# DEFORMATIONS OF SEMI-EULER CHARACTERISTICS

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Let $f : X \to S$ be a proper smooth morphism of pure relative dimension $n$ with connected fibers between Noetherian schemes. Let $\xi$ be a locally free coherent sheaf on $X$. If $s$ is a point of $S$, we have the sheaf $\xi_s = \xi|_{X_s}$ on the fiber $X_s$ of $f$ over $s$.

If $G$ is a coherent sheaf on a proper variety $Y$, the semi-Euler characteristic $\psi(G) = \Sigma_{i \text{ even}} \dim H^i(Y, G)$.

If $n$ is odd say $1 + 2m$, we will assume that we are given a non-degenerate pairing $B : \xi \otimes \xi \to \omega_{X/S}$ such that $B$ is symmetric if $n \equiv 1(4)$ or skew-symmetric if $n \equiv 3(4)$. In this situation we have

THEOREM 1. *The parity of $\psi(\xi_s)$ is locally constant on $S$ if 2 is a unit in $\mathcal{O}_S$.*

In characteristic zero this result appears in [1]. If $n = 1$, the result appears in [2]. My proof uses that the rank of a skew-symmetric matrix is even and it yields deeper results on the variation of $\dim H^i(X_s, \xi_s)$.

**1. The statement of the main results.** A Grothendieck complex for $\xi$ is a complex $K^* : 0 \to K^0 \xrightarrow{\alpha^0} K^1 \xrightarrow{\alpha^1} K^n \to 0$ of free coherent sheaves on $S$ such that the $i$ cohomology of $K^*|_s$ is isomorphic to $H^i(X_s, \xi_s)$ for all point $s$ in $S$. Grothendieck has shown that such complexes always exist locally on $S$.

We say that the complex $K^*$ is special if it has the form

$$0 \to K^0 \xrightarrow{\alpha^0} K^1 \to \cdots \to K^m \xrightarrow{\beta} \check{K}^m \xrightarrow{(-1)^*\check{\alpha}_{m-1}} \check{K}^{m-1} \to$$

$$\cdots \to \check{K}^1 \xrightarrow{(-1)^*\check{\alpha}^0} \check{K}^0 \to 0$$

where $\beta$ is skew-symmetric.

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We will prove

THEOREM 2. *Locally on $S$, $\xi$ has a special Grothendieck complex.*

*Proof that Theorem 2 ⇒ Theorem 1.* We have

$$\psi(\xi_s) = \sum_{i \text{ even}} \operatorname{rank} K^i - \operatorname{rank}(\beta(s)) - \sum_{i=0}^{m} \operatorname{rank}(\alpha^i(s)) + \operatorname{rank}(\check{\alpha}^i(s)).$$

As $\beta$ is skew-symmetric the parity of $\psi(\xi_s)$ the same as that of $\Sigma_{i \text{ even}}$ rank $K^i$ which is (locally) constant.

**2. Special complexes.** Let $L^*$ be a complex. Then $L^* \otimes L^*$ is the complex

$$(L^* \otimes L^*)^n = \bigoplus_{n_1+n_2=n} L^{n_1} \otimes L^{n_2}$$

with differential

$$d(a \otimes b) = da \otimes b + (-1)^i a \otimes db$$

if $a \in L^i$ and $b \in L^j$.

This complex has an involution $\tau : L^* \to L^*$ given by $\tau(a \otimes b) = (-1)^{\alpha\beta} b \otimes a$ where $a \in L^\alpha$ and $b \in L^\beta$.

Let $L^*$ be a complex of free coherent sheaves on $S$. We will assume

$$L^* : 0 \to L^0 \to \quad \to L^n \to 0.$$

Assume that we have a pairing

$$\gamma : L^* \otimes L^* \to \mathcal{O}_S(-n)$$

such that $\gamma \circ \tau = (-1)^m \gamma$ and such that $\gamma(s) : L^*|_s \otimes L^*|_s \to k(s)(-n)$ defines an isomorphism

$$L^*|_s \to \operatorname{Hom}(L^*|_s, k(s)(-n)).$$

LEMMA 3. *$L^*$ is a special complex in a neighborhood of $s$.*

*Proof.* Let $\gamma(a \otimes b) = R(a \otimes b)\cdot 1(-n)$ where $a \in L^p$ and $b \in L^{n-p}$. Then $R : L^p \otimes L^{n-p} \to \mathcal{O}_S$ satisfies

$$R(\delta_p(\alpha) \otimes \beta) + (-1)^p R(\alpha \otimes \delta_{n-p-1}(\beta)) = 0$$

for $\alpha \in L^p$ and $\beta \in L^{n-p+1}$.

Thus we have a commutative diagram

$$\begin{array}{ccc} L^p & \xrightarrow{\alpha_p} & L^{p+1} \\ \Big\downarrow \overline{R}_p & & \Big\downarrow \overline{R}_{p+1} \\ \check{L}^{n-p} & \xrightarrow[(-1)^{p+1}\check{\alpha}_{n-p-1}]{} & \check{L}^{n+p+1} \end{array}$$

If $\overline{R}_*$ are isomorphisms at $s$, they are isomorphisms in a neighborhood of $s$. Hence the high differentials in $L^*$ are isomorphic to the dual of the low differential upto sign.

We want to check that $R_{m+1} \circ \alpha_m : L^m \to \check{L}^m$ is skew-symmetric. This will follow if $\overline{R}_m = \check{\overline{R}}_{m+1}(-1)^m$. Now $\check{\overline{R}}_{m+1}(k)(c) = R(k \otimes c)$ and $\overline{R}_m(c)(k) = R(c \otimes k)$.

