Keizo Ushio's Sculptures, Split Tori and Möbius Bands

N. A. FRIEDMAN† and C. H. SEQUIN‡

†State University of New York, Albany. artmath@math.albany.edu

‡ University of California, Berkeley. http://www.ce.berkeley.edu/~sequin

Abstract

Keizo Ushio is a leading international stone sculptor whose work has introduced split tori and Möbius bands to the world on a grand scale. Starting with a simple circular torus or with more elaborate twisting bands, he drills a large number of closely spaced holes to form two strands, which may or may not be connected, depending on the overall rotation of the cutting void. In the case of a torus with a 360-degree rotation of the cut, he obtains two twisted, tangled loops, which can be re-arranged in an ingenious way to form a dramatic figure-8 configuration. Keizo's split loop sculptures are reviewed and classified by the topological and geometrical rules that govern how toroidal structures can be split in a regular way. The basic concepts related to these shapes are clarified and explained with computer generated renderings and through stylized plastic maquettes built on a rapid prototyping machine. These models are also used to explore possible generalizations of the underlying splitting paradigm and to realize configurations that could not easily be carved from stone.

Keywords: Geometrical sculpture, torus knots, Möbius bands, split tori.

AMS Subject Classification: 54-01

1. *Oushi Zokei* by Keizo Ushio

Keizo Ushio is a leading international stone sculptor with a large body of work that is of special interest to mathematicians. He was born in Fukusaki Town, Hyogo Perfecture, Japan, in 1951. He completed his study at Kyoto City University of Arts in 1976. Since then he has received numerous prizes and has participated in stone sculpture symposia throughout the world. His works are in public and private collections in many countries such as Australia, Denmark, Germany, Israel, Japan, Spain, and the United States [1]. His sculptures of split tori and Möbius bands invite mathematical minds to ponder the underlying geometrical and topological paradigms, without reducing the enjoyment that these sculptures create at a purely intuitive, emotional level.

Artwork can be analyzed from many different perspectives: based on its message, based on the materials or tools employed, based on its execution style and craftsmanship, or based on its historical and cultural context. In this paper we discuss a large part of Keizo Ushio's work from a mathematical perspective. We classify and analyze his sculptures that are topologically and/or geometrically equivalent to split tori or Möbius bands. We distill out the elementary structures that allow us to compare and contrast many of his sculptures with one another, and we investigate how the extracted paradigms might be extended to other forms not yet comprised in Keizo's portfolio, or which may even lie beyond what can be carved from stone.

The inspiration for this study originated from the participation of Keizo Ushio at the first conference of the International Society of the Arts, Mathematics, and Architecture in 1999 (ISAMA'99) [2] held in San Sebastian, Spain, at the University of the Basque Country. During that conference, Ushio finished carving an intriguing sculpture consisting of two intertwined loops chiselled from a single solid block of granite (Fig.1).

He started out by carving a massive circular torus with a hole equal in size to the circular cross-section of the ring. He then split that torus into two parts with a cut consisting of many drill holes that stepped around the whole ring, and while doing so, twisted through 360º (Fig.1a). This divided the torus into two identical linked loops, each having a twisting cross section in the form of a half-circle. Once the two pieces had been separated, the configuration was rearranged to make a dramatic figure-8 shape composed of the two loops placed into a position so that portions of their original toroidal surfaces became near-coincident over a significant area (Fig.1b). It is an ingenious sculpture created from a very simple geometrical starting shape. The actual execution, however, was anything but simple. The stone provided was Indian red granite. Keizo said it was the hardest granite he had ever carved. He wore out several carving tools in the process. The sculpture is now in its permanent location on the university campus.

This intriguing sculpture carries the name *Oushi Zokei*. Keizo Ushio refers to many of his other pieces also by the same name. Here we summarize his explanations: *Oushi Zokei* is obviously a cyclic permutation of the letters in his name; it was assigned as an overall name for Keizo's work by his main teacher, who was himself a pioneer in geometrical sculpture in Japan. The name has many different interpretations – just like Keizo's work itself. One translation of *Oushi* is "deep truth", another one is "bull" or "steer". *Ushi-o* also refers to the shape of a bulls back or tail; the latter thus relates to the twisted forms often found in Keizo's work. *Zo* means "creating" or "forming", while *Kei* refers to "shape" or "form". Thus overall *Oushi Zokei* alludes to the creation of twisted forms.

[INSERT FIGURE 1 ABOUT HERE]

(a) (b) (c)

Figure 1: (a) Keizo Ushio carving *Oushi Zokei* during ISAMA'99; (Photo: John Sullivan, 1999).
(b) *Oushi Zokei*, Indian red granite, 125 × 100 × 100 cm, by Keizo Ushio, San Sebastian, Spain,1999; (Photo: John Sullivan, 1999).
(c) *Oushi Zokei*, Japanese blue granite, 270 × 270 × 170 cm, by Keizo Ushio, Vesolund, Denmark, 2005.
(All photos not otherwise labeled are courtesy of Keizo Ushio).

