On Higher Dimensional Knots

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Which groups π can be fundamental group of the complement of an imbedded n-sphere in $(n + 2)$-space?

A well-known necessary condition is for instance that π/π' be isomorphic to the group of integers. (π' denotes the commutator subgroup of π.)

In the case $n = 1$, further necessary conditions have been given by various authors. (For instance, the first elementary ideal of π in the integral group ring of π/π' must be principal, generated by the "Alexander polynomial" $\Delta_\pi(t)$, where t denotes a generator of π/π'. Moreover, $\Delta_\pi(t)$ must satisfy an equation $\Delta_\pi(t^{-1}) = t^m \cdot \Delta_\pi(t)$ with m even. Compare [12], [4], [6].) However, the problem of characterizing knot groups by algebraic conditions remains unsolved.

Let us define a (differential) *n-knot* to be a differential imbedding f: $S^n \to S^{n+2}$, and the *group of the n-knot* $f: S^n \to S^{n+2}$ to be $\pi_1(S^{n+2} - f(S^n))$.

For $n \geqq 3$, we get a complete algebraic characterization of the groups of n-knots. (See Theorem 1 below.)

For $n = 2$, we only give a set of sufficient conditions on the group π for the existence of an imbedding $f: S^2 \to \Sigma^4$ into a *homotopy* 4-*sphere* with $\pi \cong \pi_1(\Sigma^4 - f(S^2))$. A homotopy k-sphere is a closed differential k-manifold with the homotopy type of S^k. (See Theorem 2.) A "good" set of algebraic conditions for a group π to be the group of a 2-knot is unknown to me. Recall that the first elementary ideal of $\pi_1(S^4 - f(S^2))$ need not be principal. Compare [5], Example 12. On the other hand, if the ideal is principal, generated by $P(t)$, S. Kinoŝhita has proved that one can prescribe $P(t)$ arbitrarily, subject to the only, obviously necessary condition, that $P(1) = \pm 1$. See [11]. I do not know whether in Theorem 2 the manifold Σ^4 can be taken to be S^4. This question may of course be related to the 4-dimensional Poincaré problem.

The case $n = 1$ will not be considered. All groups in this paper are assumed to be finitely presentable. The proofs will rely on the technique of spherical modifications as expounded in [10], §5.

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1. Statement of results

Define the *weight* $w(\pi)$ of a group $\pi \neq \{1\}$ to be the smallest integer k with the property that there exists a set of k elements $\alpha_1, \ldots, \alpha_k \in \pi$ whose normal closure equals π. By convention the trivial group $\{1\}$ has weight 0.

It is easy to prove that for any differential imbedding $f: S^n \to S^{n+2}$ the fundamental group $\pi_1(S^{n+2} - f(S^n))$ is of weight 1. (See Lemma 2 below.)

THEOREM 1. *Given $n \geq 3$. The group π is isomorphic to $\pi_1(S^{n+2} - f(S^n))$ for some differential imbedding $f: S^n \to S^{n+2}$ if and only if $\pi/\pi' \cong \mathbf{Z}$, the weight of π is 1, and $H_2(\pi) = 0$.*

(Here $H_2(\pi)$ denotes the second homology group of π with integral coefficients and trivial action of π on $\mathbf{Z}$.)

Actually, the "only if" part of this theorem holds for $n \geqq 1$:

LEMMA 1. *Let Σ^{n+2} be a homotopy $(n + 2)$-sphere, where $n \geqq 1$. If*

$$\pi \cong \pi_1(\Sigma - f(S^n))$$

for some differential imbedding $f: S^n \to \Sigma^{n+2}$, then $\pi/\pi' = \mathbf{Z}$, $w(\pi) = 1$, and $H_2(\pi) = 0$.

For $n = 2$, the argument in the proof of Theorem 1 breaks down. We shall get a partial result by strengthening the algebraic conditions. Define the *deficiency* of a presentation $(x_1, \ldots, x_q; R_1, \ldots, R_r)$ of a group to be the integer $q - r$. Clearly, a group π with a presentation of deficiency 1 and such that $\pi/\pi' \cong \mathbf{Z}$ satisfies the condition $H_2(\pi) = 0$. [Let $K^{(1)}$ be the q-fold wedge of S^1, and attach $q - 1$ two-dimensional cells to $K^{(1)}$ using $R_1, \ldots, R_{q-1}$. The resulting 2-complex K satisfies $\pi_1 K \cong \pi$ and $H_2 K = 0$. Hence $H_2(\pi) = H_2 K/\rho(\pi_2 K) = 0$, where $\rho: \pi_2 K \to H_2 K$ is the Hurewicz homomorphism. Compare Hopf [7].]

THEOREM 2. *Given a group π of weight 1, with a presentation of deficiency 1, and such that $\pi/\pi' \cong \mathbf{Z}$. There exist a homotopy 4-sphere Σ^4 and a differential imbedding $f: S^2 \to \Sigma^4$ such that $\pi \cong \pi_1(\Sigma^4 - f(S^2))$.*

The condition of deficiency 1 is definitely stronger (for groups satisfying $\pi/\pi' \cong \mathbf{Z}$) than the condition $H_2(\pi) = 0$. For example, the group $\mathbf{G}$ with the presentation $(x, a; a^3 = 1, x = axa)$ considered by R. Fox in [5], Example 12, has vanishing second homology group because it is the group of some 2-knot. (Compare Lemma 1.) However, this group does not have deficiency 1 (it has therefore deficiency 0) because its Alexander ideal, which is easily computed to be $\mathscr{E} = (3, 1 + t)$, fails to be principal.[2]

(For the notion of the Alexander ideal, compare [4]. If the group π, satisfying $\pi/\pi' \cong \mathbf{Z}$, has deficiency 1, one can find a presentation of π of the form $(x_1, \ldots, x_q; x_1C_1 = 1, \ldots, x_{q-1}C_{q-1} = 1)$, where $C_1, \ldots, C_{q-1}$ belong to the commutator subgroup of the free group on $x_1, \ldots, x_q$. It follows that an Alexander matrix of π has its last column consisting of zeros only. Hence this matrix has only one non-zero minor of order $q - 1$, i.e., the Alexander ideal $\mathscr{E}_1$ is generated by a single element.)

