Geometric and algebraic intersection numbers

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Let M be a connected $2n$-dimensional differential manifold, not necessarily compact. Let $x_0 \in M$ be a base point, and $\alpha \in \pi_n(M, x_0)$ a given homotopy class. It is well known that, unless M is simply connected, there need not exist any differentiable imbedding $\varphi : S^n \to M$ representing α.

Let $\overline{M}$ be the universal cover of M provided with an arbitrary but fixed orientation, and let $a \in H_n(\overline{M})$ be the (integral) homology class of a lifting of α.

Theorem 1. *Assuming $n > 2$, the class $\alpha \in \pi_n(M, x_0)$ is representable by a differentiable imbedding $\varphi : S^n \to M^{2n}$ if and only if for every covering transformation $\tau \neq 1$ of $\overline{M}$ the homology intersection number $a \cdot \tau(a)$ vanishes.*

If M is oriented, one can define a scalar product

$$H_q(\overline{M}) \otimes H_{m-q}(\overline{M}) \to Z[\pi]$$

with values in the integral group ring of $\pi = \pi_1(M, x_0)$. (Cf. K. Reidemeister [2] and J. Milnor [1].) Here $m = \dim M = \dim \overline{M}$ need not be even, and we assume that the projection map $p : \overline{M} \to M$ is orientation preserving. The image of $x \otimes y$ under the above pairing will be denoted as in Milnor [1] by $[x, y]$.

In terms of this scalar product Theorem 1 can be formulated as follows:

Theorem 1'. *Let M^{2n} be connected and oriented. Assuming that $n > 2$, the class $\alpha \in \pi_n(M, x_0)$ is representable by a differentiable imbedding $\varphi : S^n \to M^{2n}$ if and only if*

$$[a, a] - a \cdot a = 0 .$$

The proof is given in §1 and §2. In §3 we give conditions under which two imbeddings $\varphi : X^q \to M^m$ and $\psi : Y^{m-q} \to M^m$ representing the homology classes α, β respectively are diffeotopic to imbeddings φ_0, ψ_0 such that the cardinality of the set $\varphi_0(X^q) \cap \psi_0(Y^{m-q})$ equals the absolute value $|\alpha \cdot \beta|$ of the homology intersection number $\alpha \cdot \beta$. (Cf. Theorem 2 below.)

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The proofs of Theorem 1 and Theorem 2 depend on the following well known lemma, essentially due to H. WHITNEY.

Let B^r denote the open unit ball in R^r.

Lemma. *Let V^m be a differential manifold, not necessarily compact, and let $\varphi : B^q \to V^m$ and $\psi : B^{m-q} \to V^m$ be two differentiable imbeddings such that $\varphi(B^q)$ and $\psi(B^{m-q})$ intersect transversally at exactly two points $R = \varphi(P) = \psi(Q)$ and $R' = \varphi(P') = \psi(Q')$.*

Suppose that

(i) *both q and $m - q$ are larger than 2,*

(ii) *if $u : I \to B^q$ and $v : I \to B^{m-q}$ are paths from P to P' and Q to Q' respectively, then the loop $\varphi(u) \cdot \psi(v^{-1})$ is freely homotopic in V to a constant loop,*

(iii) *with respect to some orientation of a neighborhood of $\varphi(B^q) \cup \psi(B^{m-q})$ in V the intersection coefficients of $\varphi(B^q)$ and $\psi(B^{m-q})$ at R and R' are opposite,* i. e. $\varphi(B^q) \cdot \psi(B^{m-q}) = 0$.

Then there exists a diffeotopy $\varphi_t : B^q \to V^m$ such that $\varphi_1 = \varphi$, $\varphi_t(x)$ is independent of t for $|x| > 1 - \varepsilon$ for some positive ε and $\varphi_0(B^q) \cap \psi(B^{m-q}) = \emptyset$.

For a proof, see [3] and [4].