Thus our symmetry condition implies that $\overline{R}_m = (-1)^{m(m+1)+m}\check{R}_{m+1}(=(-1)^m\check{R}_{m+1})$. Q.E.D.

**4. The first step.** We fix a point $s$ of $S$ and freely replace $S$ by an open neighborhood of $s$. So we may assume that $S$ is affine. We have the Čech resolution

$$\xi \to \check{\mathscr{C}}^*$$

of $\xi$ with respect to some open affine cover of $X$. Then $K^* = f_*\check{\mathscr{C}}^*$ has homology sheaves $R^i f_* \xi$.

Now we have a resolution of $L \otimes L$ and a commutative diagram

$$\begin{array}{ccc} \xi \otimes \xi & \longrightarrow & \check{\mathscr{C}}^* \otimes \check{\mathscr{C}}^* \\ \Big\downarrow B & & \Big\downarrow B^* \\ \omega_{X/S} & \longrightarrow & I^* \end{array}$$

where $I^* : 0 \to I^0 \to \cdots \to I^{2n+1} \to 0$ where $I^i$ is an injective $\mathcal{O}_x$-module if $i < 2n + 1$ which is a resolution of $\omega_{X/S}$.

So we have an induced mapping

$$\alpha : K^* \otimes K^* \to f_* I^*.$$

We want to replace $\alpha : K^* \otimes K^* \to f_* I^*$ by a finite approximation.

By Grothendieck's approximation theorem we can find a complex $0 \to L^0 \to \cdots \to L^n \to 0$ of free $\mathcal{O}_S$-module of finite type together with a homomorphism $\sigma : L^* \to K^*$ such $\sigma$ is a quasi-isomorphism.

Thus we get $\beta = \alpha(\sigma \otimes \sigma) : L^* \otimes L^* \to f_* I^*$.

Using Grothendieck's proof we can find a complex $M^*$ of the same kind such that we have a commutative diagram

$$\begin{array}{ccc} L^* \otimes L^* & \xrightarrow{i} & M^* \\ \uparrow & & \downarrow \rho \\ \beta & & f_* I \end{array}$$

where $\rho$ is a quasi-isomorphism.

Let $i' = (i + (-1)^m i(\tau))/2$.

We need another kind of approximation.

**5. The second step.** Let $0 \to L^0 \to \cdots \to L^p \to 0$ be a complex of free coherent sheaves on $S$. Then replace $S$ by a neighbor of $s$ we may find normalized such complexes $M^*$ and $N^*$ together with quasi-isomorphisms $M^* \to L^* \to N^*$ where normalized means that the differential of the complex vanishes at $s$.

Once we do this we'll do the following.

We can consider the composition

$$R^* \otimes R^* \to L^* \otimes L^* \to M^* \to N^* \xrightarrow{\alpha} \mathcal{O}_S(-n)$$

where $R^*$ and $N^*$ are normalized and $\alpha$ is the isomorphism $n$-homology of $N^*$ same of $M \approx$ same of $f_* i \approx R^n f_*(\omega_{x|S}) \approx \mathcal{O}_S$ which works on $N^p = 0$ if $p > n$ by similar reasoning.

The proof of this step is easy. Let $f_{i,i}, \ldots, f_{i,i}$ be elements of $L^i$ such that their reduce in $L^i(s)$ are a basis of a maximal space which is mapped isomorphically into $\overline{L^{i+1}(s)}$. Then we have a complex $S^*$ such that $S^i$ has basis $\overline{f_{*,i}}$ and $\overline{df_{*,i-1}}$ where $a_i(\overline{f_{*,i}}) = \overline{df_{*,i}}$ and $\alpha_{c+1}(df_{*,i}) = 0$. Thus we have an obvious homomorphism $S^* \to L^*$.

Let $M^* = L^*/S^*$ is so far quotient. For $M^* = \check{N}^*$ where $\check{L}^* \to N^*$ is constructed as before.

**6. The third step.** Now take a $h : R^* \to K^*$ be a quasi-isomorphism where $R^*$ is normalized and $k : M^* \to S^*$ be a quasi-isomorphism where $S^*$ is normalized. Then $j' = k \circ i' \circ h : R^* \otimes R^* \to S^*$. Let $m : S^* \to \mathcal{O}_S(-n)$ be projection on the $n$-th component. Then we have $m \circ j' : R^* \otimes R^* \to \mathcal{O}_S(-n)$.

We want to check that this pairing satisfies the condition of Section 2.

Clearly $m \circ j'(\tau) = (-1)^m m \otimes j'$ by construction of $i'$. We need to check that

$$u^i : R^i(s) \otimes R^{n+i}(s) \to k$$

is a perfect pairing.

Now by construction $u^i$ is isomorphic $P^i : H^i(X_s, \xi_s) \otimes H^{n-1}(X_s, \xi_s) \to k$.

To check that this is a perfect pairing by Serre duality it will be enough to check that it is the usual mapping induced by $m \otimes k \circ i \circ h|_s$.

This is just that

$$\rho(\alpha \otimes \beta) = (-1)^{i(n-i)+m}(\rho(\beta \otimes \alpha))$$

but this follows from the assumption on the symmetry of the pairing $\xi \otimes \xi \to \Omega_{X/S}$.

This finishes the proof.

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## REFERENCES

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