The proof of Theorem 1 generalizes trivially to the case of links, i.e., unions of k disjointly imbedded n-spheres in S^{n+2}.

THEOREM 3. *Given $n \geqq 3$. The group π is isomorphic to $\pi_1(S^{n+2} - L_k)$, where L_k is a union of k disjointly imbedded n-spheres if and only if π/π' is free abelian of rank k, the weight of π is equal to k, and $H_2(\pi) = 0$.*

The proof follows closely the pattern of the proof of Theorem 1 and will be left to the reader.

Similarly:

THEOREM 4. *If the group π of weight k and such that π/π' is free abelian of rank k has a presentation of deficiency k, then there exists a link L_k of k disjointly imbedded 2-spheres in some homotopy 4-sphere Σ^4, such that $\pi \cong \pi_1(\Sigma^4 - L_k)$.*

2. Proofs

We begin with the proof of a slight generalization of part of Lemma 1.

LEMMA 2. *Let M^{n+2} be a simply connected differential manifold, and V^n a connected submanifold. (Either or both manifolds possibly with boundary.) Then the weight of $\pi_1(M - V)$ is at most 1.*

Let $a: S^1 \to M - V$ be an imbedding such that $a(S^1)$ bounds a small 2-disc τ in M which intersects V transversally at exactly one point. Taking a base point $z_0 \in S^1$, let $a(z_0) = x_0$ be the base point in $M - V$. Let α be the homotopy class of $a: (S^1, z_0) \to (M - V, x_0)$. I claim that the normal closure of α in $\pi_1(M - V, x_0)$ is equal to $\pi_1(M - V, x_0)$. Let $\xi \in \pi_1(M - V, x_0)$ be an arbitrary element, and let $f: (S^1, z_0) \to (M - V, x_0)$ be a differential mapping representing ξ. Since M is simply connected, we can extend f to a map $F: D^2 \to M$ which may be assumed to be differential and transversal to V. (Compare [14].) Then $F^{-1}(V)$ is a finite set of points $u_1, \ldots, u_r \in D^2 - S^1$. Let $D_1, \ldots, D_r$ be small disjoint discs around $u_1, \ldots, u_r$ contained in $D^2 - S^1$. Joining $F(u_i)$ to $y_0 = \tau \cap V$ by a path on V, we can deform F in D_i, keeping it fixed in $D^2 - D_i$, so that the deformed map F satisfies $F(D_i^*)$ [illegible] and $F(D_i - D_i^*) \cap V = \emptyset$, here represents α^{ε_i}, where $\varepsilon_i = +1$ or -1. Let z_i be a base point on the boundary of D_i^*, i.e., a point such that $F(z_i) = x_0$. A path on $D^2 - \cup_i D_i^*$ from z_i to the base point z_0 on S^1 maps by F into a loop at x_0 representing some element $T_i \in \pi_1(M - V, x_0)$. Taking these paths on D^2 to be disjoint, we clearly have $\xi = \Pi_i T_i^{-1}\alpha_i^{\varepsilon} T_i$. Hence the weight of $\pi_1(M - V, x_0)$ is at most 1.

To complete the proof of Lemma 1 it remains to show that if $f\colon S^n \to \Sigma^{n+2}$ is a differentiable imbedding, where Σ^{n+2} is a homotopy $(n+2)$-sphere, then $H_2(\pi) = 0$, where $\pi = \pi_1(\Sigma^{n+2} - f(S^n))$. This follows from the theorem of Hopf in [7] since $H_2(\Sigma^{n+2} - f(S^n)) = 0$ by Alexander duality.

The proofs of Theorem 2 and of the "if" part of Theorem 1 are based on the same construction. Thus we start proving them simultaneously. Eventually the proof will split into the two cases $n = 2$ and $n \geqq 3$.

Let π be a given group of weight 1 and such that $\pi/\pi' \cong \mathbf{Z}$.

Suppose we have succeeded constructing a manifold M^{n+2} such that $\pi_1 M \cong \pi$, and $H_q M = 0$ for $2 \leqq q \leqq n$, where $n \geqq 2$. Let $\alpha \in \pi$ be an element whose normal closure is equal to π, and let $\psi\colon S^1 \to M^{n+2}$ be a differential imbedding representing α (under some isomorphism $\pi \cong \pi_1(M, x_0)$, where $x_0 = \psi(z_0)$). Extending ψ to an imbedding $\varphi\colon S^1 \times D^{n+1} \to M^{n+2}$ and performing the spherical modification $\chi(\varphi)$, we obtain a Σ^{n+2} which is easily seen to be a homotopy sphere. We have

$$\Sigma^{n+2} = (M - \varphi(S^1 \times B^{n+1})) \cup D^2 \times S^n,$$

where B^{n+1} denotes the interior of D^{n+1}, and Σ^{n+2} is simply connected by the theorem of van Kampen. Using $\pi/\pi' \cong \mathbf{Z}$ and the fact that the homology class of α must be a generator of $H_1 M$, it follows that

$$H_2\Sigma = H_n\Sigma = 0.$$

Compare [10], §5, Lemma 5.6. The vanishing of $H_k\Sigma$ for $2 < k < n$ follows readily using a Mayer–Vietoris sequence.

