Quadratic Forms in Knot Theory

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Abstract. The purpose of this survey article is to show how quadratic and hermitian forms can give us geometric results in knot theory. In particular, we shall look at the knot cobordism groups, at questions of factorisation and cancellation of high-dimensional knots, and at branched cyclic covers.

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1. The Seifert Matrix

By an n-knot k we mean a smooth or locally-flat PL pair $\left(S^{n+2}, S^n\right)$, both spheres being oriented. In the smooth case the embedded sphere S^n may

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have an exotic smooth structure. Two such pairs are to be regarded as equivalent if there is an orientation preserving (smooth or PL) homeomorphism between them.

As shown in [**33, 48**], a regular neighbourhood of S^n has the form $S^n \times B^2$; we set $K = S^{n+2} - \operatorname{int}\left(S^n \times B^2\right)$. Then K is the *exterior* of the knot k, and since a regular neighbourhood of S^n is unique up to ambient isotopy it follows that K is essentially unique.

PROPOSITION 1.1. *S^n is the boundary of an orientable $(n+1)$-manifold in S^{n+2}.*

This is proved in [**31, 33, 48**], but in the case $n = 1$ there is a construction due to Seifert [**41**] which we now give.

PROPOSITION 1.2. *Every classical knot k is the boundary of some compact orientable surface embedded in S^3.*

PROOF. Consider a diagram of k. Starting at any point of k, move along the knot in the positive direction. At each crossing point, jump to the other piece of the knot and follow that in the positive direction. Eventually

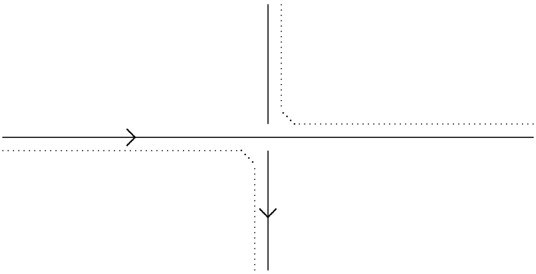


FIGURE 1.1

we return to the starting point, having traced out a *Seifert circuit*. Now start somewhere else, and continue until the knot is exhausted. The Seifert circuits are disjoint circles, which can be capped off by disjoint discs, and joined by half-twists at the crossing points. Hence we get a surface V with

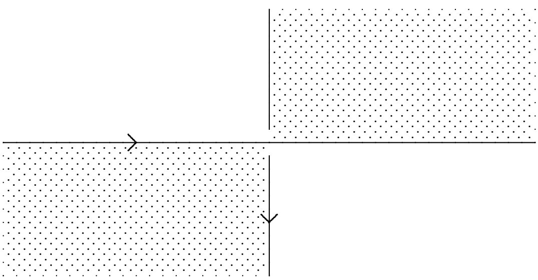


FIGURE 1.2

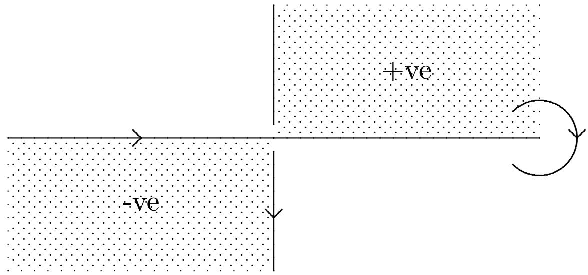


FIGURE 1.3

$\partial V = k$. To see that V is orientable, attach a normal to each disc using a right hand screw along the knot. Note that in passing from one disc to another the normal is preserved. Thus there are no closed paths on V which reverse the sense of the normal: hence V is orientable. $\square$

COROLLARY 1.3. *If there are c crossing points and s Seifert circuits, then the genus of V is $\frac{1}{2}(c - s + 1)$.*

PROOF. The *genus* of V is g where $H_1(V) = \bigoplus_1^{2g} \mathbb{Z}$. We have a handle decomposition of V with s 0-handles and c 1-handles, which is equivalent to one with a single 0-handle and $c-(s-1)$ 1-handles (by cancelling 0-handles). Thus $2g = c - s + 1$. $\square$

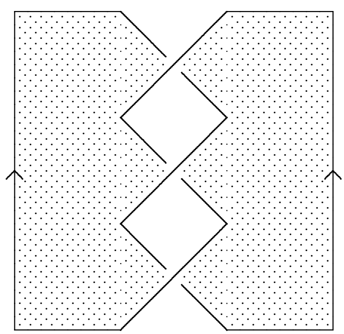
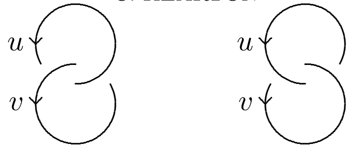


FIGURE 1.4. Trefoil knot

As an example we have a genus one Seifert surface of the trefoil knot in Figure 1.4.

DEFINITION 1.4. Let u, v be two oriented disjoint copies of S^1 in S^3 and assign a linking number as follows. Span v by a Seifert surface V and move u slightly so that it intersects V transversely. To each point of intersection we assign $+1$ or -1 according as u is crossing in the positive or negative direction, and taking the sum of these integers gives us $L(u, v)$. Two simple examples are indicated in Figure 1.5.



$L(u, v) = +1$ $\qquad$ $L(u, v) = -1$

FIGURE 1.5. The linking number

The two copies of S^1 do not have to be embedded: the general definition is in terms of cycles and bounding chains.

DEFINITION 1.5. Let $u, v \in H_1(V)$ for some Seifert surface V. Let i_+u be the result of pushing u a small distance along the positive normal to V. Then set $\theta(u, v) = L(i_+u, v)$.

LEMMA 1.6. *$\theta : H_1(V) \times H_1(V) \to \mathbb{Z}$ is bilinear.*

DEFINITION 1.7. Let $x_1, \ldots, x_{2g}$ be a basis for $H_1(V) = \bigoplus_1^{2g} \mathbb{Z}$, and set $a_{ij} = \theta(x_i, x_j)$. The matrix $A = (a_{ij})$ is a *Seifert matrix* of k.

Note that if we choose another basis for $H_1(V)$, then A is replaced by PAP', where $\det P = \pm 1$, since P is a matrix over $\mathbb{Z}$ which is invertible over $\mathbb{Z}$.

