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ON THE JONES POLYNOMIAL

Swiss Seminar in Berne

by Pierre de la Harpe, Michel Kervaire and Claude Weber

These notes are based on the talks given in the seminar mentionned in the title, held at Berne University during the summer term 1986 and organized by the Troisième Cycle Romand de Mathématiques. However, we have made no attempt to follow faithfully the oral expositions. On the contrary, we have tried to reorganize the material in a unified survey with a streamlined point of view and (hopefully) coherent notations.

We thank all the participants who attended the seminar and above all our invited speakers, Vaughan Jones, Louis Kauffman and Hugh Morton. We have also included the results of Kunio Murasugi, although his talks were given the year before at the University of Geneva, when his work had just been completed.

Table of contents

§ 1. Introduction and historical remarks

Knot theory was born around the year 1867 in Scotland from the imagination of three physicists; two Scotsmen living in Edinburgh: J. C. Maxwell and P. G. Tait and one Irishman living in Glasgow: W. Thomson (Lord Kelvin). For more details, see [Kn].

The Transactions of the Royal Philosophical Society of Edinburgh provide ample testimony of the dedication and enthusiasm of these pioneers, trying to understand the structure of matter before quantum theory was invented, and knot theory without topological invariants.

According to Thomson's theory of vortex atoms, the chemical elements are constituted by small knots formed by the vortex lines of ether. For physical reasons, these knots have to be "kinetically stable", as Thomson and Tait said. In their opinion, this condition was going to prevent many knots from giving rise to vortex atoms.

Having this in mind, Tait embarked on a quite formidable program:

(1) Try to classify knots in 3-space;

(2) Try to establish a hierarchy among knots, relying on some notion of complexity;

(3) Understand why many of the simple knots cannot occur in vortex atoms (due to the stability condition).

In Tait's paper, this last point is stated as one of the main problems of the whole subject.

(4) Explain the position of the lines in the spectrum of a chemical element from the shape of the corresponding knot.

From an epistemological point of view, this program is remarkable: Thomson and Tait (T and T' as their friends used to call them) are looking for very complicated mathematical objects, in contrast with the attitude of many scientists trying to find a simple mathematical model when they attempt to explain a new area in the natural sciences.

If one reads between the lines in Tait's paper, one can guess that he started working on (1) and (2) full of the hope that it should not be too difficult. However, he was aware of the fact that he was opening an entirely new field and that surprises might well show up. Later on, he confessed that the subject was harder than he had expected...

During the elaboration of Tait's first paper, Maxwell told him about the work on knots by C. F. Gauss and J. B. Listing who had somewhat anticipated Tait's starting point: knot projections, alternating knots, chess-board.

As to Maxwell, his interest for knots came from his theory of electro-magnetism. For instance, he gave in [Ma] a lovely interpretation of Gauss integral formula for the linking coefficient of two knots in 3-space: it is (up to a factor) equal to the work required to move a magnet pole along one knot while the other knot is run by an electric current. This interpretation is repeated by Tait in [Tai]. One can see, along the way, Seifert surfaces being introduced by Tait via the following physical argument: If one has an orientable surface Σ in 3-space whose boundary is a given knot, and if one "magnetizes the surface normally and constantly", as Tait says, then the work required to move a magnet pole on another knot will be the same as if the boundary of Σ were run by an electric current. Tait thus uses the 2-chain given by Σ to compute linking coefficients.

Note. Today, G. de Rham himself says that he chose the terminology "courant" for similar reasons. The "courants", like homology, are dual to cohomology and one can think of 1-dimensional cycles as electric currents.

Tait thought of a knot as being a rubber band in everyday 3-dimensional space. Two positions of the band represent the same knot if one can deform one position of the band into the other. In modern terms, this is non-oriented ambient isotopy.

To measure the complexity of a knot, Tait introduced what he called the (degree of) knottiness. This is called today the crossing number of the knot. By definition, it is the minimal number of double points among all projections of the knot. We shall use the notation $c(K)$.

Tait also introduced the beknottedness, now called unknotting number. He did not use it very much to measure knot complexity because he soon realized that its determination was difficult. We shall not talk about this invariant in this paper, although the second integer which appears in the inductive proof of uniqueness of the polynomial $P_K(l, m)$ in § 3 is clearly related to it.

Tait's papers contain few proofs which are acceptable by the standards of 20-th century topology. They rely on principles, not always very explicitly stated, which seemed obvious to the author, but which are in fact unproved statements. Nowadays, knot theorists have more or less agreed on the meaning of these principles and have summarized them under the name of "Tait conjectures". They are all related to the minimal crossing number of a knot. (See § 9 in this paper.)

A paradox in the achievements of 3-dimensional topology between 1965 and 1985 is the following: Knot theory was gradually embodied in the more general theory of 3-dimensional manifolds. Classifications were attempted, and sometimes attained by using very refined geometrical tools such as the Waldhausen-Jaco-Shalen-Johanson theory on the embeddings of Seifert manifolds in a Haken manifold. And yet, these refined methods could not cope with simple questions related to knot projections. In fact, during this period, the old time point of view, using projections, was almost forgotten (except by a few people, for instance John Conway).

Today, Jones polynomials and more precisely L. Kauffman's very clever and very elementary way of looking at the one-variable polynomial $V_K(t)$ have put again knot projections under the spot-light. The one-variable polynomial is the main ingredient in the proofs of several of Tait's conjectures which have remained unproved for more than a century.

This paper is devoted to a presentation of these recent achievements, mainly due to V. Jones, L. Kauffman and K. Murasugi.

We shall give the definition and prove some of the properties of the two-variable Laurent polynomial $P(K) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ associated with every oriented link K. The approach chosen here is that of V. Jones and A. Ocneanu. Another approach which uses the notion of skein invariance is due independently to many mathematicians: P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millett, J. Prztycki and P. Traczyk. Although we do discuss skein invariance in this paper, we do not go into the question of using it to define the polynomial $P(K)$.

As many mathematicians have worked simultaneously on various aspects of the definition of the polynomial, it is difficult to give proper credit to everyone. We apologize in advance for any missing ascription. We hope all will agree that V. Jones has been the one pioneer who got the subject started.

§ 2. Link diagrams

A link K in S^3 (or R^3) is a 1-dimensional compact smooth manifold without boundary. We shall use $r = r(K)$ for the number of components of K. A knot is a link with one component.

Most of the time K will be oriented.

Two oriented links K, K' are *ambient isotopic* if there exists a diffeomorphism $h: S^3 \to S^3$ of degree $+1$, such that $h(K) = K'$ and $h_{|K}$ is also of degree $+1$ on each component.

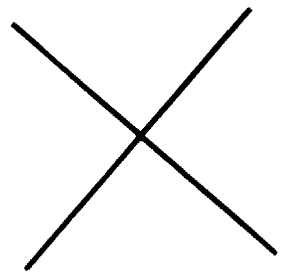
We will denote by $\mathscr{L}$ the set of ambient isotopy classes of oriented links.

A *link projection* is a generic immersion of a (finite) disjoint union of circles into the plane. (No triple point, transverse crossings only.)

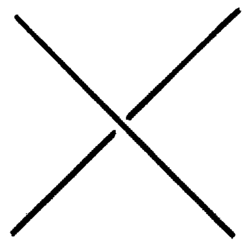
If $K \subset R^3$ is a link, an affine projection p of K on a plane $V \subset R^3$ gives a link projection if $p_{|K}$ is a generic immersion.

It is possible to recapture the isotopy class of K from the projection by specifying at each crossing point a choice of one of the two branches, singling out the branch which overcrosses the other.

A *link diagram* is a link projection together with such a choice of over/under crossing at each crossing point:



Part of a link projection



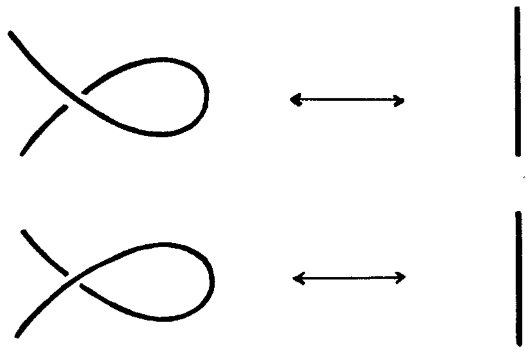
Part of a link diagram

A link diagram gives rise to a well defined ambient isotopy class of link in 3-space and every link isotopy class can be obtained in this way.

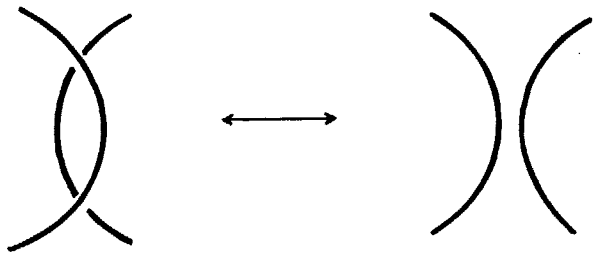
Of course many different link diagrams can give rise to isotopic links. This ambiguity is resolved by the notion of Reidemeister move:

The Reidemeister moves on link diagrams are the moves shown in the following pictures, for all possible orientations.