Let $\varphi'\colon D^2 \times S^n \to \Sigma^{n+2}$ be the inclusion map, and $f\colon S^n \to \Sigma^{n+2}$ the imbedding defined by $f(u) = \varphi'(0, u)$. Clearly, $\Sigma^{n+2} - f(S^n)$ and $\Sigma^{n+2} - \varphi'(D^2 \times S^n)$ have the same homotopy type. Hence

$$\pi_1(\Sigma^{n+2} - f(S^n)) \cong \pi_1(\Sigma^{n+2} - \varphi'(D^2 \times S^n)) \cong \pi_1(M - \varphi(S^1 \times D^{n+1})) \cong \pi_1 M \cong \pi,$$

since $\dim M \geqq 4$.

If $n = 2$, I do not know how to go any further. For $n \geqq 3$ the connected sum $\Sigma \# (-\Sigma)$ is h-cobordant, and hence diffeomorphic to S^{n+2}. (Compare is diffeomorphic to S^5 and we do not have to appeal to the connected sum $\Sigma \# (-\Sigma)$.)

It remains to construct the manifold M^{n+2} such that $\pi_1 M \cong \pi$ and $H_k M = 0$ for $2 \leqq k \leqq n$.

Let $\pi = (x_1, \ldots, x_q; R_1, \ldots, R_r)$ be a finite presentation of π (not necessarily of deficiency 1). We start with the manifold

$$M_0^{n+2} = S^1 \times S^{n+1} \# S^1 \times S^{n+1} \# \cdots \# S^1 \times S^{n+1},$$

i.e., the connected sum of q copies of $S^1 \times S^{n+1}$. Observe that M_0^{n+2} is S-parallelizable, i.e., the stabilized tangent bundle of M_0^{n+2} is trivial. (Compare [10].) It is convenient to choose as base "point" on M_0^{n+2} some contractible open subset $U \subset M_0$. The fundamental group $\pi_1(M_0, U)$ is a free group on q generators which we identify with $x_1, \ldots, x_q$. Let $f_j\colon (S^1, z_0) \to (M_0, U)$, where $j = 1, \ldots, r$ be r mutually disjoint differential imbeddings representing $R_j(x_1, \ldots, x_q)$. Since M_0 is orientable, we can extend f_j to disjoint differential imbeddings $\varphi_j\colon S^1 \times D^{n+1} \to M_0$, and then perform a polyspherical framed modification $\chi(\varphi_1, \ldots, \varphi_r)$. In other words, we consider the manifold M_1 obtained from the disjoint union $(M_0 - \bigcup_j \varphi_j(S^1 \times B^{n+1})) \cup (D^2 \times S^n)_1 \cup (D^2 \times S^n)_2 \cup \cdots \cup (D^2 \times S^n)_r$ by identifying $\varphi_j(u, v)$ for $u \in S^1$, $v \in S^n$ with $(u, v) \in (D^2 \times S^n)_j$, $j = 1, \ldots, r$. We denote by φ'_j the inclusion of $(D^2 \times S^n)_j$ into M_1. It is known that M_1 has a natural differential structure, and that the extensions φ_j of f_j can be chosen so that M_1 is S-parallelizable. (Compare [10], p. 521 and Lemma 5.4.) By the theorem of van Kampen, we have $\pi_1(M_1) \cong \pi$.

If the presentation $(x_1, \ldots, x_q; R_1, \ldots, R_r)$ of π has deficiency 1, i.e., if $r = q - 1$, then $\varphi_*\colon H_1(\bigcup_j(S^1 \times D^{n+1})) \to H_1(M_0)$ must be injective since $H_1 M_0/\mathrm{Im}\, \varphi_* \cong H_1 M_1 \cong \mathbf{Z}$. Then $H_2 M_0 = 0$ implies

$$H_2(M_0, \textstyle\bigcup_j \varphi_j(S^1 \times D^{n+1})) = 0,$$

and by excision we also have

$$H_2(M_1, \textstyle\bigcup_j \varphi'_j(D^2 \times S^n)) = 0.$$

The exact sequence

$$\Sigma_j H_2(D^2 \times S^n) \xrightarrow{\varphi'_*} H_2 M_1 \to H_2(M_1, \textstyle\bigcup_j \varphi'_j(D^2 \times S^n))$$

then yields $H_2 M_1 = 0$, at least for $n > 2$. For $n = 2$ the conclusion $H_2 M_1 = 0$ still follows since φ_* maps the generators of $\Sigma_j H_1(S^1 \times D^n)$ into primitive elements of $H_1 M_0$ and it follows that the map φ'_* of the above sequence is zero. (Compare [10], Lemma 5.6 and the "Assertion"

If the presentation of π is not necessarily of deficiency 1, and all we know is that $H_2(\pi) = 0$, we have to assume $n \geqq 3$ in order to be able to kill H_2M_1 by spherical modifications.

Let $\rho\colon \pi_2M_1 \to H_2M_1$ be the Hurewicz homomorphism. Then $\rho(\pi_2M_1)$ is the subgroup of spherical homology classes, and according to H. Hopf $H_2(\pi) = H_2M_1/\rho(\pi_2M_1)$, where the homology of π is understood with integer coefficients and trivial action of π. (Compare [7].) Hence the assumption $H_2(\pi) = 0$ implies that $\rho\colon \pi_2M_1 \to H_2M_1$ is surjective. The assumption $n \geqq 3$ guarantees that every homology class $\xi \in H_2M_1$ is representable by a differentiably imbedded 2-sphere. (Recall that according to [9] this is definitely false for $n = 2$.) Since M_1 is S-parallelizable, every differentiably imbedded 2-sphere in M_1 has a trivial normal bundle.

