KNOT COBORDISM IN CODIMENSION TWO

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A knot will be a smooth oriented submanifold Σ^n of S^{n+2}, where Σ^n is required to have the homotopy type of the n-sphere S^n. Two knots $\Sigma_o^n, \Sigma_1^n \subset S^{n+2}$ are <u>cobordant</u> if there exists an oriented smooth submanifold $V \subset I \times S^{n+2}$ of dimension n+1 such that

(1) $bV = V \cap (bI \times S^{n+2}) = \Sigma_1 \cup (-\Sigma_o)$, where Σ_i, i = 0, 1, is regarded as a submanifold of $\{i\} \times S^{n+2}$, and $-\Sigma_o$ is Σ_o with reversed orientation;

(2) The inclusions $\Sigma_o \longrightarrow V$, $\Sigma_1 \longrightarrow V$ are homotopy equivalences;

(3) V meets $bI \times S^{n+2}$ orthogonally, i.e. the intersection of V with a neighbourhood of $bI \times S^{n+2}$ is $[0,\alpha) \times \Sigma_o \cup (1-\alpha, 1] \times \Sigma_1$ for some small $\alpha > 0$.

Cobordism is an equivalence relation between knots of the same dimension. Transitivity is guaranteed by condition (3).

Let C_n be the set of cobordism classes of knots $\Sigma^n \subset S^{n+2}$. It is easy to see that the ambiant connected sum of knots induces an addition of cobordism classes which turns C_n into an abelian group. (For details, see [1], Chap.III.) The ambiant connected sum is defined as follows: Given two knots $\Sigma_i^n \subset S^{n+2}$, i = 0, 1, let $h_i : (D^{n+2}, D^n) \longrightarrow (S^{n+2}, \Sigma_i^n)$ be two embeddings such that h_o is orientation preserving and h_1 orientation reversing on both D^{n+2} and D^n. Form the disjoint union $(S^{n+2} - h_o(0)) \cup (S^{n+2} - h_1(0))$ and identify $h_o(tx)$ with $h_1((1-t)x)$ for $0 < t < 1$ and $x \in S^{n+1} =$

bD^{n+2}. This construction yields an embedding of the connected sum $\Sigma_o \# \Sigma_1$ into $S^{n+2} = S^{n+2} \# S^{n+2}$, whose cobordism class is by definition the sum of the cobordism classes of Σ_o and Σ_1. The standard embedding $S^n \subset S^{n+2}$ represents the zero element of the resulting group C_n.

THEOREM 1. $C_{2k} = 0$ <u>for</u> $k \geqq 0$, i.e. <u>every even dimensional knot is cobordant to the standardly embedded</u> $S^{2k} \subset S^{2k+2}$.

However, the groups C_{2k-1} are not finitely generated. We shall give the purely algebraic description of C_{2k-1} due to J.Levine [3]. It turns out that for $k \geqq 3$, C_{2k-1} depends only on the parity of k, so that C_n is periodic of period 4 for $n \geqq 4$. For $k \geqq 3$, the group C_{2k-1} is contained in an infinite unrestricted direct sum of cyclic groups Z, Z/2Z and Z/4Z. There are elements of each of these orders (∞ , 4 and elements of order 2 not divisible by 2) occuring in C_{2k-1}.

§ 1. <u>Seifert surfaces</u>

It is well known that a smooth closed curve $\Sigma^1 \subset S^3$ bounds an orientable surface embedded in S^3, called a Seifert surface of the knot. This fact generalizes to all dimensions: <u>Every</u> $\Sigma^n \subset S^{n+2}$ <u>is the boundary of an oriented smooth submanifold</u> $V^{n+1} \subset S^{n+2}$.

The proof is easy. Observe first that Σ^n has trivial normal bundle in S^{n+2}. Then, take an embedding $\Sigma^n \times D^2 \longrightarrow S^{n+2}$ extending the given knot $\Sigma^n \times \{0\} = \Sigma^n \subset S^{n+2}$. Now, it suffices to show that $\Sigma^n \times \{x_o\}$, where $x_o \in S^1 = bD^2$, bounds a submanifold V in the complement $X = S^{n+2} - \text{int}\,(\Sigma^n \times D^2)$. Let $\varphi : \Sigma^n \times S^1 \longrightarrow S^1$ be the projection on the second factor. Then, $\Sigma^n \times \{x_o\} = \varphi^{-1}(x_o)$ and if we can extend φ to a map $\Phi : X \longrightarrow S^1$ regular at x_o, then $\Phi^{-1}(x_o)$ will be the desired submanifold. The extension Φ exists by obstruction theory, since, at least for $n > 1$, $H^{q+1}(X, \Sigma \times S^1; \pi_q(S^1)) = 0$ for all q. In the sequel we only retain from $\Phi^{-1}(x_o)$ the connected component V containing the given knot as its boundary. Observe that

V comes equipped with a normal vector-field v.

Using the Seifert surface, we first sketch the proof of theorem 1. Let $\Sigma^{2k} \subset S^{2k+2}$ be a knot and V^{2k+1} a Seifert surface for this knot, i.e. $V \subset S^{2k+2}$ and $bV = \Sigma$. By surgery theory (see [2], § 6.), we know there exists a stably parallelizable (2k+2)-manifold W with corners, so that $bW = V \cup (I \times \Sigma) \cup V_o$, where V_o is contractible and (W, V) has a handle decomposition with handles of types $\leq$ k+1 only. (For k = 1 see the argument in [1], p.265.) By a theorem of M.Hirsch (or the direct argument in [1]), the embedding $V \subset S^{2k+2}$ extends to an immersion $W \looparrowright D^{2k+3}$. Now, since $(k+1) + (k+1) < 2k+3$, any possible intersection of the handles of (W, V) can be removed by a (small) regular homotopy. Thus we get an embedding $W \subset D^{2k+3}$ extending $V \subset S^{2k+2}$. Its restriction to the part $(I \times \Sigma) \cup V_o$ of the boundary bW yields a contractible submanifold of D^{2k+3} with boundary the given knot Σ^{2k}. From this it easily follows that the cobordism class of $\Sigma^{2k} \subset S^{2k+2}$ is zero.

The attempt to carry the same proof idea for $\Sigma^{2k-1} \subset S^{2k+1}$ fails in two places. First, one may not be able to reach a contractible manifold V_o^{2k} by performing framed surgery (in dimensions $\leq$ k) on a Seifert surface V^{2k} of the given knot. Secondly, even if one could, one still runs into the more serious trouble that the (k+1)-handles of the resulting manifold pair (W, V) will in general have non-removable intersections in D^{2k+2}. To measure these obstructions one introduces an algebraically defined cobordism group of bilinear forms.

§2. Cobordism of bilinear forms

Given $\varepsilon = \pm 1$, we define algebraically (following [3]) a group $C^{\varepsilon}(Z)$ depending on ε. Eventually, it will turn out that $C_{2k-1} \cong C^{\varepsilon}(Z)$, with $\varepsilon = (-1)^k$ for $k \geq 3$. (For results in lower dimensions, see [3].)