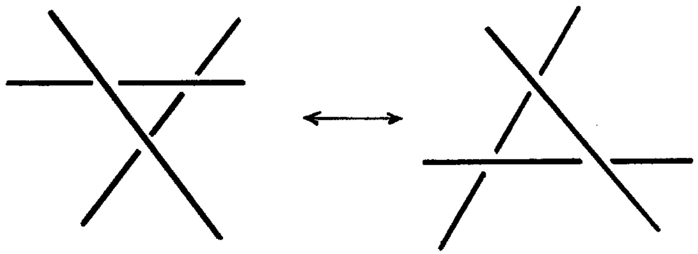
(i)



(ii)



(iii)



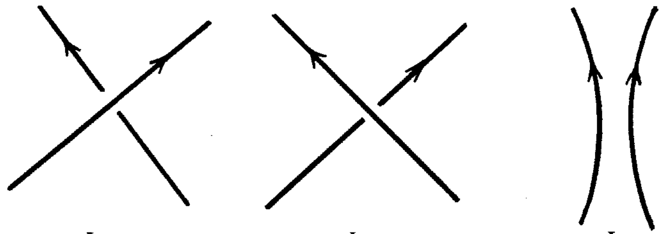
We shall take for granted without proof the classical

THEOREM. *Two oriented link diagrams represent ambient isotopic oriented links if and only if one can pass from one to the other by a finite sequence of Reidemeister moves.*

This theorem is the basis of combinatorial knot theory.

Another notion on link diagrams which will be of crucial importance in the sequel is that of *skein invariance*, due to J. Conway.

We say that 3 oriented links L_+, L_- and L_0 are *skein related* if they have diagrams which are identical except in the neighborhood of one crossing point where they look respectively as follows:



L_+ L_- L_0

Now, let $\mathscr{L}$ be the set of ambient isotopy classes of oriented links in R^3, and let A be a commutative ring. We say that a link invariant $P: \mathscr{L} \to A$ is a *linear skein invariant* if

(1) $P(\bigcirc) = 1$, where $\bigcirc$ denotes the 1-component unknot.

(2) There exist 3 invertible elements $a_+, a_-, a_0 \in A$ such that whenever L_+, L_-, L_0 are skein related, then $a_+P(L_+) + a_-P(L_-) + a_0P(L_0) = 0$.

Our objective is to define a skein invariant $P: \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ with values in the ring of Laurent polynomials in 2 variables l, m. (Standing perhaps for Lickorish and Millett.) The elements a_+, a_-, a_0 will be respectively $a_+ = l$, $a_- = l^{-1}$, $a_0 = m$.

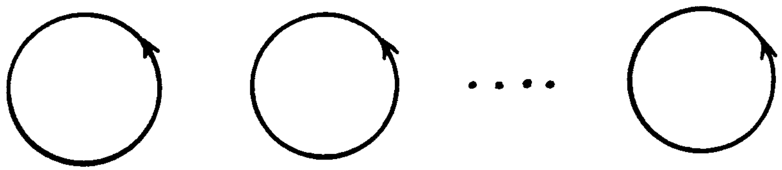
It will turn out that P is the universal linear skein invariant.

§ 3. Uniqueness and universality theorems

We prove:

Theorem 3.1. *If* $P : \mathscr{L} \to A$ *is a skein invariant, it is uniquely determined by the coefficients* a_+, a_- *and* a_0 *of the skein invariance relation.*

Proof. First note that $P(\bigcirc^r) = \left(-\frac{a_+ + a_-}{a_0}\right)^{r-1}$, where $\bigcirc^r$ denotes the unlink with r components.



(r circles)

Starting from $P(\bigcirc) = 1$, and the skein related link diagrams

L_+ L_- L_0

we see that

$$a_+ P(\bigcirc) + a_- P(\bigcirc) + a_0 P(\bigcirc^2) = 0 ,$$

and thus

$$P(\bigcirc^2) = -\frac{a_+ + a_-}{a_0} .$$

Adding $r - 1$ unknotted (and unlinked) disjoint components to each link in the above picture gives the desired formula by induction on r.

To prove the theorem, we shall use the following remark:

Lemma 3.2. *For every link projection, there is a choice of over/under crossing at each crossing point which produces the unlink* $\bigcirc^r$.

Proof. The projection is a regular immersion of a disjoint union of circles. Order these circles arbitrarily. Now, running along the images of the circles, one after the other, declare that each new crossing is an underpass. (This involves the choice of a starting point on each circle such that its image is not a crossing point.)

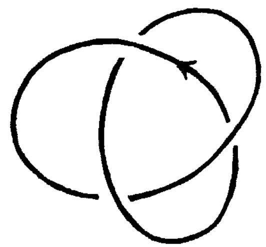
The link corresponding to this choice of crossings is $\bigcirc^r$. Indeed, it is clear that the various components are stacked, one above the other, in their chosen order, and are thus unlinked. Furthermore, it is easy to see that each component bounds a disk, and therefore is the unknot.

As a consequence of this lemma, if L is an arbitrary link diagram, there is a sequence of changes of over/under choices at the crossing points which carries the diagram into a diagram of the unlink $\bigcirc^r$ with the same number r of components.

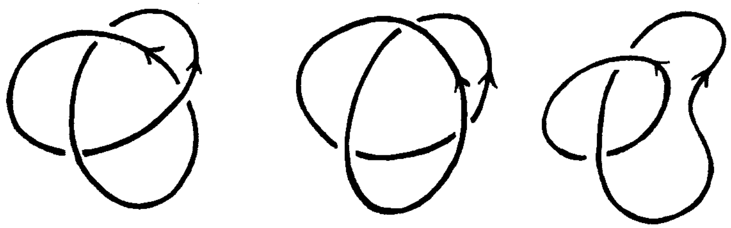
For each over/under change L_+, L_-, we get the first two terms of a skein relation (in a definite order). The third member L_0 of the skein related diagrams L_+, L_-, L_0 has one less crossing than L_+ and L_-. Thus, if we define the complexity of a link diagram to be the pair (N, S), where S is the number of crossing changes needed to get the unlink, and N is the number of crossings, and if we order the pairs by $(N, S) < (N', S')$ if $N < N'$ or $N = N'$ and $S < S'$ (alphabetical order), then we see by induction on the complexity that $P(L)$ is completely determined by $P(\bigcirc)$ and the skein invariance $a_+ P(L_+) + a_- P(L_-) + a_0 P(L_0) = 0$.

This proves theorem (3.1).

Example. Let T_+ be the right handed trefoil:



Use the skein relation

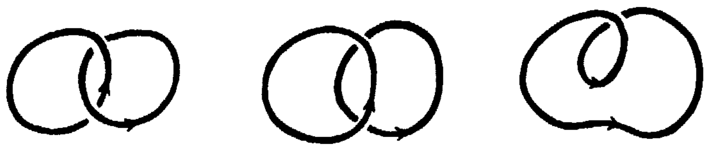


$T_+ = L_+$ $\qquad$ $L_- = \bigcirc$ $\qquad$ L_0

We get

$$a_+ P(T_+) + a_- + a_0 P(L_0) = 0\,.$$

Then, another use of a skein relation



$L_0^+ = L_0$ $\qquad$ $L_0^- = \bigcirc^2$ $\qquad$ $L_0^0 = \bigcirc$

yields the formula

$$a_+ P(L_0) + a_- \cdot \left(- \frac{a_+ + a_-}{a_0} \right) + a_0 = 0\,.$$

Solving for $P(T_+)$ in the two equations gives the result:

$$P(T_+) = -2a_- a_+^{-1} - a_-^2 a_+^{-2} + a_+^{-2} a_0^2\,.$$

Of course, we do not know yet if P calculated in this way is well defined. But if it is well defined, $P(T_+)$ must be given by the above formula.

Now, let $A = \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ be the ring of Laurent polynomials in 2 variables l, m. Suppose $P : \mathcal{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ is a skein invariant satisfying

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0$$

for any 3 skein related link diagrams L_+, L_- and L_0.

THEOREM 3.3. *If such a* P *exists it is universal in the following sense:*

(1) P *determines a unique skein invariant*

$$T : \mathcal{L} \to \mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$$

satisfying

$$xT(L_+) + yT(L_-) + zT(L_0) = 0$$

for every triple of skein related link diagrams L_+, L_- *and* L_0.

(2) *Moreover, if* $P_A : \mathcal{L} \to A$ *is any skein invariant with respect to three invertible elements* $a_+, a_-, a_0 \in A$ *as above, then* $s(T(K)) = P_A(K)$ *for all* $K \in \mathcal{L}$, *where* $s : \mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}] \to A$ *is the obvious map determined by* $s(x) = a_+$, $s(y) = a_-$ *and* $s(z) = a_0$.

For the proof of this theorem, the crucial fact is the following assertion.

LEMMA 3.4. *Let* $P : \mathcal{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ *be a skein invariant as above, i.e.* $P(\bigcirc) = 1$, *and* $lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0$, *if* L_+, L_- *and* L_0 *are skein related. Then, each monomial* $l^a m^b$ *occuring (with non-zero coefficient) in* $P(K)$ *satisfies*

$$a \equiv b \bmod 2\,.$$

The proof of this lemma will actually show that

$$a \equiv b \equiv r(K) - 1 \bmod 2$$

for each monomial $l^a m^b$ of $P(K)$, where $r(K)$ is the number of connected components of K.