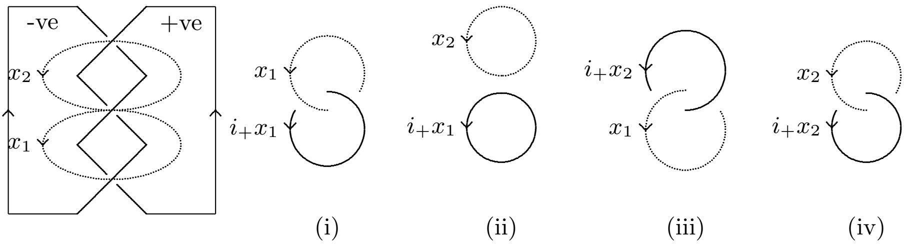


FIGURE 1.6

(i)	$a_{11} = \theta(x_1, x_1) = L(i_+x_1, x_1)$	-1
(ii)	$a_{12} = \theta(x_1, x_2) = L(i_+x_1, x_2)$	0
(iii)	$a_{21} = \theta(x_2, x_1) = L(i_+x_2, x_1)$	1
(iv)	$a_{22} = \theta(x_2, x_2) = L(i_+x_2, x_2)$	-1

TABLE 1.1

EXAMPLE 1.8. We see from Table 1.1 that the Seifert matrix of the trefoil knot in Figure 1.6 is

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

Given a knot k, there will be infinitely many Seifert surfaces of k; for example, we can excise the interiors of two disjoint closed discs from any given Seifert surface V and glue a tube $S^1 \times B^1$ to what remains of V by the boundary circles, as illustrated in Figure 1.7

DEFINITION 1.9. A Seifert surface U is obtained from a Seifert surface V by *ambient surgery* if U and V are related as in Figure 1.7 or Figure 1.8. In the first case, the interiors of two disjoint closed discs in the interior of V are excised and a tube $S^1 \times B^1$ is attached, the two attachments being made on the same side of V. In symbols, $S^0 \times B^2$ is replaced by $S^1 \times B^1$. In the second case, the procedure is reversed.

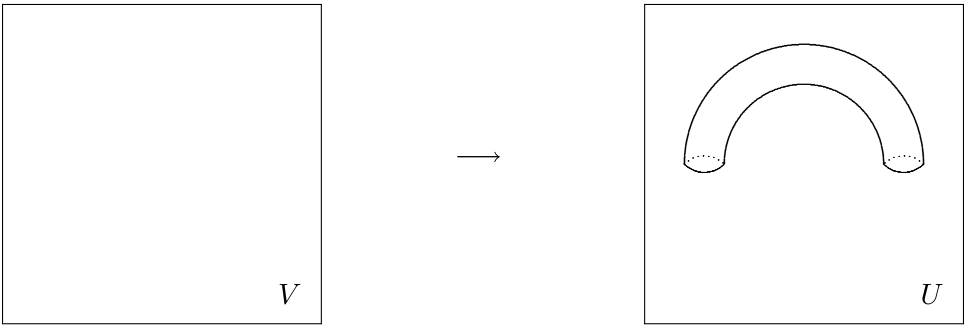


FIGURE 1.7. Ambient surgery (i)

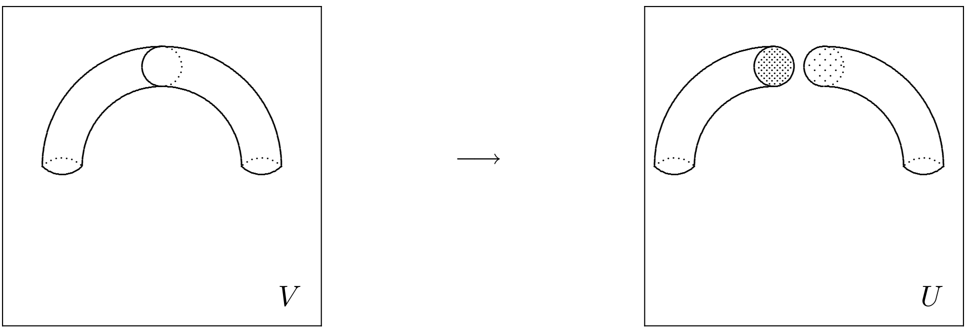


FIGURE 1.8. Ambient surgery (ii)

Note that the "hollow handle" may be knotted, and that these are inverse operations.

PROPOSITION 1.10. *For a given knot k, any two Seifert surfaces are related by a sequence of ambient surgeries.*

DEFINITION 1.11. Let A be a Seifert matrix. An *elementary S-equivalence* on A is one of the following, or its inverse.

(i) $A \mapsto PAP'$ for P a unimodular integer matrix.

(ii)
$$A \mapsto \begin{pmatrix} A & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
(iii)
$$A \mapsto \begin{pmatrix} A & \beta & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
where α is a row vector of integers and β is a column vector of integers.

Two matrices are *S-equivalent* if they are related by a finite sequence of such moves.

THEOREM 1.12. *Any two Seifert matrices of a given knot k are S-equivalent.*

PROOF. Let A be the matrix obtained from a Seifert surface U, B from V. After Proposition 1.10, it is enough to assume that V is obtained by an ambient surgery on U. Consider the diagram in Figure 1.9, and choose gen-

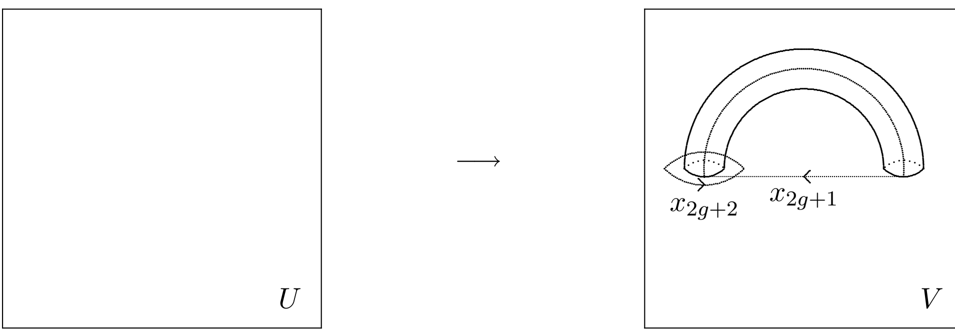


FIGURE 1.9

erators $x_1, \ldots, x_{2g}$ of $H_1(U)$, and $x_1, \ldots, x_{2g}, x_{2g+1}, x_{2g+2}$ of $H_1(V)$. Then

