A Manifold which does not admit any Differentiable Structure

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An example of a triangulable closed manifold M_0 of dimension 10 will be constructed. It will be shown that M_0 does not admit any differentiable structure. Actually, M_0 does not have the homotopy type of any differentiable manifold.

Also, a 9-dimensional differentiable manifold Σ^9 is obtained. Σ^9 is homeomorphic but not diffeomorphic to the standard 9-sphere S^9.

Use is made of a procedure for killing the homotopy groups of differentiable manifolds studied by J. MILNOR in [6]. I am indebted to J. MILNOR for sending me a copy of the manuscript of his paper.

Although much of the constructions (in particular the construction of M_0) generalizes to higher dimensions, I did not succeed disproving the existence of a differentiable structure on the higher dimensional analogues of M_0. A more general case of some of the constructions below will be published in a subsequent paper, with other applications.[1])

§ 1. Construction of an invariant

Let M^{10} be a closed triangulable manifold. Assume that M^{10} is 4-connected. (M^{10} is connected, and $\pi_i(M) = 0$ for $1 \leqq i \leqq 4$.) It follows from POINCARÉ duality and the universal coefficient theorem that $H^q(M; G) = 0$ for $5 < q < 10$, and $H^5(M)$ is free abelian of even rank $2s$, say. (If no coefficients are mentionned, integer coefficients are understood.)

Let $\Omega = \Omega S^6$ be the loop-space on the 6-sphere. It is well known that $H^5(\Omega) = Z$, $H^{10}(\Omega) = Z$, and if $\pi : \Omega \times \Omega \to \Omega$ is the map given by the product of loops, then

$$\pi^*(e_1) = e_1 \otimes 1 + 1 \otimes e_1 , \quad \text{and}$$
$$\pi^*(e_2) = e_2 \otimes 1 + 1 \otimes e_2 + e_1 \otimes e_1 ,$$

where e_1, e_2 are the generators of $H^5(\Omega)$ and $H^{10}(\Omega)$ respectively, and $H^*(\Omega \times \Omega)$ is identified with $H^*(\Omega) \otimes H^*(\Omega)$ by the KÜNNETH formula. (Compare R. BOTT and H. SAMELSON [1], Theorem 3.1.B.)

Lemma 1.1. *Let $X \in H^5(M)$ be given. There exists a map $f : M \to \Omega$ such that $f^*(e_1) = X$.*

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[1]) This paper was presented at the International Colloquium on Differential Geometry and Topology, Zurich, June 1960.

Proof. Let K be a triangulation of M. Define f by stepwise extension on the skeletons $K^{(q)}$ using obstruction theory. $f \mid K^{(4)}$ is taken to be the constant map into a base point on Ω. Let X_0 be a representative cocycle of X. For every 5-dimensional simplex s_5 of K, define $f \mid s_5$ to be a representative of $X_0[s_5]$-times the generator of $\pi_5(\Omega) \cong \pi_6(S^6) \cong Z$. The obstruction cocycle to extend $f \mid K^{(5)}$ in dimension 6 is zero. The next obstruction is in dimension 10 with values in $\pi_9(\Omega) \cong \pi_{10}(S^6) = 0$. (See [9], § 41.) Thus the lemma is proven.

Define a function $\varphi_0 \colon H^5(M) \to Z_2$ by the following device. For every $X \in H^5(M)$, take a map $f \colon M \to \Omega$ such that $f^*(e_1) = X$. Then, $\varphi_0(X) = f^*(u_2)[M]$, where $u_2 \in H^{10}(\Omega; Z_2)$ is the reduction modulo 2 of $e_2 \in H^{10}(\Omega)$, and $f^*(u_2)[M]$ is the value of the cohomology class $f^*(u_2)$ on the generator of $H_{10}(M^{10}; Z_2)$.

Lemma 1.2. *The function* $\varphi_0 \colon H^5(M) \to Z_2$ *is well defined,* i.e., $\varphi_0(X)$ *does not depend on the choice of the map* $f \colon M \to \Omega$ *such that* $f^*(e_1) = X$.

Proof. Let $f, g \colon M \to \Omega$ be two maps such that $f^*(e_1) = g^*(e_1)$. We have to show that $f^*(u_2) = g^*(u_2)$. Let K again be a triangulation of M. Since $f^*(e_1) - g^*(e_1) = 0$, it follows that f and g are 5-homotopic. (See S. T. HU [2], Chap. VI.) Since $H^q(M; \pi_q(\Omega)) = 0$ for $5 < q < 10$, it follows that f and g are 9-homotopic. Hence, we may assume that $f \mid K^{(9)} = g \mid K^{(9)}$. Let $\omega^{10}(f, g) \in C^{10}(K; \pi_{10}(\Omega))$ be the difference cochain. Then,

$$(f^*(u_2) - g^*(u_2))[s_{10}] = u_2[h\,\omega^{10}(f, g)[s_{10}]],$$

for every 10-simplex s_{10}, where $h \colon \pi_{10}(\Omega) \to H_{10}(\Omega)$ is the HUREWICZ homomorphism. According to J. P. SERRE, $u_2[h\alpha]$ is the mod. 2 HOPF invariant of the element in $\pi_{11}(S^6)$ represented by $\alpha \in \pi_{10}(\Omega S^6)$. (Compare [8], Lemme 2.) Since no element of odd HOPF invariant occurs in $\pi_{11}(S^6)$, it follows that $f^*(u_2) = g^*(u_2)$, and the proof is complete.

Lemma 1.3. *Let* $X, Y \in H^5(M)$ *be two integer cohomology classes of* M. *Then,*

$$\varphi_0(X + Y) = \varphi_0(X) + \varphi_0(Y) + x \cdot y,$$

where $x \cdot y$ *is the value on the generator of* $H_{10}(M^{10}; Z_2)$ *of the cup-product* $x \smile y$. (x, y are the mod. 2 reductions of X and Y respectively.)