Following the discussion in §5 of [10], we construct a finite sequence $M_1, M_2, \ldots, M_k, \ldots$ of $(n + 2)$-manifolds. M_{k+1} is obtained from M_k by a spherical modification $\chi(\varphi_k)$ were $\varphi_k\colon S^2 \times D^n \to M_k$ represents a suitable class $\xi_k \in H_2M_k$.

Since $M_{k+1} = (M_k - \varphi_k(S^2 \times B^n)) \cup D^3 \times S^{n-1}$, it follows from $n \geqq 3$, using the theorem of van Kampen, that $\pi_1M_{k+1} \cong \pi_1M_k \cong \pi$. Hence, at every stage of the construction, $\rho\colon \pi_2M_k \to H_2M_k$ is surjective (by Hopf's theorem [7]) and every class in H_2M_k is representable by a differentiably imbedded 2-sphere. Taking care of performing only *framed* spherical modifications (see §6 of [10]), all the manifolds M_k will be S-parallelizable, and every differentiably imbedded 2-sphere in M_k has a trivial normal bundle.

The choice of $\xi_k \in H_2M_k$ is made as follows: (1) If H_2M_k is infinite, then H_2M_k contains a primitive element ξ_k. This means that there exists an element $\eta \in H_nM_k$ such that $\xi_k \cdot \eta = 1$. Then, killing ξ_k by a spherical modification also kills η and $H_2M_{k+1} \cong H_2M_k/(\xi_k)$. For the proof, see Lemma 5.6 of [10]. (2) If H_2M_k is finite (and non-zero) then n must equal 3. Killing an arbitrary element $\xi_k \in H_2M_k$ by spherical modification introduces a new class $\xi' \in H_2M_{k+1}$ necessarily of infinite order. More precisely, the sequence

$$0 \to \mathbf{Z} \to H_2M_{k+1} \to H_2M_k/(\xi_k) \to 0$$

is exact in this case, as proved in [10], p. 16. The next modification will be taken to kill a primitive element ξ_{k+1} in the newly introduced infinite cyclic summand of H_2M_{k+1}. (In general $\xi_{k+1} \neq \xi'$.)

It is left to the reader to convince himself that the non-vanishing of the fundamental groups of M_k in the present situation does not invalidate Lemmas 5.6 and 5.8 of [10]. The proofs given in [10] apply word-for-word arbitrary generator if H_2M_k is finite, the sequence $M_1, \ldots, M_k, \ldots$ leads in a finite number of steps to a manifold $M_l = M$ such that $\pi_1 M \cong \pi$ and $H_2 M = 0$.

If $n = 3$, then $H_2M = 0$ implies $H_3M = 0$ by Poincaré duality since $H_1M \cong \pi/\pi' \cong \mathbf{Z}$ is free abelian. If $n > 3$, then $H_3M = \cdots = H_{n-1} = 0$ because then H_2M_k is torsion free, and killing a primitive class of H_2M_k leaves the q-dimensional homology group unchanged for $3 \leqq q \leqq n - 1$. Moreover, $H_nM = 0$ again by Poincaré duality.

Thus the manifold M satisfies the requirements $\pi_1M \cong \pi$ and $H_qM = 0$ for $q = 2, \ldots, n$. This completes the proof of Theorem 1.

REMARK. Of course, having an imbedding $f: S^3 \to S^5$ such that $\pi_1(S^5 - f(S^3)) \cong \pi$ it is easy to obtain an imbedding $f_n: S^n \to S^{n+2}$ with $\pi_1(S^{n+2} - f_n(S^n)) \cong \pi$ for all $n \geqq 3$ by the Artin "spinning" construction. (Compare Artin [1].)

3. Appendix

This last section contains some remarks which came up in discussions during the Symposium. They are due for the most part to M. Hirsch, J. Milnor, and J. Stallings. (However, except for Theorem III and Milnor's example under (3) below, the proofs given here are my own.)

The first question was: *What is the situation in the case of a differentiably imbedded homotopy n-sphere Σ^n in S^{n+2}?*

THEOREM I. *A homotopy n-sphere Σ^n can be imbedded in S^{n+2} (by a differential imbedding) if and only if Σ^n is the boundary of a parallelizable $(n+1)$-manifold.*

PROOF. Suppose Σ^n is imbeddable in S^{n+2}, then Σ^n bounds a parallelizable $(n+1)$-manifold. In fact, every $\Sigma^n \subset S^{n+2}$ bounds a parallelizable $V^{n+1} \subset S^{n+2}$. Let U be a tubular neighborhood of $\Sigma^n \subset S^{n+2}$ and

$$T = bU = \bar{U} - U.$$

Since Σ^n has a trivial normal bundle in S^{n+2}, we have a diffeomorphism $\Sigma^n \times D^2 \approx \bar{U}$. Let $\Sigma_1^n \subset T$ be the submanifold corresponding to $\Sigma^n \times (x_1)$, where $x_1 \in S^1 \subset D^2$. It is enough to show that Σ_1^n bounds a parallelizable $(n+1)$-manifold V in $S^{n+2} - U$. The Pontrjagin–Thom construction applied to $\Sigma_1^n \subset T^{n+1}$ and a normal vector-field to Σ_1^n in T yields a map $\Phi_1: T \to S^1$. We extend this map to a map $\Phi: S^{n+2} - U \to S^1$. The only obstruction to the extension is a cohomology class $\gamma \in H^2(S^{n+2} - U, T)$. (Coefficients in $\pi_1(S^1) \approx \mathbf{Z}$.) By the Hopf theorem [Proposition 13.1 on in T. By the Alexander duality theorem, Σ_1 is homologous to zero in $S^{n+2} - U$, hence there exists a class $\xi \in H_{n+1}(S^{n+2} - U, T)$ such that $\partial\xi = (\Sigma_1)$, the homology class of Σ_1 (in $H_n T$). Using the diagram