$$\begin{aligned} \theta\left(x_{2g+1}, x_{2g+1}\right) &= 0 \text{ if we choose the right number of twists around the} \\ &\qquad \text{handle (see Figure 1.10 for a different choice)} \\ \theta\left(x_{2g+1}, x_{2g+2}\right) &= 0 \\ \theta\left(x_{2g+2}, x_{2g+1}\right) &= 1 \\ \theta\left(x_{2g+2}, x_{2g+2}\right) &= 0 \\ \theta\left(x_i, x_{2g+2}\right) &= 0 \\ \theta\left(x_{2g+2}, x_i\right) &= 0 \text{ for } 1 \leq i \leq 2g. \end{aligned}$$

Thus
$$B = \begin{pmatrix} A & \gamma & 0 \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

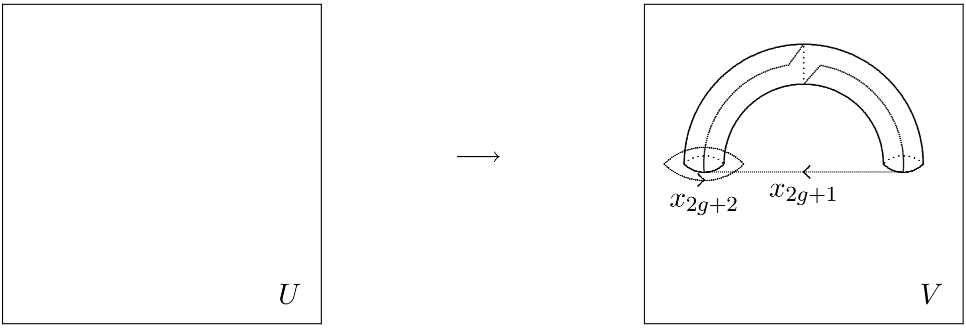


FIGURE 1.10

By a change of basis we can subtract multiples of the last row from the first $2g$ rows to eliminate γ; and the same multiples of the last column from the first $2g$ columns. Whence

$$P'BP = \begin{pmatrix} A & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Adding the handle on the other side of U gives the other kind of S-equivalence. $\square$

LEMMA 1.13. *If A is a Seifert matrix and A' is its transpose, then $A - A'$ is unimodular.*

PROOF. Recall that if $x_1, \ldots, x_{2g}$ is a basis for $H_1(V) = \bigoplus_1^{2g} \mathbb{Z}$, and $a_{ij} = \theta(x_i, x_j)$, then the matrix $A = (a_{ij})$ is a *Seifert matrix* of k. Let $i_- u$ be the result of pushing u a small distance along the negative normal to V. Then

$$\begin{aligned} a_{ij} &= \theta(x_i, x_j) \\ &= L\left(i_+ x_i, x_j\right) \\ &= L\left(x_i, i_- x_j\right) \\ &= L\left(i_- x_j, x_i\right) \end{aligned}$$

and so

$$a_{ij} - a_{ji} = L\left(i_+ x_i, x_j\right) - L\left(i_- x_i, x_j\right) = L\left(i_+ x_i - i_- x_i, x_j\right)$$

Now $i_+ x_i - i_- x_i$ is the boundary of a chain $S^1 \times I$ normal to V which meets V in x_i, and so $L\left(i_+ x_i - i_- x_i, x_j\right)$ is the algebraic intersection of this chain with x_j, i.e. the algebraic intersection of x_i and x_j in V. Hence $A - A'$ represents the intersection pairing on $H_1(V)$, whence the result. $\square$

Now let us state what happens in higher dimensions.

Definition 1.14. A $(2q-1)$-knot k is *simple* if its exterior K satisfies $\pi_i(K) \cong \pi_i(S^1)$ for $1 \leq i < q$.

The following result is proved in [**33**, Theorem 2].

Theorem 1.15. *A $(2q-1)$-knot k is simple if and only if it bounds a $(q-1)$-connected Seifert submanifold.*

Now suppose that k is a simple $(2q-1)$-knot bounding an $(q-1)$-connected submanifold V. We can repeat the construction above, using $H_q(V)$, to obtain a Seifert matrix A of k satisfying the following.

Theorem 1.16. *If A is a Seifert matrix of a simple $(2q-1)$-knot k and A' is its transpose, then $A + (-1)^q A'$ is unimodular. Moreover, if $q = 2$, then the signature of $A + A'$ is a multiple of 16.*

Theorem 1.12 remains true for simple knots. The following result is proved in [**41**] for $q = 1$, in [**36**, Theorem 2] for $q = 2$, and in [**31**, Théorème II.3] for $q > 2$.

Theorem 1.17. *Let q be a positive integer and A a square integral matrix such that $A + (-1)^q A'$ is unimodular and, if $q = 2$, $A + A'$ has signature a multiple of 16. If $q \neq 2$, there is a simple $(2q-1)$-knot k with Seifert matrix A. If $q = 2$, there is a simple 3-knot k with Seifert matrix S-equivalent to A.*

In [**36**, Theorem 3] the classification of simple knots is completed.

Theorem 1.18. *A simple $(2q-1)$-knot k, $q > 1$, is determined up to ambient isotopy by the S-equivalence class of its Seifert matrix.*

2. Blanchfield Duality

Let us set $\Lambda = \mathbb{Z}\left[t, t^{-1}\right]$, the ring of Laurent polynomials in a variable t with integer coefficients.