Proof. Let $f, g \colon M \to \Omega$ be maps such that $f^*(e_1) = X$ and $g^*(e_1) = Y$. By definition, $\varphi_0(X) = f^*(u_2)[M]$, and $\varphi_0(Y) = g^*(u_2)[M]$.

Let $f \times g \colon M \times M \to \Omega \times \Omega$ be the product of f and g. (I.e., $f \times g(u, v) = (f(u), g(v))$.) Let $D \colon M \to M \times M$ be the diagonal map. Define $F \colon M \to \Omega$

by $F = \pi \circ (f \times g) \circ D$, where $\pi : \Omega \times \Omega \to \Omega$ is given by the multiplication of loops. Since D^* maps the tensor product of cohomology classes into their cup-product, we have $F^*(e_1) = D^*(X \otimes 1 + 1 \otimes Y) = X + Y$. Therefore,

$$\varphi_0(X + Y) = F^*(u_2)[M] .$$

On the other hand,

$$\begin{aligned} F^*(u_2) &= D^*(f^*(u_2) \otimes 1 + 1 \otimes g^*(u_2) + f^*(u_1) \otimes g^*(u_1)) \\ &= f^*(u_2) + g^*(u_2) + f^*(u_1) \smile g^*(u_1) \\ &= f^*(u_2) + g^*(u_2) + x \smile y . \end{aligned}$$

(u_1 is the reduction modulo 2 of e_1.) This proves Lemma 1.3.

The function $\varphi_0 : H^5(M) \to Z_2$ induces a function $\varphi : H^5(M; Z_2) \to Z_2$ satisfying $\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y$. Indeed, if X is an integer class whose reduction modulo 2 yields $x \in H^5(M; Z_2)$, we define $\varphi(x) = \varphi_0(X)$. It follows from

$$\varphi_0(2Y) = \varphi_0(Y) + \varphi_0(Y) + y \cdot y = y \cdot y = 0 ,$$

that $\varphi(x) \in Z_2$ depends only on $x \in H^5(M; Z_2)$.

The function $\varphi : H^5(M; Z_2) \to Z_2$ is then used to construct the number $\Phi(M)$ as follows. A basis $x_1, \ldots, x_s, y_1, \ldots, y_s$ of $H^5(M; Z_2)$ as a vector space over Z_2 will be called *symplectic* if $x_i \cdot x_j = y_i \cdot y_j = 0$, and $x_i \cdot y_j = \delta_{ij}$ for all i, $j = 1, \ldots, s$. Clearly, symplectic bases always exist. Moreover, it is well known that since the function $\varphi : H^5(M; Z_2) \to Z_2$ satisfies the equation

$$\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y ,$$

the remainder modulo 2

$$\Phi(M) = \Sigma_1^s \varphi(x_i) \cdot \varphi(y_i)$$

is independent of the symplectic basis $x_1, \ldots, x_s, y_1, \ldots, y_s$.

The rest of the paper is devoted to investigating the properties of the invariant Φ.

Clearly, Φ is an invariant of the homotopy type of 4-connected closed manifolds of dimension 10.

Our objective is the proof of the following theorems.

Theorem 1. *If M^{10} has the homotopy type of a C^1-differentiable 4-connected closed manifold, then* $\Phi(M) = 0$.

(It can be shown that the converse of this theorem would follow from the conjecture that the cohomology ring $H^*(M)$ and $\Phi(M)$ are a complete set of invariants of the homotopy type of the triangulable 4-connected closed manifold M of dimension 10.)

Theorem 2. *There exists a closed 4-connected combinatorial manifold M_0 of dimension 10 for which $\Phi(M_0) = 1$.*

(In fact a specific example will be constructed.)

In § 2, the proof of Theorem 1 will be carried out taking Lemmas 4.2 and 5.1 for granted. (Lemma 4.2 is used in the proof of Lemma 2.2, and Lemma 5.1 is used to deduce Theorem 1 from Lemma 2.4.) The Lemmas 4.2 and 5.1 are proved at the end of the paper, in § 4 and § 5. Theorem 2 will be proved in § 3.

§ 2. Proof of Theorem 1

Let M^{10} be a closed C^1-differentiable manifold which is 4-connected.

Lemma 2.1. *M^{10} is a π-manifold.*

Proof. Let $M^{10} \subset R^{n+10}$ be an imbedding with n large. We have to show that the normal bundle ν is trivial. This is done by constructing a field of normal n-frames f_n. Let K be a triangulation of M^{10}. Since $\pi_4(SO_n) = 0$, and M^{10} is 4-connected, it follows that $H^{q+1}(M; \pi_q(SO_n)) = 0$ for $0 \leqq q < 9$. Thus, there is only one possibly non-vanishing obstruction $\mathfrak{o}(\nu, f_n) \in H^{10}(M; \pi_9(SO_n)) \cong \pi_9(SO_n)$ to the construction of the field f_n of normal n-frames. By Lemma 1 of [7], $\mathfrak{o}(\nu, f_n)$ is in the kernel of the HOPF-WHITEHEAD homomorphism $J_9: \pi_9(SO_n) \to \pi_{n+9}(S^n)$. But J_9 is a monomorphism. (Compare proof of Lemma 1.2 of [4].) Hence, $\mathfrak{o}(\nu, f_n) = 0$, and the lemma is proved. (Recall that the proof of the assertion: J_9 is a monomorphism, was based on Corollary 2.6 of J. F. ADAMS paper *On the structure and applications of the STEENROD algebra*, Comm. Math. Helv. 32 (1958), 180–214. This statement also follows from the portion of the POSTNIKOV decomposition mod. 2 of S^n given below in § 5.)