Proof of the lemma. True for the unknot, and more generally for the unlink with r components, since

$$P(\bigcirc^r) = \left(- \frac{l + l^{-1}}{m} \right)^{r-1}$$

as we have seen earlier.

Now, suppose that L_+, L_- and L_0 are 3 skein related link diagrams. Then, we have

$$P(L_+) = -l^{-2}P(L_-) - l^{-1}mP(L_0).$$

Hence, the claim follows by induction on the complexity of the link diagram, observing that $r(L_+) = r(L_-) = r(L_0) \pm 1$.

This completes the proof of lemma 3.4.

Now, given the skein invariant $P: \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$, define $T: \mathscr{L} \to \mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$ by replacing each monomial $l^a m^b$ of $P(K)$ by $x^i y^j z^k$, where

$$k = b,$$
$$i + j + k = 0,$$
$$i - j = a,$$

i.e. $i = 1/2(a-b), j = -1/2(a+b), k = b$.

By the above assertion ($a \equiv b \bmod 2$), T is a Laurent polynomial in x, y, z.

Perhaps more explicitly, we have

$$T(x, y, z) = P((x/y)^{1/2}, z \cdot (xy)^{-1/2}).$$

Observe that T is homogeneous of degree 0. This certainly is a necessary condition for T to be a skein invariant. (Exercise!)

It is clear that $T(\bigcirc) = 1$. We have to verify that

$$xT(L_+) + yT(L_-) + zT(L_0) = 0,$$

if L_+, L_- and L_0 are skein related.

Substituting $(x/y)^{1/2}$ for l and $z \cdot (xy)^{-1/2}$ for m in the relation

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0,$$

we obtain

$$(x/y)^{1/2}T(L_+) + (y/x)^{1/2}T(L_-) + z(xy)^{-1/2}T(L_0) = 0$$

which yields the desired formula after multiplying by $(xy)^{1/2}$.

Further, if $P_A: \mathscr{L} \to A$ is any skein invariant (with respect to invertible elements a_+, a_-, a_0 of some commutative ring A) and if we define $s: \mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}] \to A$ by $s(x) = a_+$, $s(y) = a_-$, $s(z) = a_0$, then $s(T(L)) = P_A(L)$ follows for all link diagrams L by uniqueness, since both sT and P_A satisfy the same skein invariance with respect to a_+, a_-, a_0.

The existence of $P: \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ will be proved in § 6, after some preliminaries on Hecke algebras in the next two paragraphs.

§ 4. Hecke algebras

In this section we isolate the classical facts about Hecke algebras which we will need in the next two sections in order to prove the existence of P. The knowledgeable reader can thus skip this paragraph and proceed directly to § 5.

Let K be a field and let $q \in K$ be some element of K.

The Hecke algebra H_n over K corresponding to q is the associative K-algebra with unit 1, generated by $T_1, \ldots, T_{n-1}$ subject to the following relations

$$T_iT_j = T_jT_i \quad \text{whenever} \quad |i-j| \geqslant 2,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad \text{and}$$
$$T_i^2 = (q-1)T_i + q$$

for all $i, j \in \{1, \ldots, n-1\}$, with of course $i \leqslant n-2$ for the second family of relations.

We see that there is a natural map $H_n \to H_{n+1}$ of K-algebras which make H_{n+1} a (H_n, H_n)-bimodule. We think of $q \in K$ as being fixed once and for all.

Consider also the (H_n, H_n)-bimodule $H_n \oplus H_n \otimes_{H_{n-1}} H_n$.

Proposition 4.1. *There is a natural map of* (H_n, H_n)*-bimodules*

$$\varphi: H_n \oplus H_n \otimes_{H_{n-1}} H_n \to H_{n+1}$$

given by $\varphi(a + \Sigma_i b_i \otimes c_i) = a + \Sigma_i b_i T_n c_i$.

Moreover, φ *is an isomorphism.*

The proof of this proposition will occupy the remainder of this section. We have divided it into seven claims.

Claim 1. *The map* φ *is well defined.*

Proof. If $u \in H_{n-1}$, then

$$\varphi(bu \otimes c) = buT_nc, \quad \text{and} \quad \varphi(b \otimes uc) = bT_nuc.$$

But u is a K-linear combination of monomials in $T_1, \ldots, T_{n-2}$ which commute with T_n in H_{n+1}. Hence, $buT_nc = bT_nuc$, and so φ is well defined.

CLAIM 2. *The map* φ *is surjective.*

We have to show that H_{n+1} is generated as a vector space over K by the monomials with at most one occurence of T_n.

The proof will be by induction on n. Let M be a monomial in $T_1, \ldots, T_n$ with two occurences of T_n at least. Displaying two consecutive occurences of T_n in M, we write $M = M_1 T_n M_2 T_n M_3$, where we can assume that M_2 is a monomial in $T_1, \ldots, T_{n-1}$ only. Assume *by induction* that M_2 contains T_{n-1} at most once. If M_2 does not contain T_{n-1} at all, then

$$M = M_1 M_2 T_n^2 M_3 = (q-1) M_1 M_2 T_n M_3 + q M_1 M_2 M_3 ,$$

reducing the number of occurences of T_n in each new monomial. If M_2 contains T_{n-1} exactly once, we can write $M_2 = M' T_{n-1} M''$, with M', M'' monomials in $T_1, \ldots, T_{n-2}$ and then,

$$M = M_1 M' T_n T_{n-1} T_n M'' M_3 ,$$

using the fact that $T_1, \ldots, T_{n-2}$ commute with T_n. But now, $T_n T_{n-1} T_n = T_{n-1} T_n T_{n-1}$ yields

$$M = M_1 M' T_{n-1} T_n T_{n-1} M'' M_3 ,$$

reducing again the number of occurences of T_n.

Hence, every element of H_{n+1} is a sum $a + \Sigma_i b_i T_n c_i$ with a, b_i, c_i coming from H_n and it is now clear that φ is surjective.

CLAIM 3. *Monomials in normal form generate* H_{n+1} *over* K.

We have actually proved a little more than was stated in Claim 2. Consider the following lists of monomials:

$$S_1 = \{1, T_1\} ,$$
$$S_2 = \{1, T_2, T_2 T_1\} ,$$
$$S_3 = \{1, T_3, T_3 T_2, T_3 T_2 T_1\} ,$$
$$\ldots$$
$$S_i = \{1, T_i, T_i T_{i-1}, \ldots, T_i T_{i-1} \ldots T_1\} ,$$
$$\ldots$$
$$S_n = \{1, T_n, T_n T_{n-1}, \ldots, T_n T_{n-1} \ldots T_1\} .$$

Note the property that $V_i \in S_i$ implies $T_{i+1} V_i \in S_{i+1}$.

Consider the set of monomials $M = U_1 . U_2 . \ldots . U_n$ for all possible choices of $U_i \in S_i$, $i = 1, \ldots, n$. We shall say that these monomials are in *normal form*. There are $(n+1)!$ of them.

We claim that these monomials M generate H_{n+1} as a K-space. Consequently, $\dim_K H_{n+1} \leqslant (n+1)!$ and also $\dim_K \{H_n \oplus H_n \otimes H_n\} \leqslant (n+1)!$, where the tensor product is over H_{n-1} as above.

Proof. We may assume *by induction* that the claim holds for H_n. As H_{n+1} is generated over K by monomials M_0 and $M = M_1 T_n M_2$, where M_0, M_1, M_2 are monomials in $T_1, \ldots, T_{n-1}$, and as the induction hypothesis makes the case of M_0 clear, we concentrate on $M = M_1 T_n M_2$. By induction, M_2 is a K-linear combination of monomials of the form $V_1 . V_2 . \ldots . V_{n-1}$, with $V_i \in S_i$ for $i = 1, \ldots, n-1$. We have

$$M_1 T_n V_1 V_2 \ldots V_{n-1} = M'_1 T_n V_{n-1} = M'_1 U_n ,$$

with $U_n = T_n V_{n-1} \in S_n$. By induction again, M'_1 is a K-linear combination of monomials of the form $U_1 . U_2 . \ldots U_{n-1}$ with $U_i \in S_i$. Thus M is a K-linear combination of monomials $U_1 . U_2 . \ldots U_n$ as desired and $\dim_K H_{n+1} \leqslant (n+1)!$.

This shows also that $H_n \otimes_{H_{n-1}} H_n$ is spanned over K by the subspaces $H_n \otimes U_{n-1}$ with $U_{n-1} \in S_{n-1}$. Therefore, its K-dimension is at most $n! \, . \, n$, so that the proof of claim 3 is complete.

Remark. Let $\mathfrak{S}_{n+1}$ be the symmetric group on $\{1, \ldots, n+1\}$, and denote by s_i the transposition $(i, i+1)$. The same argument as above shows that any $\pi \in \mathfrak{S}_{n+1}$ can be written as a product $w_1 . w_2 \ldots . w_n$, where

$$w_i \in \{1, s_i, s_i s_{i-1}, \ldots, s_i s_{i-1} \ldots s_1\} .$$

We shall use this remark presently in the proof of the following claim 4.

