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# DEFORMATIONS OF SEMI-EULER CHARACTERISTICS

By GEORGE R. KEMPF

Let  $f : X \rightarrow S$  be a proper smooth morphism of pure relative dimension  $n$  with connected fibers between Noetherian schemes. Let  $\xi$  be a locally free coherent sheaf on  $X$ . If  $s$  is a point of  $S$ , we have the sheaf  $\xi_s = \xi|_{X_s}$  on the fiber  $X_s$  of  $f$  over  $s$ .

If  $G$  is a coherent sheaf on a proper variety  $Y$ , the semi-Euler characteristic  $\psi(G) = \sum_{i \text{ even}} \dim H^i(Y, G)$ .

If  $n$  is odd say  $1 + 2m$ , we will assume that we are given a non-degenerate pairing  $B : \xi \otimes \xi \rightarrow \omega_{X/S}$  such that  $B$  is symmetric if  $n \equiv 1(4)$  or skew-symmetric if  $n \equiv 3(4)$ . In this situation we have

**THEOREM 1.** *The parity of  $\psi(\xi_s)$  is locally constant on  $S$  if 2 is a unit in  $\mathbb{O}_s$ .*

In characteristic zero this result appears in [1]. If  $n = 1$ , the result appears in [2]. My proof uses that the rank of a skew-symmetric matrix is even and it yields deeper results on the variation of  $\dim H^i(X_s, \xi_s)$ .

**1. The statement of the main results.** A Grothendieck complex for  $\xi$  is a complex  $K^* : 0 \rightarrow K^0 \xrightarrow{\alpha^0} K^1 \xrightarrow{\alpha^1} K^2 \rightarrow 0$  of free coherent sheaves on  $S$  such that the  $i$  cohomology of  $K^*|_s$  is isomorphic to  $H^i(X_s, \xi_s)$  for all point  $s$  in  $S$ . Grothendieck has shown that such complexes always exist locally on  $S$ .

We say that the complex  $K^*$  is special if it has the form

$$0 \rightarrow K^0 \xrightarrow{\alpha^0} K^1 \rightarrow \dots \rightarrow K^m \xrightarrow{\beta} \check{K}^m \xrightarrow{(-1)^* \alpha_{m-1}} \check{K}^{m-1} \rightarrow \dots \rightarrow \check{K}^1 \xrightarrow{(-1)^* \alpha^0} \check{K}^0 \rightarrow 0$$

where  $\beta$  is skew-symmetric.

We will prove

**THEOREM 2.** *Locally on  $S$ ,  $\xi$  has a special Grothendieck complex.*

*Proof that Theorem 2  $\Rightarrow$  Theorem 1.* We have

$$\psi(\xi_s) = \sum_{i \text{ even}} \text{rank } K^i - \text{rank}(\beta(s)) - \sum_{i=0}^m \text{rank}(\alpha^i(s)) + \text{rank}(\alpha^i(s)).$$

As  $\beta$  is skew-symmetric the parity of  $\psi(\xi_s)$  the same as that of  $\sum_{i \text{ even}} \text{rank } K^i$  which is (locally) constant.

**2. Special complexes.** Let  $L^*$  be a complex. Then  $L^* \otimes L^*$  is the complex

$$(L^* \otimes L^*)^n = \bigoplus_{n_1 + n_2 = n} L^{n_1} \otimes L^{n_2}$$

with differential

$$d(a \otimes b) = da \otimes b + (-1)^i a \otimes db$$

if  $a \in L^i$  and  $b \in L^j$ .

This complex has an involution  $\tau : L^* \rightarrow L^*$  given by  $\tau(a \otimes b) = (-1)^{\alpha\beta} b \otimes a$  where  $a \in L^\alpha$  and  $b \in L^\beta$ .