The THOM construction associates with every framed manifold $(M; f_n)$, where $M \subset R^{n+\dim M}$, an element $\alpha(M; f_n) \in \pi_{n+\dim M}(S^n)$. We say that $(M^{10}; f_n)$ is *homotopic to zero* if the corresponding element $\alpha(M; f_n)$ is the neutral element of $\pi_{n+10}(S^n)$.

Lemma 2.2. *If $(M^{10}; f_n)$ is homotopic to zero, where M^{10} is 4-connected, then $\Phi(M) = 0$.*

Proof. The assumption that $(M; f_n)$ is homotopic to zero implies the existence of a framed manifold $(V^{11}; F_n)$ with boundary M^{10}. (Compare R. THOM [10].) We may assume that V is connected, and hence has a trivial tangent bundle. We can therefore apply to $V - M$ the procedure for killing the homotopy groups of a differentiable manifold studied by J. MILNOR. Specifically, using Theorem 3 of [6], we obtain a new 11-dimensional differentiable manifold with boundary M^{10} which is also 4-connected. This new 4-connected manifold will again be denoted by V^{11}. We can now forget about the fields of normal frames.

We proceed to compute $\Phi(M)$. Consider the cohomology exact sequence of the pair (V, M) with coefficients in Z_2,

$$\cdots \to H^5(V) \xrightarrow{i^*} H^5(M) \xrightarrow{\delta} H^6(V, M) \to \cdots .$$

Using relative Poincaré-Lefschetz duality (over Z_2), and the formula

$$u \smile \delta x[V, M] = i^*(u) \smile x[M] ,$$

where $u \in H^5(V)$, $x \in H^5(M)$ and $[V, M]$, $[M]$ are the generators of $H_{11}(V, M; Z_2)$ and $H_{10}(M; Z_2)$ respectively, it follows that $H^5(M; Z_2)$ has a symplectic basis $x_1, \ldots, x_s, y_1, \ldots, y_s$ say, such that $x_1, \ldots, x_s$ is a vector basis of Ker δ. Consequently, in order to prove $\Phi(M) = 0$, it is sufficient to show that $\varphi(x) = 0$ for every $x \in \text{Ker}\ \delta$.

Let $X \in H^5(M)$ be an integer class whose reduction modulo 2 is x, and let $f: M^{10} \to \Omega = \Omega S^6$ be a map such that $f^*(e_1) = X$. We have to show that $f^*(u_2) = 0$, where u_2 generates $H^{10}(\Omega; Z_2)$. Let Ω^* be the space obtained from Ω by attaching a cell of dimension 6 by a map $S^5 \to \Omega$ of degree 2. By Lemma 4.2 in § 4, below, for every map $g: S^{10} \to \Omega^*$, one has $g^*(u_2) = 0$, where we denote by $u_2 \in H^{10}(\Omega^*; Z_2)$ again the class corresponding to the old $u_2 \in H^{10}(\Omega; Z_2)$ under the canonical isomorphism $H^{10}(\Omega; Z_2) \cong H^{10}(\Omega^*; Z_2)$.

We attempt to extend $f: M \to \Omega^*$ to a map of V into Ω^*. Let (K, L) be a triangulation of (V, M). The stepwise extension of f on the skeletons $K^{(q)} \smile L$ leads to obstructions in the groups $H^{q+1}(K, L; \pi_q(\Omega^*))$. For $q < 5$, $\pi_q(\Omega^*) = 0$. We meet a first obstruction for $q = 5$ in $H^6(K, L; Z_2)$. By the Hopf theorem, this obstruction is δx. (See S. T. Hu [2].) Since $\delta x = 0$, it is possible to extend f on $K^{(6)} \smile L$. Using $H^{q+1}(K, L; G) = 0$ for $5 < q < 10$ (since V is 4-connected), it follows that there exists a map $F: K - \tau \to \Omega^*$, where τ is some 11-dimensional simplex in $K - L$, such that $F \mid L = f$. Let S^{10} denote the boundary of τ, and let $g: S^{10} \to \Omega^*$ be the restriction of F on S^{10}. Since $\partial(K - \tau) = L - S^{10}$, and $g^*(u_2) = 0$, it follows that $f^*(u_2) = 0$. The proof of Lemma 2.2 is complete.

Corollary 2.3. *If two 4-connected framed manifolds $(M; f_n)$ and $(M'; f'_n)$ of dimension 10 define the same element $\alpha = \alpha(M; f_n) = \alpha(M'; f'_n)$ by the Thom construction, then $\Phi(M) = \Phi(M')$.*

This is obtained by observing that Φ is additive with respect to the connected sum of manifolds.

It follows that Φ provides a homomorphism from a subgroup of $\pi_{n+10}(S^n)$

into Z_2. We denote this homomorphism by Φ again. Actually, Φ is defined on every element of $\pi_{n+10}(S^n)$. Indeed, using spherical modifications [6], it is easy to see that every element $\alpha \in \pi_{n+10}(S^n)$ is obtainable from a 4-connected framed manifold by the THOM construction. This remark will not be used in the present paper.

It follows from Corollary 2.3 that Theorem 1 is equivalent to the statement that $\Phi(\alpha) = 0$ for every $\alpha \in \pi_{n+10}(S^n)$, provided $\Phi(\alpha)$ is defined.

Since $\Phi(\alpha)$ is obviously zero for every element α of odd order, and by J. P. SERRE's results $\pi_{n+10}(S^n)$ contains no element of infinite order, it is sufficient to show that Φ annihilates the 2-component of the group $\pi_{n+10}(S^n)$. By Lemma 5.1 in § 5 below, every element α in the 2-component of $\pi_{n+10}(S^n)$ is representable in the form

$$\alpha = \beta \circ \eta \, ,$$

where $\eta \in \pi_{n+10}(S^{n+9})$ is the generator of the stable 1-stem, and $\beta \in \pi_{n+9}(S^n)$. Hence, Theorem 1 will follow from the

Lemma 2.4. *Every element* $\alpha \in \pi_{n+10}(S^n)$ *of the form* $\alpha = \beta \circ \eta$, *with* $\eta \in \pi_{n+10}(S^{n+9})$, *and* $\beta \in \pi_{n+9}(S^n)$ *is obtainable by the* THOM *construction from a framed manifold* $(\Sigma^{10}; f_n)$, *where* Σ^{10} *has the homotopy type of the* 10-*sphere* S^{10}.