Most people who see the manoeuvre of re-arranging the two stone loops for the first time are amazed and delighted. But few people can fully understand how this transformation works without the help of a small model that they can manipulate themselves. Thus, inspired by Keizo's work, many small plastic maquettes of the prototypical forms underlying Keizo's geometrical sculptures have been created by the second author. Today it is easy to create such tangible models. One starts by describing the desired shapes as simple sweeps in a computer-aided design (CAD) program and then sends the boundary representation of these shapes in the form of a collection of some ten thousand triangles to a rapid prototyping machine. Most of the models depicted in this paper were made on a small Fused Deposition Modelling (FDM) machine from Stratasys [3]. This machine deposits under computer control thin beads of semi-liquid ABS (acrylonitrile-butadiene-styrene) plastic in consecutive layers, one hundredth of an inch thick. This machine can easily build convoluted forms with narrow gaps that would be difficult to create in a subtractive process, where the carving tool needs to reach every single point on the surface. The ABS models are also robust enough for further refinement, for sanding and painting, and for extensive manipulation by many people. The models can even be made attractive enough to serve as little desktop sculptures.

These interlocking split half-tori have become one of the signature pieces of Keizo Ushio. During the last several years he has carved that shape in many different types of stone. While the two halves of the torus remain linked, they retain enough mobility so that they can be placed in different relations to one another and installed in different positions as a final sculpture. A different, more vertical configuration of the final piece is shown in Figure 1c, yielding a very dramatic look. In this sculpture, the drill marks have been enhanced and refined to become an important feature of the sculpture, strongly contrasting with the texture of the original torus surface. Sunlight dramatically interacts with the emphasized drill marks and further enhances the appeal of the sculpture. Yet another configuration is shown in Figure 3b, where the two halves have been moved apart by only a small amount from their original positions in the torus. As the scale of the sculpture is enhanced, e.g., as for his 2005 sculpture *Dream Dance* (not pictured) [4], which has a maximal extent of 3.2 meters, the fabrication process presents an even bigger challenge. Not only must the drill holes be very carefully aligned, so that they properly join when drilled from opposite sides, but moving the two halves into their final position requires sturdy cranes and has to be done with utmost prudence, so as not to damage the carefully finished surfaces.

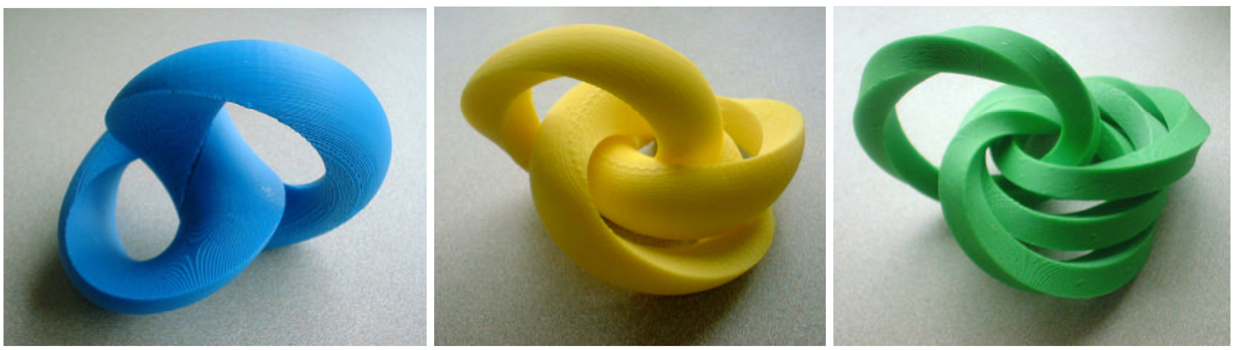
These sculptures are based on an ingenious original idea by Keizo Ushio. A circular torus is cut into two equal parts with a helically twisting sweep surface. When using this motif for his sculptures, the results are astonishingly varied, depending on the stone used, the texturing and finishes of the surfaces, and the final positioning of the two pieces. Keizo Ushio has split tori and Möbius bands in several other ways, creating a wide variety of visual effects. His sculptures are not only aesthetically pleasing, they also invite conscious analysis. Looking at one of Keizo's looping sculptures, one is compelled to trace along one of the edges or faces, trying to find out how many passes it takes before one arrives back at the same location. Other questions one might ask are: Does the sculpture form a truly knotted structure, or is it just some kind of twisted loop? How many different colours would it take, if one were to paint in a different colour every one of the apparent “faces” between the sharp edges of the individual strands?