Consider integral valued bilinear forms on finitely generated free Z-modules. If A is such a form on the free Z-module H, denote by

A' the transpose of A defined by $A'(x, y) = A(y, x)$ for all $x, y \in H$. We shall say that A is an ε-form if the form $A + \varepsilon A'$ is unimodular. (See the first few lines of § 4 below for the definition.)

DEFINITION. An ε-form A defined on H is null-cobordant if there exists a Z-submodule H_o of H such that

(1) $2.\mathrm{rank}\ (H_o) = \mathrm{rank}\ (H)$;

(2) A vanishes on H_o, i.e. $A(x, y) = 0$ for all $x, y \in H_o$.

Remarks. Observe that such an H_o can then be taken to be a direct summand of H. Observe also that if A is an ε-form on H, then the rank of H must be even. Indeed, the form $S = A + \varepsilon A'$ induces a symmetric, unimodular form S^* on the F_2-space $H/2H$ such that $S^*(x, x) = 0$ for all $x \in H/2H$. Take a maximal F_2-subspace U of $H/2H$ such that $S^*(u, v) = 0$ for all $u, v \in U$. Let V be the orthogonal complement of U, i.e. $V = \{ v \in H/2H \mid S^*(u, v) = 0 \text{ for all } u \in U\}$. Then, $\dim U + \dim V = \dim H/2H = \mathrm{rank}\ (H)$, since S^* is unimodular. On the other hand, the maximality of U implies $U = V$. Indeed, if $v \in V$, then $S^*(v, v) = 0$ together with $S^*(u, v) = 0$ for all $u \in U$ imply that $S^*(w_1, w_2) = 0$ for all $w_1, w_2 \in F_2 v + U$. Since U is a maximal subspace with this property, it follows that $v \in U$. Thus $\mathrm{rank}\ (H) = 2.\dim U$.

If A is an ε-form on G and B an ε-form on H, we denote by $A \oplus B$ the ε-form on $G \oplus H$ given by

$$(A \oplus B)\ (x \oplus u,\ y \oplus v) = A(x, y) + B(u, v),$$

where $x, y \in G$ and $u, v \in H$.

DEFINITION. Let A, B be two ε-forms defined on G and H respectively. We say that A and B are cobordant if the ε-form $A \oplus (-B)$ on $G \oplus H$ is null-cobordant.

Cobordism of ε-forms is an equivalence relation. Reflexivity and symmetry are both trivial. Transitivity follows from a cancellation lemma:

LEMMA. _If_ A _and_ B _are_ ε-_forms and both_ $A \oplus B$ _and_ B _are null-cobordant, then so is_ A.

Transitivity is then immediate. If $A_1 \oplus (-A_2)$ and $A_2 \oplus (-A_3)$ are null-cobordant, so is $A_1 \oplus (-A_3) \oplus A_2 \oplus (-A_2)$ and the lemma applies with $A = A_1 \oplus (-A_3)$, $B = A_2 \oplus (-A_2)$ and yields that $A_1 \oplus (-A_3)$ is null-cobordant, i.e. A_1 and A_3 are cobordant if A_1, A_2 and A_2, A_3 are.

Proof of the lemma. Let A be defined on G and B on H. By assumption there exists $L \subset G \oplus H$ so that $2.\text{rank } L = \text{rank } G + \text{rank } H$ and $A \oplus B$ vanishes on L. Also, there is $H_o \subset H$ with $2.\text{rank } H_o = \text{rank } H$ and B vanishes on H_o. Set $L_o = L \cap (G \oplus H_o)$ and let G_o be the projection of L_o on G. If $x, y \in G_o$, there exist $u, v \in H_o$ such that $x \oplus u, y \oplus v \in L$. Then,

$$A(x, y) = A(x, y) + B(u, v) = (A \oplus B)(x \oplus u, y \oplus v) = 0.$$

It remains to prove that G_o has the right rank. Take H_o to be a direct summand of H and write $H = H_o \oplus H_1$. Projecting L on H_1 gives an exact sequence

$$0 \longrightarrow L_o \longrightarrow L \longrightarrow L_1 \longrightarrow 0.$$

Observe that $L \cap H_o$ and $H_o \oplus L_1$ are orthogonal under $B + \varepsilon B'$. For $L \cap H_o$ and H_o this is obvious. Let $u \in L \cap H_o$ and $w \in L_1$. There exists $x \oplus v \in G \oplus H_o$ such that $x \oplus (v + w) \in L$. Then,

$$\begin{aligned}(B+\varepsilon B')(u, w) &= (B+\varepsilon B')(u, v + w) \\ &= (A+\varepsilon A')(0, x) + (B+\varepsilon B')(u, v + w) \\ &= (A \oplus B)(u, x+v+w) + \varepsilon.(A \oplus B)(x+v+w, u) = 0,\end{aligned}$$

since u is also in L. Therefore, since $B + \varepsilon B'$ is unimodular,

$$\text{rank } (L \cap H_o) + \text{rank } (H_o \oplus L_1) \leqq \text{rank } H.$$

Now, $L \cap H_o = L_o \cap H$ and there is an exact sequence

$$0 \longrightarrow L_o \cap H \longrightarrow L_o \longrightarrow G_o \longrightarrow 0.$$

The two exact sequences give

$$\text{rank } G_o = \text{rank } L_o - \text{rank } (L_o \cap H),$$

rank L_o = rank L - rank L_1, and so

rank G_o = rank L - (rank $(L \cap H_o)$ + rank L_1).

Pluging in the above inequality and using the hypotheses on rank L and rank H_o, one gets rank $G_o \geqq \frac{1}{2}$.rank G. Of course, equality must hold since $A+\varepsilon A'$ which is unimodular vanishes on G_o. This completes the proof of the lemma.

It is clear that the direct sum of ε-forms induces an addition of the cobordism classes and turns the set $C^{\varepsilon}(Z)$ of cobordism classes of ε-forms into an abelian group. We have included reference to Z in the notation because similarly defined groups over other coefficient domains will be introduced later.

§ 3. The transition theorem

Using a Seifert surface, we shall now associate with every knot $\Sigma^{2k-1} \subset S^{2k+1}$ an $(-1)^k$-form. Let V^{2k} be a Seifert surface for a given knot $\Sigma \subset S^{2k+1}$, i.e. $V^{2k} \subset S^{2k+1}$ and $bV = \Sigma$. Let v be the normal vector-field to V in S^{2k+1} and i_+, i_- : $V \longrightarrow S^{2k+1} - V$ the maps defined by $i_{\pm}(P) = P \pm \alpha.v(P)$ for small $\alpha > 0$, $P \in V$.
Let $H = H_k(V^{2k})/(\text{torsion})$. If $x, y \in H$, the linking number $L(x, i_+(y)) \in Z$ in S^{2k+1} is well defined. We set

$$A(x, y) = L(x, i_+(y)).$$

Observing that $L(x, i_+(y)) - L(x, i_-(y)) = I(x, y)$, where I denotes the intersection number in V, we have

$$L(y, i_+(x)) = L(i_-(y), x) = (-1)^{k+1}L(x, i_-(y))$$
$$= (-1)^{k+1}.(L(x, i_+(y)) - I(x, y)), \text{ and thus}$$
$$A(x, y) + (-1)^k A(y, x) = I(x, y).$$

By Poincaré duality, the intersection form I on H is unimodular, and so the form A is a $(-1)^k$-form.