Theorem 2.1. *If A is a Seifert matrix of a simple $(2q-1)$-knot k, then the Λ-module M_A presented by the matrix $tA + (-1)^q A'$ depends only on the S-equivalence class of A, and so is an invariant of k. Moreover, there is a non-singular $(-1)^{q+1}$-hermitian pairing*

$$\langle\ ,\ \rangle_A : M_A \times M_A \to \Lambda_0/\Lambda$$

given by the matrix $(1-t)\left(tA + (-1)^q A'\right)^{-1}$ which is also an invariant of k. Conjugation is the linear extension of $t \mapsto t^{-1}$, Λ_0 is the field of fractions of Λ, and non-singular *means that the adjoint map $M_A \to \overline{\mathrm{Hom}}\,(M_A, \Lambda_0/\Lambda)$ is an isomorphism.*

Definition 2.2. The Λ-module in Theorem 2.1 is called the *knot module* of k, and has a geometric significance which is explained in §6. The determinant of $tA + (-1)^q A'$ is the *Alexander polynomial* of k, and is defined up to multiplication by a unit of Λ. The hermitian pairing is due to R.C. Blanchfield [**9**]. The formula given here was discovered independently in [**22, 44**].

The following two results are proved in [**22, 23, 44, 45**].

Theorem 2.3. *If A is a Seifert matrix of a simple $(2q-1)$-knot k, then the module and pairing $(M_A, \langle\ ,\ \rangle_A)$ satisfy:*

(i) M_A is a finitely-generated Λ-torsion-module;
(ii) $(t-1) : M_A \to M_A$ is an isomorphism;
(iii) $\langle\ ,\ \rangle_A : M_A \times M_A \to \Lambda_0/\Lambda$ is a non-singular $(-1)^{q+1}$-hermitian pairing.

For $q = 2$ the signature is divisible by 16. Moreover, for $q > 1$, the module and pairing determine the knot k up to ambient isotopy.

Theorem 2.4. *Suppose that $(M, \langle\ ,\ \rangle)$ satisfies*

(i) M is a finitely-generated Λ-torsion-module;
(ii) $(t-1) : M \to M$ is an isomorphism;
(iii) $\langle\ ,\ \rangle : M \times M \to \Lambda_0/\Lambda$ is a non-singular $(-1)^{q+1}$-hermitian pairing.

and that, for $q = 2$, the signature is divisible by 16. Then for $q \geq 1$, $(M, \langle\ ,\ \rangle)$ arises from some simple $(2q-1)$-knot as $(M_A, \langle\ ,\ \rangle_A)$.

In [**43**, pp 485-489] Trotter proves the following.

Proposition 2.5. *If A is a Seifert matrix, then A is S-equivalent to a matrix which is non-degenerate; that is, to a matrix with non-zero determinant.*

Proposition 2.6. *If A and B are non-degenerate Seifert matrices of a simple $(2q-1)$-knot k, then A and B are congruent over any subring of $\mathbb{Q}$ in which* $\det A$ *is a unit. (Of course,* $\det A$ *is the leading coefficient of the Alexander polynomial of k.)*

Definition 2.7. Let $\varepsilon = (-1)^q$ and let A be a non-degenerate Seifert matrix of a simple $(2q-1)$-knot k. Set $S = (A + \varepsilon A')^{-1}$ and $T = -\varepsilon A' A^{-1}$.

Proposition 2.8. *The pair (S, T) have the following properties:*

(i) S is integral, unimodular, ε-symmetric;
(ii) $(I-T)^{-1}$ exists and is integral;
(iii) $T'ST = S$;
(iv) $A = (I-T)^{-1}S^{-1}$.

The following result is proved in [**44**, p179]

Theorem 2.9. *The matrix S gives a $(-1)^q$-symmetric bilinear pairing $(\ ,\ )$ on M_A on which T (i.e. t) acts as an isometry. The pair $(M_A, (\ ,\ ))$ determines and is determined by the S-equivalence class of A.*

3. Factorisation of Knots

If we have two classical knots, there is a natural way to take their sum: just tie one after another in the same piece of string. Alternatively, we can think of each knot as a knotted ball-pair and identify the boundaries so that the orientations match up. The latter procedure generalises to higher dimensions.

DEFINITION 3.1. Let k_1, k_2 be two n-knots, say $k_i = \left(S_i^{n+2}, S_i^n\right)$. Choose a point on each S_i^n and excise a tubular neighbourhood, i.e. an unknotted ball-pair, leaving a knotted ball-pair $\left(B_i^{n+2}, B_i^n\right)$. Identify the boundaries so that the orientations match up, giving a sphere-pair $k_1 + k_2$.

If k_1, k_2 are simple knots with Seifert matrices A_1, A_2, then clearly $k_1 + k_2$ is also simple and has a Seifert matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

and the pairings in §2 are given by the orthogonal direct sum.

For the case $n = 1$, H. Schubert showed in [**40**] that every knot factorises uniquely as a sum of irreducible knots. In [**25**] it is shown that unique factorisation fails for $n = 3$, and in [**1**] E. Bayer showed that it fails for $n = 5$ and for $n \geq 7$. The following example is contained in [**1**], although presented here in a slightly different way.

Let $\Phi_{15}(t)$ denote the 15^{th} cyclotomic polynomial, which we normalise so that $\Phi_{15}(t) = \Phi_{15}\left(t^{-1}\right)$, and let $\zeta = e^{\frac{2\pi i}{15}}$. Then $\mathbb{Z}[\zeta] \cong \mathbb{Z}\left[t, t^{-1}\right] / \left(\Phi_{15}(t)\right)$ is the ring of integers, and conjugation in $\mathbb{Z}\left[t, t^{-1}\right]$ corresponds to complex conjugation in $\mathbb{Z}[\zeta]$. Moreover, $\zeta - 1$ is a unit in $\mathbb{Z}[\zeta]$, and so we can think of $\mathbb{Z}[\zeta]$ as a Λ-module satisfying properties (i) and (ii) of Theorem 2.3. If $u \in U_0$, the set of units of $\mathbb{Z}[\zeta + \overline{\zeta}]$, then we can define a hermitian pairing on $\mathbb{Z}[\zeta]$ by $(x, y) = ux\overline{y}$. This corresponds to a hermitian pairing as in Theorem 2.3(iii) by

$$\langle x(t), y(t) \rangle = \frac{u(t)x(t)y\left(t^{-1}\right)}{\Phi_{15}(t)} \longleftrightarrow (x(\zeta), y(\zeta)) = ux(\zeta)\overline{y(\zeta)}.$$

The case of skew-hermitian pairings is dealt with by using $\left(\zeta - \overline{\zeta}\right) u$ in place of u. Note that $\zeta - \overline{\zeta}$ is a unit:

$$\left(\zeta - \overline{\zeta}\right)^2 \left(\zeta + \overline{\zeta}\right) \left(1 - \zeta - \overline{\zeta}\right) = 1. \tag{3.1}$$

LEMMA 3.2. *Let $u_r = \zeta^r + \zeta^{-r} - 1$ for $r = 0, 1, 2, 7$. Then $u_r \in U_0$ and*

$$\begin{pmatrix} u_r & 0 \\ 0 & -u_r \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

represent isometric pairings on $\mathbb{Z}[\zeta] \times \mathbb{Z}[\zeta]$.