While such questions may appear confusing and difficult to answer at first glance, one can gain a lot of insight with just a little bit of mathematical analysis. Such an analysis can also provide a conceptual framework by which Keizo's work can be ordered into some distinct families and thereby be understood at a deeper intellectual level. This does not detract from their intuitively sensed beauty; on the contrary: through this understanding of their structure, one becomes more intimately familiar with them, and on revisiting them, one starts to see them as old friends. In this paper we review some of the signature sculptures of Keizo Ushio that emerge from longitudinal cuts in tori and in twisted bands and analyze the underlying geometrical structures.

2. Spliting a Torus

Keizo Ushio's sculpture shown in Figure 1 was created in painstaking hard labour by first carving a torus from hard granite, and then drilling about one hundred holes into it. But conceptually this operation amounts to sweeping a ‘knife’ once around a torus, and in doing so, letting it execute one full twist around the curved toroidal axis. This cuts the torus into two identical parts, which however remain interlocked. If the two radii of the torus are suitably chosen, then the two rings can be re-arranged so that the outer curvature of one fits snugly into the inner curvature of the toroidal swept loop. In this arrangement, the two rings seem to form a new symbiotic structure. A model of this configuration (Fig.2a) can easily be made on a rapid-prototyping (RP) machine. The two loops were created simultaneously in a single run on a Fused Deposition Modelling (FDM) machine [3]. A narrow twisting gap was left in the part description of the torus, which was then filled in by the FDM machine with some scaffolding consisting of filler material. This material, being differently colored and more brittle than the material that forms the actual part, can easily be removed with a scalpel after the two halves of the split torus have been pried apart.

[INSERT FIGURE 2 ABOUT HERE]



(a) (b) (c)

Figure 2: Maquettes of interlocking congruent rings cut from a single toroid by Carlo Séquin, 1999:
(a) Two loops with the same basic geometry as *Oushi Zokei* 1999 (Figure 1b), FDM maquette 7 × 10 × 8 cm;
(b) Three mutually interlocking loops that assemble into a similar torus, FDM maquette 7 × 11 × 8 cm;
(c) Four loops mutually interlocking loops that assemble into a twisted square toroid, FDM maquette 8 × 12 × 9 cm;
(Photos: Carlo Séquin, 2002).

2.1 The Basic Splitting Paradigm

This fascinating way of splitting a torus into two parts naturally invites inquisitive minds to ponder whether a torus could also be split into three or more identical loops, and what interesting configurations these interlocking rings might assume. Since it is very difficult to do this investigation in one's head, the second author started to build small plastic maquettes of these alternative configurations and to explore possible extensions of the original splitting paradigm [5][6]. If we want to split the torus into three identical parts, the cutting 'knife' must have the shape of a three-spoked star, where three straight blades join together at angles of 120º at the toroidal axis. This 3-blade knife is then rotated through 360º as it is swept once around the toroidal ring. The result can be seen in Figure 2b. The three mutually interlocking rings form a physical maquette that is very satisfying to manipulate and to play with. However, they are less suitable to yield a good sculptural configuration that matches the dramatic harmony of *Oushi Zokei* (Fig.1b,c).

Similarly, if the cutting 'knife' has four 'spokes' (equivalent to two cuts at right angles), then four interlocking loops will result (Fig.2c). It is quite clear how this paradigm can be extended to more than four parts. N interlocking rings can be generated by a knife with N blades joining together at angles of 360º/N at the toroidal axis. Partitioning the torus into 4 and into 6 parts is particularly attractive. In the first case, the cross sections of the individual rings can be made square, which then leads to a toroid that also has a twisting square cross section (Fig.2c). In the second case, if the individual rings have cross sections equal to an equilateral triangle, then the assembled toroid will have a hexagonal profile [6]. The symmetry of the cross sections of the individual rings makes these puzzles particularly intriguing – and for some people quite challenging to put back together again, since it is not immediately obvious what faces need to be joined against one another and what faces are part of the original torus surface. These bundles of mutually interlocking rings are less suitable to configure as permanent large-scale sculptures.

2.2 Torus Knots and Interlocking Rings

The torus can also be cut by twisting a single straight ‘knife’ through angles other than 360º. Keizo Ushio has explored some of the possibilities in his work. Figure 3 shows sculptures where the cut is rotated through 180º, 360º, and 540º, respectively. The second option represents the case discussed above and shown in Figure 1. But for the other two twisting angles a new configuration arises: The torus is no longer cut into two parts, but remains connected as a single strand with a semi-circular cross section that loops twice around the central hole, and while doing so executes either a single twist of 360º (Fig.3a) or three full twists (Fig.3c). Figure 4a gives a schematic computer rendering of the geometry of the sculpture depicted in Figure 3a.