THEOREM 2. <u>The above construction provides a well-defined map</u>

$$L : C_{2k-1} \longrightarrow C^{\varepsilon}(Z), \text{ with } \varepsilon = (-1)^k.$$

<u>For</u> $k \geqq 3$, L <u>is an isomorphism</u>.

<u>Sketch of proof</u>. If Seifert surfaces V_1, V_2 have been chosen for 2 knots Σ_1, Σ_2, their ambiant boundary connected sum V is a Seifert surface for $\Sigma_1 \# \Sigma_2$. Moreover, the $(-1)^k$-form associated with V is clearly the direct sum of the $(-1)^k$-forms associated with V_1 and V_2. Thus, in order to show L well-defined, it suffices to consider the case of a null-cobordant knot $\Sigma^{2k-1} \subset S^{2k+1}$ and to prove that the $(-1)^k$-form A associated with any of its Seifert surfaces is null-cobordant. Let $\Delta \subset D^{2k+2}$ be a 2k-disc with boundary Σ. The union $M = \Delta \cup V$, where V is a Seifert surface for is a closed manifold (with corners along Σ) embedded in D^{2k+2}. An obstruction theory argument similar to the one in § 1, shows that M is the boundary of an oriented submanifold $W \subset D^{2k+2}$. Consider $H_o = \text{Ker } j_*$, where $j_* : H_k M \longrightarrow H_k W$. If x, $y \in H_o$, and say ξ, η are representative cycles, then ξ, η bound (k+1)-chains α, β in W. We can view $L(x, i_+(y))$ as the intersection coefficient $I(\alpha, i_+(\beta))$ which is clearly zero since W and i_+W are disjoint. (i_+ is here the extension to W of the map defined above on V using the normal vector-field.) Thus A vanishes on $H_o = \text{Ker } j_*$. It then follows that A is null-cobordant by showing that rank $H_o = \frac{1}{2}$.rank $H_k M$. For this, write the homology exact sequence of (W, M):

$$0 \longrightarrow H_{2k+1}(W, M) \longrightarrow H_{2k}(M) \longrightarrow H_{2k}(W) \longrightarrow \dots$$

$$\dots \longrightarrow H_{k+1}(W) \longrightarrow H_{k+1}(W, M) \xrightarrow{d} \text{Ker } j_* \longrightarrow 0,$$

breaking it at $H_k M$. It yields

$$\begin{aligned}\text{rank Ker } j_* &= \text{rank } H_{k+1}(W, M) - \text{rank } H_{k+1}(W) + \text{rank } H_{k+1}(M) \\ &\quad - \text{rank } H_{k+2}(W, M) + \dots \\ &= (-1)^{k+1}.\tfrac{1}{2}E(M) + \tfrac{1}{2}.\text{rank } H_k M + (-1)^k.E(W),\end{aligned}$$

where E denotes the Euler characteristic. Since the "double" of W is a closed odd-dimensional manifold, $0 = E(W \cup W) = 2.E(W) - E(M)$, and so rank Ker $j_* = \frac{1}{2}$.rank $H_k(M)$ as desired. Hence, L is a well-defined homomorphism.

Proving the surjectivity of L is straightforward. Given A so that $A + (-1)^k A'$ is unimodular, first take a matrix representative which we denote again by A. Construct a 2k-dimensional stably parallelizable manifold V with boundary whose intersection matrix is $A + (-1)^k A'$ in dimension k. The manifold V is obtained by attaching k-handles to D^{2k} with linkings of the attaching (k-1)-spheres dictated by the entries in $A + (-1)^k A'$. (See [1], p.256 for details.) Then, $bV = \Sigma$ is a homotopy (2k-1)-sphere as a consequence of the assumption that $A + (-1)^k A'$ is unimodular.($k \geqq 3$ is needed here to quarantee $\pi_1 \Sigma = 0$.) Take a random embedding $V \subset S^{2k+1}$. Its restriction to bV yields a knot for which V is a Seifert surface, and thus to which there corresponds some $(-1)^k$-form B of the same rank as A. Actually, $B + (-1)^k B' = A + (-1)^k A'$, since both are equal to the intersection form on $H_k V$. One can then change B to A by readjusting the mutual linking of the handles of V.

The proof of injectivity of L proceeds in two steps: (1) One shows that every knot $\Sigma^{2k-1} \subset S^{2k+1}$ is cobordant to a simple knot $\Sigma_o^{2k-1} \subset S^{2k+1}$, i.e. Σ_o is the boundary of a (k-1)-connected manifold $V_o \subset S^{2k+1}$; (2) It is then enough to show for a simple knot that null-cobordism of the associated $(-1)^k$-form, constructed using a (k-1)-connected Seifert surface, implies null-cobordism of the knot.

To prove step (1), use surgery on a Seifert surface V^{2k} for the knot $\Sigma^{2k-1} \subset S^{2k+1}$ to produce a manifold W with $bW = V \cup (I \times \Sigma) \cup V_o$ (corners along $bV = \{0\} \times \Sigma$ and $\{1\} \times \Sigma = bV_o$), where V_o is (k-1)-connected and (W, V) has a handle decomposition with all handles of type $\leqq$ k. As in the proof of theorem 1, it is easy to extend $V \subset S^{2k+1}$ to an embedding $W \subset D^{2k+2}$. The problem is then to embed a (2k+2)-disc D_o^{2k+2} in D^{2k+2} so that $(\text{int } D_o) \cap W = \emptyset$, in such a way as to engulf $V_o \subset bD_o = S_o^{2k+1}$. If this can be done, then

D^{2k+2} - int D_o^{2k+2} which is diffeomorphic to $I \times S^{2k+1}$ contains a cobordism between $\Sigma \subset S^{2k+1}$ and $\Sigma_o = bV_o \subset S_o^{2k+1}$. Since V_o is (k-1)-connected, $\Sigma_o \subset S_o^{2k+1}$ is a simple knot. The existence of an embedding $D_o^{2k+2} \subset D^{2k+2}$ with $V_o \subset bD_o$ follows from the engulfing theorem of M.Hirsch. (Theorem 2 of "Embeddings and compression of polyhedra and smooth manifolds". Topology, Vol.4 (1966), 361-369.)

Remark. It was pointed out to me that the existence of a PL-embedding of D_o^{2k+2} with the desired properties is fairly obvious, taking for granted the relative regular neighbourhood theorem. Indeed, since V_o has k-dimensional spine, the cone over spine (V_o) embeds disjointly form int (W), observing that (W, V) has only h-handles with $h \leqq k$. Thus D_o can be taken to be a regular neighbourhood of such an embedded cone.

