SOME NON-FIBRED 3-KNOTS

C. KEARTON

It is well known that a simple n-knot, $n > 3$, is fibred if and only if the leading coefficient of its Alexander polynomial is ± 1. This can be proved directly by doing ambient surgery on a Seifert surface of the knot, in the manner of J. Levine [**6**], or by using the fibration theorem of W. Browder and J. Levine [**2**, Corollary 1.6]. Using a recent result of S. K. Donaldson [**3**], concerning the intersection form on a smooth closed simply-connected 4-manifold, we are able to construct examples of simple 3-knots which satisfy the condition above but are not fibred. As a corollary, we see at once that the Browder–Levine fibration theorem does not extend to dimension 5. Furthermore, it follows easily that the quasi-fibring conjecture for 5-manifolds (with infinite-cyclic fundamental group), and the smoothing conjecture for simply-connected finite Poincaré complexes in dimension 4 also fail. For details of both these conjectures the reader is referred to [**7**].

An *n-knot* k is a smooth pair (S^{n+2}, Σ^n) where Σ^n is homeomorphic to the n-sphere S^n. Let K denote the exterior of k; that is, the complement of an open tubular neighbourhood of Σ^n in S^{n+2}. The knot k is said to be *fibred* if K fibres smoothly over S^1. The $(2q-1)$-knot k is *simple* if $\pi_i(K) \cong \pi_i(S^1)$ for $1 \leqslant i < q$; for $q \geqslant 2$, such knots have been classified in terms of the S-equivalence class of the Seifert matrix (see [**6**] for details).

There exists a 16×16 integer matrix A with the following properties

(i) $\det(A + A') = 1$

(ii) $\operatorname{signature}(A + A') = 16$

(iii) $\det A = 1$.

Examples can be found in [**1**, p. 502]. By the results of J. Levine [**6**], there exists a unique simple 3-knot $k = (S^5, S^3)$ with Seifert matrix S-equivalent to A. The leading coefficient for the Alexander polynomial of k is $\det A = 1$, so one might expect k to be fibred. Suppose that this were so, and let V be a Seifert surface of k diffeomorphic to the fibre. If B is a Seifert matrix of k arising from V, then $B = P'AP$ for some unimodular integer matrix P (see [**6**, Proposition 2]). But $B + B'$ represents the intersection pairing on $H_2(V)$, and since $\partial V = S^3$ this contradicts the result of S. K. Donaldson [**3**], that if the intersection pairing on $H_2(M\,;\mathbb{Z})$ is positive definite, where M is a closed smooth simply-connected 4-manifold, then it can be represented by the identity matrix. Hence k is not fibred, and the Browder–Levine fibration theorem does not extend to dimension 5. We remark that $\sigma(k)$, the 4-knot obtained by spinning k, satisfies the Browder–Levine conditions and hence is fibred. (For details of spinning and the algebraic invariants of $\sigma(k)$, see [**5**].)

Let k be the exterior of the knot above, and set $L = K \bigcup_{S^1 \times S^3} (S^1 \times B^4)$ where $\partial K = S^1 \times S^3$. Then L is a closed 5-manifold with $\pi_1(L) \cong \mathbb{Z}$, and the higher homotopy groups of L are finitely-generated since the homology groups of $\tilde{L}$, the universal cover of L, are finitely-generated. The only group that needs checking is

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$H_2(\tilde{L})$, which is a free abelian group of rank 16 since $\det A = 1$. Hence $\tilde{L}$ has the homotopy type of a finite complex.