$$\begin{array}{ccccc}
H^1(S^{n+2} - U) & \xrightarrow{i^*} & H^1 T & \longrightarrow & H^2(S^{n+2} - U, T) \\
\updownarrow & & \updownarrow & & \updownarrow \\
H_{n+1}(S^{n+2} - U, T) & \xrightarrow{\partial} & H_n T & \longrightarrow & H_n(S^{n+2} - U)
\end{array}$$

where the vertical isomorphisms are given by Poincaré duality, we see that the dual of ξ maps by i^* onto $\pm\sigma$. (The diagram is commutative up to sign.) Hence, $\delta\sigma = \gamma = 0$. Keeping $\Phi|T = \Phi_1$ fixed, we can approximate $\Phi\colon S^{n+2} - U \to S^1$ by a differential map, and assume that

$$\Phi(\Sigma_1) = a \in S^1$$

is a regular value of Φ. Then $\Phi^{-1}(a) = V^{n-1} \subset S^{n+2}$ or at least the component of Σ_1 in $\Phi^{-1}(a)$ is the desired manifold.

In particular, this shows that every higher knot has some sort of "generalized genus."

The converse is clear for $n = 1$ or 2. For $n = 3$, every compact orientable 3-manifold in imbeddable in S^5. [M. Hirsch, *Ann. of Math.*, 74 (1961), 494–497. Theorem 3.] If $n = 4$ we argue as follows. Every homotopy 4-sphere Σ^4 is the boundary of a contractible 5-manifold V. (Compare [10], Theorem 6.6.) Let Σ^5 be the double of V. The homotopy 5-sphere Σ^5 is the boundary of a contractible 6-manifold W, and according to Smale [13] W is diffeomorphic to the 6-disc D^6. Hence, Σ^5 is diffeomorphic to S^5, and so every homotopy 4-sphere is imbeddable in S^5. Let $n \geqq 5$, and suppose that the homotopy n-sphere Σ^n is the boundary of a parallelizable $(n+1)$-manifold. If n is even, then Σ^n is h-cobordant to S^n. (See [10], Theorem 6.6.) Hence, according to Smale [13], Σ^n is diffeomorphic to S^n. If n is odd, Σ^n is h-cobordant and hence diffeomorphic to the boundary of a $\frac{1}{2}(n-1)$-connected manifold W^{n+1}, where W is the connected sum along the boundary of a finite number of copies of some manifold W_0. In Part II of [10] we give an explicit construction of W_0 as a "thickened" wedge of k-spheres, where $2k = n + 1$. Each S^k imbedded in W_0 has a stably trivial normal bundle. Now any wedge of k-spheres can be imbedded in S^{n+2}, and the imbeddability of W in S^{n+2} follows if we prove that every stably trivial SO_k-bundle over S^k can be realized by a normal field of k-planes over S^k in S^{2k+1}. Let f^{k+1} be a trivialization of the normal bundle of $S^k \subset S^{2k+1}$. If v is the field of normal vectors over S^k corresponding to then the SO_k-bundle over S^k given by the field of normal k-planes orthogonal to v has a characteristic element $\chi \in \pi_{k-1}(SO_k)$ given by $\chi = \tau(\alpha)$, where $\tau\colon \pi_k(S^k) \to \pi_{k-1}(SO_k)$ is the transgression homomorphism of the fibration $SO_k \to SO_{k+1} \to S^k$ and (α) is the homotopy class of α. Stably trivial SO_k-bundles over S^k are exactly those whose characteristic element is in the image of τ. Hence, by a proper choice of (α), one can realize any stably trivial SO_k-bundle over S^k by a field of normal k-planes in S^{2k+1}. This completes the proof of Theorem I.

THEOREM II. (J. Milnor.) *If Σ^n is the boundary of a parallelizable manifold, there exists a differential imbedding $f_0\colon \Sigma^n \to S^{n+2}$ such that $\pi_1(S^{n+2} - f_0(\Sigma^n)) \cong \mathbf{Z}$.*

PROOF. The statement is trivial for $n = 1$, 2 and n even $\geqq 6$ since then Σ^n is diffeomorphic to S^n as observed before. For $n = 4$ it follows since every homotopy 4-sphere is imbeddable in S^5. (Compare the proof of Theorem I.) If n is odd, let $W^{2k} \subset S^{2k+1}$, where $2k = n + 1$, be a connected submanifold of S^{2k+1} such that $bW = \Sigma^n$ and $S^{2k+1} - W$ is simply connected. We shall prove that for such an imbedding $\Sigma^n \subset S^{n+2}$, we have $\pi_1(S^{n+2} - \Sigma^n) \cong \mathbf{Z}$. The existence of a submanifold $W^{2k} \subset S^{2k+1}$ of the required kind will be shown at the end of the proof. Let ℓ be a loop in the complement of Σ^n. We can assume that ℓ is a differentiable curve, transversal to W. Let $A(\ell)$ be the algebraic intersection number $I(\ell, W) \in Z$. Clearly, $A(\ell)$ depends only on the homotopy class of ℓ in $\pi_1(S^{n+2} - \Sigma^n, x_0)$, where $x_0 = \ell(0)$, and provides a homomorphism $A\colon \pi_1(S^{n+2} - \Sigma^n, x_0) \to \mathbf{Z}$. It is obvious that A is surjective. Suppose that $A(\ell) = 0$. We show that ℓ is homotopic in $S^{n+2} - \Sigma^n$ in a loop in $S^{n+2} - W$. Unless ℓ is already a loop in $S^{n+2} - W$, there exist a pair $t < t'$ of values of t and small positive numbers ε, ε' such that $\ell[t + \varepsilon, t' - \varepsilon']$ is contained in $S^{n+2} - W$, and