Step (2) is an application of surgery. We have $\Sigma^{2k-1} \subset S^{2k+1}$ a simple knot bounding $V^{2k} \subset S^{2k+1}$, where V is (k-1)-connected. The problem is to perform ambiant surgery on V in D^{2k+2} so as to produce a contractible submanifold of D^{2k+2} with boundary Σ^{2k-1}. We are assuming that the associated $(-1)^k$-form is null-cobordant, so by hypothesis there exists a basis $x_1, \ldots, x_r, x_{r+1}, \ldots, x_{2r}$ of H_kV with the property that $L(x_\alpha, i_+(x_\beta)) = 0$ for all $\alpha, \beta \leqq r$. Clearly then, $I(x_\alpha, x_\beta) = 0$ for all $\alpha, \beta \leqq r$ and since V is (k-1)-connected (and $k \geqq 3$), we can take disjoint embeddings $f_\alpha : S^k \longrightarrow V$ $\alpha = 1, \ldots, r$ representing x_α. The conditions $L(x_\alpha, i_+(x_\beta)) = 0$ for $\alpha \neq \beta$ then mean that the f_α's can be extended to mutually disjoint embeddings $F_\alpha : D^{k+1} \longrightarrow D^{2k+2}$. Moreover, $L(x_\alpha, i_+(x_\alpha)) = 0$ implies that F_α is extendible to an embedding $F_\alpha : D^{k+1} \times D^k \longrightarrow D^{2k+2}$ such that $F_\alpha \mid S^k \times D^k$ is a tubular neighbourhood of $f_\alpha(S^k)$ in V. It is well known then, and easy to check that surgery on $x_1, \ldots, x_r \in H_k(V)$ (as just shown possible) replaces V by a contractible manifold with the same boundary.

§ 4. Algebraic study of $C^{\varepsilon}(Z)$

Let A be a bilinear form on H. Then A defines a homomorphism $H \longrightarrow \mathrm{Hom}(H, Z)$ which to $x \in H$ associates the homomorphism $H \longrightarrow Z$ defined by $y \longmapsto A(x, y)$. We say A is non-singular if the associated map $H \longrightarrow \mathrm{Hom}(H, Z)$ which we denote by A again is injective. (A is unimodular iff $A : H \longrightarrow \mathrm{Hom}(H, Z)$ is an isomorphism.)

Prop. 1. Every ε-form is cobordant to a non-singular one.

Proof. Let A be an ε-form on the Z-module H. Suppose A is singular, i.e. there exists $e_1 \in H$, $e_1 \neq 0$, so that $A(e_1, x) = 0$ for all $x \in H$. We may assume that e_1 is not divisible in H and thus Ze_1 is a direct summand in H. Then, since $S = A + \varepsilon A'$ is unimodular, there exists $e_2 \in H$ with $S(e_1, e_2) = \varepsilon$, or $A(e_2, e_1) = 1$. Let G be the orthogonal complement of e_1, e_2 under S, i.e. $G = \{ x \in H \mid S(e_1, x) = S(e_2, x) = 0\}$. It is easy to check that $H = Ze_1 \oplus Ze_2 \oplus G$. Now, let B be the restriction of A on G. Claim: B is an ε-form on G which is cobordant to A. The first statement is trivial. For the second, let D be the diagonal in $G \oplus G$, i.e. $D = \{(x, x) \in G \oplus G \mid x \in G\}$. Then, $A \oplus (-B)$ vanishes on $Ze_1 + D \subset H \oplus G$, and rank $(Ze_1 + D) = \frac{1}{2}(\mathrm{rank}\ H + \mathrm{rank}\ G)$. The proposition follows by induction on rank H.

As above, let $A : H \longrightarrow \mathrm{Hom}(H, Z)$, and similarly, $S = A + \varepsilon A' : H \longrightarrow \mathrm{Hom}(H, Z)$. Since S is an isomorphism, we have a Z-map $s = S^{-1}A : H \longrightarrow H$. Note that s is characterized by

$$A(x, y) = S(sx, y)$$

for all $x, y \in H$.

Prop. 2. If L is a non-zero Z-submodule of H on which the ε-form A vanishes and such that $s(L) \subseteq L$, then A is cobordant to an ε-form of strictly smaller rank.

Proof. We may assume that L is pure and hence a direct summand in H. Set $M = \{ x \in H \mid S(x, y) = 0 \text{ for all } y \in L\}$, where as before $S = A + \varepsilon A'$. Then, $A(x, y) = 0$ and $A'(x, y) = 0$ for all $x \in M$, $y \in L$.

Indeed, $A'(x, y) = A(y, x) = S(sy, x) = 0$ since $sy \in L$. Then, $A(x, y) = 0$ for $x \in M$, $y \in L$ follows from $S = A + \varepsilon A'$.

Therefore, A induces a form B on M/L. Since M is a pure submodule, and hence a direct summand in H, it follows that $T = B + \varepsilon B'$ is unimodular. Thus B is an ε-form. Consider $A \oplus (-B)$ on $H \oplus (M/L)$. It vanishes on the submodule D of all (x, x^*), where $x \in M$ and x^* = class of x mod L. Now, rank D = rank M = rank H - rank L since M is the orthogonal complement of L under the non-singular form S. Thus rank $(H + (M/L))$ = rank H - rank L + rank M = 2.rank D. Hence, A is cobordant to B of strictly smaller rank.

<u>Exercise</u>. Define A to be <u>reduced</u> if A is non-singular and there is no non-zero submodule invariant by s on which A vanishes. Prove that if A and B are reduced ε-forms on G and H respectively and are cobordant, then there exists an isometry $f : G_Q \longrightarrow H_Q$, i.e. a Q-isomorphism $(G_Q = G \oplus_Z Q)$ such that $B(fx, fy) = A(x, y)$. On the other hand, the example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix} \qquad , \qquad B = \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix}$$

shows that cobordism of reduced forms does not imply isomorphism over Z. (In this example, $A \oplus (-B)$ vanishes on $Zx_1 \oplus Zx_2$, where $x_1 = (3, 1, 3, 0)$ and $x_2 = (-6, 1, 0, 3)$. However, A and B are not isomorphic over Z since $B(e_1, e_1) = 2$, where $e_1 = (1, 0)$, but $A(x, x) \neq 2$ for all x.) Conclude that the determinant of a <u>reduced</u> representative of a cobordism class is an invariant of the class. I do not know whether this can be exploited to clarify the structure of $C^{\varepsilon}(Z)$.

In order to state the next proposition we have to define auxiliary cobordism groups $C^{\varepsilon}_{\lambda}(Z)$ and $C^{\varepsilon}_{\lambda}(Q)$. A polynomial f will be said to be <u>reciprocal</u> if $f(X) = X^n f(X^{-1})$, where n = deg f. For example, if A is an $\mathcal{E}$-form, then $\Delta_A(X) = \det(A.X + \mathcal{E}A')$ is a (well-defined) reciprocal polynomial in $Z[X]$ called the Alexander polynomial of A. Let A be defined on H say, and set $s = S^{-1}A$, where as above $S = A + \mathcal{E}A'$, i.e. s is defined by $A(x, y) = S(sx, y)$ for all $x, y \in H$. Suppose that A is non-singular. Then $t = 1 - s^{-1}$: $H \oplus Q \longrightarrow H \oplus Q$ is an automorphism. (It is easily verified that $t = -\mathcal{E}A^{-1}A'$.) Moreover, t is an isometry for A and S extended to $H \oplus Q$. The polynomial $\Delta_A(X) = \det(A.X + \mathcal{E}A')$ is the characteristic polynomial of t (up to the non-zero factor det A).