§ 1. Proof of Theorem 1

It is easy to see that $a \cdot \tau(a) = 0$ for all $\tau \neq 1$ is a necessary condition for the existence of an imbedding $\varphi : S^n \to M^{2n}$ representing $\alpha \in \pi_n(M, x_0)$. (Recall that a denotes the homology class of a lifting of α in the universal cover $\overline{M}$ of M.) For let $\varphi : S^n \to M^{2n}$ be a *mapping* representing α, and $f : S^n \to \overline{M}$ a lifting of φ. Let $\tau : \overline{M} \to \overline{M}$ be a covering transformation, $\tau \neq 1$. We show that if P is a point in $f(S^n) \cap \tau f(S^n)$ then φ is not an imbedding. Let $Q = \tau^{-1} P \in f(S^n)$. Since $\tau \neq 1$, we have $Q \neq P$. Choose $Q', P' \in S^n$ such that $f(Q') = Q$ and $f(P') = P$. Then $Q' \neq P'$ but $\varphi(Q') = \varphi(P')$ since f is a lifting of φ. Hence φ is not bijective. Now, if φ is an *imbedding*, it follows that $f(S^n) \cap \tau f(S^n) = \emptyset$ for every $\tau \neq 1$, and a fortiori $a \cdot \tau(a) = 0$.

Conversely, suppose that $a \cdot \tau(a) = 0$ for every $\tau \neq 1$. Since $\overline{M}$ is simply connected and $n > 2$, WHITNEY's lemma (cf. introduction) implies that a can be represented by a differentiable imbedding $f : S^n \to \overline{M}$. The projection map $p : \overline{M} \to M$ is an immersion. Hence f projects to an immersion $p \circ f = \varphi : S^n \to M$. We may assume without loss of generality that $\varphi(S^n)$ intersects itself transversally in a finite number of points where only two sheets of $\varphi(S^n)$ cross each other. In other words, we may assume that φ is a completely regular immersion in the sense of [4]. (This can be obtained by an arbitrarily close approximation to φ in the C^2-topology so that the new φ still lifts to an imbedding.)

Let S_φ be the set of pairs of (distinct) points (P, Q) on S^n such that $\varphi(P) = \varphi(Q)$. If $S_\varphi \neq \emptyset$, pick a pair $(P, Q) \in S$. *Claim*: There exists another pair $(P', Q') \in S_\varphi$ and φ is regularly homotopic to an immersion $\Phi : S^n \to M$ such that

(i) any lifting $F : S^n \to \overline{M}$ of Φ is an imbedding,
(ii) $S_\Phi = S_\varphi - \{(P, Q), (P', Q')\}$,
(iii) Φ and φ coincide outside some neighborhood of a path joining P to P' on S^n.

This will prove the theorem by induction on the number of self-intersection points.