[INSERT FIGURE 3 ABOUT HERE]

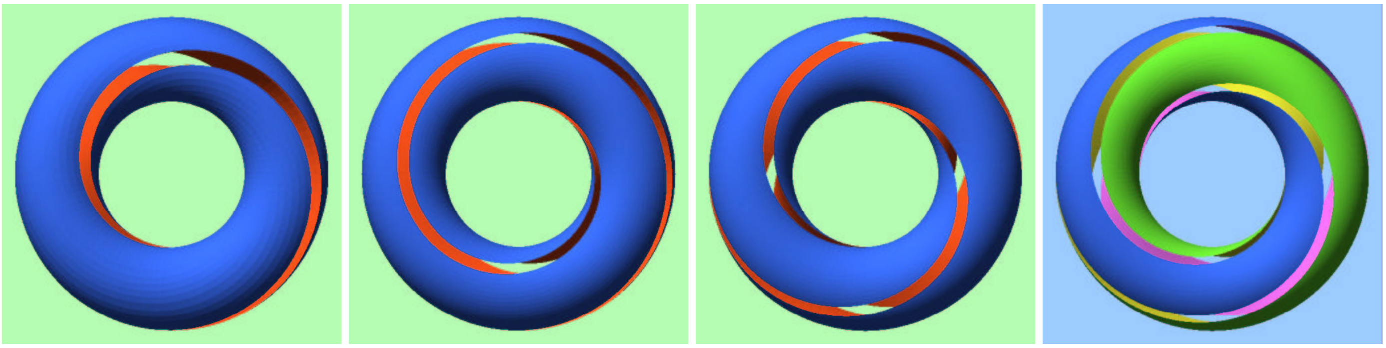
(a) (b) (c)

Figure 3: Split tori by Keizo Ushio:
(a) 180º twist: *Möbius in Space*, white granite, 220 × 220 × 63 cm, Kitaki Okayama 1989.
(b) 360º twist: *Samsara,* white granite, 165 × 170 × 80 cm, Narita 1992.
(c) 540º twist: *Oushi Zokei Ikiru (Life),* Japanese white granite, 150 × 170 × 50 cm, Ikuno Ginzan Mineral Park 1996.

Again we can generalize this cutting action by using a ‘knife’ that has three or more blades joining on the toroidal axis, and which executes twists of various amounts as it sweeps around the toroidal loop. Figure 4b and 4c show the results for a 3-blade knife executing twists of 120º and 240º, respectively. In both cases the result is a single connected strand looping around the central hole three times. If we use a 4-blade knife, we can generate a quadruple loop for twist amounts of ±90º or ±270º. But if we use ±180º of twist, then the torus is actually divided into two double loops (Fig.4d). These two loops, however, are tightly nested and, unlike the case of Figure 2a, cannot be separated and put into a different sculptural configuration. Of course, in all those cases, in order to make the individual branches quite visibly distinct, one should use a ‘thick knife’ that produces sizeable gaps between the individual branches of the multi-loop.

In summary, we use a knife with n blades and apply a total twist angle of t*(360º/n), where t is an integer that indicates through how many “sectors” the knife is twisted in its journey around the toroidal loop. Under these conditions, the cut line on the surface forms a (t, n)-torus link, and the solid parts after the cut has been executed also form a (t, n)-torus link. It can be shown that the solid parts form g connected components, where g is the greatest common divisor of (t, n), and each link component is a (t/g, n/g)-torus knot. When t and n are relatively prime, there is only a single connected component.

[INSERT FIGURE 4 ABOUT HERE]



(a) (b) (c) (d)

Figure 4: Computer generated examples of connected multi-loops cut from a toroid to form (t, n)-torus links: (a) t=1, n=2; (b) t=1, n=3; (c) t=2, n=3; (d) t=2, n=4, resulting in two linked, inseparable double loops. (Carlo Séquin 2004).

2.3 Keizo's Torus Knot Sculptures

Keizo Ushio has carved a variety of sculptures (Fig.5) that result from dividing a solid three-dimensional torus with a simple straight rotating cutting line (n=2). Actually, at every longitudinal location Keizo will drill half-way through the torus from both sides to prevent breakout of the stone where the drill would emerge from the torus, if the hole were drilled in a single pass. Each sculpture obtains its own personality from the treatment of this cutting surface and the texturing of the original torus surface. In *Ikiru* (Fig.3c) the very rough surface contrasts with the more precisely defined space curve winding around the torus, formed by the intersection with the splitting surface.

A particularly intriguing example is *Dream Lens* (Fig.5b). It seems to consist of three intertwined pieces with different textures. Why do the three pieces not fall out of position? What holds them together? Actually, this is just a single strand forming a torus knot (t=3, n=2). Then, how can this strand be textured in three different ways? The answer is that the texture gradually transitions from one type to the next as the strand winds through the inner hole of the torus!

[INSERT FIGURE 5 ABOUT HERE]

(a) (b)

Figure 5: Torus Knot Sculptures by Keizo Ushio:

(a) *Mugen (Dream Field),* Chinese white granite, 360 × 320 × 180 cm, Yachiyo Hyogo, 1999.
(b) *Yume Lenz (Dream Lens),* Japanese blue granite, 320 × 300 × 200 cm, Maiko Park, Kobe, 2004.