Now, let $\mathcal{E} = \pm 1$ be given together with an irreducible, reciprocal polynomial of even degree $\lambda \in Z[X]$ such that $\lambda(1) = (-\mathcal{E})^m$, where $2m = \deg \lambda$.

Define an isometric structure over Z or Q associated with $\mathcal{E}$, λ to be an $\mathcal{E}$-symmetric unimodular bilinear form S on a Z-space (resp. Q-space) V together with an injection $s : V \longrightarrow V$ satisfying

(1) $S(sx, y) + S(x, sy) = S(x, y)$,

(2) $\varphi(X) = X^{2m}\lambda(1-X^{-1})$ is the minimal polynomial of s.

Note. Condition (1) is equivalent with $S(tx, ty) = S(x, y)$ for $t = 1 - s^{-1}$. We have stated it in the more complicated form of condition (1) because t is in general only defined over Q. Also, condition (2) simply says λ is minimal polynomial of t.

There are obvious definitions of cobordism: An isometric structure (S, s) on V is null-cobordant if there exists an s-invariant subspave $V_o \subset V$ of half the rank of V on which S vanishes. Further, (S_1, s_1) and (S_2, s_2) are cobordant if $(S_1 \oplus (-S_2), s_1 \oplus s_2)$ is null-cobordant. Trivial modifications in the proof of the lemma in § 2 show that cobordism of isometric structures (associated with the same $\mathcal{E}$, λ) is an equivalence relation. The set of equivalence classes $C^{\mathcal{E}}_{\lambda}(Z)$, resp. $C^{\mathcal{E}}_{\lambda}(Q)$ is a group under direct sum.

Prop. 3. There are injections

$$\Sigma_\lambda \; C^\varepsilon_\lambda(Z) \xrightarrow{\;i\;} C^\varepsilon(Z) \xrightarrow{\;j\;} \Sigma_\lambda C^\varepsilon_\lambda(Q),$$

<u>where the extreme terms are restricted direct sums over all irreducible, reciprocal</u> $\lambda \in Z[X]$ <u>of even degree</u> 2m <u>satisfying</u> $\lambda(1) = (-\varepsilon)^m$.

Proof. Let (S, s) be an isometric structure on a free Z-module H representing an element σ in some $C^\varepsilon_\lambda(Z)$. Define A by A(x, y) = S(sx, y) for all x, y $\in$ H. Condition (1) on S and s implies that $A + \varepsilon A' = S$. Thus A is an ε-form on H. Set $i(\sigma)$ = cobordism class of A. It is immediate that we get a well-defined map $i : \Sigma_\lambda \; C^\varepsilon_\lambda(Z) \longrightarrow C^\varepsilon(Z)$. Next we show that i is injective. Let $(S_k, s_k) \in C^\varepsilon_{\lambda_k}(Z)$, k = 1, ..., n, be such that $A = A_1 \oplus \ldots \oplus A_n$ is null-cobordant, where $A_k(x, y) = S_k(s_k x, y)$ on H_k. Thus we assume that $H = H_1 \oplus \ldots \oplus H_n$ contains a Z-subspace $N \subset H$ with A(x, y) = 0 for all x, y $\in$ N and rank H = 2.rank N. We can assume that N is pure. Observe first that $sN \subseteq N$, where $s = s_1 \oplus \ldots \oplus s_n$. Indeed, since A(x, y) = 0 and thus A'(x, y)= 0 for all x, y $\in$ N, we have S(x, y) = 0 for all x, y $\in$ N. If now x is a fixed element of N, then S(sx, N) = A(x, N) = 0. Therefore, N + Zsx is S-orthogonal to N. Since S is unimodular, rank (N + Zsx) $\leqq$ rank H - rank N = rank N. Thus sx $\in$ N, since N is pure. It follows next that the projection N_k of N into H_k is contained in N. (k = 1, ..., n.) To see this, let $\psi_k = \prod_{\ell \neq k} \varphi_\ell$ where φ_k is the minimal polynomial of s_k. There exist polynomials $u_k \in Z[X]$ such that $\Sigma_k u_k \psi_k = c$, a non-zero integer. Using $\psi_k(s_\ell) = 0$ for $k \neq \ell$, we have, with $x = x_1 + \ldots + x_n$,

$$cx_k = \Sigma_\ell \; u_\ell(s) \; \psi_\ell(s)x_k = \Sigma_\ell \; u_\ell(s_k) \; \psi_\ell(s_k)x_k = u_k(s_k) \; \psi_k(s_k)x_k$$
$$= \Sigma_\ell \; u_k(s_\ell) \; \psi_k(s_\ell)x_\ell = \Sigma_\ell \; u_k(s) \, \psi_k(s)x_\ell = u_k(s) \; \psi_k(s)x,$$

showing that if x $\in$ N, then $cx_k = u_k(s) \, \psi_k(s)x \in N$, and since N is pure this implies $x_k \in N$.

From this it follows that each A_k is null-cobordant. Since

$A_k = A|H_k$ and $N_k \subseteq N$, we have $A_k|N_k = 0$. It is trivial that $N \subseteq N_1 \oplus N_2 \oplus \ldots \oplus N_n$. We have just proved that $N_1 \oplus \ldots \oplus N_n \subseteq N$. So, $N = N_1 \oplus \ldots \oplus N_n$, and then

$\frac{1}{2}$rank H = rank $N = \Sigma_k$ rank $N_k \leqq \frac{1}{2}\Sigma_k$ rank H_k = $\frac{1}{2}$rank H,

where rank $N_k \leqq \frac{1}{2}$rank H_k for every k because A_k is non-singular and vanishes on N_k. Thus, rank $N_k = \frac{1}{2}$.rank H_k for every k.

Now, N_k is clearly s_k-invariant, and $S_k = A_k + \varepsilon A_k'$ vanishes on it, so (S_k, s_k) is null-cobordant. This completes the proof of injectivity of $i : \Sigma_\lambda C_\lambda^\varepsilon(Z) \longrightarrow C^\varepsilon(Z)$.

The map $j : C^\varepsilon(Z) \longrightarrow \Sigma_\lambda C_\lambda^\varepsilon(Q)$ is defined as follows: Let A be a <u>reduced</u> ε-form on H, and consider A as defined on $V = H \otimes Q$. The Alexander polynomial $\Delta_A(X) = \det(A.X + \varepsilon A')$ splits over $Z[X]$ as a product $\Delta_A = \lambda_1^{e_1}. \ldots . \lambda_r^{e_r}$, with $\lambda_\alpha \in Z[X]$ irreducible. Then, as it turns out, each λ_α must be reciprocal. Its sign can be fixed so that $\lambda_\alpha(1) = (-\varepsilon)^m$, where $2m_\alpha = \deg \lambda_\alpha$. Further $S = A + \varepsilon A'$ and the isometry $t = -\varepsilon A^{-1}A'$ split as direct sums of isometric structures (S_α, s_α) on the eigenspaces V_α. (In particular the minimal polynomial of $t_\alpha = t|V_\alpha$ is λ_α.) The class of $\Sigma_\alpha(S_\alpha, s_\alpha)$ is then by definition the j-image of A.