We now prove the claim. Since $\varphi(P) = \varphi(Q)$ and $f(P) \neq f(Q)$, there exists a covering transformation $\tau \neq 1$ such that $f(P) = \tau f(Q) = A$, say. The point A is a transversal intersection point of $f(S^n)$ and $\tau f(S^n)$. Let $\varepsilon (\varepsilon = \pm 1)$ be the intersection coefficient. Since $a \cdot \tau(a) = f(S^n) \cdot \tau f(S^n) = 0$ by assumption, there exists another intersection point A' of $f(S^n)$ and $\tau f(S^n)$ with intersection coefficient $-\varepsilon$. Let P', $Q' \in S^n$ be such that $f(P') = \tau f(Q') = A'$, and let $u : I \to S^n$ be a path on S^n from P to P', and similarly $v : I \to S^n$ a path on S^n from Q to Q' such that $u(I) \cap v(I) = \emptyset$. We may assume moreover that $u(I)$ and $v(I)$ are disjoint from the points of the pairs in S_φ except for $u(0) = P, u(1) = P'$, $v(0) = Q$ and $v(1) = Q'$. Then $\varphi u(I)$ and $\varphi v(I)$ intersect only at $\varphi u(0) = \varphi v(0)$ and $\varphi u(1) = \varphi v(1)$. Since $\overline{M}$ is simply connected, fu and $\tau f v$ are two homotopic paths from A to A'. Hence φu and φv are homotopic paths on M from $\varphi(P) = \varphi(Q) = R$ to $\varphi(P') = \varphi(Q') = R'$. Take disjoint open neighborhoods N_u and N_v of $u(I)$ and $v(I)$ respectively with diffeomorphisms $h_u : B^n \to N_u$ and $h_v : B^n \to N_v$, and such that P, Q, P', Q' are the only points from S_φ in $N_u \cup N_v$. Let $V = M - \varphi(S^n - N_u \cup N_v)$ and set $\varphi_1 = \varphi h_u | B^n$ and $\psi = \varphi h_v | B^n$. We are now in a position to apply WHITNEY's lemma. The loop $\varphi_1 h_u^{-1}(u) \cdot \psi h_v^{-1}(v^{-1})$ is homotopic to a constant loop in V because it is homotopic to a constant loop in M and the inclusion $V \to M$ induces an isomorphism $\pi_1 V \cong \pi_1 M$. Thus φ_1 is diffeotopic in V, relative to a neighborhood of the boundary of D^n, to an imbedding $\varphi_0 : B^n \to V$ such that $\varphi_0(B^n) \cap \psi(B^n) = \emptyset$. Define the immersion $\Phi : S^n \to M$ by

$$\Phi(x) = \begin{cases} \varphi_0 h_u^{-1}(x) & \text{if } x \in N_u, \text{ and} \\ \varphi(x) & \text{if } x \in S^n - N_u. \end{cases}$$

It is easily checked that Φ satisfies the conditions (ii) and (iii) of the above claim. To see that condition (i) stating that Φ lifts to an imbedding $F : S^n \to \bar{M}$ is also satisfied, let A, B be points on S^n with $F(A) = F(B)$. A path w from A to B on S^n maps to a loop Fw, and the loop Φw is homotopic to a constant loop in M. Since $\Phi(A) = \Phi(B)$ we have $\varphi(A) = \varphi(B)$ because the self-intersection points of Φ are self-intersection points of φ. In fact φw is also homotopic to a constant loop in M. (We may assume $\varphi w = \Phi w$ because, unless $A = B$, we have $A, B \in S^n - N_u$ and we can take $w(I) \subset S^n - N_u$.) But then, this means that $f(A) = f(B)$, and since f is an imbedding by construction, it follows that $A = B$. So F is bijective, and hence an imbedding. The proof of Theorem 1 is thus complete.

§ 2. The scalar product

Let M be a connected, oriented, differential manifold of dimension m, not necessarily even. Suppose M is triangulated as a regular cell complex. The triangulation of M lifts to a triangulation of $\bar{M}$ invariant under the covering transformations. We denote by ξ_i^q, $i = 1, \ldots, \alpha_q$ the q-cells of M and choose for each i a lifting x_i^q of ξ_i^q. Let η_i^{m-q} be the dual cell to ξ_i^q. ($\xi_i^q \cdot \eta_i^{m-q} = 1$.) A lifting y_j^{m-q} of η_j^{m-q} is then determined by the condition $x_i^q \cdot y_j^{m-q} = \delta_{ij}$. When we change the lifting x_i^q of ξ_i^q we demand that y_i^{m-q} be changed accordingly so that $x_i^q \cdot y_j^{m-q} = \delta_{ij}$ remains valid.