While mathematicians refer to the topology of the sculpture in Figure 5a as the (1,2) torus knot, the result is not really a knot in the traditional sense. If it were made from soft material, it could readily be opened up into an un-tangled loop. However, if the cutting surface is rotated through 540° as it sweeps around the torus (Fig.5b), then a true knot results. The resulting (3,2) torus knot is equivalent to a clover leaf knot or trefoil knot. Ushio has carved several versions of this knot (Fig.3c).

2.4 Möbius Spaces

When examining these single-strand rigid sculptures emerging from splitting a torus with a 'thick knife', it becomes clear that the twisting void is a defining feature of the resulting art work. As many sculptors realize, and as the first author has pointed out on several occasions, the spaces where material is missing may be even more important visually than the remaining material itself. We note that if the space in a torus is split with a 180° rotation of the cutting tool, then the resulting cutting surface has the shape of a Möbius band. Therefore the resulting gap is a 'Möbius space' into which one could snuggly fit a Möbius band. We investigated ways to enhance the visual impact of this space. At the Bridges 2000 conference [5] the sculpture maquette *Möbius Space* (Fig.6a) was shown. Following a suggestion by the first author, its interior space had been hollowed out dramatically. Its visual impact was further enhanced by giving it a shining, silvery, almost mirror-like surface, while the outer torus surface was left dark and more textured. Clearly, in a large-scale sculpture of that kind, the viewer would be drawn primarily to this inside space. In one of his more recent split-torus sculptures, Keizo Ushio also dramatically enhanced the visibility of the inner space by coloring it orange and by strongly emphasizing the drill marks (Fig.6b). This paradigm opens a whole new way of creating additional interesting variations of sculptures based on the idea of splitting a torus or a twisted band. The maquette displayed in Figure 6c shows what happens when this space is equivalent to a Möbius band with three half-twists [6].

[INSERT FIGURE 6 ABOUT HERE]

(a) (b) (c)

Figure 6: Möbius Spaces:
(a) FDM maquette 12 × 10 × 5 cm, by Carlo Séquin, Berkeley, 2000; (Photo: Carlo Séquin 2000);
(b) *Möbius in Space* , granite and paint, 320 × 300 × 200 cm, by Keizo Ushio, Cottesloe, 2005;
(c) FDM maquette of a triply twisted space, 12 × 10 × 5 cm, by Carlo Séquin, Berkeley, 2005; (Photo: Carlo Séquin 2005).

3. Splitting Twisted Bands

If the original toroidal loop does not have a simple circular cross section, but has distinct edges and faces, as in the case of a twisted band, the interactions between the cutting surface and the possibly twisted features on the toroid can become more complicated. In the simplest case, though, the cutting surface twists in the same way that the overall prismatic structure twists as it follows the toroidal loop. The basic paradigm is then one of a prismatic band split into two halves, which twist around each other by varying amounts. If that twist is an odd multiple of 180°, then one strand merges into the other after one lap around the toroidal loop, and overall we obtain just one single strand. For even multiples of 180°, the split creates two separate loops. If the starting shape has a built-in twist of exactly ±180°, we obtain a Möbius band. This split Möbius band is another one of Keizo Ushio's signature shapes (Fig.7). Over the last two decades he has sculpted dozens of variations. In the following we analyze the geometrical principles behind these sculptures.

[INSERT FIGURE 7 ABOUT HERE]

(a) (b) (c)

Figure 7: Split Möbius bands by Keizo Ushio:
(a) *Oushi Zokei*, Chinese granite, 180 × 90 × 50 cm, Fukuyama Hiroshima, 2000.
(b) *Oushi Zokei*, Japanese white granite, 220 × 210 × 100 cm, Sanda Hyogo, 1991.
(c) *Möbius in Space*, African black granite, 200 × 200 × 100 cm, Mihama Fukui, 1990.

3.1 Simple Möbius Loops

A quick way to obtain a Möbius band is to start with a narrow strip of paper, giving one end a half-twist, and then joining the two ends together. The resulting Möbius band is one-sided and has only a single edge in the shape of a warped figure-8. If this paper band is again cut apart along its centre line, it will then open into a single two-sided band with a full 360° twist in it. We note that it takes a great deal of technical ability to divide a stone Möbius band without breaking it. Keizo Ushio has developed the required technical ability to a high degree, and he has carved an impressive variety of granite sculptures based on this concept (Figs.7,8). He normally drills holes halfway from 'both sides' of the band along its centre line, as for his toroidal sculptures. The emerging form is a single rigid shape. The volume of stone removed in this splitting process also has form of a Möbius band and thus leaves behind a Möbius space, as discussed in the previous section. The resulting sculpture thus can be understood in two ways. Firstly, it can be seen as a split granite Möbius band with a cut that follows the twisting of the band. Secondly, it can be seen as a double-length curled prism that loops back along itself, and in doing so defines a Möbius space. The treatment of the drill marks and of the original band surface may emphasize one way of viewing or the other.