Proof. We first show that $\beta \in \pi_{n+9}(S^n)$ is obtainable by the THOM construction from a framed manifold $(\Sigma^9; f_n)$, where Σ^9 has the homotopy type of the 9-sphere.

It is well known that β is obtainable by the THOM construction from some framed manifold $(M^9; f_n)$. We have to show that $(M^9; f_n)$ is homotopic to a framed manifold $(\Sigma^9; f_n)$, where Σ^9 is a homotopy sphere. This is done by simplifying M^9 by a series of spherical modifications. (See J. MILNOR [6].)

Assuming by induction that M^9 is $(p-1)$-connected $(0 \leqq p \leqq 4)$, we have to prove that $(M; f_n)$ is homotopic to a p-connected framed manifold $(M'; f'_n)$. Recall that a spherical modification of type $(p+1, q+1)$ applied to a class $\lambda \in \pi_p(M^9)$ consists of the following construction. Represent λ by an imbedding

$$f: S^p \times D^{q+1} \to M^9 \, ,$$

with $p + q + 1 = 9$. (This is possible for $p \leqq 4$ since M^9 is a π-manifold and the normal bundle of any imbedding $S^p \to M^9$ is stable in this range of dimensions.) The manifold M is then replaced by

$$M' = (M - f(S^p \times D^{q+1})) \cup (D^{p+1} \times S^q) \, ,$$

under identification of $f(S^p \times S^q)$ regarded as the boundary of $f(S^p \times D^{q+1})$ with $S^p \times S^q$ regarded as the boundary of $D^{p+1} \times S^q$. By Theorem 2 of

[6], the manifolds M and M' bound a 10-dimensional differentiable manifold $\omega = \omega(M, f)$, and $f: S^p \times D^{q+1} \to M^9$ can be chosen such that the field f_n (over M) is extendable over ω as a field of normal n-frames. (We can think of ω as imbedded in R^{n+10} with $M \subset R^{n+9} \times (0)$ and $M' \subset R^{n+9} \times (1)$ since n can be taken as large as we please.) Hence spherical modifications of type $(p+1, q+1)$ with $0 \leqq p \leqq 4$ can be performed so as to carry $(M; f_n)$ into a homotopic framed manifold. It is known (Theorem 3 of [6]) that for $p < 4$, spherical modifications simplify the manifold. More precisely $\pi_p(M')$ is isomorphic to the quotient of $\pi_p(M)$ by the subgroup generated by λ, and $\pi_i(M) \cong \pi_i(M') = 0$ for $i < p$. Hence, it is easy, using [6], to obtain a 3-connected framed manifold homotopic to $(M^9; f_n)$. The case $p = 4$ requires special care. If $\lambda \in \pi_4(M^9)$ is the class we want to kill, there exists an imbedding $f: S^4 \times D^5 \to M^9$ such that $f \mid S^4 \times (0)$ represents λ. Let $M' = \chi(M, f)$ be the 9-dimensional manifold obtained from M and f by spherical modification. (f is supposed to be chosen so that $(M'; f'_n)$ with some f'_n is homotopic to $(M; f_n)$.) In general, however, $f \mid x_0 \times (bdry\, D^5)$ represents a *non-zero* element of $\pi_4(M')$. Thus, it is not clear a priori that a series of spherical modifications of type (5, 5) will carry M into a 4-connected manifold, and hence a homotopy sphere.

If λ is a generator of the free part of $\pi_4(M) \cong H_4(M)$, there exists by Poincaré duality a class $\mu \in H_6(M)$ whose intersection coefficient with λ (or $h\lambda$ rather, where h is the Hurewicz homomorphism) is 1. It follows that in this case the cycle given by $f \mid x_0 \times (bdry\, D^5)$ is homologous to zero in $M - f(S^4 \times D^5)$, and hence in M'. Thus $H_4(M') \cong \pi_4(M')$ has strictly smaller rank than $H_4(M) \cong \pi_4(M)$, and the torsion subgroup is unchanged.

I claim that if $\lambda \in \pi_4(M)$ is a torsion element, the homology class of the cycle $f \mid x_0 \times (bdry\, D^5)$ *is of infinite order for any f representing* λ. Hence, one more spherical modification will lead to a manifold with 4-dimensional homology group of not bigger rank than $H_4(M)$ and with a strictly smaller torsion subgroup. (I.e., a series of spherical modifications will lead to a 4-connected framed manifold homotopic to $(M^9; f_n)$. By Poincaré duality, a closed 4-connected manifold of dimension 9 has the homotopy type of S^9.)

Since the Betti numbers p_4, p'_4 of M and M' (in dimension 4) differ at most by 1, and differ indeed by 1 if and only if λ' (represented by $f \mid x_0 \times (bdry\, D^5)$) in M' is of infinite order, it is sufficient to show that $p'_4 + p_4 \equiv 1 \bmod. 2$. Since $p'_i = p_i$ for $0 \leqq i \leqq 3$, this is equivalent to showing that the semi-characteristics $E^*(M)$ and $E^*(M')$ of M and M' (over the rationals, say) satisfy $E^*(M') + E^*(M) \equiv 1 \bmod. 2$. We use the formula

$$E^*(M') + E^*(M) \equiv E(\omega) + r \qquad \text{mod. } 2,$$

where $E(\omega)$ is the EULER characteristic of the manifold ω with boundary $\dot{\omega} = M' - M$, and r is the rank of the bilinear form on $H^5(\omega, \dot{\omega}; Q)$ defined by the cup-product. (Compare M. A. KERVAIRE [3], § 8, formula (8.9).) It is easily seen that $E(\omega) = 1$, up to sign, and since $u \cdot u = 0$ for every $u \in H^5(\omega, \dot{\omega}; Q)$, the rank r must be even: $r \equiv 0 \pmod 2$. Hence, $E^*(M') + E^*(M) \equiv 1$ mod. 2.