Let $\pi = \pi_1(M, x_0)$ be the fundamental group of M at x_0 which we identify with the group of covering transformations of $\bar{M}$. A q-dimensional chain x of (the triangulation of) $\bar{M}$ has a unique expression as $x = \Sigma_i \lambda_i x_i^q$, where $\lambda_i \in Z[\pi]$ and almost all λ_i's are zero. If $y = \Sigma_j \mu_j y_j^{m-q}$, define

$$[x, y] = [\Sigma_i \lambda_i x_i^q, \Sigma_j \mu_j y_j^{m-q}] = \Sigma_i \lambda_i \bar{\mu}_i ,$$

where $\mu \to \bar{\mu}$ is the anti-ringhomomorphism of $Z[\pi]$ onto itself determined by $\tau \to \tau^{-1}$ for $\tau \in \pi$.

Theorem. *The scalar product $[x, y]$ is independent of the choice of the liftings x_i^q and induces a pairing*

$$H_q \bar{M} \otimes H_{m-q} \bar{M} \to Z[\pi] .$$

The first statement follows by an easy calculation, using $\overline{\lambda \cdot \mu} = \bar{\mu} \cdot \bar{\lambda}$. The bilinearity of the product is obvious, and the second statement follows from the formula $[x, dy] = \pm [dx, y]$, where $\dim x + \dim y = m + 1$, and d is the boundary $Z[\pi]$-homomorphism. (Compare [1].)

We now derive a formula which will yield a translation of the condition "$a \cdot \tau(a) = 0$ for all $\tau \neq 1$" in terms of the scalar product.

Let $a_0 = \Sigma_i \alpha_i x_i^q$, $\alpha_i \in Z[\pi]$, be a representative q-chain for $a \in H_q \overline{M}$ and $b_0 = \Sigma_j \beta_j y_j^{m-q}$ a representative of $b \in H_{m-q} \overline{M}$ in terms of the dual subdivision of $\overline{M}$. Then,

$$[a_0, b_0] = \Sigma_i \alpha_i \overline{\beta}_i .$$

We calculate $a_0 \cdot \tau(b_0)$, writing x_i and y_j for x_i^q and y_j^{m-q} respectively to simplify the notation. We have $\alpha_i = \Sigma_{\varrho \in \pi} a_{i,\varrho} \varrho$ and $\beta_j = \Sigma_{\varrho \in \pi} b_{j,\varrho} \sigma$, where $a_{i,\varrho}, b_{j,\sigma} \in Z[\pi]$ are almost all 0. Then,

$$a_0 \cdot \tau(b_0) = (\Sigma_{i,\varrho} a_{i,\varrho} \varrho x_i) \cdot \tau(\Sigma_{j,\sigma} b_{j,\sigma} \sigma y_j) = \Sigma_{i,\varrho} a_{i,\varrho} b_{i,\tau^{-1}\varrho} ,$$

and

$$\Sigma_i \alpha_i \overline{\beta}_i = \Sigma_{i,\varrho,\sigma} a_{i,\varrho} b_{i,\sigma} \varrho \sigma^{-1} = \Sigma_\tau (\Sigma_{i,\varrho} a_{i,\varrho} b_{i,\tau^{-1}\varrho}) \tau .$$

In other words, the integer $a_0 \cdot \tau(b_0)$ is just the coefficient of τ in the scalar product $[a_0, b_0]$. In formula,

$$(*) \qquad [a, b] = \Sigma_{\tau \in \pi} (a \cdot \tau(b)) \tau .$$

(Compare Milnor [1] where (*) is taken as definition of $[a, b]$.)

Now, if $m = 2n$, $q = n$ and $a = b$, it follows that the condition "$a \cdot \tau(a) = 0$ for all $\tau \neq 1$" is equivalent to

$$[a, a] - a \cdot a = 0 .$$

This proves Theorem 1′.