[INSERT FIGURE 8 ABOUT HERE]

(a) (b) (c)

Figure 8: Split Möbius bands by Keizo Ushio:
(a) *Oushi Zokei*, African black granite, 230 × 250 × 60 cm, Bondi Beach, Sidney, Australia, 2003.
(b) *Kakehashi (Bridge)*, Japanese pink granite, 200 × 300 × 100 cm, Takino Hyogo, 2000.
(c) *Möbius in Space ORC*, African black granite, 180 × 80 × 60 cm, Osaka, 1993.

While the sculptures shown in Figure 7 start from a circular, regularly twisting Möbius band, much more variety can be achieved if that starting shape is altered. In Figure 8a the original Möbius loop has been stretched into a vertically oriented oval. The upper half of the sculpture has highly polished surfaces, while the lower parts are kept in the rough, as if they were just emerging from the earth. In *Kakehashi* (Fig.8b) half the Möbius band is kept straight and flat, forming a bridge, and all the necessary twist is accommodated in the loop above that bridge. Again, this sculpture is partly polished and mostly rough in the lower portion with a smooth transition of the texture from one part to the other – as was employed in *Dream Lens* (Fig.5b). Finally, in the sculpture shown in Figure 8c all the twist of a Möbius band is concentrated in a short section of the overall loop, while the rest of the loop is kept untwisted.

3.2 *Full-Twist Bands*

If the paper strip used in Section 3.1 is given a full 360° twist before its ends are joined, we obtained a two-sided surface with two edges. If this loop is cut along its centre line, two interlocking fully twisted double-sided loops will result. Even if the original band was made form a rigid material, the two resulting loops can now be moved apart some limited distance, since they are no longer connected to each other. Keizo Ushio has exploited this idea in several granite sculptures. In *Oushi Zokei* (Fig.9a) he uses a form already seen in *Kakehashi* (Fig.8b) where all the twist is accommodated in the return loop above a straight planar base. The two separated segments are then moved apart by a significant amount. His 1996 *Oushi Zokei* (Fig.9b) consists of two rather ‘free-form’ loops, and it is not immediately obvious that they are the two halves of one and the same initial band. In the vertical form of *Kyousei* (Fig.9c) the two half-twists are distinctly separated and concentrated in two short stretches in otherwise straight segments of the toroidal loop.

[INSERT FIGURE 9 ABOUT HERE]

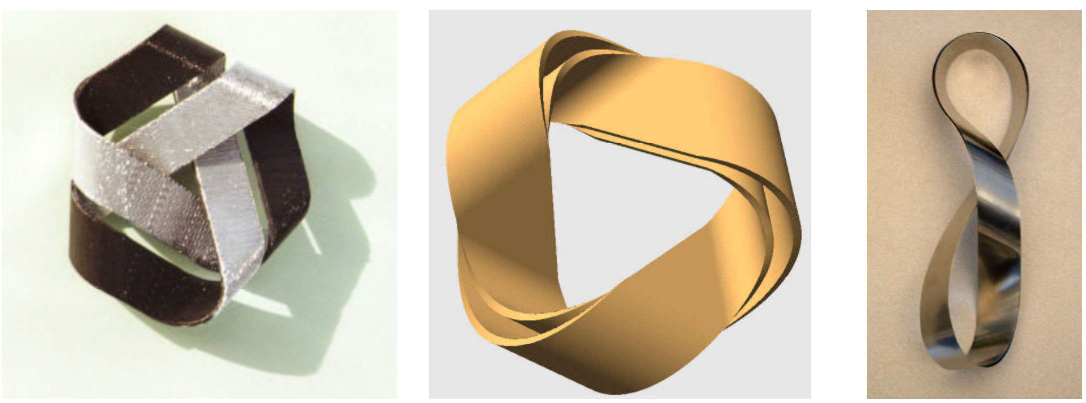
(a) (b) (c)

Figure 9: Interlocking loops by Keizo Ushio:
(a) *Oushi Zokei*, black granite, 200 × 300 × 110 cm, Kobe Suma-Rikyu Park, 1992.
(b) *Oushi Zokei Kyousei*, black granite, 260 × 300 × 110 cm, Kakogawa City Hall, Hyogo, 1996.
(c) *Kyousei (Symbiosis)*, blue granite, 320 × 150 × 80 cm, Vesolund, Denmark, 2006.

3.3 Triple-Twist Möbius Loops

If our paper strip is given three half-twists before its ends are joined, we obtain a triple-twist Möbius loop, which can be configured to have 3-fold symmetry, as exhibited in one form of the international symbol for recycling. (One often also sees a version where one arrow is flipped, thus corresponding to an ordinary Möbius loop with just one flip). The 3-fold symmetric version is the basis for M. C. Escher's drawing of a split twisted band, called *Möbius I* [7]. This sketch inspired the maquette shown in Figure 10a. This particular geometry would be too thin and too fragile to be carved from stone. Keizo Ushio uses much thicker bands in his sculptures (Fig.11). For completeness it should be pointed out, that the Möbius band cannot only be split 'sideways', but can also be split into two thin layers (Fig.10b); this configuration is most suitable for a realization in metal. A related puzzle is available from Conrad Valett in Germany [8]. It springs open into a twisted, knotted double-size loop, and the challenge is then to put it back together into its double-layered form (Fig.10c).