Suppose that L were quasi-fibred over the circle; that is, there exists a smooth closed simply-connected submanifold N of codimension one such that the sequences

$$0 \longrightarrow \pi_i(N) \longrightarrow \pi_i(L) \xrightarrow{g_*} \pi_i(S^1) \longrightarrow 0$$

are exact where $g: L \to S^1$ represents a generator of $H^1(L;\mathbb{Z})$. Then we may assume that $N = g^{-1}(*)$ for some $* \in S^1$, by the Thom–Pontryagin construction. Let $U = V \bigcup_{S^3} B^4 \subset L$ where V is any Seifert surface of k. Then $U = f^{-1}(*)$ where $f: L \to S^1$ represents the same generator of $H^1(L;\mathbb{Z})$ as g does. Hence f and g are homotopic, and so U and N are framed cobordant. Since $H_2(L;\mathbb{Z}) \cong H_3(L;\mathbb{Z}) = 0$, we can associate a linking number to any pair of disjoint 2-cycles in L. Hence U and its framing yields a matrix of integers just as the Seifert surface V does; in fact, we obtain the Seifert matrix associated with the same choice of basis of $H_2(V) \cong H_2(U)$. Similarly we obtain a matrix C from N. The argument of [**6**, §§7, 6, 8] applies to show that these matrices are S-equivalent. But as above, this means that $C+C'$ is positive definite of rank 16, and represents the intersection pairing on $H_2(N;\mathbb{Z})$, which is a contradiction. Hence, L is a counter-example to the quasi-fibring conjecture for 5-manifolds (with $\pi_1 = \mathbb{Z}$) in [**7**], and hence the smoothing conjecture for simply-connected finite Poincaré complexes in dimension 4 also fails. For details of both these conjectures, and their equivalence, the reader is referred to [**6**].

The referee has pointed out that it is possible to construct other counterexamples to the conjectures above by appealing to recent work of M. H. Freedman. In [**4**], Freedman gives a complete classification of closed simply-connected topological 4-manifolds: such a manifold M is specified up to homeomorphism by a unimodular quadratic form over $\mathbb{Z}$ (which can be arbitrary) and, if the form is odd, an invariant in $\mathbb{Z}_2$ which is the obstruction to smoothing $M \times S^1$. In the even case, M is a closed spin manifold and the obstruction to smoothing $M \times S^1$ is (signature M)/8, mod 2. Taking the form $A+A'$ yields a topological manifold M such that $M \times S^1$ is a counterexample to the conjectures above.

Added in proof, March 1983. That the knot k fibres topologically over the circle can be seen as follows. By [**4**] the isometry $A^{-1}A'$ of $A+A'$ is induced by a homeomorphism $h: M \to M$, where M is the unique closed simply-connected topological 4-manifold with intersection form represented by $A+A'$. Replacing h by an isotopic map if necessary we may assume that it is the identity on a 4-ball B^4 (this is possible because of the topological annulus conjecture settled in [**8**]). Set $V = M - \operatorname{int} B^4$, and construct K by using $h \mid V$ to identify the two ends of $V \times I$. Then K is the exterior of a locally-flat topological 3-knot $k = (S^5, S^3)$ with Seifert surface homeomorphic to V and Seifert matrix A. Since signature $(A+A') = 16$, k can be smoothed, and hence is the same as the knot k considered above. Hence k is a smooth 3-knot which fibres topologically but not smoothly.

References

1. E. Bayer, J. A. Hillman and C. Kearton, 'The factorization of simple knots', *Math. Proc. Camb. Phil. Soc.*, 90 (1981), 495–506.
2. W. Browder and J. Levine, 'Fibering manifolds over a circle', *Comm. Math. Helv.*, 40 (1965), 153–160.

3. S. K. DONALDSON, 'Self-dual connections and the topology of smooth 4-manifolds', *Bull. Amer. Math. Soc.*, 8 (1983), 81–83.
4. M. H. FREEDMAN, 'The topology of 4-dimensional manifolds', *J. Differential Geom.*, 17 (1982), 357–453.
5. C. MCA. GORDON, 'Some higher-dimensional knots with the same homotopy groups', *Quart. J. Math. Oxford Ser.* (2), 24 (1973), 411–422.
6. J. LEVINE, 'An algebraic classification of some knots of codimension two', *Comm. Math. Helv.*, 45 (1970), 185–198.
7. J. L. SHANESON, 'Non-simply-connected surgery and some results in low dimensional topology', *Comm. Math. Helv.*, 45 (1970), 333–352.
8. F. QUINN, 'Ends of maps, III', *J. Differential Geom.*, 17 (1982), 503–521.

Department of Mathematical Sciences,
University of Durham,
Durham DH1 3LE,
England.