Exercise. Deduce from the remark that $\mathfrak{S}_{n+1}$ has a presentation on generators $s_1, \ldots, s_n$ with the relations

$$s_i s_j = s_j s_i \quad \text{whenever} \quad |i-j| \geqslant 2 \quad \text{with} \quad i, j = 1, \ldots, n ,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for} \quad i = 1, \ldots, n-1 ,$$
$$s_i^2 = 1 \quad \text{for} \quad i = 1, \ldots, n .$$

CLAIM 4. *The monomials in normal form* $M = U_1 . U_2 \ldots . U_n$, *with* $U_i \in S_i$ *for* $i = 1, \ldots, n$ *are K-linearly independent. Also, the map* φ *is an isomorphism.*

Proof. Denote by $l: \mathfrak{S}_{n+1} \to N$ the word length in $\mathfrak{S}_{n+1}$, relative to the generators $\{s_1, s_2, ..., s_n\}$. For $i \in \{1, ..., n\}$, define $L_i \in \mathrm{End}_K(K\mathfrak{S}_{n+1})$ by

$$L_i(\pi) = \begin{cases} s_i\pi & \text{if} \quad l(s_i\pi) > l(\pi), \\ qs_i\pi + (q-1)\pi & \text{if} \quad l(s_i\pi) < l(\pi), \end{cases}$$

for every $\pi \in \mathfrak{S}_{n+1}$.

The crucial fact is the following

ASSERTION. *There is an algebra map* $L: H_{n+1} \to \mathrm{End}_K(K\mathfrak{S}_{n+1})$ *such that* $L(T_i) = L_i$ *for* $i = 1, ..., n$.

To prove the assertion, we have to check that the endomorphisms $L_i \in \mathrm{End}_K(K\mathfrak{S}_{n+1})$ satisfy the defining relations of the Hecke algebra H_{n+1}. For this, see the following three claims.

Assuming the assertion, consider a monomial in normal form $M = U_1 . U_2 ... . U_n$ as above. Then, $L(M)$ maps $1 \in K\mathfrak{S}_{n+1}$ to $w_1 . w_2 ... . w_n$, where $w_i = s_i s_{i-1} ... s_{i-j}$ if $U_i = T_i T_{i-1} ... T_{i-j}$. The remark after claim 3 now shows that any of the $(n+1)!$ elements of $\mathfrak{S}_{n+1}$ is of the form $w_1 . w_2 ... . w_n$, so that these elements are K-linearly independent in $K\mathfrak{S}_{n+1}$. But, as the map from H_{n+1} to $K\mathfrak{S}_{n+1}$ which sends x to $L(x)(1)$ is K-linear, this implies that the elements $M = U_1 . U_2 ... . U_n$ in normal form must also be linearly independent. Hence, $\dim_K H_{n+1} = (n+1)!$.

Now, a dimension count shows that the surjective map φ is an isomorphism.

It remains to prove the above assertion: The L_i's satisfy the defining relations for H_{n+1}.

CLAIM 5. $L_i^2 = (q-1)L_i + q$ *for* $i = 1, ..., n$.

Proof. Let $\pi \in \mathfrak{S}_{n+1}$. If $l(s_i\pi) > l(\pi)$, then

$$L_i^2(\pi) = L_i(s_i\pi) = qs_i^2\pi + (q-1)s_i\pi$$
$$= (q-1)s_i\pi + q\pi = ((q-1)L_i + q)(\pi).$$

If on the other hand, $l(s_i\pi) < l(\pi)$, set $\pi' = s_i\pi$ and observe that $l(s_i\pi') > l(\pi')$. Thus,

$$L_i^2(\pi) = L_i(qs_i\pi + (q-1)\pi) = L_i(q\pi' + (q-1)\pi)$$
$$= qs_i\pi' + (q-1)L_i(\pi) = ((q-1)L_i + q)(\pi).$$

The next claim will be used in proving the last two types of relations for the endomorphisms L_i.

CLAIM 6. *For* $j = 1, ..., n$ *define* $R_j \in \mathrm{End}_K(K\mathfrak{S}_{n+1})$ *by*

$$R_j(\pi) = \begin{cases} \pi s_j & \textit{if} \quad l(\pi s_j) > l(\pi), \\ q\pi s_j + (q-1)\pi & \textit{if} \quad l(\pi s_j) < l(\pi). \end{cases}$$

Then, $L_i R_j = R_j L_i$ *for all* $i, j \in \{1, ..., n\}$.

Proof. Choose $i, j \in \{1, ..., n\}$ and $\pi \in \mathfrak{S}_{n+1}$. The proof that $L_i R_j(\pi) = R_j L_i(\pi)$ is by direct verification from the definitions of the operators L_i, R_j and is divided into six cases.

(6.1) $\quad l(s_i\pi s_j) = l(\pi) + 2$,

(6.2) $\quad l(s_i\pi s_j) = l(\pi) - 2$,

(6.3)-(6.6) $\quad l(s_i\pi s_j) = l(\pi)$ and
$l(s_i\pi) = l(\pi) + \varepsilon$, where $\varepsilon = \pm 1$,
$l(\pi s_j) = l(\pi) + \varepsilon'$, where $\varepsilon' = \pm 1$.

The first two cases are straightforward calculations.

Among the last four cases, two are also trivial, namely those with $\varepsilon \neq \varepsilon'$. There remain the two cases with $\varepsilon = \varepsilon' = \pm 1$. Then, the *exchange lemma* applied to the symmetric group viewed as a Coxeter group (on the generators $s_1, ..., s_n$) implies that in these cases we have $s_i\pi = \pi s_j$. (If $\varepsilon = \varepsilon' = +1$, this equality is given as property C in Bourbaki, Groupes et Algèbres de Lie, Chap. IV, n° 1.7. If $\varepsilon = \varepsilon' = -1$, the same property yields $s_i(\pi s_j) = (\pi s_j)s_j$.) This is just what is needed to complete the verification of $L_i R_j(\pi) = R_j L_i(\pi)$.

CLAIM 7. $L_i L_j = L_j L_i$ *whenever* $|i-j| \geqslant 2$,

$$L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}.$$

Proof. Let $\pi \in \mathfrak{S}_{n+1}$. Write $\pi = s_{i_1} . s_{i_2} ... . s_{i_r}$ in reduced form, i.e. with $r = l(\pi)$. We thus have $\pi = R_{i_r} R_{i_{r-1}} ... R_{i_1}(1)$.

Setting $R = R_{i_r} ... R_{i_1}$, we have

$$L_i L_j(\pi) = L_i L_j R(1) = R L_i L_j(1) \quad \text{by claim 6},$$
$$= R(s_i s_j) = R(s_j s_i) \quad \text{since} \quad |i-j| \geqslant 2, \quad \text{and thus}$$
$$L_i L_j(\pi) = L_j L_i(\pi).$$

Since this holds for every $\pi \in \mathfrak{S}_{n+1}$, one has $L_i L_j = L_j L_i$.

A similar calculation, based on the same principle, proves that $L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}$ for $i = 1, ..., n-1$.

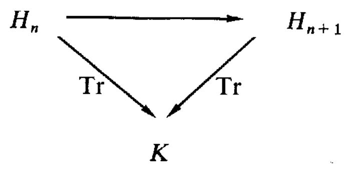
This completes the proof of Proposition 4.1.

§ 5. The trace

The fundamental idea of V. Jones which led him to the definition of his original one-variable polynomial is the construction of the trace. Originally, V. Jones used algebras which are quotients of the algebras H_n. The lifting of the trace to the Hecke algebras H_n was observed by A. Ocneanu.

The trace will commute with the inclusion $H_n \to H_{n+1}$ and therefore yield a trace on the direct limit of the H_n's. (Compare with the discussion in § 12.)

THEOREM. *Let K be a field and let $q, z \in K$ be two elements of K. Let H_n be the* Hecke *algebra over K corresponding to q. There exists a trace $\mathrm{Tr}: H_n \to K$ compatible with the inclusion $H_n \to H_{n+1}$,* i.e. *the diagram*



commutes, and such that

(1) $\mathrm{Tr}(1) = 1$,

(2) Tr *is K-linear and* $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$,

(3) *If $a, b \in H_n$, then* $\mathrm{Tr}(aT_n b) = z\mathrm{Tr}(ab)$.

Notice that the last property enables us to calculate $\mathrm{Tr}(x)$ for an arbitrary $x \in H_n$ by using the fact that monomials in normal form generate H_n over K. For instance,

$$\begin{aligned}\mathrm{Tr}(T_1) &= z\,,\\ \mathrm{Tr}(T_1T_2) = \mathrm{Tr}(T_2T_1) &= z^2\,,\\ \mathrm{Tr}(T_1T_2T_1) = z\mathrm{Tr}(T_1^2) &= z((q-1)z+q)\,.\end{aligned}$$

Proof. The K-linear map $\mathrm{Tr}: H_{n+1} \to K$ is defined by induction on n, using the structure lemma of § 4 (Proposition 4.1):

$$\varphi: H_n \oplus H_n \otimes_{H_{n-1}} H_n \xrightarrow{\sim} H_{n+1}\,.$$

Starting with $\mathrm{Tr}: H_0 = K \to K$ the identity, one defines $\mathrm{Tr}: H_{n+1} \to K$ by $\mathrm{Tr}(x) = \mathrm{Tr}(a) + \Sigma_i\, z\mathrm{Tr}(b_ic_i)$, if $\varphi(a+\Sigma_i b_i \otimes c_i) = x$.