Remarks. (1) If we replace a by τa in Theorem 1′, $[a, a] - a \cdot a$ becomes $\tau([a, a] - a \cdot a)\tau^{-1}$. Hence, only the conjugacy class of $[a, a] - a \cdot a$ is well determined by α. It seems therefore adequate to choose once and for all a base point $z_0 \in \overline{M}$ above $x_0 \in M$ and require all liftings to be liftings at z_0. We then have a function

$$Q : \pi_n(M, x_0) \to Z[\pi] ,$$

$Q(\alpha) = [a, a] - a \cdot a$, and α is representable by an imbedded n-sphere if and only if $Q(\alpha) = 0$. The function Q is somewhat like a quadratic form. Let

$\langle\alpha,\beta\rangle = [a,b] + (-1)^n\overline{[a,b]}$ and let λ_1 denote the coefficient of $1\in\pi$ in $\lambda\in Z[\pi]$. Then

$$Q(\tau\alpha) = \tau Q(\alpha)\bar{\tau} \quad (\tau\in\pi)\,,$$

$$Q(\alpha+\beta) = Q(\alpha) + Q(\beta) + B(\alpha,\beta)\,,$$

where $B(\alpha,\beta) = \langle\alpha,\beta\rangle - \langle\alpha,\beta\rangle_1$ is bilinear and symmetric.

(2) We have proved a slightly stronger statement than Theorem 1′. A class $\alpha\in H_n M^{2n}$ is representable by an imbedded manifold $A^n \subset M^{2n}$ such that $\pi_1 A \to \pi_1 M$ is trivial if and only if α is the projection of a class $a\in H_n(\overline{M})$ which is representable by a submanifold $B^n \subset \overline{M}$ and $[a,a] - a\cdot a = 0$. This gives a hint for the study of the intersection of submanifolds of M in §3.

Example. Take M^{2n} to be the connected sum of $S^1\times S^{2n-1}$ with k copies of $S^n\times S^n$. Let $x_i, y_i\in\pi_n(M, x_0)$ be represented by $S^n\times$ (point) and (point) $\times S^n$ in the i-th copy of $S^n\times S^n$ (suitably joined to the base point x_0). Then, $\pi_n(M, x_0)$ is the free $Z[J]$-module generated by $x_1,\ldots,x_k$, $y_1,\ldots,y_k$, where J denotes the (multiplicative) infinite cyclic group. If $\alpha = \Sigma_i\alpha_i x_i + \Sigma_j\beta_j y_j\in\pi_n(M, x_0)$ is a given homotopy class, where α_i, $\beta_j\in Z[J]$, it follows from the formula (*) above that

$$Q(\alpha) = \Sigma_i(\alpha_i\bar{\beta}_i + (-1)^n\beta_i\bar{\alpha}_i) - \Sigma_i(\alpha_i\bar{\beta}_i + (-1)^n\beta_i\bar{\alpha}_i)^0\,,$$

where λ^0 is the integer obtained by substituting the value 1 for every element of J in $\lambda\in Z[J]$.

For instance, if we let t denote a generator of J, the class $x + (t + t^{-1})y$ in $\pi_n M$, where $M = S^1\times S^{2n-1} \# S^n\times S^n$, is representable by a differentiable imbedding of S^n into M if n is odd, but is not representable if n is even. The same statement holds for $x + (t - t^{-1})y$ interverting even and odd.

§ 3. Reducing the geometric intersection of submanifolds

Let $\varphi : X^q \to M^m$ and $\psi : Y^{m-q} \to M^m$ be differentiable immersions, resp. imbeddings, where X, Y, M are connected differential manifolds. We assume X, Y to be compact, without boundary.

Roughly speaking, the problem is to use a deformation of φ so as to reduce the intersection $\varphi(X)\cap\psi(Y)$ to consist of a number of points equal to the algebraic intersection number of $\varphi(X)$ and $\psi(Y)$.

We assume that M is oriented. There are then two cases:

Case 1. X and Y are oriented. Denoting by $\alpha \in H_q M$ and $\beta \in H_{m-q} M$ the integral homology classes represented by $\varphi : X^q \to M$ and $\psi : Y^{m-q} \to M$ respectively, the algebraic (homology) intersection number $\alpha \cdot \beta$ is then an integer.