[INSERT FIGURE 10 ABOUT HERE]



(a) (b) (c)

Figure 10: Split triply-twisted Möbius bands:
(a) Painted FDM maquette, 10 × 8 × 3 cm, by Carlo Séquin, 1999; (Photo: Carlo Séquin 2004).
(b) A different way to split a Möbius band: computer generated rendering by Carlo Séquin, 2004.
(c) Related metal puzzle, 30 × 9 × 6 cm, by Conrad Valett; (Photo: Carlo Séquin 2006).

A triple-twist Möbius loop is among one of the earliest large stone sculptures by Keizo Ushio (Fig.11a). In later versions he has placed the resulting shape vertically on end (Fig.11b) and has also experimented with different texture combinations for the surface and for the cut. Note, that in all cases the resulting strand forms a true knot – the trefoil or clover leaf knot.

[INSERT FIGURE 11 ABOUT HERE]

(a) (b)

Figure 11: Split triply-twisted Möbius bands by Keizo Ushio:
(a) *Oushi Zokei 540° Twist,* African black granite , 110 × 270 × 270 cm, Tokiwa Park, Ube, Yamaguchi, 1993.
(b) *Oushi Zokei 540° Twist,* blue granite, 200 × 120 × 50 cm, Bondi Beach, Sidney, Australia, 2001.

4. Splitting Knots

In Section 2.2 we have seen that when the torus is cut with properly chosen values of t and n, one obtains a single torus knot. The simplest such torus knot is the trefoil knot. If the sweep along this knot curve does not use a circular cross section but rather forms a 'band' with distinctly different 'width' and 'thickness' values, then two other degrees of freedom appear in the definition of this shape: *twist* and *azimuth*. Changing them can dramatically alter the look and feel of a sculpture. In particular, there are only some twist values that maintain the three-fold symmetry of the trefoil knot, and only very few choices will allow the ribbon to curve around itself smoothly and organically.

If we try to form a trefoil knot from a flat ribbon, we find that the tightest configuration forms a one-sided loop; but it does not maintain 3-fold symmetry! In order to obtain a 3-fold symmetrical shape, we may give the ribbon either zero twist (seen in a projection along the symmetry axis) or impose three half-twists, as depicted in Figure 12a. Now we can split this ribbon into two 'fibres' by letting the 'knife' follow the curving and twisting of the band. Because of the built-in one-sidedness, the cut will not result in two separate components, but will produce a more complicated knot formed by a half-ribbon with twice the band's original length. The second author has experimented with ways to turn this geometric form into an aesthetically pleasing sculpture [6]; one result is *Infinite Duality* (Fig. 12b). In 2005 this basic form was used in the entry *Knot Divided* to the annual snow sculpting competition in Breckenridge, Colorado [9]. In this case, the 3-fold symmetry was abandoned in order to obtain a more dramatic looking sculpture (Fig.12c) and to make the best possible use of the 12 feet tall snow blocks made available to the participants. For the mathematically inclined, this sculpture also presents an interesting puzzle. While the original ribbon forms the simplest possible knot, the 3-crossing trefoil knot, the final structure forms a much more complicated knot. The reader is invited to figure out its crossing number [10].

[INSERT FIGURE 12 ABOUT HERE]

(a) (b) (c)