$$I(\ell[t - \varepsilon, t + \varepsilon], W) = -I(\ell[t' - \varepsilon', t' + \varepsilon'], W) \neq 0,$$

where the sets $\ell[t - \varepsilon, t + \varepsilon] \cap W$ and $\ell[t' - \varepsilon', t' + \varepsilon'] \cap W$ consist of a single point. Since $S^{n+2} - W$ is simply connected, $\ell|[t + \varepsilon, t' - \varepsilon']$ is homotopic in $S^{n+2} - W$ (and *a fortiori* in $S^{n+2} - \Sigma^n$) to a path ℓ' contained in a neighborhood of W. (Use a path on W connecting $\ell(t)$ and $\ell(t')$.) Replacing the portion of ℓ between $t + \varepsilon$ and $t' - \varepsilon'$ by ℓ' and using a field of normal vectors to W in S^{n+2} we can "push" the path

$$\ell([t - \varepsilon, t + \varepsilon]) \cdot \ell'([t + \varepsilon, t' - \varepsilon']) \cdot \ell([t' - \varepsilon', t' + \varepsilon'])$$

across W (keeping its endpoints fixed) without intersecting Σ^n, thus follows by induction on the cardinality of the finite set $\ell(I) \cap W$. Since $S^{n+2} - W$ is simply connected, $A(\ell) = 0$ implies $\ell \simeq 1$, hence A is an isomorphism and $\pi_1(S^{n+2} - \Sigma^n) \simeq \mathbf{Z}$.

If $n \geqq 5$, we take for W a finite connected sum along the boundary of thickened wedge of k-spheres, and the imbedding $W^{2k} \subset S^{2k+1}$ whose existence has been shown in the proof of Theorem I. Since W retracts by deformation on a wedge of k-spheres, we clearly have

$$\pi_1(S^{2k+1} - W^{2k}) = \{1\}.$$

If $n = 3$, take an imbedding $\Sigma^3 \subset S^5$ which bounds a simply connected $W^4 \subset S^5$. (Compare Hirsch, *Ann. of Math.*, *74* (1961), 494–497, Theorem 3.) I claim that $S^5 - W$ is simply connected. Let $f: S^1 \to S^5 - W$ be a given differential imbedding and denote again by f an extension $f: (D^2, S^1) \to (S^5, S^5 - W)$. Firstly, we can push $f(D^2)$ away from $\Sigma^3 = bW^4$. To do this assume that f is transversal to W and Σ^3. If x, x' are two intersection points of $f(D^2)$ and Σ^3 with opposite intersection numbers, then x, x' are the endpoints of a path w in $f(D^2) \cap W$ and also endpoints of a path w' on Σ^3. Let $\varphi: Q \to W^4$ be an immersion of the 2-disc with two corners Q into W such that $\varphi | bQ = w' \cdot w^{-1}$. ($W$ is simply connected.) Keeping $\varphi | bQ$ fixed we can approximate φ by an imbedding into S^5 such that $\varphi(\operatorname{int} Q) \cap \Sigma^3 = \emptyset$. Using a field of normal 3-frames on $\varphi(Q)$ we can push f along $\varphi(Q)$ using the method of H. Whitney [*Ann. of Math.*, 45 (1944), 220–246]. The effect of this operation is to remove the intersection points x, x' from $f(D^2) \cap \Sigma^3$. After a finite number of such operations we obtain a new mapping $f: (D^2, S^1) \to (S^5 - \Sigma^3, S^5 - W^4)$ with $f | S^1$ unchanged.

Now, assuming f still transversal to W, the intersection $f(D^2) \cap W$ consists of a finite number of disjoint closed curves. Let $C_1, \ldots, C_r$ be the corresponding curves in D^2. Let C_1 bound a disc D_1 on D^2. Since $f(C_1)$ is homotopic to a point in W, we can replace $f | D_1$ by a mapping into W. Using a normal vector-field on W, f can be pushed away from W in a neighborhood of D_1, thus reducing the number of components of $f(D^2) \cap W$ by at least 1. Hence the intersection $f(D^2) \cap W$ can also be removed in a finite number of steps, proving that $f | S^1$ is homotopic to a point in $S^5 - W$. Thus $\pi_1(S^5 - W) = \{1\}$. This completes the proof of Theorem II.

THEOREM III. (M. Hirsch.) *Let $f: \Sigma^n \to S^{n+2}$ be a differential imbedding with $n \geqq 5$. If $S^{n+2} - f(\Sigma^n)$ has the homotopy type of S^1, then Σ^n is diffeomorphic to S^n.*

set $V^{n+2} = S^{n+2} - U$. Extending g to an imbedding $g: S^1 \times D^{n+1} \to V^{n+2}$ using a trivialization of the normal bundle of $g(S^1)$. Then

$$V' = V - g(S^1 \times \operatorname{int} D^{n+1})$$

provides a simple h-cobordism between $g(S^1 \times S^n)$ and $bU = bV$. It is easily seen that inclusions $g(S^1 \times S^n) \subset V'$ and $bU \subset V'$ are homotopy equivalences, and since the fundamental group is infinite cyclic, they are simple homotopy equivalences (compare J. H. C. Whitehead, *Proc. London Math. Soc.*, 45 (1939), 243–327). It follows by a theorem of B. Mazur [*Ann. of Math.*, 77 (1963), 232–249] that bU is diffeomorphic to $S^1 \times S^n$. Since Σ^n has a trivial normal bundle, bU is diffeomorphic to $S^1 \times \Sigma^n$. Hence there exists a diffeomorphism $h: S^1 \times \Sigma^n \to S^1 \times S^n$. Lifting h to the universal coverings, we get a diffeomorphism $H: R^1 \times \Sigma^n \to R^1 \times S^n \subset R^{n+1}$. Hence Σ^n can be differentially imbedded into R^{n+1}. The compact region W bounded by $H(0 \times \Sigma^n)$ in R^{n+1} is a contractible manifold with boundary Σ^n. Using the assumption $n \geqq 5$, it follows by Smale [13] that W is diffeomorphic to D^{n+1}. In particular $\Sigma^n = bW$ is diffeomorphic to S^n.