PROOF. Since ζ^2, ζ^7 are also primitive 15^{th} roots of unity, (3.1) shows that $u_r \in U_0$ for $r = 1, 2, 7$. Of course, $u_0 = 1$. Now consider

$$\begin{pmatrix} \bar{a} & \bar{b} \\ b & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} = \begin{pmatrix} a\bar{a} - b\bar{b} & 0 \\ 0 & b\bar{b} - a\bar{a} \end{pmatrix} = \begin{pmatrix} u_r & 0 \\ 0 & -u_r \end{pmatrix}$$

For each r, we can write

$$u_r = \zeta^r + \zeta^{-r} - 1 = 1 - (\zeta^r - 1)\left(\zeta^{-r} - 1\right)$$

and so we can take $a = 1, b = (\zeta^r - 1)$. □

LEMMA 3.3. *The hermitian forms on $\mathbb{Z}[\zeta]$ given by $\pm u_r$ for $r = 0, 1, 2, 7$ are distinct.*

PROOF. Two such forms represented by $u, v \in U_0$ are equivalent if and only if $uv^{-1} \in N(U)$, where $N(c) = c\bar{c}$; write $u \sim v$ to denote this equivalence. Then $u_1 > 0$ but u_1 is conjugate to $u_7 < 0$, so $\pm u_1 \notin N(U)$ and hence the forms represented by $\pm 1, \pm u_1$ are distinct. Similarly $u_2/u_1 > 0$ but u_2/u_1 is conjugate to $u_4/u_2 < 0$, so $\pm u_2/u_1 \notin N(U)$. Similar arguments apply in the other cases. □

COROLLARY 3.4. *For each $q > 2$ there exist eight distinct irreducible simple $(2q-1)$-knots k_r, k_r^-, $r = 0, 1, 2, 7$, such that $k_r + k_r^- = k_s + k_s^-$ for all $r, s \in \{0, 1, 2, 7\}$.*

PROOF. By Theorem 2.3 there exist unique simple $(2q-1)$-knots k_r, k_r^- corresponding to $u_r, -u_r$ respectively. These are irreducible because in each case the Alexander polynomial is $\Phi_{15}(t)$. □

REMARK 3.5. It is shown in [**1**] that $U_0/N(U)$ has exactly eight elements, so that these are all the forms there are in this case.

There is another method, due to J.A. Hillman, depending upon the module structure, and this can be used to show that unique factorisation fails for $n \geq 3$ (see [**5**]). There are further results on this topic in [**18, 17, 19**]. It is known from [**10**] that every n-knot, $n \geq 3$, factorises into finitely many irreducibles, and that a large class of knots factorise into irreducibles in at most finitely many different ways (see [**6, 8**]).

I should mention that the method of [**25**] relies on the signature of a smooth 3-knot being divisible by 16, and hence does not generalise to higher dimensions. The work of Hillman in [**20**] shows that the classification theorems 1.16, 1.17, 1.18, and 2.3 hold for locally-flat topological 3-knots without any restriction on the signature.

4. Cancellation of Knots

In [**3**] Eva Bayer proves a stronger result, that cancellation fails for simple $(2q-1)$-knots, $q>1$.

Example 4.1. Let A be the ring of integers associated with Φ_{12}, and define Γ_{4n} to be the following lattice given in [**39**]. We take $\mathbb{R}^{4n}$ to denote the euclidean space, and let $e_1,\ldots,e_{4n}$ be an orthonormal basis with respect to the usual innerproduct. Then Γ_{4n} is the lattice spanned by the vectors e_i+e_j and $\frac{1}{2}(e_1+\cdots+e_{4n})$. Let $A\Gamma_{4n}$ be the corresponding hermitian lattice. It is shown in [**3**] that the hermitian form $A\Gamma_{4n}$ is irreducible if $n>1$. Moreover

$$A\Gamma_8 \perp A\Gamma_8 \perp < -1 > \cong A\Gamma_{16} \perp < -1 >; \tag{4.1}$$

indeed, this holds already over $\mathbb{Z}$ by [**39**, Chap. II, Proposition 6.5]. But $A\Gamma_8 \perp A\Gamma_8 \ncong A\Gamma_{16}$ because the latter is irreducible. By Theorem 2.1 this shows that cancellation fails for q odd, $q\neq 1$. For q even, just multiply 4.1 by the unit $\tau-\tau^{-1}$ where τ is a root of Φ_{12}.

In [**4**], examples are given where the failure of cancellation depends on the structure of the knot module, and by the device known as spinning this result is extended to even dimensional knots. (See §7 for the definition of spinning.)

5. Knot Cobordism

There is an equivalence relation defined on the set of n-knots as follows.

Definition 5.1. Two n-knots $k_i=\left(S_i^{n+2},S_i^n\right)$ are *cobordant* if there is a manifold pair $\left(S^{n+2}\times I,V\right)$ such that

$$V\cap\left(S^{n+2}\times\{i\}\right)=\partial V\cap\left(S^{n+2}\times\{i\}\right)=S_i^n$$

(with the orientations reversed for $i=0$) and $S_i^n\hookrightarrow V$ is a homotopy equivalence for $i=0,1$.

Remark 5.2. For $n\geq 6$ the manifold V is a product, i.e. it is homeomorphic to $S^n\times I$, by the h-cobordism theorem.

This equivalence relation respects the operation of knot sum, and the equivalence classes form an abelian group C_n under this operation, with the trivial knot as the zero. In [**31**] M.A. Kervaire shows that $C_{2q}=0$ for all q.

To tackle the odd-dimensional case, we begin by quoting a result of Levine: [**35**, Lemma 4].

