# S-EQUIVALENCE OF KNOTS

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ABSTRACT. S-equivalence of classical knots is investigated, as well as its relationship with mutation and the unknotting number. Furthermore, we identify the kernel of Bredon's double suspension map, and give a geometric relation between slice and algebraically slice knots. Finally, we show that every knot is S-equivalent to a prime knot.

## 1. INTRODUCTION

An *oriented knot* $k$ is a smooth (or PL) oriented pair $(S^3, S^1)$; two knots are regarded as the same if there is an orientation preserving diffeomorphism sending one onto the other. An *unoriented knot* $k$ is defined in the same way, but without regard to the orientation of $S^1$. Every oriented knot is spanned by an oriented surface, a Seifert surface, and this gives rise to a matrix of linking numbers called a Seifert matrix. Any two Seifert matrices of the same knot are S-equivalent: the definition of S-equivalence is given in, for example, [14, 21, 11]. It is the equivalence relation generated by ambient surgery on a Seifert surface of the knot. In [19], two oriented knots are defined to be *S-equivalent* if their Seifert matrices are S-equivalent, and the following result is proved.

**Theorem 1.** *Two oriented knots are S-equivalent if and only if they are related by a sequence of* doubled-delta moves *shown in Figure 1.*

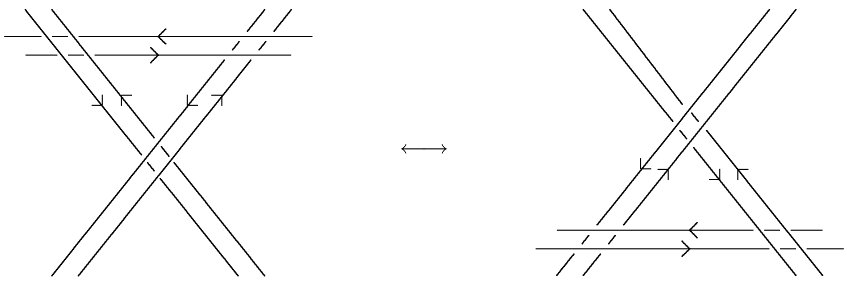

FIGURE 1. The doubled-delta move

It is, of course, from this result that the concept of S-equivalence of knots derives its interest. Note that, as pointed out in [19], a knot is S-equivalent to the trivial knot if and only if it has Alexander polynomial one.

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It has long been known that mutation of a knot preserves the Alexander, Jones, Homfly and Kauffman polynomials (see [7, 17, 18, 16]), and it was shown in [10] that it does not preserve the knot module. Here we investigate some properties of S-equivalence of knots and its relationship with mutation. We show that cancellation does not hold for S-equivalence, that S-equivalence does not imply mutation, and that mutation does not imply S-equivalence.

For a knot $k$ S-equivalent to the trivial knot we define $\delta(k)$ to be the least number of doubled-delta moves needed to unknot $k$, and by relating this to the unknotting number we show that $\delta(k)$ can be arbitrarily large.

We also show that two knots are sent to the same image under Bredon's double suspension map (see [2]) if and only if they are S-equivalent. Moreover we show that a knot is algebraically slice if and only if it is S-equivalent to a slice knot, with a corresponding result for doubly-slice knots.

Finally we use results of Lickorish [15] to show that every knot is S-equivalent to a prime knot.

## 2. S-equivalence

**Theorem 2.** *There exist oriented knots $k, l, m$ such that $k + m$ is S-equivalent to $l + m$ but $k$ is not S-equivalent to $l$.*

*Proof.* Examples are given in [1] of simple $(4\,q+1)$-knots $K, L, M$ such that $K+M = L + M$ but $K \neq L$, for each $q \geq 1$. Since simple knots in higher dimensions are in one-one correspondence with the S-equivalence classes of Seifert matrices, we can let $U, V, W$ be Seifert matrices of $K, L, M$ respectively, and then $U \oplus W$ is S-equivalent to $V \oplus W$ but $U$ is not S-equivalent to $V$. Now as pointed out in [8], the Seifert matrices of classical knots are the same as those that occur with simple $(4\,q + 1)$-knots, so there exist classical knots $k, l, m$ with Seifert matrices $U, V, W$ respectively: this establishes the result. □

Suppose that $k$ is S-equivalent to the trivial knot: we define $\delta(k)$ to be the least number of doubled-delta moves needed to change $k$ to the trivial knot.