Summarizing, we have proved so far that every $\beta \in \pi_{n+9}(S^n)$ is obtainable by the THOM construction from a framed manifold $(\Sigma^9; f_n)$, where the manifold Σ^9 has the homotopy type of S^9.

Taking a representative $f: S^{n+10} \to S^{n+9}$ of η such that $f^{-1}(S^{n+9} - x_0)$ is diffeomorphic to $S^1 \times (S^{n+9} - x_0)$, we obtain that $\alpha = \beta \circ \eta$ is obtainable by the THOM construction from $(S^1 \times \Sigma^9; f_n)$.

It remains to show that $(S^1 \times \Sigma^9; f_n)$ is homotopic to a framed manifold $(\Sigma^{10}; f'_n)$, where Σ^{10} is a homotopy sphere.

Apply once more the spherical modification theorems (Theorems 2 and 3 of [6]), this time to the class $\lambda \in \pi_1(S^1 \times \Sigma^9)$ represented by $S^1 \times (z_0)$. The resulting framed manifold is homotopic to $(S^1 \times \Sigma^9; f_n)$ and has the homotopy type of the 10-sphere. This completes the proof of Lemma 2.4.

To complete the proof of Theorem 1 it remains to prove the Lemmas 4.2, and 5.1. This is done in § 4 and § 5.

§ 3. Construction of M_0

This section relies on J. MILNOR's paper [5]. Let $f_0: S^4 \to SO_4$ be a differentiable map whose homotopy class (f_0) satisfies

$$i_*(f_0) = \partial i_5 ,$$

where $\partial: \pi_5(S^5) \to \pi_4(SO_5)$ is taken from the homotopy exact sequence of SO_6/SO_5, and $i: SO_4 \to SO_5$ is the usual inclusion. Define $f_1 = f_2 = i \circ f_0$. Using $f_1, f_2: S^4 \to SO_5$, a diffeomorphism $f: S^4 \times S^4 \to S^4 \times S^4$ is given by $f(x, y) = (x', y')$, where $y' = f_1(x) \cdot y$, and $x = f_2(y') \cdot x'$. Let $M(f_1, f_2)$ be the MILNOR manifold obtained from the disjoint union of $D^5 \times S^4$ and $S^4 \times D^5$ by identifying each point (x, y) in the boundary of $D^5 \times S^4$ with $f(x, y)$, considered as a point on the boundary of $S^4 \times D^5$. By Lemma 1 of [5], together with the remark at the bottom of page 963 in the proof of Lemma 1 in [5], it follows that the differentiable manifold $M(f_1, f_2)$ is homeomorphic to the 9-sphere. It will follow from Theorem 1 in this paper, that $M(f_1, f_2)$ is not diffeomorphic to the standard S^9. Let W^{10} be the differentiable manifold with boundary $M(f_1, f_2)$ obtained using the construction on page 964 of [5]. W can alternately be described as follows. Let U be a tubular neighborhood of the diagonal Δ in $S^5 \times S^5$. It is well known that U is the space of the fibre bundle $p: U \to S^5$ with fibre D^5 associated with the tangent bundle of S^5. The differentiable manifold W is obtained by straightening the angles of the quotient space of the disjoint union of two copies U' and U'' of U under an identification of $p'^{-1}(V)$ with $p''^{-1}(V)$ such that the images of Δ' and Δ'' in W have intersection number 1. (V is an imbedded 5-disc on S^5, and $p'^{-1}(V) \cong D^5 \times D^5$ is identified with $p''^{-1}(V) \cong D^5 \times D^5$ under $(u, v) \leftrightarrow (v, u)$, $u, v \in D^5$.)

Since W is a 10-dimensional manifold whose boundary $M(f_1, f_2)$ is homeomorphic to S^9, the union of W with the cone over the boundary is a 10-dimensional closed manifold M_0. Since $M(f_1, f_2)$ is combinatorially equivalent to S^9, it follows that M_0 possesses a combinatorial structure. (Compare J. Milnor, *On the relationship between differentiable manifolds and combinatorial manifolds*, mimeographed notes 1956, § 4.)

It is easily seen that M_0 is 4-connected.

We proceed to compute $\Phi(M_0)$. Let x, $y \in H^5(M_0; Z_2)$ be the cohomology classes dual to the homology classes of the imbedded spheres $j', j'': S^5 \to M_0$ given by the images in W of the diagonals Δ' and Δ'' in U' and U'' respectively. Clearly, x, y is a symplectic basis of $H^5(M_0; Z_2)$. (I.e., $x \cdot x = y \cdot y = 0$, and $x \cdot y = 1$.) To show that $\varphi(x) = \varphi(y) = 1$, observe that the normal bundles of j' and j'' (regarded as imbeddings of S^5 in the *differentiable* manifold W) are non-trivial. These bundles are isomorphic to $p: U \to S^5$. Let K be the Thom complex of this bundle. (I.e., the space obtained by collapsing the boundary of U to a point.) It is well known that K admits a cell decomposition $S^5 \cup e^{10}$, where the attaching map $S^9 \to S^5$ is a representative of the Whitehead product $[i_5, i_5]$. On the other hand, the Thom construction provides a map $f_0: M_0 \to K$ such that $f_0^*(e_1) = X$, the dual class of $j': S^5 \to M_0$, and $f_0^*(u_2)[M_0] = 1$, where e_1 generates $H^5(K; Z)$ and u_2 generates $H^{10}(K; Z_2)$. A map $f: M_0 \to \Omega S^6$ is obtained by composition of f_0 with the usual inclusion $S^5 \cup e^{10} \to \Omega S^6$. (Recall that ΩS^6 has a cell decomposition $\Omega S^6 = S^5 \cup e^{10} \cup e^{15} \cup e^{20} \cup \ldots$, where the attaching map of e^{10} represents $[i_5, i_5]$.) Then, $f: M_0 \to \Omega S^6$ has the properties $f^*(e_1) = X$, $f^*(u_2) = 1$, showing that $\varphi(x) = 1$. The same construction applied to Y, the dual class of $j'': S^5 \to M_0$ yields $\varphi(y) = 1$. Hence $\Phi(M_0) = \varphi(x) \cdot \varphi(y) = 1$.