Lemma 5.3. *Every $(2q-1)$-knot is cobordant to a simple knot.*

DEFINITION 5.4. Let A_0, A_1 be two Seifert matrices of simple $(2q-1)$-knots. If $\begin{pmatrix} -A_0 & 0 \\ 0 & A_1 \end{pmatrix}$ is congruent to one of the form $\begin{pmatrix} 0 & N_1 \\ N_2 & N_3 \end{pmatrix}$ where each of the N_i are square of the same size then A_0, A_1 are said to be *cobordant.* This is an equivalence relation for Seifert matrices, and the set of equivalence classes forms a group under the operation induced by taking the block sum:

$$(A_0, A_1) \mapsto \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}.$$

LEMMA 5.5. *Let k_0, k_1 be simple $(2q-1)$-knots which are cobordant, with Seifert matrices A_0, A_1 respectively. Then A_0, A_1 are cobordant.*

From Proposition 2.5 we deduce:

COROLLARY 5.6. *Every Seifert matrix is cobordant to a non-degenerate matrix.*

DEFINITION 5.7. Setting $\varepsilon_q = \text{sign}\,(-1)^q$, the group obtained from the Seifert matrices of simple $(2q-1)$-knots is denoted by G_{ε_q}. The subgroup of G_+ given by matrices A such that the signature of $A + A'$ is divisible by 16 is denoted G_+^0. The map $\varphi_q : C_{2q-1} \to G_{\varepsilon_q}$ is induced by taking a simple knot to one of its Seifert matrices, and is a homomorphism.

The definition above appears in [**35**], where the following result is proved.

THEOREM 5.8. *The map φ_q is*

(a) an isomorphism onto G_{ε_q} for $q \geq 3$;
(b) an isomorphism onto G_+^0 for $q = 2$;
(c) an epimorphism onto G_- for $q = 1$.

The proof of [**34**, Lemma 8] shows that two Seifert matrices are cobordant if and only if they are cobordant over the rationals. This leads to the idea of Witt classes for the $(-1)^q$-symmetric forms and isometries in Theorem 2.9, and this is the strategy that Levine uses to investigate G_ε, and to prove Theorem 5.9 below (see [**35**, p 108]).

THEOREM 5.9. *G_ε is the direct sum of cyclic groups of orders 2, 4 and ∞, and there are an infinite number of summands of each of these orders.*

Both Levine's treatment and that of Kervaire in [**32**] rely on Milnor's classification of isometries of innerproduct spaces in terms of hermitian forms in [**38**].

I shall not attempt to prove any of these results, but it is easy to give Milnor's proof in [**37**] of infinitely many summands of infinite order for q odd, and at the same time to suggest an alternative way of looking at G_{ε_q}. First make the following definition, taken from [**24**].

DEFINITION 5.10. A module and pairing $(M_A, <\ ,\ >_A)$ as in Theorem 2.1 is *null-cobordant* if there is a submodule of half the dimension of M_A which is self-annihilating under $<\ ,\ >_A$. And $(M_A, <\ ,\ >_A)$, $(M_B, <\ ,\ >_B)$ are *cobordant* if the orthogonal direct sum $A \perp (-B)$ is null-cobordant. The *dimension* of M_A is the dimension over $\mathbb{Q}$ of M_A after passing to rational coefficients.

It is shown in [**24**] that $(M_A, <\ ,\ >_A)$ is null-cobordant if and only if A is null-cobordant. If we use rational coefficients, which we may as well do in light of [**34**, Lemma 8], the proof is even easier. We shall treat $(M_A, <\ ,\ >_A)$ in this way for the rest of this section, and set $\Gamma = \mathbb{Q}\left[t, t^{-1}\right]$.

DEFINITION 5.11. Given $p(t) \in \Lambda$, define $p^*(t) = t^{\deg(p(t))} p\left(t^{-1}\right)$. And for an irreducible $p(t) \in \Lambda$ define $M_A(p(t))$ to be the p-primary component of M_A, i.e. the submodule annihilated by powers of $p(t)$.

The following result is essentially Cases 1 and 3 of [**38**, p93, Theorem 3.2]: note that Case 2 does not arise here since we are dealing with knot modules, i.e. $p(t) \neq t \pm 1$.

PROPOSITION 5.12. *$(M_A, <\ ,\ >_A)$ splits as the orthogonal direct sum of $M_A(p(t))$ where $p(t) = p^*(t)$, and of $M_A(p(t)) \oplus M_A(p^*(t))$ where $p(t) \neq p^*(t)$. Furthermore, for $p(t) = p^*(t)$ the space $M_A(p(t))$ splits as an orthogonal direct sum $M^1 \oplus M^2 \oplus \ldots$ where M^i is annihilated by $p(t)^i$ but is free over the quotient ring $\Gamma/p(t)^i\Gamma$.*

It is slightly easier to prove [**38**, Theorem 3.3] here.

THEOREM 5.13. *When $p(t) = p^*(t)$, for each i, the vector space*

$$H^i = M^i / p(t) M^i$$

over the field $E = \Gamma/p(t)\Gamma$ admits one and only one hermitian inner product $((x), (y))$ such that

$$\left\langle p(t)^{i-1} x, y \right\rangle = \frac{a(t)}{p(t)} \longleftrightarrow ((x), (y)) = a(\zeta) \tag{5.1}$$

where (x) denotes the image of $x \in M^i$ in H^i. The sequence of these hermitian inner product spaces determines $(M_A(p(t)), <\ ,\ >)$ up to isometry.

The following is easy to prove, where we think of the hermitian form on H^i as taking values in Γ_0/Γ.

LEMMA 5.14. *For i even, M^i is null-cobordant. For i odd, M^i is cobordant to H^i.*

This enables us to make the following definition of the Milnor signatures (compare [**38**]).

Definition 5.15. For each $p(t) = p^*(t)$, define

$$\sigma_p = \sum_{i \text{ odd}} \sigma_i,$$

where σ_i is the signature of H^i.