In order to prove these statements, let $\mu_\alpha = \Delta_A / \lambda_\alpha^{e_\alpha}$. We have $V_\alpha = \mu_\alpha(t)V \neq 0$. One proves that if λ_α was not reciprocal then A would vanish on the t-invariant subspace V_α, whereas A was assumed to be reduced. Indeed, letting $\bar{f}$ denote $\bar{f}(X) = X^n.f(X^{-1})$, where $n = \deg f$, we have $\bar{\lambda}_\alpha = \pm \lambda_\beta$ for some β. If $\beta \neq \alpha$, then $\bar{\mu}_\alpha \cdot \mu_\alpha$ is divisible by Δ_A and so for all x_α, $y_\alpha \in V_\alpha$, we have

$$A(x_\alpha, y_\alpha) = A(\mu_\alpha(t), \mu_\alpha(t)y) = A(x, t^*. \bar{\mu}_\alpha(t). \mu_\alpha(t)y) = 0.$$

where $* = -\deg \mu_\alpha$. This proves that each λ_α is reciprocal.

The statement about the sign follows from $\Delta_A(1) = (-\varepsilon)^m$, where $2m = \deg \Delta_A$. [This in turn, because if $\varepsilon = -1$, $\Delta_A(1)$ is the determinant of the skew-symmetric matrix $A - A'$. Thus it must be a square. If $\varepsilon = +1$, then $\Delta_A(1) = \det(A + A') = (-1)^{\frac{1}{2}(\text{rank-signature})}$. Now since $A + A'$ is unimodular and even, the signature is divisible by 8.]

It remains to verify that λ_α is the minimal polynomial of $t_\alpha = t|V_\alpha$. But if not, there is an integer e, $1 < e \leqq e_\alpha$, such that $W_\alpha = \lambda_\alpha^{e-1}(t_\alpha)V_\alpha \neq 0$ and $\lambda_\alpha^e(t_\alpha)V_\alpha = 0$. Then, for arbitrary x_α, $y_\alpha \in W_\alpha$ one has

$$\begin{aligned} A(x_\alpha, y_\alpha) &= A(\lambda_\alpha^{e-1}(t_\alpha)u_\alpha, \lambda_\alpha^{e-1}(t_\alpha)v_\alpha) \\ &= A(u_\alpha, t_\alpha^*. \lambda_\alpha^{2e-2}(t_\alpha)v_\alpha) = 0, \end{aligned}$$

since $2e-2 \geqq e$. Hence, again, A would not be reduced.

It is clear that the map j is well-defined since the cobordism class of a reduced ε-form A determines its isomorphism class over Q. To see that j is a homomorphism, observe that there is an obvious definition of a reduced isometric structure, i.e. (S, s) defined on H is reduced if s is injective and there is no non-zero subspace $H_o \subseteq H$ such that $S|H_o = 0$ and $s(H_o) \subseteq H_o$. The reduction processes for an ε-form described in Prop.1 and Prop.2 yield reduction processes for isometric structures. Hence, first reducing $A_1 \oplus A_2$ and then applying j yields the same as gotten by reduction of $j(A_1) \oplus j(A_2)$. We leave the details to the reader.

It is easily seen that none of the maps i nor j is surjective. So,

$$\Sigma_\lambda C_\lambda^\varepsilon(Z) \subset C^\varepsilon(Z) \subset \Sigma_\lambda C_\lambda^\varepsilon(Q)$$

holds with proper inclusions. (Of course, the maps $C_\lambda^\varepsilon(Z) \subset C_\lambda^\varepsilon(Q)$ are merely the ones gotten by extension of scalars from Z to Q.)

Very little is known about these groups and even less is known on the cokernels of i and j.

We conclude with a few remarks on $C^\varepsilon(Q)$ due to J. Levine [4] and J. Milnor [5].

Recall that an element of $C_\lambda^\varepsilon(Q)$ is represented by an ε-symmetric non-degenerate bilinear form S on a finite dimensional Q-space V together with an isometry $t : V \longrightarrow V$, i.e. $S(tx, ty) = S(x, y)$, such that λ is the minimal polynomial of t. (The polynomial $\lambda \in Z[X]$ is given as irreducible, reciprocal and satisfying $\lambda(1) = \pm 1$. This implies that the degree of λ is even. The sign can then be fixed so that $\lambda(1) = (-\varepsilon)^m$, where $2m = \deg\lambda$.)

Prop.4. <u>The group</u> $C_\lambda^\varepsilon(Q)$ <u>is independent of</u> ε, i.e. $C_\lambda^{+1}(Q)$ <u>and</u> $C_{\lambda'}^{-1}(Q)$ <u>are isomorphic, where</u> $\lambda' = (-1)^m\lambda$.

Suppose (S_o, t) is an isometric structure on a Q-space V with S_o symmetric. We associate with (S_o, t) the structure (S_1, t), where S_1 is defined by $S_1(x, y) = S_o(x, (t - t^{-1})y)$. It is easy to check that S_1 is skew-symmetric. The conditions on λ imply that $t - t^{-1}$ is an isomorphism, and so S_1 is non-degenerate. Clearly, t is an isometry for S_1. It is immediate that the map $(S_o, t) \longrightarrow (S_1, t)$ is compatible with cobordism and induces an isomorphism $C_\lambda^{+1}(Q) \longrightarrow C_{\lambda'}^{-1}(Q)$. For the balance of the paragraph we let $C_\lambda(Q)$ stand for $C_\lambda^{+1}(Q)$. The study of this group can be attacked in the following way: Let $K = Q[X]/(\lambda(X))$ and let $\tau \in K$ be the element corresponding to X. Since λ is reciprocal, $\tau^{-1} \in K$ is also a root of λ and there is a Q-automorphism of K sending τ into τ^{-1}. We denote it with a bar: $a \longmapsto \bar{a}$. Let F be its fixed field: $x \in F$ iff $\bar{x} = x$. Now, suppose V is a (finite dimensional) Q-space with an isometric structure (S, t) associated with $(\varepsilon = +1, \lambda \in Z[X])$ as above. Following J.Milnor [5], we can equip V with a K-space structure and a hermitian form (,) with respect to the involution on K. To give V a K-space structure it is enough to define $\tau.x = t(x)$. (Recall we are assuming that λ is minimal polynomial of t.) To define the hermitian form (,), fix a pair $x, y \in V$. Then S defines a Q-linear map $K \longrightarrow Q$ via $a \longrightarrow S(ax, y)$. Since K/Q is separable, there exists a unique element, (x, y) by definition, of K such that

$$\mathrm{trace}_{K/Q}\{a.(x, y)\} = S(ax, y),$$

for all $a \in K$. It is easy to verify that (x, y) is linear in the first variable and $(y, x) = \overline{(x, y)}$. (For details, see J.Milnor [5].)