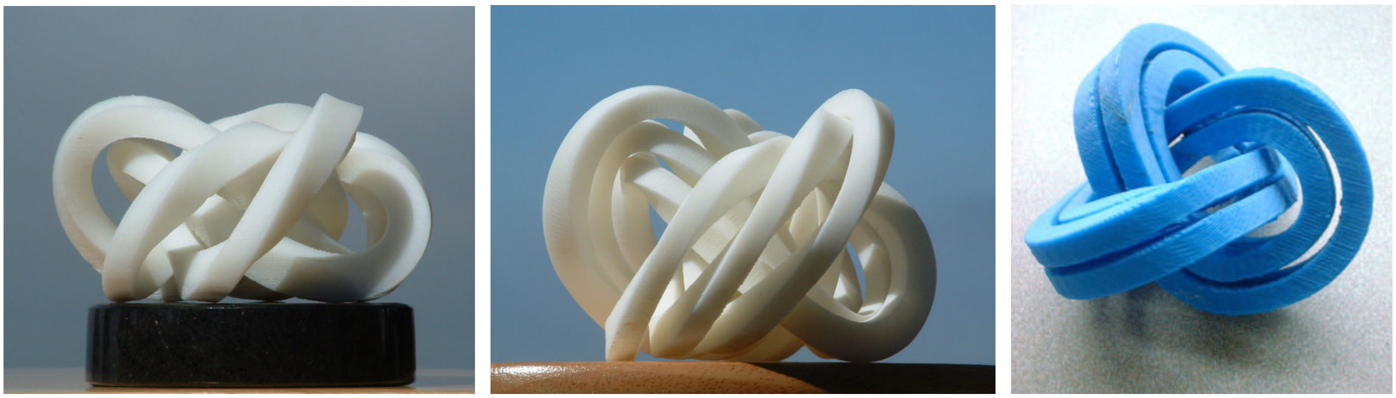
Figure 12: Split Möbius bands in trefoil knot configuration:
(a) FDM maquette, 5 × 11 × 10 cm, by Carlo Séquin 1999; (Photo: Carlo Séquin 2004).
(b) *Infinite Duality* , bronze cast, 6 × 12 × 11 cm, by Carlo Séquin 2004; (Photo: Carlo Séquin 2004).
(c) *Knot Divided*, snow sculpture, 350 × 300 × 300 cm, by Team Minnesota, Breckenridge, USA, 2005. (Photo: Carlo Séquin 2005).

Many artists have been fascinated by knots and tangles, and they deserve their own future review article. In this paper we have focused on the geometries that result when simple loops or knots are split lengthwise. That process results in objects of a related kind, but of higher complexity. In principle, the splitting process could then be repeated. For sculptures that have to be created by manual labour, this recursion very quickly reaches a practical limit. On the other hand, objects designed procedurally on a computer can go much further, and the results may even be fabricated with a layered manufacturing technology. Here we want to conclude with the discussion of some shapes that are a natural extension of the paradigms discussed in this paper, and which should be realizable as large scale sculptures – some in stone, others in metal.

All shapes are based on a prismatic ribbon wound into a trefoil knot (Fig.13). If the ribbon executes three half-twists, then the configurations shown in Figure 12 result. If the ribbon has an even number of half-twists, the split will separate it into two interlocking trefoils. A challenge is to carefully adjust the twist along the ribbon in such a way that that the two trefoils can be moved away from each other by an appreciable distance. Figure 13a shows a solution based on a ribbon with three full twists that allows both trefoils to stay in contact with the ground plane as they are rotated apart by about 30 degrees around the central vertical axis. To make this possible the original ribbon lays flat against the ground in the lower bends and assumes a vertical orientation at the top bends, allowing the separated trefoils to cross over one another at these locations. In this orientation, the ribbon could even be cut into three side-by side strands so that three independently movable trefoils result (Fig.13b).

If we try to form a tight and compact trefoil knot from an *n*-sided prism, we find that for $n = 4$ we can join the ends of the prismatic strand with almost no apparent twisting (Fig.13c). However, it turns out that where one lobe transitions into the next one, the Frenet frame that defines the osculating plane at each curve point exhibits a quick 90º torsional twist. Thus when we follow one of the prism edges, we find that it will jog to an adjacent position as we travel once around the whole knot, and we only return to the starting point after four passes around the knot. Splitting the 4-sided prismatic strand into four square fibres will thus lead to a single knotted loop of four times the length of the original strand. Since this lengthy loop has to carry its weight through four passes around the trefoil knot, this is not a candidate for a stone sculpture, but might be realized in COR-TEN steel in the style of the giant knot sculptures by Greg Johns [11].

[INSERT FIGURE 13 ABOUT HERE]



(a) (b) (c)

Figure 13: Split trefoil knots:
(a) Split into two separate movable trefoils, FDM maquette, 6 × 9 × 9 cm, by Carlo Séquin, Berkeley, 2006;
(b) Split into three separate movable trefoils, FDM maquette, 9 × 13 × 13 cm, by Carlo Séquin, Berkeley, 2006;
(c) Split into 4-strand continuous loop, FDM maquette 4 × 7 × 6 cm, by Carlo Séquin, Berkeley, 2004;
(Photos: Carlo Séquin 2006).

5. Conclusions

We have discussed a representative selection of sculptures by Keizo Ushio that emerge from splitting toroidal or other closed-loop structures, including ribbons exhibiting twists ranging from 180° to 540°. In the latter case, the resulting forms are topologically knotted. In Ushio's hands, a simple mathematical paradigm has blossomed into a plethora of fascinating stone sculptures. We find these sculptures very satisfying at various emotional and intellectual levels. One may marvel at the overall symmetry and balance of the final shape or the skill and courage needed to chisel the raw granite into these elegant forms. Then again one may be intrigued by the topological issues brought forth by these twisted ribbons or by the enclosed single- or double-sided spaces. Ushio's work is a wonderful demonstration that a gifted artist can use a simple mathematical paradigm and enhance it in many different ways to create a rich body of artistic work.

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