If $M(f_1, f_2)$, with the differentiable structure induced by W (of which $M(f_1, f_2)$ is the boundary) were diffeomorphic to S^9 with the standard differentiable structure, the differentiable structure on W could be extended to a differentiable structure over the cone $CM(f_1, f_2)$, providing a differentiable structure on M_0. However, $\Phi(M_0) = 1$ and Theorem 1 show that a differentiable structure on M_0 does not exist. Hence, $M(f_1, f_2)$, homeomorphic to S^9, is not diffeomorphic to S^9.

§ 4. The auxiliary space Ω^*

Let $Y = S^5 \cup_{2i_5} e^6$ be the space obtained by attaching a 6-cell to S^5 by a map $S^5 \to S^5$ of degree 2.

Lemma 4.1. *Let* $\alpha \in \pi_5(Y) \cong Z_2$ *be the generator, then* $[\alpha, \alpha] \neq 0 \in \pi_9(Y)$.

Proof. We identify Y with the STIEFEL manifold $V_{7,2}$. Consider the exact sequence

$$\cdots \to \pi_{10}(S^6) \to \pi_9(S^5) \xrightarrow{i_*} \pi_9(V_{7,2}) \to \cdots .$$

Since $\pi_{10}(S^6) = 0$, and $[i_5, i_5]$ is non-zero in $\pi_9(S^5)$, it follows that $i_*[i_5, i_5] = [i_*(i_5), i_*(i_5)] = [\alpha, \alpha] \neq 0$.

Let $Y^* = Y \cup e^{10}$ be the space obtained from Y by attaching a 10-cell e^{10} using a representative $f: S^9 \to Y$ of $[\alpha, \alpha]$. Since Y is 4-connected, the characteristic map $\hat{f}: (D^{10}, S^9) \to (Y^*, Y)$ of e^{10} induces an *isomorphism*

$$\hat{f}_* : \pi_{10}(D^{10}, S^9) \to \pi_{10}(Y^*, Y) .$$

(Compare J. H. C. WHITEHEAD [12], Theorem 1.) Thus the relative HUREWICZ homomorphism $h_R : \pi_{10}(Y^*, Y) \to H_{10}(Y^*, Y) \cong Z$ is an isomorphism. Consider the homotopy-homology ladder of (Y^*, Y):

$$\begin{array}{ccccccccc} \cdots \to \pi_{10}(Y) & \to & \pi_{10}(Y^*) & \xrightarrow{j_0} & \pi_{10}(Y^*, Y) & \xrightarrow{\partial} & \pi_9(Y) & \to \cdots \\ \downarrow & & \downarrow h & & \downarrow h_R & & \downarrow & \\ \cdots \to 0 & \to & H_{10}(Y^*) & \xrightarrow{j_*} & H_{10}(Y^*, Y) & \to & 0 & \to \cdots . \end{array}$$

Since ∂ sends the generator of $\pi_{10}(Y^*, Y)$ into $[\alpha, \alpha] \neq 0$, and $2[\alpha, \alpha] = 0$, it follows that every element in $\mathrm{Im}\{h : \pi_{10}(Y^*) \to H_{10}(Y^*)\}$ can be halved.

It follows that for every map $g_0 : S^{10} \to Y^*$, the induced homomorphism $g_0^* : H^{10}(Y^*; Z_2) \to H^{10}(S^{10}; Z_2)$ is zero.

Let Ω be the space of loops over S^6. Up to homotopy type $\Omega = S^5 \cup e^{10} \cup e^{15} \cup \ldots$, with e^{10} attached by a map of class $[i_5, i_5]$. Let $\Omega^* = \Omega \cup e^6$, where e^6 is attached by a map of degree 2 on $S^5 \subset \Omega$. There is a natural inclusion $Y^* \to \Omega^*$ which induces an isomorphism on cohomology groups in dimension 10. Hence, we have the

Lemma 4.2. *Let* $g: S^{10} \to \Omega^*$ *be a map, and let* u_2 *be the generator of* $H^{10}(\Omega^*; Z_2) \cong Z_2$. *Then,* $g^*(u_2) = 0$.

§ 5. A lemma on homotopy groups of spheres

Lemma 5.1. *The map* $\pi_{n+9}(S^n) \to \pi_{n+10}(S^n)$, *for* $n \geqq 12$, *defined by composition with the generator* η *of* $\pi_{n+10}(S^{n+9})$ *is surjective on the* 2*-component.*

This lemma was communicated to me without proof by H. TODA who has also proved that the 2-component of $\pi_{n+10}(S^n)$ is Z_2. (See H. TODA [11], Corollary to Proposition 4.10.)

We give a sketch of proof by computation of the POSTNIKOV decomposition modulo 2 of S^n for large n, up to dimension $n+10$.