Case 2. At least one of the manifolds X, Y is non-orientable. Then $\varphi : X^q \to M$ and $\psi : Y^{m-q} \to M$ still represent mod 2 homology classes $\alpha \in H_q(M; Z_2)$ and $\beta \in H_{m-q}(M; Z_2)$. In this case the intersection number $\alpha \cdot \beta$ is an integer modulo 2.

We use $|\alpha \cdot \beta|$ to mean the absolute value in Case 1. In Case 2, $|\alpha \cdot \beta|$ is the *integer* 0 or 1 depending on whether $\alpha \cdot \beta = 0 \in Z_2$ or $\alpha \cdot \beta = 1 \in Z_2$.

In either case we shall assume that φ and ψ satisfy the following hypothesis:

(H) *The induced homomorphisms* $\varphi_* : \pi_1 X \to \pi_1 M$ *and* $\psi_* : \pi_1 Y \to \pi_1 M$ *are trivial.*

It follows that φ and ψ can be lifted to differentiable immersions, resp. imbeddings $f : X^q \to \overline{M}$ and $g : Y^{m-q} \to \overline{M}$, where as before $\overline{M}$ is the universal cover of M. We let a and b denote the homology classes represented by f and g. In Case 1, $a \in H_q \overline{M}$ and $b \in H_{m-q} \overline{M}$. In Case 2, $a \in H_q(\overline{M}; Z_2)$ and $b \in H_{m-q}(\overline{M}; Z_2)$.

We also assume

$$(H')\; q > 2 \quad \textit{and} \quad m - q > 2\,.$$

Finally, we use the following notation. If $\lambda \in Z[\pi]$, $\lambda = \Sigma_\tau n_\tau \tau$, set $w\lambda = \Sigma_\tau |n_\tau|$.

Theorem 2. *With the above notations and hypotheses, including* (H) *and* (H'), *the immersion, resp. imbedding* $\varphi : X^q \to M^m$ *is regularly homotopic, resp. diffeotopic to an immersion, resp. imbedding* $\varphi_0 : X^q \to M^m$ *such that* $\varphi_0(X) \cap \psi(Y)$ *consists of* $|\alpha \cdot \beta|$ *points if and only if*

in Case 1, $w[a, b] - |\alpha \cdot \beta| = 0$,

in Case 2, $[a, b] - a \cdot b = 0$ *in* $Z_2[\pi]$ *for some liftings* a, b *of* α, β.

Remark. Observe that

$$\alpha \cdot \beta = \Sigma_{\tau \in \pi} a \cdot \tau(b)\,.$$

Thus, in view of (*), the equation $w[a, b] - |\alpha \cdot \beta| = 0$ in Case 1 is equivalent to the statement that $a \cdot \tau(b)$ does not change sign as τ runs over π. (More precisely, $(a \cdot \sigma b)(a \cdot \tau b) \geqq 0$ for all $\sigma, \tau \in \pi$.) Obviously the condition is independent of the choice of liftings.

An important special case in practice seems to be Case 1 when $|\alpha \cdot \beta| = 0$ or 1. Then, $w[a, b] - |\alpha \cdot \beta| = 0$ is equivalent to the nicer looking condition

$$[a, b] - a \cdot b = 0 \text{ for some liftings } a, b \text{ of } \alpha, \beta.$$

(However, in general this last condition is definitely stronger than the former.)

As an illustration to the theorem, let $M = S^1 \times S^{2n-1} \# S^n \times S^n$, and $x, y \in \pi_n(M, x_0)$ be the elements represented by $S^n \times Q$ and $P \times S^n$ respectively, with $x_0 = P \times Q$. Let $u, v \in \pi_n(M, x_0)$ be given elements and A be the 2 by 2 matrix over $Z[J]$ such that $\binom{u}{v} = A\binom{x}{y}$, where J is the multiplicative infinite cyclic group.