It is clear that if $a, b \in H_n$, then

$$\mathrm{Tr}(aT_nb) = z\mathrm{Tr}(ab)\,,$$

since $\varphi(a \otimes b) = aT_nb$.

The only statement to be proved is then:

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx) \quad \text{for all} \quad x, y \in H_{n+1}\,.$$

This is proved by induction on n.

We may assume that x and y are monomials containing T_n at most once.

If y does not contain T_n at all, then writing $x = x'T_nx''$, where x', x'' are monomials in $T_1, \ldots, T_{n-1}$, one has

$$\mathrm{Tr}(xy) = z\mathrm{Tr}(x'x''y) = z\mathrm{Tr}(yx'x'') = \mathrm{Tr}(yx'T_nx'') = \mathrm{Tr}(yx)\,.$$

If y contains T_n, it suffices to check the case where $x = aT_nb$ and $y = T_n$, as is easily verified. (Here $a, b \in H_n$.)

There are various cases depending on whether or not the elements a and b actually contain T_{n-1}. The worst case is the one in which $a = a'T_{n-1}a''$, $b = b'T_{n-1}b''$ with a', a'', b', b'' belonging to H_{n-1}. We have then

$$\begin{aligned}\mathrm{Tr}(aT_nbT_n) &= z((q-1)\mathrm{Tr}(ab)+q\mathrm{Tr}(ab'b''))\\ \mathrm{Tr}(T_naT_nb) &= z((q-1)\mathrm{Tr}(ab)+q\mathrm{Tr}(a'a''b))\,.\end{aligned}$$

But

$$\mathrm{Tr}(ab'b'') = \mathrm{Tr}(a'T_{n-1}a''b'b'') = z\mathrm{Tr}(a'a''b'b'')\,,$$

and

$$\mathrm{Tr}(a'a''b) = \mathrm{Tr}(a'a''b'T_{n-1}b'') = z\mathrm{Tr}(a'a''b'b'')\,.$$

Hence,

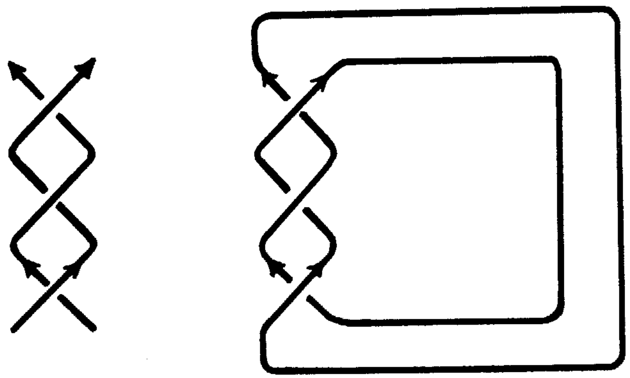
$$\mathrm{Tr}(aT_nbT_n) = \mathrm{Tr}(T_naT_nb)$$

as desired.

§ 6. Existence of the two-variable polynomial

The polynomial will be defined as a braid invariant.

A braid α gives rise to a link $K(\alpha)$ by the "closing" operation, as shown in the picture



Closing a braid

Recall that every oriented link is ambient isotopic to a closed braid, as was already known to Alexander [Al]. (See also [Mo].)

Now, let $K = \mathbf{C}(q, z)$ be the rational field in 2 variables q, z over the complex numbers and let $w = 1 - q + z \in K$.

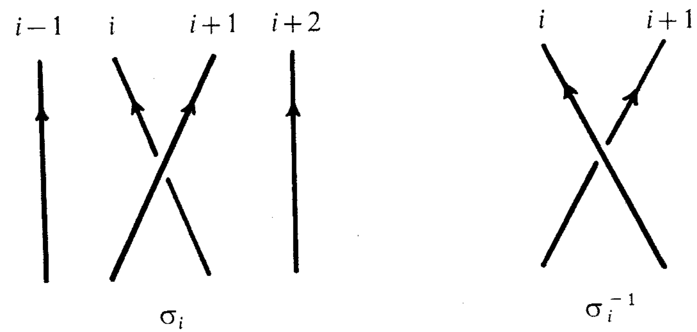
With every braid $\alpha \in B_n$ we will associate an element $V_\alpha(q, z)$ in the quadratic extension $K(\sqrt{q/zw})$ of K.

It is a quite remarkable fact that $V_\alpha(q, z)$ will depend only on the link $K(\alpha)$ obtained from α by closing the braid as shown above, and not on α itself.

Thus, we will be able to define $V_K(q, z) = V_\alpha(q, z)$, where α is any braid such that K is ambient isotopic to $K(\alpha)$.

In order to define $V_\alpha(q, z)$ we now proceed to fix some notations.

We use the following conventions regarding the generators $\sigma_1, ..., \sigma_{n-1} \in B_n$ of the braid group on n strings B_n.



Recall that B_n has a presentation on the generators $\sigma_1, ..., \sigma_{n-1}$ with relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i - j| \geqslant 2,$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $i = 1, ..., n-2$.

Note that there is a well defined homomorphism $e: B_n \to \mathbf{Z}$ given by $e(\sigma_i) = 1, i = 1, ..., n-1$, on the generators. We call e the exponent sum.

There is also an obvious representation $\rho: B_n \to H_n$ determined by

$$\rho(\sigma_i) = T_i .$$

Note that $T_i \in H_n$ is invertible in H_n: $T_i^{-1} = \frac{1}{q}(1 - q + T_i)$. Now, let $\alpha \in B_n$. The corresponding element $V_\alpha(q, z) \in K(\sqrt{q/zw})$ is defined by the formula

$$V_\alpha(q, z) = (1/z)^{(n+e(\alpha)-1)/2} \, . \, (q/w)^{(n-e(\alpha)-1)/2} \, . \, \mathrm{Tr}(\rho(\alpha)) ,$$

where $w = 1 - q + z$, $\rho: B_n \to H_n$ is as above, and $e(\alpha)$ is the exponent sum of α.

In order to show that $V_\alpha(q, z)$ depends only on the link $K(\alpha)$, we appeal to Markov's theorem which gives necessary and sufficient conditions for 2 braids $\alpha \in B_m$, $\beta \in B_n$ to produce isotopic links $K(\alpha)$, $K(\beta)$ by closing.

Define a Markov move of type 1 to be the operation of replacing a braid $\alpha \in B_n$ by a conjugate $\gamma \alpha \gamma^{-1} \in B_n$ with $\gamma \in B_n$.

A Markov move of type 2 consists in replacing $\alpha \in B_n$ by $\alpha . \sigma_n$ or $\alpha . \sigma_n^{-1}$ in B_{n+1}. Or, replacing $\alpha . \sigma_n \in B_{n+1}$, resp. $\alpha . \sigma_n^{-1} \in B_{n+1}$ by $\alpha \in B_n$ if α is a word in the generators $\sigma_1, \ldots, \sigma_{n-1}$ only.

THEOREM (Markov). *Let $\alpha \in B_m$, $\beta \in B_n$ be two braids. Then, $K(\alpha)$ and $K(\beta)$ are ambient isotopic as oriented links iff there exists a finite sequence of Markov moves carrying α to β.*

For a proof, see [Mo].

Thus, we have to show that $V_\alpha(q, z)$ is unchanged by Markov moves on α.

Let $\alpha \in B_n$, $\gamma \in B_n$ and $\beta = \gamma\alpha\gamma^{-1}$. Then the string numbers of α and β are the same. Also $e(\alpha) = e(\beta)$, and $\mathrm{Tr}(\rho(\beta)) = \mathrm{Tr}(\rho(\gamma)\rho(\alpha)\rho(\gamma)^{-1}) = \mathrm{Tr}(\rho(\alpha))$. Hence, $V_\beta(q, z) = V_\alpha(q, z)$.

If $\alpha \in B_n$ and $\beta = \alpha . \sigma_n \in B_{n+1}$, we have $e(\beta) = e(\alpha) + 1$, $n(\beta) = n + 1$ (where $n = n(\alpha)$). Thus,

$$\begin{aligned} V_\beta(q, z) &= (1/z)^{(n+e(\alpha)-1)/2} . (q/w)^{(n-e(\alpha)-1)/2} . (1/z) . \mathrm{Tr}(\rho(\alpha) . T_n) \\ &= V_\alpha(q, z), \text{ as desired} . \end{aligned}$$

If $\alpha \in B_n$ and $\beta = \alpha\sigma_n^{-1} \in B_{n+1}$, then $e(\beta) = e(\alpha) - 1$, $n(\beta) = n + 1$ and

$$V_\beta(q, z) = (1/z)^{(n+e(\alpha)-1)/2} . (q/w)^{(n-e(\alpha)-1)/2} . (q/w) . \mathrm{Tr}\{\rho(\alpha) . T_n^{-1}\} .$$

Now,

$$\rho(\alpha) . T_n^{-1} = \frac{1}{q} \rho(\alpha) . (1 - q + T_n) ,$$

and

$$\mathrm{Tr}\{\rho(\alpha) . T_n^{-1}\} = \frac{1}{q}(1 - q + z) . \mathrm{Tr}(\rho(\alpha)) = (w/q) . \mathrm{Tr}(\rho(\alpha)) .$$

Hence, again $V_\beta(q, z) = V_\alpha(q, z)$.