We begin with a remark which will yield Lemma 5.1 whenever a long enough portion of the POSTNIKOV decomposition of S^n is obtained. Let $X = K(Z_2, n+9) \times_k K(Z_2, n+10)$ be the space of the fibration over $K(Z_2, n+9)$ associated with the k-invariant $k \in H^{n+11}(Z_2, n+9; Z_2)$. Let $f: S^{n+9} \to X$ be a map representing the generator of $\pi_{n+9}(X) \cong Z_2$. Then, the composition

$$f \circ \eta: S^{n+10} \to X, \quad \text{where } \eta: S^{n+10} \to S^{n+9}$$

represents the generator of $\pi_{n+10}(S^{n+9})$, is essential if and only if $k = Sq^2(\varepsilon)$, where ε is the fundamental class of $H^{n+9}(Z_2, n+9; Z_2)$.

Since $Sq^2(\varepsilon)$ generates $H^{n+11}(Z_2, n+9; Z_2)$, it follows that $k \neq Sq^2(\varepsilon)$ implies $k = 0$. Hence, $f \circ \eta$ is inessential if $k \neq Sq^2(\varepsilon)$.

If $k = Sq^2(\varepsilon)$, let $\hat{f}: S^{n+9} \cup_\eta e^{n+11} \to X \cup_{f \circ \eta} e^{n+11}$ be the map induced by f. Let $s \in H^{n+9}(S^{n+9} \cup_\eta e^{n+11}; Z_2)$ be the generator. We identify $H^{n+9}(X \cup e^{n+11}; Z_2)$ and $H^{n+9}(X; Z_2)$ with $H^{n+9}(Z_2, n+9; Z_2)$. Since $\hat{f}^*(\varepsilon) = s$, and $Sq^2(s) \neq 0$, it follows that $Sq^2(\varepsilon) \neq 0$ in $H^{n+11}(X \cup e^{n+11}; Z_2)$. To show that $f \circ \eta$ is essential, it is therefore sufficient to show that $Sq^2(\varepsilon) = 0$ in $H^{n+11}(X; Z_2)$. This follows from the commutativity of the diagram

$$\begin{array}{ccccc} 0 \leftarrow H^{n+9}(X; Z_2) & \leftarrow & H^{n+9}(Z_2, n+9; Z_2) & \leftarrow & 0 \\ \downarrow Sq^2 & \approx & \downarrow Sq^2 & & \\ H^{n+11}(X; Z_2) & \leftarrow & H^{n+11}(Z_2, n+9; Z_2) & \overset{\tau}{\leftarrow} & H^{n+10}(Z_2, n+10; Z_2), \end{array}$$

where the rows are taken from the exact sequence of the fibration defining X (in the stable range), and τ is the transgression.

Let $Y_{10} \to Y_9 \to \cdots \to Y_i \to Y_{i-1} \to \cdots \to Y_0 = K(Z, n)$ be the modulo 2 POSTNIKOV decomposition of S^n. (I.e., $p_i: Y_i \to Y_{i-1}$ is a fibration with fibre $F_i = K(\pi_i, n+i)$, where π_i is the 2-component of the stable group $\pi_{n+i}(S^n)$, and $H^*(Y_i; Z_2)$ contains Z_2 in dimension 0 and n, $H^q(Y_i; Z_2) = 0$ for $0 < q < n$, and $H^{n+k}(Y_i; Z_2) = 0$ for $0 < k < i+2$.) By the $\mathfrak{C}$-theory with $\mathfrak{C}$ = the class of finite groups whose order is prime to

2, a map $S^n \to Y_i$ inducing an isomorphism $H^n(Y_i; Z_2) \cong H^n(S^n; Z_2)$ induces an isomorphism of the 2-component of $\pi_{n+k}(S^n)$ with $\pi_{n+k}(Y_i)$ for $k \leqq i$. (Compare J. P. SERRE [8].) We have $\pi_9 \cong Z_2 + Z_2 + Z_2$ and $\pi_{10} \cong Z_2$ as will be seen below, thus

$$F_9 = K(Z_2, n+9) \times K(Z_2, n+9) \times K(Z_2, n+9),$$

and Lemma 5.1 follows by showing that the restriction of the fibration $Y_{10} \to Y_9$ over one of the factors of F_9 is $K(Z_2, n+9) \times_k K(Z_2, n+10)$ with $k = Sq^2$. This is equivalent to showing that $H^{n+11}(Y_9; Z_2) \cong Z_2$ is generated by a class u_9 such that $i_9^*(u_9) = Sq^2(\varepsilon_9)$, where ε_9 is one of the fundamental classes of $H^9(F_9; Z_2)$, and $i_9: F_9 \to Y_9$ is the inclusion.

In a similar way, it can be read off from the tables below that composition with η provides *injective* maps $\pi_{n+7}(S^n) \otimes Z_2 \to \pi_{n+8}(S^n)$ and $\pi_{n+8}(S^n) \to \pi_{n+9}(S^n)$ in the stable range. Using $\pi_7(SO_n) \cong Z$, $\pi_8(SO_n) \cong Z_2$, and $\pi_9(SO_n) \cong Z_2$, this implies that $J_9: \pi_9(SO_n) \to \pi_{n+9}(S^n)$ is a monomorphism.

We proceed to a partial description of the modulo 2 cohomology of the spaces Y_7.

$H^*(Y_0)$ is given by J. P. SERRE in [9]. This result of J. P. SERRE and the ADEM relations between the STEENROD squares are the essential tools in computing $H^*(Y_k; Z_2)$ for $k > 0$. Since we stay in the stable range, the spectral sequences of $p_k: Y_k \to Y_{k-1}$ reduce to exact sequences

$$\cdots \leftarrow H^{n+q+1}(Y_{k-1}) \overset{\tau}{\leftarrow} H^{n+q}(F_k) \overset{i_k^*}{\leftarrow} H^{n+q}(Y_k) \overset{p_k^*}{\leftarrow} H^{n+q}(Y_{k-1}) \leftarrow \cdots.$$

It is therefore sufficient to determine at each step the kernel and the image of the transgression τ. Since the cohomology of Y_k is independent of k up to dimension n, we omit to mention the non-vanishing cohomology groups in dimension $\leqq n$. The direct sum of the subgroups of $H^*(Y_k; Z_2)$ in dimensions $> n$ is denoted $H^+(Y_k)$.