Conversely, given a (non-singular) hermitian K-space V, the formulae $t(x) = \tau.x$ and $S(x, y) = \mathrm{trace}_{K/Q}(x, y)$ define an isometric structure (S, t) on V, viewed as a Q-space now, where t has minimal polynomial λ. Thus, if we define a hermitian K-space V to be <u>null-cobordant</u> if it contains a K-subspace U such that $\dim U = \frac{1}{2}.\dim V$ on which the hermitian form vanishes, we get a one-to-one correspondence between the elements of $C_\lambda(Q)$ and the cobordism classes of (non-singular) hermitian K-spaces.

Hence, we can derive some information on $C_\lambda(Q)$ from the classification of hermitian spaces. Let V_f, $f \in F$, be the 1-dimensional hermitian K-space defined by $(x, y) = x.\bar{y}.f$. If $f' = fk\bar{k}$ for some $k \in K$, then $V_{f'} \cong V_f$. Now, every hermitian K-space is the direct sum of 1-dimensional spaces V_f. Thus, there is a surjection

$$Z[F^{\cdot}/N_{K/F}K^{\cdot}] \longrightarrow C_\lambda(Q),$$

where $N_{K/F}$ is the norm from K to F and the dot means that we remove the 0-element. ($Z[F^{\cdot}/N_{K/F}K^{\cdot}]$ is the integral group ring of the multiplicative group $F^{\cdot}/N_{K/F}K^{\cdot}$. The map is the linear extension of $f \longrightarrow V_f$.) Actually, if we define a product in $C_\lambda(Q)$ by the tensor product <u>over</u> K of the associated hermitian K-spaces, the above map is a ring homomorphism. Its kernel can probably be described in terms of symbols. We do not pursue here and, following J.Levine [4], shall only describe invariants for the elements of $C_\lambda(Q)$ obtained by embedding $C_\lambda(Q)$ into a direct sum of cobordism groups for isometric structures over the real and p-adic completions of Q.

More precisely, let V be a hermitian space over $K = Q[X]/(\lambda(X))$, where $\lambda \in Z[X]$ is irreducible, reciprocal of degree 2m, and

$\lambda(1) = (-1)^m$. Denote again by F the fixed field of the involution on K determined by $\tau \longrightarrow \overline{\tau} = \tau^{-1}$, where τ is the image of X in K. We can view V as an F-space with symmetric, non-singular, F-valued bilinear form β defined by $\beta(x, y) = \mathrm{trace}_{K/F}(x, y)$ and an isometry t defined by $t(x) = \tau.x$. Set $c = \tau + \overline{\tau} \in F$. Then, t (or τ) satisfies the equation $t^2 - c.t + 1 = 0$. We shall denote by μ the polynomial $\mu(X) = X^2 - cX + 1$. Now, for every prime $\wp$ in F, we can extend the coefficients from F to the $\wp$-adic completion F . We get a map

$$C_\lambda(Q) \longrightarrow \Sigma_{\wp} C_\mu(F_\wp),$$

where $C_\mu(F_\wp) = 0$ if the polynomial μ is reducible over $F_\wp$, and otherwise $C_\mu(F_\wp)$ is the cobordism group of hermitian $K_\wp$-spaces ($K_\wp$ being the completion of K at a prime extending $\wp$, or equivalently $K_\wp = F_\wp[X]/(\mu(X))$.) Equivalently, $C_\mu(F_\wp)$ can be viewed as the cobordism group of symmetric bilinear forms on $F_\wp$-spaces with an isometry t satisfying $\mu(t) = 0$.

Prop.5. <u>The map</u>

$$C_\lambda(Q) \longrightarrow \Sigma_\wp C_\mu(F_\wp)$$

<u>is injective</u>.

Proof. Let V be an F-space with isometric structure (β, t) such that the isometry t satisfies $t^2 - c.t + 1 = 0$. We can assume that V is reduced, i.e. there is no non-zero t-invariant subspace of V on which β vanishes. If (β, t) maps to 0 in $\Sigma_\wp C_\mu(F_\wp)$, then the form β represents 0 locally at every prime $\wp$, i.e. there exists $x_\wp \neq 0$ in $V \otimes F_\wp$ such that $\beta(x_\wp, x_\wp) = 0$. By a well-known theorem on quadratic forms over global fields the form β represents 0 over F, i.e. there exists an $x \in V$, $x \neq 0$, such that $\beta(x, x) = 0$. (See O'Meara[6], p.187.) But then, it follows that β vanishes on $Fx + Ftx$, which is t-invariant, contradicting the assumption that V was reduced. Indeed, $\beta(x, tx) = \beta(t^{-1}x, x)$, and thus $\beta(x, tx) = \frac{1}{2}.\beta(x, (t+t^{-1})x) = \frac{c}{2} . \beta(x, x) = 0$, since $t+t^{-1} = c$.

Of course, $\beta(tx, tx) = \beta(x, x) = 0$. This proves proposition 5.

It remains to calculate $C_\mu(F_\wp)$. There are 3 cases apart from the trivial case where μ is reducible in $F_\wp[X]$.

Prop.6. <u>Suppose μ is irreducible in the completion</u> $F_\wp$. <u>Then</u>,

(I) <u>If $\wp$ is a real prime</u>, $C_\mu(R) \cong Z$;

(II) <u>If $\wp$ is a finite prime and</u> -1 <u>is norm from</u> $K_\wp = F_\wp[X]/(\mu(X))$ <u>to</u> $F_\wp$, <u>then</u> $C_\mu(F_\wp) \cong Z/2Z \oplus Z/2Z$;

(III) <u>If $\wp$ is a finite prime and</u> -1 <u>is not a norm from</u> $K_\wp$ <u>to</u> $F_\wp$, <u>then</u> $C_\mu(F_\wp) \cong Z/4Z$.

Observe first that given an isometric structure (β, t) on an $F_\wp$-space V, where $t^2 - ct + 1 = 0$, the cobordism class of the form β alone determines the cobordism class of the isometric structure. Indeed, if (β, t) is reduced and β is null-cobordant, then V must be 0. Otherwise, there exists $x \in V$, $x \neq 0$ such that $\beta(x, x) = 0$, and the argument at the end of the proof of Prop.5 shows that x and tx span a t-invariant non-zero subspace of V on which β vanishes, contradicting the assumption that (β, t) was reduced.

Now, in case I, i.e. $F_\wp = R$, the cobordism class of a symmetric bilinear form is determined by its signature. It follows that $C_\mu(R) = Z$ generated by V_1 in the above notation, i.e. C viewed as R-space with $\beta(x, y) = x\bar{y} + \bar{x}y$ and $t(x) = \tau.x$, where τ is a root of $\mu(X) = X^2 - cX + 1$. (Observe that the involution defined above does coincide with complex conjugation in this case. What else could it be?)