**Theorem 3.** *For any $k$ S-equivalent to the trivial knot, $u(k) \leq 8\,\delta(k)$ where $u(k)$ is the unknotting number of $k$. In particular, $\delta(k)$ can be arbitrarily large.*

*Proof.* Clearly a doubled-delta move can be accomplished by eight unknotting moves: in Figure 1 simply change the four crossings in the top left-hand part, slide the horizontal band down and then change the four crossings in the bottom right-hand part. This shows that $u(k) \leq 8\,\delta(k)$. By [6, p 354] there are knots with Alexander polynomial one and arbitrarily large unknotting number. The result follows. □

In [2] Bredon defines a semigroup homomorphism $\omega^2 : K_1 \to K_5$, where $K_n$ denotes the semigroup of isotopy classes of smooth simple $n$-knots. Bredon's definition and results are more general, but this is all we shall need here. Bredon shows that $\omega^2$ preserves the Seifert matrix; since the simple 5-knots are classified by the S-equivalence class of the Seifert matrix, we have the following.

**Theorem 4.** *$\omega^2(k) = \omega^2(l)$ if and only if $k$ and $l$ are S-equivalent.*

## 3. S-equivalence and Mutation

If the oriented knots $k, l$ are positive mutants, i.e. if they are related by a mutation which does not require any changes in orientation, then by [12, Theorem 2.1], they have the same Seifert matrix and so they are S-equivalent. The converse fails, as the following result shows.

**Theorem 5.** *There exist oriented knots $k, l$ such that $k$ is S-equivalent to $l$ but $k$ and $l$ are not equivalent by any sequence of mutations even as unoriented knots.*

*Proof.* Take $k$ to be the trivial knot and $l$ to be the Kinoshita-Terasaka knot. Both have trivial Alexander polynomial and are therefore S-equivalent, but $l$ has non-trivial Jones polynomial and so cannot be equivalent by mutation to $k$. Alternatively one can take $l$ to be the Conway knot. These knots appear as $11n_{42}$ and $11n_{34}$ respectively in the Knotscape table, available from the website of Morwen Thistlethwaite: *www.math.utk.edu/˜morwen/* . □

**Theorem 6.** *There exist unoriented knots $k, l$ such that $k$ is a mutant of $l$ but $k$ is not S-equivalent to $l$ however the orientations are chosen.*

*Proof.* In [10] it is shown that if $h'$ denotes the reverse of $h$, i.e. with the orientation of $S^1$ reversed, then $h+h$ and $h+h'$ are mutants as unoriented knots. If we take $h$ to be the pretzel knot $(25, -3, 13)$, it is shown in [9] that $h$ has Steinitz-Fox-Smythe row ideal class $\rho$ which does not satisfy $\rho^2 = 1$. (The arguments used are those of [3].) Furthermore, the row ideal class of $h'$ is $\tau$, the column ideal class of $h$, so that the row ideal class of $h+h$ is $\rho^2 \neq 1$, whereas the row ideal class of $h+h'$ is $\rho\tau = 1$. Thus $k = h + h$ and $l = h + h'$ have non-isomorphic knot modules. If we give $k$ the opposite orientation, then its row ideal class is $\tau^2 \neq 1$, so that $k' = h' + h'$ and $l = h+h'$ have non-isomorphic knot modules. Thus neither $k$ nor $k'$ is S-equivalent to $l$ however the orientations are chosen. □