The symbol q_k stands for the composition $p_1 \circ p_2 \circ \cdots \circ p_k$, and ε_k denotes the fundamental class of $H^{n+k}(G, n+k; G)$.

I omit Y_1 and Y_2 whose cohomology is straightforward, but has to be computed up to dimension $n + 17$ and $n + 16$ respectively. $H^{n+4}(Y_2; Z_2)$ is generated by $q_2^*(Sq^4 \varepsilon_0)$, and $H^{n+5}(Y_2; Z_2)$ by a class u_2 such that $i_2^*(u_2) = Sq^3(\varepsilon_2)$.

$F_3 = K(Z_8, n+3)$, with $\tau(\varepsilon_3') = q_2^*(Sq^4 \varepsilon_0)$ and $\tau(\beta \varepsilon_3) = u_2$, where β is the BOCKSTEIN operator associated with the sequence of coefficients $0 \to Z_2 \to Z_{16} \to Z_8 \to 0$, and ε_3' is the mod. 2 reduction of ε_3.

$H^+(Y_3)$ has a basis consisting of
u_3 in dimension $n+7$, such that $i_3^*(u_3) = Sq^4\varepsilon_3'$;
$Sq^1(u_3)$, $q_3^*(Sq^8\varepsilon_0)$; $Sq^2(u_3)$, v_3 such that $i_3^*(v_3) = Sq^5\beta\varepsilon_3$; $Sq^3(u_3)$;
$Sq^4(u_3)$; $Sq^5(u_3)$, $Sq^4Sq^1(u_3)$, $q_3^*(Sq^{12}\varepsilon_0)$; $Sq^6(u_3)$, $Sq^4Sq^2(u_3)$, $Sq^4(v_3)$;
$Sq^6Sq^1(u_3)$, $Sq^5Sq^2(u_3)$, $q_3^*(Sq^{14}\varepsilon_0)$;
$Sq^8(u_3)$, $Sq^7Sq^1(u_3)$, $Sq^6Sq^2(u_3)$, $Sq^6(v_3)$, $q_3^*(Sq^{15}\varepsilon_0)$; ...

$Y_4 = Y_5 = Y_3$. $(\pi_4 = \pi_5 = 0.)$

$F_6 = K(Z_2, n+6)$ with $\tau(\varepsilon_6) = p_5^* p_4^*(u_3)$.
$H^+(Y_6)$ has a basis consisting of
$q_6^*(Sq^8\varepsilon_0)$; $p_6^*p_5^*p_4^*(v_3)$, u_6 such that $i_6^*(u_6) = Sq^2Sq^1\varepsilon_6$;
$Sq^1(u_6)$; nothing in dimension $n+11$; $q_6^*(Sq^{12}\varepsilon_0)$, $Sq^2Sq^1(u_6)$;
$p_6^*p_5^*p_4^*(Sq^4v_3)$, $Sq^4(u_6)$, v_6 such that $i_6^*(v_6) = Sq^7\varepsilon_6$;
$q_6^*(Sq^{14}\varepsilon_0)$, $Sq^5(u_6)$; $q_6^*(Sq^{15}\varepsilon_0)$, $p_6^*p_5^*p_4^*(Sq^6v_3)$, ...
(and possibly other classes of dimension $n+15$).

$F_7 = K(Z_{16}, n+7)$ with $\tau(\varepsilon_7') = q_6^*(Sq^8\varepsilon_0)$ and $\tau(\beta'\varepsilon_7) = p_6^*p_5^*p_4(v_3)$, where β' is the Bockstein operator of $0 \to Z_2 \to Z_{32} \to Z_{16} \to 0$, and ε_7' is the reduction modulo 2 of ε_7.

$H^+(Y_7)$ has a basis consisting of
u_7 in dimension $n+9$, such that $i_7^*(u_7) = Sq^2(\varepsilon_7')$, $p_7^*(u_6)$;
$Sq^1(u_7)$, $p_7^*(Sq^1u_6)$, v_7 such that $i_7^*(v_7) = Sq^2\beta'\varepsilon_7$;
$Sq^1(v_7)$; $Sq^2Sq^1(u_7)$, $p_7^*(Sq^2Sq^1u_6)$, ... $(Sq^2(v_7) = 0.)$

$F_8 = K(Z_2 + Z_2, n+8)$ with $\tau(\varepsilon_8') = u_7$, $\tau(\varepsilon_8'') = p_7^*(u_6)$, where ε_8' and ε_8'' are the two fundamental classes in $H^{n+8}(F_8; Z_2)$.

$H^+(Y_8)$ has a basis consisting of
$p_8^*(v_7)$, u_8, v_8, where $i_8^*(u_8) = Sq^2(\varepsilon_8')$ and $i_8^*(v_8) = Sq^2(\varepsilon_8'')$;
$Sq^1(u_8)$, $Sq^1(v_8)$, $p_8^*(Sq^1v_7)$;
$Sq^2(u_8)$, $Sq^2(v_8)$, ...

$F_9 = K(Z_2 + Z_2 + Z_2, n+9)$ with fundamental classes ε_9, ε_9', ε_9'' which are send by transgression on $p_8^*(v_7)$, u_8, v_8 respectively.

$$H^{n+11}(Y_9; Z_2) \cong Z_2(u_9), \quad \text{where} \quad i_9^*(u_9) = Sq^2(\varepsilon_9).$$

We have seen that this statement implies Lemma 5.1, hence the proof is complete.

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