Next, suppose that $\wp$ is a finite prime and -1 is a norm from $K_\wp$ to $F_\wp$. Since $K_\wp/F_\wp$ is a finite extension of local fields, $[F_\wp^\bullet : NK_\wp^\bullet] = [K_\wp : F_\wp] = 2$, where for simplicity N denote the norm from $K_\wp$ to $F_\wp$. Let 1, g be representatives of the 2 classes in $F_\wp^\bullet/NK_\wp^\bullet$. Then, $C_\mu(F_\wp) = Z/2Z \oplus Z/2Z$ generated by V_1 and V_g, where as above V_f denotes the $F_\wp$-space of dimension 2, which we identify with $K_\wp$, together with the isometric structure given by

$\beta(x, y) = \text{trace}\,(x.\bar{y}.f)$ and $t(x) = \tau x$, where the trace is from $K_{\mathscr{P}}$ to $F_{\mathscr{P}}$.

It is clear that for every $f \in F_{\mathscr{P}}^{\bullet}$, $V_f \oplus V_f$ is null-cobordant. Proof: Choose $k \in K$ so that $-1 = k.\bar{k}$. Then $U = \{(x, xk) \in V_f \times V_f\}$ has half the dimension of $V_f \oplus V_f$ and $\beta \oplus \beta$ vanishes on U. It is also obvious that V_1 and V_g are reduced and thus neither one is null-cobordant. Finally, V_1 and V_g are not cobordant, otherwise, being reduced, they would be isomorphic. It is easy to check that V_1 and V_f are isomorphic if and only if $f \in N(K_{\mathscr{P}}^{\bullet})$.

It remains to handle the case where $\mathscr{P}$ is a finite prime and -1 is not a norm from $K_{\mathscr{P}}$ to $F_{\mathscr{P}}$. We still have $[F_{\mathscr{P}}^{\bullet} : NK_{\mathscr{P}}^{\bullet}] = 2$, and in this case we can take $+1$ and -1 as representatives of the two elements in $F_{\mathscr{P}}^{\bullet}/NK_{\mathscr{P}}^{\bullet}$. Obviously, $V_1 \oplus V_{-1}$ is null-cobordant. Thus in this case V_1 suffices to generate $C_{\mu}(F_{\mathscr{P}})$. Claim: V_1 is precisely of order 4. Observe first that $V_1 \oplus V_1$ is not null-cobordant. If it was, then V_1 and V_{-1} would be cobordant, hence isomorphic, since they are reduced. But $V_1 \cong V_{-1}$ implies that -1 is a norm. It remains to show that $V_1 \oplus V_1 \oplus V_1 \oplus V_1$ is null-cobordant. This relies on the fact that -1 is a sum of norms from $K_{\mathscr{P}}$ to $F_{\mathscr{P}}$. Assume for a moment that $-1 = k_1.\bar{k}_1 + k_2.\bar{k}_2$. An easy calculation shows that the form $\beta \oplus \beta \oplus \beta \oplus \beta$ vanishes on the subspace of $V_1 \oplus V_1 \oplus V_1 \oplus V_1$ consisting of all $(x, y, k_1x + k_2y, \bar{k}_2x - \bar{k}_1y)$, where $x, y \in K_{\mathscr{P}}$. Hence, $-1 = k_1.\bar{k}_1 + k_2.\bar{k}_2$ implies that V_1 is of order 4 in $C_{\mu}(F_{\mathscr{P}})$. In order to show that $-1 = k_1.\bar{k}_1 + k_2.\bar{k}_2$ we can either appeal to a general theorem stating that a regular quaternary space over a local field is universal. (See O'Meara, [6], § 63C. In particular Remark 63:18.) Or, we can see this directly in the case of interest to us as follows: Let Q_p be the completion of the rationals contained in F. Assume first $p \neq 2$. Then, -1 is a sum of squares in Q_p and so a fortiori a sum of norms in every quadratic extension $K_{\wp}/F_{\wp}$. Sketch of proof: First, an easy counting argument shows that -1 is a sum of squares in F_p, the field with p elements. Indeed, let S be the set of non-zero squares in F_p. If $0 \in -1 - S$, then -1 is a square and we are trough. Otherwise, $-1 - S \subseteq F_p^{\cdot} = S \cup (-S)$. We have $-S \neq -1 - S$ because $-1 \notin -1 - S$. It follows that $-1 - S \not\subseteq -S$ because the inclusion would imply equality since the two sets have the same cardinality $\frac{1}{2}(p-1)$. Thus, $-1 - S$ intersects the complement S of $-S$, i.e. $(-1-S) \cap S \neq \emptyset$. Thus there exist $\alpha^2, \beta^2 \in S$ so that $-1-\beta^2 = \alpha^2$.

Now it is easy to solve $-1 = x^2 + y^2$ in Q_p. Choose $x_o, y_o \in Z$ so that $-1 = x_o^2 + y_o^2 \bmod p$. We can actually take $0 \leqq x_o < p$, $0 \leqq y_o < p$. Assuming by induction that we can find $s_n = x_o + x_1 p + \dots + x_n p^n$ $(0 \leqq x_i < p)$ and $t_n = y_o + y_1 p + \dots + y_n p^n$ $(0 \leqq y_i < p)$ such that

$$-1 = s_n^2 + t_n^2 \bmod p^{n+1},$$

set $s_{n+1} = s_n + x_{n+1}p^{n+1}$, $t_{n+1} = t_n + y_{n+1}p^{n+1}$ with unknown x_{n+1}, y_{n+1}. Writing $1 + s_n^2 + t_n^2 = k_n p^{n+1}$, we find that $1 + s_{n+1}^2 + t_{n+1}^2 = 0 \bmod p^{n+2}$ if (and only if) x_{n+1}, y_{n+1} satisfy the congruence

$$k_n + 2(x_o x_{n+1} + y_o y_{n+1}) = 0 \bmod p.$$

Solving this congruence in x_{n+1}, y_{n+1} is clearly possible since $p \neq 2$ and x_o, y_o cannot both be 0 mod p since $x_o^2 + y_o^2 = -1 \bmod p$. The sequences $\{s_n\}$, $\{t_n\}$ clearly converge in Q_p to x, resp. y, satisfying $x^2 + y^2 = -1$.

The case $p = 2$ would be more difficult. However, it does not arise here. Recall that $(1-\tau)^{-1}$ has to be integral. (τ is a root of the polynomial $X^2 - cX + 1$ defining the extension $K_{\wp}/F_{\wp}$.) This implies that $\frac{1}{2-c}$ and hence c^{-1} must be integral in $F_{\wp}$. (Here $\wp | 2$ of course.) But then, using the fact that the multiplicative group of the residue class field of F has odd order (finite field of characteristic 2), one can find an integral element α of F such that $\alpha^2 + c^{-2} = 0 \bmod \mathfrak{P}$. Setting $d = c^2 - 4$ (the discriminant of $X^2 - cX + 1$), we then have

$$(2\alpha)^2 - d.c^{-2} = -1 \mod 4\mathfrak{P}.$$

From there it is easy to find $x, y \in F_{\mathfrak{P}}$ such that $x^2 - d.y^2 = -1$ using the successive approximations as in the case $p \neq 2$. This last equation however means that -1 is a norm from $K_{\mathfrak{P}}$ to $F_{\mathfrak{P}}$.

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