## 4. S-equivalence and Algebraically Slice Knots

Recall that a knot $k = \left(S^3, S^1\right)$ is *slice* if it is the boundary of a smooth ball-pair $\left(B^4, B^2\right)$. Every Seifert matrix of a slice knot is unimodularly congruent to one of the form

$$A = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}, \tag{4.1}$$

all the blocks being $g \times g$ for some $g$ (see [13, Lemma 2]). This is a necesssary but not a sufficient condition for $k$ to be slice, so we say that $k$ is *algebraically slice* if it has a Seifert matrix S-equivalent to one of the form in (4.1).

**Theorem 7.** *A knot $k$ is algebraically slice if and only if it is S-equivalent to a slice knot.*

*Proof.* Clearly if $k$ is S-equivalent to a slice knot then $k$ is algebraically slice.

Conversely, suppose that $k$ is algebraically slice, and let $A$ be a Seifert matrix of the form (4.1) which is in the S-equivalence class given by $k$; then

$$A - A' = \begin{pmatrix} 0 & B \\ -B' & D \end{pmatrix},$$

where $|A - A'| = |B|^2$, so $|B| = \pm 1$. Thus there exist square unimodular integer matrices $X, Y$ such that $X'BY = I$, the identity matrix. Then we have

$$\begin{pmatrix} X' & 0 \\ 0 & Y' \end{pmatrix} \begin{pmatrix} 0 & B \\ -B' & D \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} 0 & X'BY \\ -Y'B'X & Y'DY \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I' & C \end{pmatrix},$$

where $C = Y'DY$. So we may assume that $A$ satisfies

$$A - A' = \begin{pmatrix} 0 & I \\ -I' & C \end{pmatrix} \tag{4.2}$$

whilst retaining the form in (4.1). Taking transposes in (4.2) we find that $C' = -C$, so the diagonal entries of $C$ are all zero. By (integer) row moves on $A$, i.e. by adding appropriate integer multiples of the first $g$ rows to the last $g$ rows, we can kill off the entries of $C$ above the diagonal and by the corresponding column moves we kill off the entries of $C$ below the diagonal. This gives a unimodular congruence of $A$ which allows us to assume that

$$A - A' = \begin{pmatrix} 0 & I \\ -I' & 0 \end{pmatrix}, \tag{4.3}$$

$A$ still having the form in (4.1). Let the corresponding basis of $H_1(V)$ be

$$f_1, f_3, \ldots, f_{2g-1}, f_2, f_4, \ldots, f_{2g}.$$

Then we can construct a knot $l$ having $A$ as its Seifert matrix by modifying the