Let  $L^*$  be a complex of free coherent sheaves on  $S$ . We will assume

$$L^* : 0 \rightarrow L^0 \rightarrow \cdots \rightarrow L^n \rightarrow 0.$$

Assume that we have a pairing

$$\gamma : L^* \otimes L^* \rightarrow \mathcal{O}_S(-n)$$

such that  $\gamma \circ \tau = (-1)^n \gamma$  and such that  $\gamma(s) : L^*|_s \otimes L^*|_s \rightarrow k(s)(-n)$  defines an isomorphism

$$L^*|_s \rightarrow \text{Hom}(L^*|_s, k(s)(-n)).$$

**LEMMA 3.**  *$L^*$  is a special complex in a neighborhood of  $s$ .*

*Proof.* Let  $\gamma(a \otimes b) = R(a \otimes b) \cdot 1(-n)$  where  $a \in L^p$  and  $b \in L^{n-p}$ . Then  $R : L^p \otimes L^{n-p} \rightarrow \mathbb{C}_S$  satisfies

$$R(\delta_p(\alpha) \otimes \beta) + (-1)^p R(\alpha \otimes \delta_{n-p-1}(\beta)) = 0$$

for  $\alpha \in L^p$  and  $\beta \in L^{n-p+1}$ .

Thus we have a commutative diagram

$$\begin{array}{ccc} L^p & \xrightarrow{\alpha_p} & L^{p+1} \\ \downarrow \bar{R}_p & & \downarrow \bar{R}_{p+1} \\ \check{L}^{n-p} & \xrightarrow{(-1)^{p+1} \check{\alpha}_{n-p-1}} & \check{L}^{n+p+1} \end{array}$$

If  $\bar{R}_*$  are isomorphisms at  $s$ , they are isomorphisms in a neighborhood of  $s$ . Hence the high differentials in  $L^*$  are isomorphic to the dual of the low differential upto sign.

We want to check that  $R_{m+1} \circ \alpha_m : L^m \rightarrow \check{L}^m$  is skew-symmetric. This will follow if  $\bar{R}_m = \check{R}_{m+1}(-1)^m$ . Now  $\check{R}_{m+1}(k)(c) = R(k \otimes c)$  and  $\bar{R}_m(c)(k) = R(c \otimes k)$ .

Thus our symmetry condition implies that  $\bar{R}_m = (-1)^{m(m+1)+m} \check{R}_{m+1} = (-1)^m \check{R}_{m+1}$ . Q.E.D.

**4. The first step.** We fix a point  $s$  of  $S$  and freely replace  $S$  by an open neighborhood of  $s$ . So we may assume that  $S$  is affine. We have the Čech resolution

$$\xi \rightarrow \check{\mathcal{C}}^*$$

of  $\xi$  with respect to some open affine cover of  $X$ . Then  $K^* = f_* \check{\mathcal{C}}^*$  has homology sheaves  $R^i f_* \xi$ .

Now we have a resolution of  $L \otimes L$  and a commutative diagram

$$\begin{array}{ccc} \xi \otimes \xi & \longrightarrow & \check{\mathcal{C}}^* \otimes \check{\mathcal{C}}^* \\ \downarrow B & & \downarrow B^* \\ \omega_{X/S} & \longrightarrow & I^* \end{array}$$

where  $I^* : 0 \rightarrow I^0 \rightarrow \cdots \rightarrow I^{2n+1} \rightarrow 0$  where  $I^i$  is an injective  $\mathcal{O}_x$ -module if  $i < 2n + 1$  which is a resolution of  $\omega_{X/S}$ .

So we have an induced mapping

$$\alpha : K^* \otimes K^* \rightarrow f_* I^*.$$

We want to replace  $\alpha : K^* \otimes K^* \rightarrow f_* I^*$  by a finite approximation.

By Grothendieck's approximation theorem we can find a complex  $0 \rightarrow L^0 \rightarrow \cdots \rightarrow L^n \rightarrow 0$  of free  $\mathcal{O}_S$ -module of finite type together with a homomorphism  $\sigma : L^* \rightarrow K^*$  such  $\sigma$  is a quasi-isomorphism.