Suppose that $u \cdot v = 1$, and let I_n denote the matrix

$$I_n = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

The classes u and v can be represented by imbedded spheres with just one intersection point if and only if

$$I_n A I_n^* A^* = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}$$

for some integer r. (t denotes a generator of J and A^* is the conjugate transposed of $A = (a_{ij})$, i. e. $A^* = (a_{ij}^*)$, where $a_{ij}^* = \bar{a}_{ji}$.)

Proof of Theorem 2. We may assume that $\varphi(X)$ and $\psi(Y)$ intersect transversally in a finite number of points $S_1, \ldots, S_k$ and $\varphi^{-1}(S_i), \psi^{-1}(S_i)$ each consists of a single point for every $i = 1, \ldots, k$. Let $f: X \to \overline{M}$ and $g: Y \to \overline{M}$ be arbitrary liftings of φ and ψ respectively. (Hypothesis (H).)

Case 1. The manifolds X and Y are oriented. If $\tau \in \pi$, we have $f(X) \cdot \tau g(Y) = \varepsilon_\tau |a \cdot \tau(b)|$ for some $\varepsilon_\tau = \pm 1$. For each $\tau \in \pi$ we can select intersection points $R_{\tau,j}, j = 1, \ldots, |a \cdot \tau(b)|$, of $f(X)$ and $\tau g(Y)$ so that the intersection coefficient of $f(X)$ and $\tau g(Y)$ at $R_{\tau,j}$ is equal to ε_τ, and thus independent of j. (If $a \cdot \tau(b) = 0$, the set $\{R_{\tau,j}\}$ is empty.) Let $(P_{\tau,j}, Q_{\tau,j}) \in X \times Y$ be the uniquely determined pair such that $f(P_{\tau,j}) = \tau g(Q_{\tau,j}) = R_{\tau,j}$.

Now, let $(P, Q) \in X \times Y$ be such that $\varphi(P) = \psi(Q)$ and $P \neq P_{\tau,j}$ for all τ, j, if any such pair exists. Then, there exists a covering transformation $\sigma \in \pi$ such that $f(P) = \sigma g(Q) = R$, say, and $R \neq R_{\sigma,i}$ for all $i \cdot (1 \leqq i \leqq |a \cdot \sigma b|.)$ Actually $R \neq R_{\tau,j}$ for all τ, j. Since $f(X) \cdot \sigma g(Y) = \varepsilon_\sigma |a \cdot \sigma(b)|$, there must exist another pair $(P', Q') \in X \times Y$ with $f(P') = \sigma g(Q') = R'$, where $R' \neq R_{\sigma,i}$ for all i and the intersection coefficients of $f(X)$ and $\sigma g(Y)$ at

R and R' are opposite. Again, since $\psi^{-1}(pR')$ consists of a single point, we actually have $R' \neq R_{\tau,j}$ for all τ, j.

We choose a path $u: I \to X$ from P to P' such that $u(I)$ and some neighborhood N_u of $u(I)$ in X are disjoint from any other φ-preimage of an intersection point of $\varphi(X)$ and $\psi(Y)$. Using WHITNEY's lemma as in §1, we can eliminate the intersection points $\varphi(P) = \psi(Q)$ and $\varphi(P') = \psi(Q')$ by a diffeotopy of $\varphi | N_u$ keeping φ fixed near the boundary of N_u.

It follows that φ is always (i. e. without condition on $[a, b]$) regularly homotopic, resp. diffeotopic, to an immersion, resp. an imbedding φ_0 such that

$$|\varphi_0(X) \frown \psi(Y)| = \Sigma_\tau |a \cdot \tau(b)| ,$$

where $|\varphi_0(X) \frown \psi(Y)|$ denotes the cardinality of the finite set $\varphi_0(X) \frown \psi(Y)$.