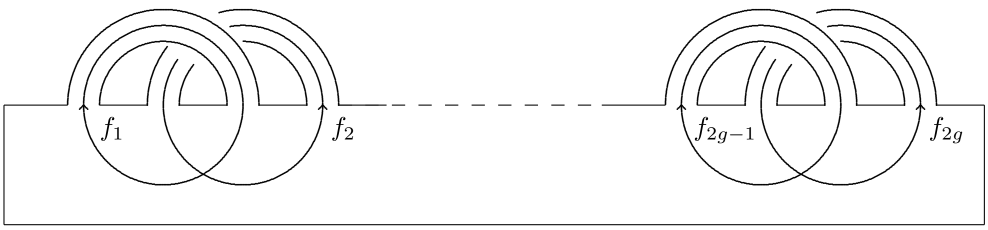


FIGURE 2. A standard circle basis

trivial knot shown in Figure 2. We do not alter the $f_{odd}$, but change the embeddings of the $f_{even}$ to fit the entries in $A$. We can now do ambient surgery on the $f_{odd}$ in $B^4$ to show that $l$ is a slice knot: take a product $V \times [0,1] \subset \partial B^4 \times [0,1] \subset B^4$, where $V \times \{0\} \subset \partial B^4$. Then attaching a 2-handle to $V \times [0,1]$ along each of $f_1 \times \{1\}, f_3 \times \{1\}, \ldots, f_{2g-1} \times \{1\}$ gives a 3-manifold $W$ embedded in $B^4$ such that $\partial W = V \cup B^2$ and $\partial\left(B^4, B^2\right) = \left(S^3, S^1\right) = l$. (We have performed $g$ surgeries which would reduce a closed orientable surface of genus $g$ to $S^2$, and hence reduce $V$ to a disc.) Thus $k$ is S-equivalent to $l$ which is slice. □

A knot $k = \left(S^3, S^1\right)$ is *doubly-slice* if it is the intersection of an unknotted 2-knot $\left(S^4, S^2\right)$ with the equatorial $S^3 \subset S^4$. Every Seifert matrix of a doubly-slice knot is S-equivalent to one of the form

$$A = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \tag{4.4}$$

all the blocks being square (see [20, Theorem 2.3]). Again, this is a necesssary but not a sufficient condition for $k$ to be doubly-slice, so we say that $k$ is *algebraically doubly-slice* if $k$ has a Seifert matrix S-equivalent to one of the form (4.4).

**Theorem 8.** *A knot is algebraically doubly-slice if and only if it is S-equivalent to a doubly-slice knot.*

*Proof.* Clearly if $k$ is S-equivalent to a doubly-slice knot then $k$ is algebraically doubly-slice.

Conversely, suppose that $k$ is algebraically doubly-slice; then the proof above yields a Seifert matrix $A$ of the form (4.4) which is in the S-equivalence class given by $k$ and such that $A - A'$ has the form (4.3.) Let $B^4_+/B^4_-$ be the northern/southern hemispheres of $S^4$, and construct $W_+/W_-$ as above in $B^4_+/B^4_-$ using $\{f_1, f_3, \ldots, f_{2g-1}\}/\{f_2, f_4, \ldots, f_{2g}\}$ respectively. Then $W_+ \cup W_- \cong B^3$ and $l$ is doubly-slice. □

The reef knot is cobordant to the trivial knot, indeed it is doubly-slice, but is not S-equivalent to the trivial knot, for it has Alexander polynomial $\left(1 - t + t^2\right)^2$. On the other hand, if a knot is S-equivalent to the trivial knot then its Alexander polynomial is 1, so by [4] or [5, Theorem 11.7B] it is topologically slice. But by [6] we know that there are infinitely many such knots which are not slice in the smooth category.

## 5. S-equivalence and Prime Knots

In this section we use results due to Lickorish in order to show that every S-equivalence class contains a prime knot.

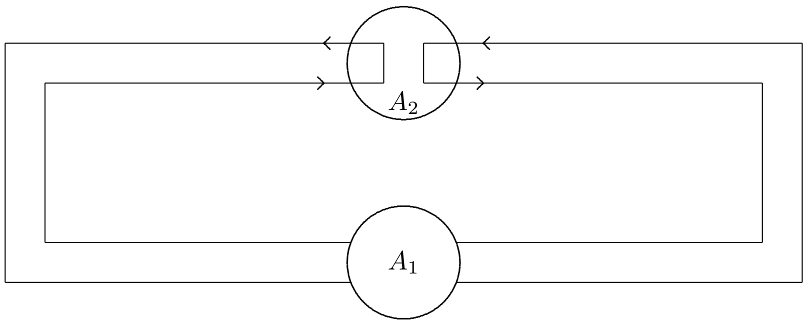


Figure 3. Prime tangle plus trivial tangle

**Theorem 9.** *Every knot $k$ is S-equivalent to a prime knot.*

*Proof.* First note that by the results of [22], every untwisted double of a non-trivial knot has genus 1, hence is prime, and has Alexander polynomial one, so is S-equivalent to the trivial knot. Thus we can assume that $k$ is non-trivial.