Thus we get  $\beta = \alpha(\sigma \otimes \sigma) : L^* \otimes L^* \rightarrow f_* I^*$ .

Using Grothendieck's proof we can find a complex  $M^*$  of the same kind such that we have a commutative diagram

$$\begin{array}{ccc} L^* \otimes L^* & \xrightarrow{i} & M^* \\ \uparrow \beta & & \downarrow \rho \\ & & f_* I \end{array}$$

where  $\rho$  is a quasi-isomorphism.

Let  $i' = (i + (-1)^m i(\tau))/2$ .

We need another kind of approximation.

**5. The second step.** Let  $0 \rightarrow L^0 \rightarrow \cdots \rightarrow L^p \rightarrow 0$  be a complex of free coherent sheaves on  $S$ . Then replace  $S$  by a neighbor of  $s$  we may find normalized such complexes  $M^*$  and  $N^*$  together with quasi-isomorphisms  $M^* \rightarrow L^* \rightarrow N^*$  where normalized means that the differential of the complex vanishes at  $s$ .

Once we do this we'll do the following.

We can consider the composition

$$R^* \otimes R^* \rightarrow L^* \otimes L^* \rightarrow M^* \rightarrow N^* \xrightarrow{\alpha} \mathcal{O}_S(-n)$$

where  $R^*$  and  $N^*$  are normalized and  $\alpha$  is the isomorphism  $n$ -homology of  $N^*$  same of  $M \approx$  same of  $f_* i \approx R^n f_*(\omega_{x|S}) \approx \mathcal{O}_S$  which works on  $N^p = 0$  if  $p > n$  by similar reasoning.

The proof of this step is easy. Let  $f_{i,i}, \dots, f_{i,i}$  be elements of  $L^i$  such that their reduce in  $L^i(s)$  are a basis of a maximal space which is

mapped isomorphically into  $L^{i+1}(s)$ . Then we have a complex  $S^*$  such that  $S^i$  has basis  $\overline{f_{*,i}}$  and  $\overline{df_{*,i-1}}$  where  $a_i(\overline{f_{*,i}}) = \overline{df_{*,i}}$  and  $\alpha_{c+1}(df_{*,i}) = 0$ . Thus we have an obvious homomorphism  $S^* \rightarrow L^*$ .

Let  $M^* = L^*/S^*$  is so far quotient. For  $M^* = \check{N}^*$  where  $\check{L}^* \rightarrow N^*$  is constructed as before.

**6. The third step.** Now take a  $h : R^* \rightarrow K^*$  be a quasi-isomorphism where  $R^*$  is normalized and  $k : M^* \rightarrow S^*$  be a quasi-isomorphism where  $S^*$  is normalized. Then  $j' = k \circ i' \circ h : R^* \otimes R^* \rightarrow S^*$ . Let  $m : S^* \rightarrow \mathbb{O}_s(-n)$  be projection on the  $n$ -th component. Then we have  $m \circ j' : R^* \otimes R^* \rightarrow \mathbb{O}_s(-n)$ .

We want to check that this pairing satisfies the condition of Section 2.

Clearly  $m \circ j'(\tau) = (-1)^m m \otimes j'$  by construction of  $i'$ . We need to check that

$$u^i : R^i(s) \otimes R^{n+i}(s) \rightarrow k$$

is a perfect pairing.

Now by construction  $u^i$  is isomorphic  $P^i : H^i(X_s, \xi_s) \otimes H^{n-1}(X_s, \xi_s) \rightarrow k$ .

To check that this is a perfect pairing by Serre duality it will be enough to check that it is the usual mapping induced by  $m \otimes k \circ i \circ h|_s$ .

This is just that

$$\rho(\alpha \otimes \beta) = (-1)^{i(n-i)+m}(\rho(\beta \otimes \alpha))$$

but this follows from the assumption on the symmetry of the pairing  $\xi \otimes \xi \rightarrow \Omega_{X/S}$ .

This finishes the proof.

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