Thus $V_\alpha(q, z)$ is well defined as an invariant of oriented links.

On the face of it, $V_\alpha(q, z)$ does not look much so far like a polynomial with integral coefficients. However, as it turns out, a slick change of variables will do the trick. We have:

PROPOSITION 6.1. *There exists for each link K, a unique* Laurent *polynomial $P_K(l, m) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$, such that*

$$P_K(i(z/w)^{1/2}, i(q^{-1/2} - q^{1/2})) = V_\alpha(q, z) ,$$

whenever α is a braid giving rise (up to ambient isotopy, of course) to the link K by closing.

Thus, $V_\alpha(q, z)$ becomes the Laurent polynomial $P_{K(\alpha)}(l, m)$ in

$$\mathbf{Z}[l, l^{-1}, m, m^{-1}]$$

after the change of variables

$$l = i(z/w)^{1/2} , \quad m = i(q^{-1/2} - q^{1/2}) .$$

The key to the proof of this fact is the skein invariance of $V_\alpha(q, z)$ which we now proceed to show.

Let $\beta, \gamma \in B_n$ be two braids and let $\alpha_+, \alpha_-, \alpha_0$ be the three braids

$$\alpha_+ = \beta\sigma_k\gamma , \quad \alpha_- = \beta\sigma_k^{-1}\gamma , \quad \alpha_0 = \beta\gamma ,$$

for some index $k \leqslant n - 1$.

For any braid $\alpha \in B_n$, with exponent sum $e = e(\alpha)$, define

$$W_\alpha(q, z) = (1/z)^{(n+e-1)/2} . (q/w)^{(n-e-1)/2} . \rho(\alpha) \in H_n ,$$

where H_n is now the Hecke algebra over $K(\sqrt{q/zw})$ with $K = \mathbf{C}(q, z)$, corresponding to q.

SKEIN INVARIANCE LEMMA. *Set $l = i(z/w)^{1/2}$ and $m = i(q^{-1/2} - q^{1/2})$. Then, we have the relation*

$$lW_{\alpha_+} + l^{-1}W_{\alpha_-} + mW_{\alpha_0} = 0 .$$

Taking the trace, we obtain from this lemma the skein invariance of $V_\alpha(q, z)$: With the same notations as above

$$lV_{\alpha_+} + l^{-1}V_{\alpha_-} + mV_{\alpha_0} = 0 .$$

Proof of the lemma. Set $e = e(\alpha_0)$, and observe that we have

$$\begin{aligned} W_{\alpha_+} &= (1/z)^{1/2}(q/w)^{-1/2}(1/z)^{(n+e-1)/2}(q/w)^{(n-e-1)/2} . \rho(\beta)T_k\rho(\gamma) , \\ W_{\alpha_-} &= (1/z)^{-1/2}(q/w)^{1/2}(1/z)^{(n+e-1)/2}(q/w)^{(n-e-1)/2} . \rho(\beta)T_k^{-1}\rho(\gamma) . \end{aligned}$$

An easy calculation now gives for $lW_{\alpha_+} + l^{-1}W_{\alpha_-} + mW_{\alpha_0}$ an expression of the form

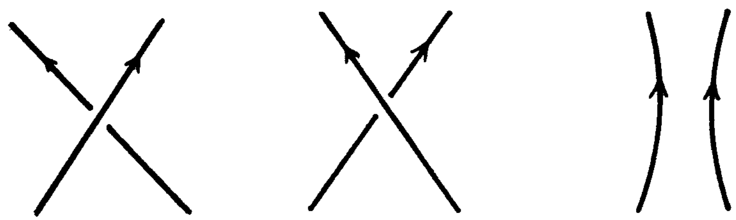
$$i(1/z)^{(n+e-1)/2}(q/w)^{(n-e-1)/2} . \rho(\beta) . C . \rho(\gamma) ,$$

where

$$C = q^{-1/2}T_k - q^{1/2}T_k^{-1} + q^{-1/2} - q^{1/2}.$$

Recalling that $T_k^{-1} = q^{-1}(1-q+T_k)$, it is easy to verify that $C = 0$.

Proof of proposition 6.1. Let K_+, K_- and K_0 be three skein related links.



K_+ K_- K_0

It is an obvious consequence of the classical proof of Alexander's theorem that the three links can be presented as closed braids of the form

$$K_+ = K(\alpha_+), \quad K_- = K(\alpha_-), \quad K_0 = K(\alpha_0),$$

with $\alpha_+ = \beta\sigma_k\gamma$, $\alpha_- = \beta\sigma_k^{-1}\gamma$, $\alpha_0 = \beta\gamma$ for some braids β, $\gamma \in B_n$ and some index $k \leqslant n-1$.

Writing $V(K)$ for V_α if $K = K(\alpha)$, it follows from the skein invariance lemma above, that

$$lV(K_+) + l^{-1}V(K_-) + mV(K_0) = 0,$$

if K_+, K_- and K_0 are skein related.

It follows now by induction on the link complexity, as in the proof of uniqueness in § 3, that $V(K)$ is actually a Laurent polynomial with integer coefficients in the variables l and m.

We change notation and set $P_K(l, m) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$, where

$$P_{K(\alpha)}\left(i(z/w)^{1/2}, i(q^{-1/2}-q^{1/2})\right) = V_\alpha(q, z)$$

Since it is obvious that $P_K(l, m) = 1$ if K is the unknot $\bigcirc$, we have shown that $P : \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ exists as a skein invariant. It is universal by what we saw before in § 3.

§ 7. Some properties of $P_K(l, m)$

In this paragraph we gather some of the basic properties of the polynomial $P_K(l, m)$, also denoted $P(K)$ if the variables are understood.

Let K' be the oriented link obtained from K by reversing the orientations of all the components. Then, we have

Property 7.1. $P(K') = P(K)$.

Proof. Let K_+, K_-, K_0 be three skein related links. We see that K'_+, K'_- and K'_0 are also skein related. Hence,

$$lP(K'_+) + l^{-1}P(K'_-) + mP(K'_0) = 0.$$

By uniqueness, this implies $P(K') = P(K)$ for all K. (Of course $\bigcirc' = \bigcirc$.)

Property 7.1 can also be proved from the definition given in § 6 as follows. If $K = K(\alpha)$, then $K' = K(\alpha')$, where $\alpha' = \sigma_{i_r}^{\varepsilon_r} \ldots \sigma_{i_1}^{\varepsilon_1}$ if $\alpha = \sigma_{i_1}^{\varepsilon_1} \ldots \sigma_{i_r}^{\varepsilon_r}$. Observe that the operation $\alpha \mapsto \alpha'$ is a well defined antiautomorphism of B_n. There is an analogous antiautomorphism of H_n, sending the monomial $M = T_{i_1} \ldots T_{i_r}$ to $M' = T_{i_r} \ldots T_{i_1}$ and it is easily checked that for all $x \in H_n$, $\mathrm{Tr}(x) = \mathrm{Tr}(x')$.

Next, let $K^\times$ be the mirror image of K. Then we have

Property 7.2. $P_{K^\times}(l, m) = P_K(l^{-1}, m)$.

Proof. Observe that if K_+, K_-, and K_0 are skein related, then so are $K_-^\times$, $K_+^\times$ and $K_0^\times$ *in this order*, i.e.

$$lP(K_-^\times) + l^{-1}P(K_+^\times) + mP(K_0^\times) = 0.$$

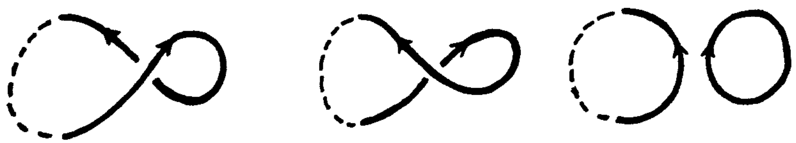
The property follows by uniqueness applied to $P_{K^\times}(l, m) = P_K(l^{-1}, m)$.

We shall skip the alternative proof of that property based on braid presentations.

If K_1 and K_2 are two links and $K_1 \amalg K_2$ their distant union (disjoint, unlinked), then we have

Property 7.3. $P(K_1 \amalg K_2) = -\dfrac{l + l^{-1}}{m} \cdot P(K_1) \cdot P(K_2)$.

Proof. If $K_2 = \bigcirc$, this follows from the skein invariance as shown in the following picture



$L_1 = L_+$ $\qquad$ $L_1 = L_-$ $\qquad$ $L_0 = L_1 \amalg \bigcirc$

which yields

$$lP(K_1) + l^{-1}P(K_1) + mP(K_1 \amalg \bigcirc) = 0\,,$$

and therefore

$$P(K_1 \amalg \bigcirc) = -\frac{l+l^{-1}}{m}\,.\,P(K_1)\,.$$

If K_2 is more complicated, use induction on the complexity of one of its diagrams L_2. If L_2^+, L_2^-, L_2^0 are skein related, so are $L_1 \amalg L_2^+$, $L_1 \amalg L_2^-$, $L_1 \amalg L_2^0$ for any diagram L_1 of K_1 and Property 7.3 follows.