Lemma 5.16. *The signature σ_p is additive over knot composition and zero when k is null-cobordant, and so is a cobordism invariant.*

Example 5.17 (Milnor, [**37**]). For each positive integer m, let $p_m(x) = mt+(1-2m)+mt^{-1}$. Then p_m is irreducible and is the Alexander polynomial of a simple $(4q+1)$-knot k_m for each $q \geq 1$ having $\begin{pmatrix} m & 1 \\ 0 & 1 \end{pmatrix}$ as a Seifert matrix. Then $\sigma_{p_m} = \pm 2$ for each m, and so for each $q \geq 1$ we have infinitely many independent knots of infinite order in C_{4q+1}.

6. Branched Cyclic Covers

Recall that if k is an n-knot $\left(S^{n+2}, S^n\right)$, then a regular neighbourhood of S^n has the form $S^n \times B^2$, and the *exterior* of k is $K = S^{n+2} - \operatorname{int}\left(S^n \times B^2\right)$.

Choose a base-point $* \in K$. By Alexander-Poincaré duality, K has the homology of a circle, and so the Hurewicz theorem gives a map $\pi_1(K,*) \twoheadrightarrow H_1(K)$ whose kernel is the commutator subgroup $[\pi_1(K,*), \pi_1(K,*)]$ of the group $\pi_1(K,*)$. We write the infinite cyclic group $H_1(K)$ multiplicatively, as $(t:\ )$; the generator t is represented by $\{a\} \times S^1 \subset S^n \times S^1 = \partial K$ for some point $a \in S^n$, and is chosen so that the oriented circle has linking number $+1$ with S^n.

Let $\tilde{K} \to K$ be the infinite cyclic cover corresponding to the kernel of the Hurewicz map. A triangulation of K lifts to a triangulation of $\tilde{K}$ on which $(t:\ )$ acts as the group of covering transformations. This induces an action of $(t:\ )$ on the chain complex $C_*(\tilde{K})$, which extends by linearity to make $C_*(\tilde{K})$ a $\Lambda = \mathbb{Z}[t,t^{-1}]$-module. The Λ-module $C_*(\tilde{K})$ is finitely-generated because the original triangulation of K is finite. Passing to homology we obtain $H_*(\tilde{K})$ as a finitely-generated, indeed finitely-presented, Λ-module. For a simple $(2q-1)$-knot, the only non-trivial module is $H_q(\tilde{K})$, and this is in fact the same as the Λ-module M_A presented by the matrix $tA+(-1)^qA'$ in Theorem 2.1.

To recover K from $\tilde{K}$ all we do is identify x with tx, for each $x \in \tilde{K}$.

Compose the Hurewicz map with the map sending $(t:\ )$ onto the finite cyclic group of order r, and denote the r-fold cover of K corresponding to the kernel of this map by $\tilde{K}^r$. Since $\partial\tilde{K}^r \cong S^n \times S^1$, being an r-fold cover of $S^n \times S^1$, we may set $K^r = \tilde{K}^r \cup_\partial \left(S^n \times B^2\right)$ to obtain the r-fold cover of S^{n+2} branched over S^n. It may happen that $K^r \cong S^{n+2}$, in which case we have another n-knot k^r, which we refer to as the *r-fold branched cyclic* *cover* of k. In this case $H_*(\tilde{K})$ is a $\mathbb{Z}[t^r, t^{-r}]$-module when $\tilde{K}$ is regarded as the infinite cyclic cover of $\tilde{K}^r$. Note that k^r is the fixed point set of the $\mathbb{Z}_r$ action on $S^{n+2} = K^r$ given by the covering transformations.

Let k be a simple $(2q-1)$-knot giving rise to a pair of matrices (S, T) as in Proposition 2.8, and define U, V by

$$U = \begin{pmatrix} 0 & \dots & 0 & T \\ I & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & I & 0 \end{pmatrix} \qquad V = \begin{pmatrix} S & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & S \end{pmatrix}$$

there being $r \times r$ blocks in each case. It is not hard to show that the pair (V, U) satisfies the conditions of Proposition 2.8, and so corresponds to a unique simple $(2q-1)$-knot k_r if $q \geq 2$ (see Theorem 2.9). Moreover,

$$U^r = \begin{pmatrix} T & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & T \end{pmatrix}$$

from which it follows without much difficulty that the r-fold branched cyclic cover of k_r is $\#_1^r k$, the sum of r copies of k.

The topological construction of k_r may be described as follows. Take a Seifert surface W of k which meets a tubular neighbourhood N of k in a collar neighbourhood of $k = \partial W$. Take r close parallel copies $W_1, \dots, W_r$ of W, so that the space between W_i and W_{i+1} is diffeomorphic to $W \times [0,1]$ for $i = 1, \dots, r-1$, and that between W_r and W_1 is diffeomorphic to the exterior of k split open along W. We can join ∂W_i to ∂W_{i+1} for $1 \leq i \leq r-1$ by bands within N to get the the boundary connected sum of the W_i. Then it is not hard to see that the Seifert surface we have constructed has V, U as above. Moreover, for $q > 1$, the resulting knot k_r is independent of the bands used, since the bands unknot in these dimensions.

In [**29**] examples are given of simple $(2q-1)$-knots k, l, $q \geq 2$, for which $\#_1^r k = \#_1^r l$ but $k_r \neq l_r$. It is known that there are examples for any odd r. The argument runs as follows.

Let $\Phi_m(t)$ denote the m^{th} cyclotomic polynomial, where m is divisible by at least two distinct odd primes, and let ζ be a primitive m^{th} root of unity. We write $\mathbb{K} = \mathbb{Q}(\zeta)$ and $\mathbb{F} = \mathbb{Q}\left(\zeta + \zeta^{-1}\right)$. Let $h_{\mathbb{K}}$ denote the class number of $\mathbb{K}$, $h_{\mathbb{F}}$ that of $\mathbb{F}$, and $h_- = h_{\mathbb{K}}/h_{\mathbb{F}}$. According to the work of Eva Bayer in [**2**], the number of distinct simple $(2q-1)$-knots, $q \geq 3$, with Alexander polynomial $\Phi_m(t)$ is $h_- 2^{d-1}$ where $2d = \varphi(m) = [\mathbb{K} : \mathbb{Q}]$. The factor h_- represents the number of isomorphism classes of Λ-modules supporting a Blanchfield pairing [**2**, Corollary 1.3], and the factor 2^{d-1} represents the number of pairings (up to isometry) which a given module supports. The latter is in one-one correspondence with $U_0/N(U)$ where U is the group of units in (the ring of integers of) $\mathbb{K}$, U_0 the group of units of $\mathbb{F}$, and $N : \mathbb{K} \to \mathbb{F}$ is the norm.