THEOREM IV. *Given a homotopy n-sphere Σ^n which is the boundary of a parallelizable manifold, $n \geqq 3$, and a finitely presentable group π of weight* 1, *such that $\pi/\pi' \cong \mathbf{Z}$ and $H_2(\pi) = 0$, there exists a differential imbedding $f: \Sigma^n \to S^{n+2}$ with $\pi \cong \pi_1(S^{n+2} - f(\Sigma^n))$.*

PROOF. Let $\Sigma^n \subset S^{n+2}$ be an imbedding such that $\pi_1(S^{n+2} - \Sigma^n) \cong \mathbf{Z}$ (Theorem II). Let $f_0: S^n \to S^{n+2}$ be an n-knot with $\pi_1(S^{n+2} - f_0(S^n)) \cong \pi$ (Theorem 1). Take the relative connected sum $(S^{n+2}, \Sigma^n) \# (S^{n+2}, f_0(S^n))$, and let $f: \Sigma^n \to S^{n+2}$ be the resulting differential imbedding. Then $\pi_1(S^{n+2} - f(\Sigma^n)) \cong \pi * Z/(\alpha^{-1}\gamma)$, where γ generates $\mathbf{Z}$, and α is some element of π. Hence, $\pi_1(S^{n+2} - f(\Sigma^n)) \cong \pi$.

If the complement of a 1-knot $f: S^1 \to S^3$ has an infinite cyclic fundamental group, then $S^3 - f(S^1)$ and S^1 have the same homotopy type. This is a simple consequence of the Lemma of Dehn-Papakyriakopoulos [*Ann. of Math.*, *66* (1957), 1–26].

For which values of n can one imbed S^n into S^{n+2} so that $\pi_1(S^{n+2} - f(S^n))$ is infinite cyclic but $S^{n+2} - f(S^n)$ does not have the homotopy type of S^1?

THEOREM V. (J. Stallings) *For $n \geqq 3$ there exist smooth imbeddings $f: S^n \to S^{n+2}$ such that $\pi_1(S^{n+2} - f(S^n)) \cong \mathbf{Z}$ and $\pi_2(S^{n+2} - f(S^n)) \neq 0$.*

For $n = 2$ the question remains open.

PROOF. Let $M = S^1 \times S^{n+1} \# S^2 \times S^n$, and let L denote the multiwith J. Then, $\pi_2(M_0, x_0)$ is a free $\mathbf{Z}[J]$-module on one generator, where $\mathbf{Z}[J]$ is the integral group ring of J. Choosing a generator of $\pi_2(M_0, x_0)$, we can identify $\pi_2(M_0, x_0)$ with $\mathbf{Z}[J]$. Let $\varphi_0: S^2 \to M_0$ be a differential imbedding representing the element $(2 - t) \in \pi_2(M_0, x_0) = \mathbf{Z}[J]$. Clearly, φ_0 has a trivial normal bundle and we can extend φ_0 to an imbedding $\varphi: S^2 \times D^n \to M_0$. Let $M = \chi(M_0, \varphi)$ be the manifold obtained from M_0 by spherical modification. $M = (M_0 - \varphi(S^2 \times \text{int } D^n)) \cup D^3 \times S^{n-1}$. Since the homology class of $\varphi_0(S^2)$ is a generator of H_2M_0, it follows that $H_2M = \cdots = H_nM = 0$. Since $n \geqq 3$, $\pi_1 M \cong \mathbf{Z}$ and it is easily seen that $\pi_2 M$ is non-trivial. Indeed, M_0 and M are the two components of the boundary of a manifold W which is homotopy equivalent to $M_0 \cup e^3$ and $M \cup e^n$. Hence, the inclusions $M_0 \to W$ and $M \to W$ induce epimorphisms $i_*: \pi_2 M_0 \to \pi_2 W$ and $\pi_2 M \to \pi_2 W$. The kernel of i_* is the ideal of $\mathbf{Z}[J]$ generated by $2 - t$. Hence, $\pi_2 W = \text{Im } i_* \neq 0$, and therefore $\pi_2 M \neq 0$.

Let $\psi: S^1 \times D^{n+1} \to M$ be an imbedding representing a generator of $\pi_1 M \cong \mathbf{Z}$. Then, the manifold

$$\chi(M, \psi) = (M - \psi(S^1 \times \text{int } D^{n+1})) \cup D^2 \times S^n$$

is a homotopy $(n + 2)$-sphere Σ^{n+2}. Let $f: S^n \to \Sigma^{n+2}$ be the imbedding $S^n \to 0 \times S^n \subset D^2 \times S^n \subset \Sigma^{n+2}$. We have

$$\pi_1(\Sigma^{n+2} - f(S^n)) \cong \pi_1 M \cong \mathbf{Z},$$

and

$$\begin{aligned}\pi_2(\Sigma^{n+2} - f(S^n)) &\cong \pi_2(\Sigma^{n+2} - D^2 \times S^n) \\ &\cong \pi_2(M - \psi(S^1 \times D^{n+1})) \cong \pi_2 M \neq 0.\end{aligned}$$

By suitably changing the differentiable structure of Σ^{n+2} in the complement of $f(S^n)$ we obtain a differential imbedding $f: S^n \to S^{n+2}$ with

$$\pi_1(S^{n+2} - f(S^n)) \cong \mathbf{Z} \qquad \text{and} \qquad \pi_2(S^{n+2} - f(S^n)) \neq 0.$$

Finally, we discuss the independence of the algebraic conditions:

(i) $\pi/\pi' \cong \mathbf{Z}$;
(ii) There exists an element $\alpha \in \pi$ whose normal closure is π;
(iii) $H_2(\pi) = 0$.