Second proof. If $K_1 = K(\alpha)$ with $\alpha \in B_m$ and $K_2 = K(\beta)$, with $\beta \in B_n$, then $K_1 \amalg K_2 = K(\alpha\,.\,s(\beta))$ with $\alpha\,.\,s(\beta) \in B_{m+n}$, where $s\colon B_n \to B_{m+n}$ shifts all indices of the generators $\sigma_1, \ldots, \sigma_{n-1}$ by m, i.e. $s(\sigma_i) = \sigma_{m+i}$. It follows that α and $s(\beta)$ commute in B_{m+n}, and it is easily verified that $\mathrm{Tr}(\rho(\alpha\,.\,s(\beta))) = \mathrm{Tr}(\rho(\alpha))\,.\,\mathrm{Tr}(\rho(\beta))$. Then,

$$V_{\alpha s(\beta)}(q, z) = (q/zw)^{1/2}\,.\,V_\alpha(q, z)\,.\,V_\beta(q, z)\,.$$

With $l = i(z/w)^{1/2}$ and $m = i(q^{-1/2} - q^{1/2})$, we have

$$-\frac{l+l^{-1}}{m} = -\frac{(z/w)^{1/2} - (z/w)^{-1/2}}{q^{-1/2} - q^{1/2}} = -\left(\frac{zq}{w}\right)^{1/2}\frac{1-\dfrac{w}{z}}{1-q}$$

$$= -\left(\frac{zq}{w}\right)^{1/2}\frac{z-(1-q+z)}{z(1-q)} = \left(\frac{q}{zw}\right)^{1/2}$$

Therefore,

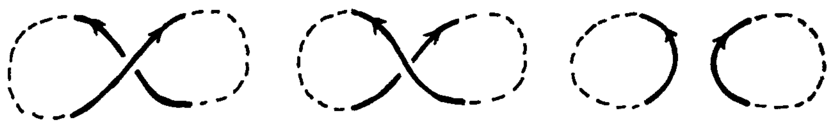
$$V_{\alpha s(\beta)}(q, z) = -\frac{l+l^{-1}}{m}\,.\,V_\alpha(q, z)\,.\,V_\beta(q, z)$$

as required.

If K_1, K_2 are 2 links, denote by $K_1 \# K_2$ a connected sum of K_1 and K_2 performed from the unlinked union on any choice of components.

PROPERTY 7.4. $P(K_1 \# K_2) = P(K_1)\,.\,P(K_2)$.

Proof. We use the skein relation



$L_+ = L_1 \# L_2$ $\qquad$ $L_- = L_1 \# L_2$ $\qquad$ $L_0 = L_1 \amalg L_2$

where L_1 and L_2 are diagrams of K_1 and K_2.

This gives the formula

$$lP(L_1 \# L_2) + l^{-1}P(L_1 \# L_2) + mP(\overline{L_1 \amalg L_2}) = 0\,.$$

Solving for $P(L_1 \# L_2)$ and using property 7.3, the factor $-(l+l^{-1})/m$ cancels out and the result follows.

The proof using braid presentations is more complicated and will be omitted.

Since $P\colon \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ is the universal skein invariant, it must specialize to the Alexander polynomial and to the one-variable Jones polynomial.

Specifically, define

$$\Delta_K(t) = P_K(i, i(t^{1/2} - t^{-1/2}))\,,$$

then we have

PROPERTY 7.5. $\Delta_K(t)$ *satisfies the skein invariance*

(1) $\Delta_{\bigcirc}(t) = 1$,

(2) $\Delta(K_+) - \Delta(K_-) + (t^{1/2} - t^{-1/2})\Delta(K_0) = 0$,

which characterizes the Alexander *polynomial as normalized by* J. Conway. (See L. Kauffman, [Ka$_1$].)

Recall from § 3 that the exponent of m in each monomial of $P_K(l, m)$ is congruent mod 2 to $r(K) - 1$, where $r(K)$ is the number of components of K. Hence, for a knot, a link with a single component, the exponent of m in $P_K(l, m)$ is even and therefore $\Delta_K(t) = P_K(i, i(t^{1/2} - t^{-1/2}))$ is indeed a Laurent polynomial in t.

To obtain the one-variable Jones polynomial we use the substitution $l = it, m = i(t^{1/2} - t^{-1/2})$. Explicitly,

$$V_K(t) = P_K(it, i(t^{1/2} - t^{-1/2}))$$

Then we have

PROPERTY 7.6. *$V_K(t)$ satisfies the skein invariance*

$$tV(K_+) - t^{-1}V(K_-) + (t^{1/2} - t^{-1/2})V(K_0) = 0,$$

which (together with $V(\bigcirc)=1$) characterizes Jones *one-variable polynomial, with the sign conventions used in reference* [Jo_3].

Whereas $P_K(l, m)$ determines $\Delta_K(t)$ and $V_K(t)$, it is known that there are no other relations between these polynomials. More precisely:

(1) The Alexander polynomial $\Delta_K(t)$ does not determine Jones polynomial $V_K(t)$ because the trivial knot $\bigcirc$ and Conway's eleven crossing knot 11_{471} have $\Delta(t) = 1$, but $V_K(t) \neq 1$ for $K = 11_{471}$.

(2) $V_K(t)$ does not determine $\Delta_K(t)$: The knots 4_1 and 11_{388} have the same $V(t)$ but different $\Delta(t)$.

(3) $V_K(t)$ and $\Delta_K(t)$ together do not determine $P_K(l, m)$: The knot 11_{388} and its mirror image have the same $V(t)$ and $\Delta(t)$ but different $P(l, m)$.

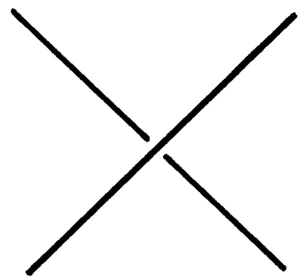
For more details on these questions, see [L.-M.].

We now turn to L. Kauffman's definition of the one-variable Jones polynomial $V_K(t)$ directly from the link diagram.

§ 8. L. Kauffman's approach to V. Jones' one-variable polynomial

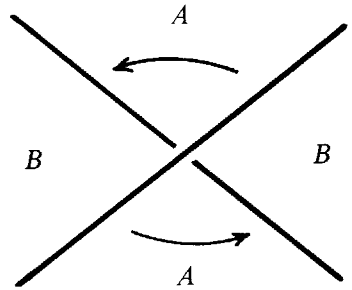
The importance of Kauffman's approach [Ka_3] is that it gives a new way to define and compute Jones polynomial $V_K(t)$. It is by using this definition that Kauffman and Murasugi prove their theorems about alternating links (see § 10 and 11).

Let L be an *unoriented* link diagram. Look at a double point; with no string orientation, they all look the same, up to a local homeomorphism:

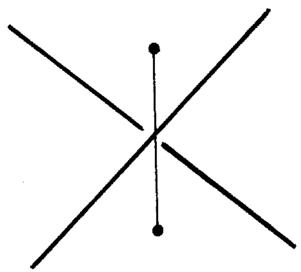


Locally, the plane $\mathbf{R}^2$ is divided into four regions.

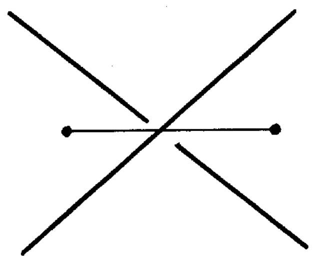
Look at the quarter turn the "over" line must make, in the positive sense, in order to coincide with the "under" line. Call "A" the two regions which are swept by the over line during the trip. Call "B" the other two.



Definition. A *marker* for a double point is a choice of "A" or "B" for this double point. It is symbolised like that:

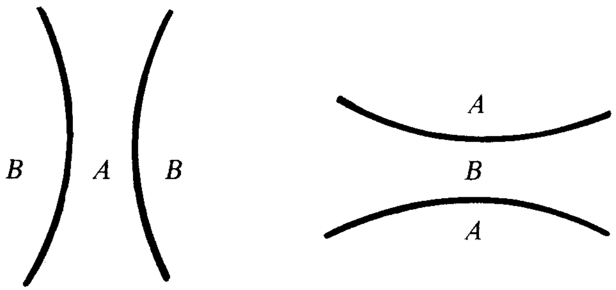


Marker A



Marker B

Now, if a marker is chosen, one can split the link diagram by connecting the two opposite regions whose name has been elected. Here are the pictures.



splitting if marker A is chosen | splitting if marker B is chosen

Definition. A *state* S for L is a choice of a marker at every double point of L.

Suppose now that a state S for L is given. Make the correct splitting at every double point of L. The underlying knot projection Γ is transformed into a bunch Γ_S of disjoint simple closed curves in S^2. Let $|S|$ be the number of curves in Γ_S.

Write $a(S)$ for the number of markers A in the state S and write $b(S)$ for the number of B's.