If h_- has an odd factor $r > 1$ coprime to m, then there exists an ideal a of $\mathbb{Q}(\zeta)$ which has order r in the class group and supports a unimodular hermitian pairing h, i.e. as a Λ-module it supports a Blanchfield pairing. Then $\perp_1^r (a, h)$ has determinant (I, u) for some $u \in U_0/N(U)$, where I denotes a principal ideal. Since r is odd and $|U_0/N(U)| = 2^{d-1}$, there exists $v \in U_0/N(U)$ such that $v^r = u$.

Let k, l be the simple $(2q-1)$-knots, $q \geq 2$, corresponding to $\kappa = (a, h) \perp (a, -h)$, $\lambda = (I, v) \perp (I, -v)$ respectively. Then $\perp_1^r \kappa$, $\perp_1^r \lambda$ are indefinite and have the same rank, signatures and determinant. Hence by [**2**, Corollary 4.10] they are isometric, and so $\#_1^r k = \#_1^r l$. But κ is not isometric to λ, for the determinant of κ is $\left(a^2, \alpha\right)$ for some α, and a^2 is non-zero in the ideal class group since r is odd. Hence $k \neq l$. A similar, but more involved, argument shows that $k_r \neq l_r$.

Many examples may be obtained from the tables in [**46**] or [**47**].

7. Spinning and Branched Cyclic Covers

First we recall a definition of spinning. Let k be the n-knot $\left(S^{n+2}, S^n\right)$ and let B be a regular neighbourhood of a point on S^n such that $(B, B \cap S^n)$ is an unknotted ball pair. Then the closure of the complement of B in S^{n+2} is a knotted ball pair $\left(B^{n+2}, B^n\right)$, and $\sigma(k)$ is the pair $\partial\left[\left(B^{n+2}, B^n\right) \times B^2\right]$. The following is proved in [**28**, Theorem 3].

Theorem 7.1. *Let k, l be simple $(2q-1)$-knots, $q \geq 3$; then $\sigma(k) = \sigma(l)$ if and only if $H_q(\tilde{K}) \cong H_q(\tilde{L})$.*

Note that [**28**] only covers the case $q \geq 5$; the theorem is extended to $q \geq 3$ by the results of [**21**].

In [**7**] it is shown that the following holds.

Proposition 7.2. *If $q \geq 4$, then the map σ acting on simple $(2q-1)$-knots is finite-to-one.*

The following is easy to prove (see [**30**, Lemma 3.1]).

Lemma 7.3. *Let k be an n-knot and r an integer such that the r-fold cyclic cover of S^{n+2} branched over k is a sphere. Then the r-fold cyclic cover of S^{n+3} branched over $\sigma(k)$ is also a sphere, and $\sigma\left(k^r\right) = \sigma(k)^r$.*

In [**42**], Strickland proves the following result.

THEOREM 7.4. *Let k be a simple $(2q-1)$-knot, $q \geq 2$. Then k is the r-fold branched cyclic cover of a knot if and only if there exists an isometry u of $\left(H_q(\tilde{K}), <\ ,\ >\right)$ such that $u^r = t$.*

A careful reading of [**42**] shows that the same proofs apply, almost verbatim, to yield the following result (see [**30**, Theorem 2.5]).

THEOREM 7.5. *Let k be a simple $(2q-1)$-knot, $q \geq 5$. Then $\sigma(k)$ is the r-fold branched cyclic cover of a knot if and only if there is a Λ-module isomorphism $u\colon H_q(\tilde{K}) \to H_q(\tilde{K})$ such that $u^r = t$.*

Thus if we can find a simple $(2q-1)$-knot k, $q \geq 5$, such that there is a Λ-module isomorphism $u\colon H_q(\tilde{K}) \to H_q(\tilde{K})$ with $u^r = t$, but no such isometry of $H_q(\tilde{K})$, then $\sigma(k)$ will be the r-fold branched cyclic cover of a knot but k will not. Examples of such knots are given in [**30**] for all $q \geq 5$ and all even r. Here is an example, due to S.M.J. Wilson, much simpler than the ones in [**30**] but not capable of generalising to the $2r$-fold case.

EXAMPLE 7.6. Let $f(t) = t^2 - 3t + 1$, which has roots $\frac{3\pm\sqrt{5}}{2}$. Set $\tau = \frac{3+\sqrt{5}}{2}$, $\xi = \frac{1+\sqrt{5}}{2}$, and note that $\xi^2 = \tau$. Put $R = \mathbb{Z}[\xi] = \mathbb{Z}\left[\tau, \tau^{-1}\right]$ and define conjugation in the obvious way by $\tilde{\xi} = \frac{1-\sqrt{5}}{2}$. Think of R as an R-module, and put a hermitian form on it by setting $(x, y) = x\tilde{y}$. Since $\xi\tilde{\xi} = -1$, ξ is an isomorphism on R but not an isometry. Since τ only has two square roots, $\pm\xi$, there are no isometries whose square is τ. In the usual way, $(R, (\ ,\ ))$ corresponds to a knot module and pairing for a simple $(4q+1)$-knot, $q > 1$.

8. Concluding Remarks

So far we have only dealt with odd dimensional knots, but much of what has been said can be extended to even dimensions. By Proposition 1.1, every $2q$-knot k bounds an orientable $(2q+1)$-manifold V in S^{2q+2}. A *simple* $2q$-knot k is one for which there is a $(q-1)$-connected V, so that $H_q(V)$ and $H_{q+1}(V)$ are the only non-trivial homology groups. Then $H_q\left(\widetilde{K}\right)$, $H_{q+1}\left(\widetilde{K}\right)$ are the only non-trivial homology modules. There is a sesquilinear duality pairing on $H_q\left(\widetilde{K}\right) \times H_{q+1}\left(\widetilde{K}\right)$, and this, together with some more algebraic structure connecting them, can be used to classify the simple $2q$-knots in high dimensions. See [**13, 14, 15, 26, 27**] for details.

Classification results have also been obtained for more general classes of knots; see [**11, 12, 16**].

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