(1) Any finite cyclic group satisfies (ii) and (iii) and does not satisfy (i).
(2) Let G be a non-trivial group such that $G = G'$, and suppose that G

the group with the presentation $G = (\alpha, \beta; \alpha^2 = \beta^3 = (\alpha^{-1}\beta)^5)$. A representation $\rho: G \to U(5)$ into the group of 5×5 unitary matrices is given by

$$\rho(\alpha) = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix} \qquad \rho(\beta) = \begin{Vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

Under the above assumptions on G, the group $\pi = G * \mathbf{Z}$ satisfies (i) and (iii). It even satisfies the requirement stronger than (iii) of having deficiency 1. I claim that the weight of π is bigger than 1. We prove this for the above group $G = (\alpha, \beta; \alpha^2 = \beta^3 = (\alpha^{-1}\beta)^5)$. In the general case the proof is quite similar. Let $W(\alpha, \beta, \gamma)$ be a word representing an element ξ in $\pi = G * \mathbf{Z}$. (γ denotes a generator of $\mathbf{Z}$.) Claim: $\pi/(\xi) \neq \{1\}$.

The statement is trivial unless the sum of exponents of γ in $W(\alpha, \beta, \gamma)$ is equal to ± 1. (Look at the abelianized group of $\pi/(\xi)$.) We can then assume that the exponent-sum of γ in $W(\alpha, \beta, \gamma)$ is $+1$. Regard $W(A, B, X) = E$, where $A = \rho(\alpha)$ and $B = \rho(\beta)$ as an equation in $U(5)$ for the unknown matrix X. (E denotes the unit matrix of $U(5)$.) Following M. Gerstenhaber and O. Rothaus, consider the mapping $w: U(5) \to U(5)$ defined by $w(X) = W(A, B, X)$. (Compare *Proc. Nat. Acad. Sci.*, *48* (1962), 1531–1533. I am very grateful to W. Magnus and G. Baumslag for pointing out this paper to me.) Since $U(5)$ is connected, w is homotopic to the mapping $w_0: U(5) \to U(5)$ defined by

$$w_0(X) = W(E, E, X).$$

Since X enters into $W(A, B, X)$ with exponent-sum 1, we have

$$W(E, E, X) = X.$$

Hence, w_0 is the identity mapping, and w must be a surjective map since $U(5)$ is a finite dimensional manifold. In other words, the equation $W(A, B, X) = E$ has a solution $C \in U(5)$, i.e., $W(A, B, C) = E$. Now, let $\pi_* \subset U(5)$ be the subgroup of $U(5)$ generated by A, B, C. Then $\pi_* \neq \{1\}$ and $\pi/(\xi)$ maps surjectively onto π_* by $\rho_*: \pi/(\xi) \to U(5)$ defined by $\rho_*(\alpha) = A$, $\rho_*(\beta) = B$, $\rho_*(\gamma) = C$. It follows that $\pi/(\xi) \neq \{1\}$. Since this argument applies to any group element $\xi \in \pi$, the group π does not satisfy (ii).

and let $\alpha: S^1 \times S^1 \to S^1 \times S^1$ be the corresponding automorphism of the torus:

$$\alpha(z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d),$$

where

$$A = \left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\|.$$

$(a, b, c, d \in \mathbf{Z})$. Let E_A be the closed orientable 3-manifold obtained from $S^1 \times S^1 \times I$ by identifying $S^1 \times S^1 \times (0)$ with $S^1 \times S^1 \times (1)$ using α. Then E_A is the total space of a (locally trivial) fibration over S^1 with fibre $S^1 \times S^1$. It follows that $\pi_2 E_A = 0$. Since $\alpha(1, 1) = (1, 1)$, we can define a section $S^1 \to E_A$ by mapping each $z \in S^1$ into the point $(1, 1)$ of the fibre over z. This defines an element $\tau \in \pi_1(E_A, x_0)$, where x_0 is the point $(1, 1) \in S^1 \times S^1$ of the fibre over $1 \in S^1$. Let ξ, η be the generators of the fundamental group of the fibre over $1 \in S^1$ represented by $\xi(t) = (e^{2\pi i t}, 1)$ and $\eta(t) = (1, e^{2\pi i t})$. The group $\pi_1(E_A, x_0)$ is generated by τ, ξ, η and has the presentation $(\tau, \xi, \eta; [\xi, \eta] = 1,\ \tau^{-1}\xi\tau = \xi^a\eta^b,\ \tau^{-1}\eta\tau = \xi^c\eta^d)$. Hence, denoting $\pi_1(E_A, x_0)$ by π, we have $\pi/\pi' \cong \mathbf{Z}$ if and only if $(a-1)(d-1) - bc = \pm 1$. Then, $H_2(E_A) \cong \mathbf{Z}$ by Poincaré duality, and since $\pi_2(E_A) = 0$, it follows that

$$H_2(\pi) = H_2(E_A)/\rho\pi_2(E_A) \cong \mathbf{Z}.$$

Moreover, if $(a-1)(d-1) - bc = \pm 1$, the normal closure of τ in π is equal to π.

A simple example of a unimodular matrix satisfying the above requirements is

$$\left\| \begin{matrix} 1 & 1 \\ -1 & 0 \end{matrix} \right\|$$

The corresponding group, satisfying (i), (ii) but not (iii), has the presentation $(\tau, \xi, \eta; \xi^{-1}\eta^{-1}\xi\eta = 1, \tau^{-1}\xi\tau = \xi\eta, \tau^{-1}\eta\tau = \xi^{-1})$.

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