By the proof of [15, Theorem 4], $k$ is the sum of a prime (two-string) tangle with the trivial tangle (see [15] for definitions). We can redraw [15, Figure 4] as Figure 3 above. Here $A_1, A_2$ are 3-balls, $(A_1, k \cap A_1)$ is a prime tangle and $(A_2, k \cap A_2)$ is the trivial tangle. Strictly speaking $A_2$ should be the closure of $S^3 - A_1$, but it is more convenient to draw $A_1$ and $A_2$ with a product neighbourhood between them. The tangle in Figure 4 is shown to be prime in [15, §2]; let us call it $(A_3, t_3)$.

Let $l$ be the knot formed by replacing $(A_2, k \cap A_2)$ by $(A_3, t_3)$ in Figure 3; put another way, let $l$ be formed by adding the prime tangle $(A_1, k \cap A_1)$ to the prime tangle $(A_3, t_3)$. By [15, Theorem 1], $l$ is a prime knot (it is easily checked that it

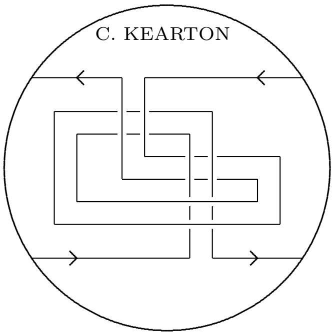

FIGURE 4. The prime tangle $(A_3, t_3)$

has only one component). As explained in [15], there are many ways of adding two given tangles; we shall specify below which way to choose.

We need a Seifert surface for $l$. First note that applying Seifert's algorithm to $k$ shows that there is a Seifert surface $V$ for $k$ contained in $A_1$ except for the two outlying discs. The first part of our surface is shown in Figure 5, and is formed by attaching a disc to $V$ along two arcs in its boundary (of course, $V \cap A_1$ is not shown). We are looking at the positive side of the surface. The second part is shown in Figure 6; here we are looking at the negative side. These parts are joined along the arcs $ab$ and $cd$ to form $U$, which is a Seifert surface of $l$. The disc in the first part of the surface is lying in the plane and the second part is above the plane (except for the edges $ab$ and $cd$). The genus of $U$ is one greater than that of $V$, and the diagrams show two extra generators needed for $H_1(U)$, i.e. $f, g$.

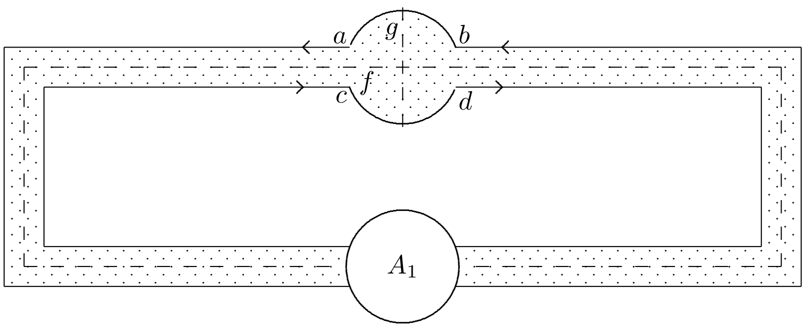


FIGURE 5. First part of the Seifert surface $U$

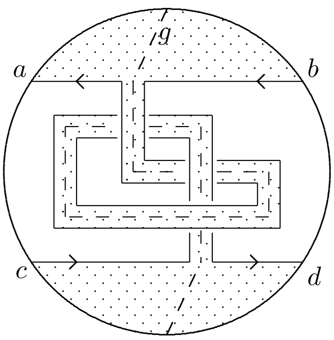


FIGURE 6. Second part of the Seifert surface $U$

Let $i_+$ denote the map which pushes a cycle off $U$ for a small distance in the positive normal direction. By putting the appropriate number of twists in the bands of $U$ we may arrange for $L(i_+f, f) = 0$ and $L(i_+g, g) = 0$, where $L(u, v)$ denotes the linking number in $S^3$ of the two disjoint cycles $u, v$. Note that although this changes $l$, we are simply taking a different sum of the two tangles, achieved by using a different homeomorphism between their boundaries (see [15] for details). Moreover $L(i_+g, f) = 0$, and we can orient the cycles so that $L(i_+f, g) = +1$.