If $a \cdot \tau(b)$ does not change sign, we then have

$$|\varphi_0(X) \frown \psi(Y)| = |\Sigma_\tau a \cdot \tau(b)| = |\alpha \cdot \beta| .$$

Conversely, if $|\varphi_0(X) \frown \psi(Y)| = |\alpha \cdot \beta|$, then $|\Sigma_\tau a \cdot \tau(b)| = \Sigma_\tau |a \cdot \tau(b)|$, and it follows that $a \cdot \tau(b) = \varepsilon |a \cdot \tau(b)|$ for all τ, where $\varepsilon = \pm 1$ is independent of τ.

Case 2. The manifold X, say, is non-orientable. Then, $a \in H_q(\overline{M}; Z_2)$ and we also regard b as a class in $H_{m-q}(\overline{M}; Z_2)$. For those $\tau \in \pi$ such that $a \cdot \tau(b) \neq 0$ (mod 2), we choose an intersection point R_τ of $f(X)$ and $\tau g(Y)$, and let $(P_\tau, Q_\tau) \in X \times Y$ be the uniquely determined pair such that $f(P_\tau) = \tau g(Q_\tau) = R_\tau$. Let $(P, Q) \in X \times Y$ be a pair such that $\varphi(P) = \psi(Q)$ and $P \neq P_\tau$ for all $\tau \in \pi$. Then $f(P) = \sigma g(Q) = R$, say, for some $\sigma \in \pi$, and since either P_σ does not exist or $P \neq P_\sigma$, there exist another pair (P', Q') with $f(P') = \sigma g(Q') = R'$, say, where $P' \neq P_\tau$ for all $\tau \in \pi$. Let v be a path on Y from Q to Q' such that $v(I) \frown \{Q_\tau\} = \emptyset$. Choose an orientation of a neighborhood N_v of $v(I)$ in Y. Since X is non-orientable, there exists a path u in X from P to P' with $u(I) \frown \{P_\tau\} = \emptyset$, and an orientation of a neighborhood N_u of $u(I)$ in X such that the intersection number of $\varphi(N_u)$ and $\psi(N_v)$ is the integer 0. We can then use WHITNEY's lemma again, as in §1, and eliminate the pairs (P, Q) and (P', Q') from the set of intersection pairs by a diffeotopy of $\varphi | N_u$ keeping φ fixed near the boundary of N_u. Thus φ can be replaced by φ_0 such that

$$|\varphi_0(X) \frown \psi(Y)| = \Sigma_\tau |a \cdot \tau(b)| ,$$

where $|a \cdot \tau(b)|$ is the integer 0 if $a \cdot \tau(b) = 0$ mod 2 and the integer 1 if $a \cdot \tau(b) = 1$ mod 2.

If $[a, b] = a \cdot b$ in $Z_2[\pi]$ for some liftings a, b, then $a \cdot \tau(b) = 0$ for $\tau \neq 1$, and therefore, using these liftings in the above argument, $|\varphi_0(X) \frown \psi(Y)|$ is equal to 0 or 1 depending on whether $a \cdot b = 0$ or 1 mod 2 respectively. In other words, $|\varphi_0(X) \frown \psi(Y)| = |a \cdot b|$. Since $a \cdot \tau(b) = 0$ for $\tau \neq 1$ implies $\alpha \cdot \beta = \Sigma_\tau a \cdot \tau(b) = a \cdot b$, we have $|\varphi_0(X) \frown \psi(Y)| = |\alpha \cdot \beta|$ as desired.

Conversely, if $\varphi(X) \frown \psi(Y) = \emptyset$, we obviously have $[a, b] - a \cdot b = 0$ for any liftings of φ and ψ since both terms are then 0. If $|\varphi(X) \frown \psi(Y)| = 1$, we take a, b to be the classes of liftings of φ and ψ whose images intersect each other. Then $a \cdot \tau(b) = \delta_{\tau,1}$. Hence $[a, b] - a \cdot b = 0$ in this case too. This completes the proof of Theorem 2.

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