{p}+\frac{b}{q}\right) > 0 \quad \text{iff} \quad \frac{a}{p}+\frac{b}{q} < \frac{1}{2}.$$

Set $\nabla_{a,b}(t) = t - 2\cos 2\pi\left(\frac{a}{p}+\frac{b}{q}\right) + t^{-1}$, let $\nabla(t) = \Delta(t).t^{-n}$ where $n = (p-1)(q-1)/2$, and let $\nabla^{a,b}(t) = \nabla(t)/\nabla_{a,b}(t)$. Let $V = H_1(\tilde{K}; \mathbb{R})$ and let σ be the image of $\tilde{z}$(and $\tilde{z}'$) in V. Then $\nabla^{a,b}(t)V$ is the subspace of V annihilated by $\nabla_{a,b}(t)$. Referring to Section 9, it is easy to see that this is a 1-dimensional vector space over $\mathbb{C}$, with basis given by $\nabla^{a,b}(t)\sigma$, and that the signature of the corresponding Hermitian form is sign $c_{a,b}$.

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College of St Hild and St Bede
Durham, DH1 1SZ
England.