Then if $V$ gives rise to a Seifert matrix $A$, $U$ yields a Seifert matrix of the form

$$B = \begin{pmatrix} A & \alpha & 0 \\ \beta & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha$ ($\beta$) is a column (row) vector over $\mathbb{Z}$. The obvious column and corresponding row operations show that $B$ is congruent by a unimodular matrix to

$$\begin{pmatrix} A & \alpha & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which is S-equivalent to $A$. Thus $l$ is S-equivalent to $k$, and by [15, Theorem 1] $l$ is prime. □

## References


1. E. Bayer, *Definite hermitian forms and the cancellation of simple knots*, Archiv der Math. **40** (1983), 182–185. 2
2. G.E. Bredon, *Regular $O(n)$-manifolds, suspension of knots, and knot periodicity*, Bull. Amer. Math. Soc. **79** (1973), 87–91. 1, 2
3. R.H. Fox and N. Smythe, *An ideal class invariant of knots*, Proc. Amer. Math. Soc. **15** (1964), 707–709. 3
4. M.H. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357–453. 4
5. M.H. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, no. 39, Princeton University Press, Princeton, 1990. 4
6. R.E. Gompf, *Smooth concordance of topologically slice knots*, Topology **25** (1986), 353–373. 2, 4
7. A. Kawauchi, *A survey of knot theory*, Birkhäuser Verlag, Basel-Boston-Berlin, 1996. 1
8. C. Kearton, *Classification of simple knots by Blanchfield duality*, Bull. Amer. Math. Soc. **79** (1973), 952–955. 2
9. ______, *Non-invertible knots of codimension two*, Proc. Amer. Math. Soc. **40** (1973), 274–276. 3
10. ______, *Mutation of knots*, Proc. Amer. Math. Soc. **105** (1989), 206–208. 1, 3
11. ______, *Quadratic forms in knot theory*, Quadratic Forms 1999 (E. Bayer-Fluckiger et al., ed.), Contemporary Mathematics, vol. 272, AMS, 2000, pp. 135–154. 1
12. P. Kirk and C. Livingston, *Concordance and mutation*, Geometry & Topology **5** (2001), 831–883. 3
13. J. Levine, *Knot cobordism groups in codimension two*, Comm. Math. Helv. **44** (1969), 229–244. 4
14. ______, *An algebraic classification of some knots of codimension two*, Comm. Math. Helv. **45** (1970), 185–198. 1
15. W.B.R. Lickorish, *Prime knots and tangles*, Trans. Amer. Math. Soc. **267** (1981), 321–332. 1, 5, 5, 5
16. ______, *Polynomials of links*, Bull. London Math. Soc. **20** (1988), 558–588. 1
17. ______, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. 1
18. W.B.R. Lickorish and K.C. Millett, *A polynomial invariant of oriented links*, Topology **26** (1987), 107–141. 1

19. Swatee Naik and Theodore Stanford, *A move on diagrams that generates S-equivalence of knots*, arXiv:math.GT/9911005 (2nd Nov 1999). 1, 1
20. D.W. Sumners, *Invertible knot cobordisms*, Comm. Math. Helv. **46** (1971), 240–256. 4
21. H.F. Trotter, *On S-equivalence of Seifert matrices*, Invent. math. **20** (1973), 173–207. 1
22. J.H.C. Whitehead, *On doubled knots*, Jour. London Math. Soc. **12** (1937), 63–71. 5

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