If $c(L)$ denotes the number of crossings (double points) of L, one clearly has $2^{c(L)}$ states.

L being given, Kauffman defines a polynomial $<L> \in \mathbf{Z}[A, B, d]$ in the following way:

$$<L> = \sum_S A^{a(S)} B^{b(S)} d^{|S|-1}$$

the summation being taken over the $2^{c(L)}$ states.

Notations. Write "○" for an unoriented, connected, simple closed curve in $\mathbf{R}^2$ and write ○ ∐ L for a disjoint union of such a diagram and an unoriented link diagram L.

Property 1. $<○> = 1$.

Property 2. $<○ \amalg L> = d<L>$ if L is non empty.

Property 3. Let L be an unoriented link diagram. Select a crossing $\times$ and write L_A for the diagram obtained from L by connecting the two regions A at $\times$, and write L_B for the diagram obtained by connecting the two B's. Then:

$$<L> = A<L_A> + B<L_B> .$$

Proposition 8.1. $< >$ *is the unique function from the set of unoriented link diagrams to* $\mathbf{Z}[A, B, d]$ *which satisfies properties 1, 2 and 3.*

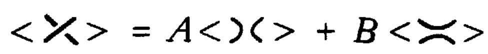
The proof is straightforward.

Proposition 8.2. *If one sets* $B = A^{-1}$ *and* $d = -(A^2+A^{-2})$, *one gets a function into* $\mathbf{Z}[A^{\pm 1}]$ *which is invariant under Reidemeister moves (ii) and (iii).*

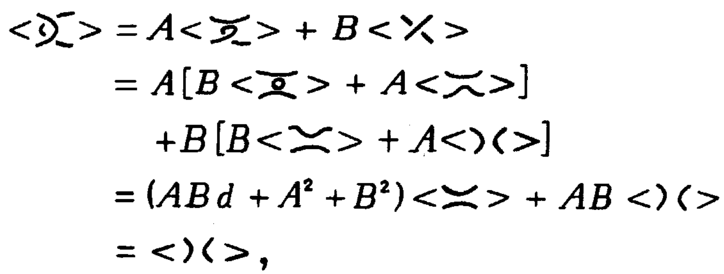
Notations. Following Kauffman, we shall still write $< >$ for the function into $\mathbf{Z}[A^{\pm 1}]$. From now on, only this function will be used.

We now recall briefly Kauffman's proof of proposition 8.2.

First of all, we shall use Kauffman's schematic way of writing property 3:

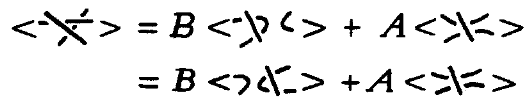


Invariance under move (ii):

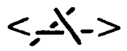


since we have set $B = A^{-1}$ and $d = -(A^2+A^{-2})$.

Invariance under move (iii):



by invariance under move (ii)

=  .

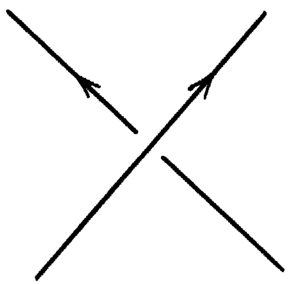
Q.E.D.

This seems to be as far as one can get without orienting link diagrams, because < > is *not* invariant under Reidemeister move (i).

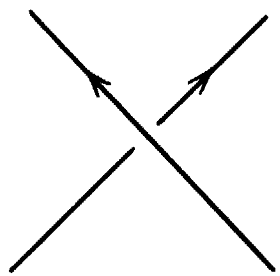
To remedy this state of affairs, Kauffman proceeds like this:

Let L be an *oriented* link diagram.

Recall now that, up to a rotation in $\mathbf{R}^2$, there are two types of double points:



Sign +1



Sign −1

Definition. The writhe number $w(L)$ is the sum of the signs of the double points of L.

This number is also called twist number. It was known to Tait and much used by Little. See § 9 of these notes.

Kauffman's polynomial $f_L(A) \in \mathbf{Z}[A^{\pm 1}]$ is then defined in the following way:

$$f_L(A) = (-A)^{-3w(L)} <L> .$$

PROPOSITION 8.3. *The polynomial* f *is invariant under Reidemeister moves (i), (ii) and (iii).*

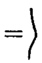
Proof of proposition 8.3. The writhe number is unchanged by the moves (ii) and (iii). Hence proposition (8.2) implies the invariance of f under the moves (ii) and (iii).

We now prove the invariance under move (i).

Let $\hat{L}$ be a link diagram with a portion looking like this:

and let L be the link diagram obtained from $\hat{L}$ by removing the loop.

It is immediate that

$\hat{L}_A =$  and $\hat{L}_B =$  ○

If we apply property 3 for < > we get

$$<\hat{L}> = A<L> + A^{-1}<L \amalg \bigcirc> .$$

By property 2

$$<\hat{L}> = A<L> + A^{-1}(-A^2-A^{-2}) <L> .$$

So $<\hat{L}> = (A-A^{-1}(A^2+A^{-2})) <L> = (-A)^{-3} <L>$.

Now, for any orientation of the string, the sign of the double point is −1.

Hence $w(\hat{L}) = w(L) - 1$.

Going back to the definition,

$$\begin{aligned} f_{\hat{L}} &= (-A)^{-3w(\hat{L})} <\hat{L}> = (-A)^{-3w(L)+3} <\hat{L}> \\ &= (-A)^{-3w(L)+3} (-A)^{-3} <L> \\ &= (-A)^{-3w(L)} <L> = f_L . \end{aligned}$$

The proof for the other loop is similar. Q.E.D.

From proposition 8.3 we deduce that Kauffman's polynomial induces a map $f: \mathscr{L} \to \mathbf{Z}[A^{\pm 1}]$.

THEOREM 8.4. *The map* $f: \mathscr{L} \to \mathbf{Z}[A^{\pm 1}]$ *satisfies:*

1. $f(\bigcirc) \equiv 1$.
2. *If* L_+, L_- *and* L_0 *are skein related (see § 3), then:*
$$A^4 f_{L_+} - A^{-4} f_{L_-} = (A^{-2}-A^2) f_{L_0} .$$

From the universality of Jones polynomial, we obtain:

COROLLARY 8.5. *Let K be an oriented link in* $\mathbf{R}^3$ *and let L be an oriented diagram of K. Then:*

$$V_K(t) = f_L(t^{1/4}) .$$

Recall that we use Jones definition in the Bulletin AMS [Jo$_3$] for V_K. If we were to use Jones definition in the Notices AMS [Jo$_4$], we would set $A = t^{-1/4}$.

Proof of theorem 8.4. The proof of 1. is straightforward from the definition. For 2., using Kauffman's notations one has:

$$< \times > = A < \asymp > + A^{-1} <)(>$$

and

$$< \times > = A^{-1} < \asymp > + A <)(>$$

Hence:

$$A < \times > - A^{-1} < \times > = (A^2 - A^{-2}) < \asymp >$$

If we orient the strings and put the writhe number in the picture, we get the formula 2. Q.E.D.

Using L. Kauffman's definition of Jones polynomial, the following properties are easily proved (enjoyable exercise left to the reader):

I. If K_1 and K_2 are two oriented links in S^3, let $K_1 \amalg K_2$ denote their distant union (one in each hemisphere). Then:

$$V_{K_1 \amalg K_2} = \mu V_{K_1} \cdot V_{K_2}$$

where $\mu = -(t^{1/2} + t^{-1/2})$.

II. Let $K_1 \# K_2$ denote any connected sum of K_1 and K_2 as in § 7 prop. 4. Then:

$$V_{K_1 \# K_2} = V_{K_1} \cdot V_{K_2} .$$

III. Let $K^\times$ denote the mirror image of K. Then:

$$V_{K^\times}(t) = V_K(t^{-1}) .$$

The first three formulas are rather straightforward from the definitions.

IV. (Jones reversing result). Let K be an oriented link in S^3 and let γ be a component of K. Let λ be the linking coefficient of γ with what is left of K when we remove γ. (We suppose that this is not empty!) Let $\hat{K}$ be the oriented link obtained from K by changing the orientation of γ, while keeping the others fixed. Then:

$$V_{\hat{K}}(t) = t^{3\lambda} V_K(t) .$$

Proof. Of course, we have $<K> = <\hat{K}>$, because, for the polynomial $<\ >$, orientations do not matter.

Now: $w(K) = w(\gamma) + 2\lambda$.
So: $w(\hat{K}) = w(\gamma) - 2\lambda$.
Hence: $w(\hat{K}) = w(K) - 4\lambda$.

We substitute and get:

$$f_{\hat{K}}(A) = (-A)^{-3w(\hat{K})} <\hat{K}> = (-A)^{-3w(K)+12\lambda} <K>$$
$$= (-A)^{12\lambda}(-A)^{-3w(K)} <K> = (-A)^{12\lambda} f_K(A) = A^{12\lambda} f_K(A) .$$

As one substitutes $t^{1/4}$ for A to get Jones 1-variable polynomial, the result follows.

To finish this paragraph, we illustrate quickly Kauffman's definition by computing Jones one variable polynomial for the right-handed trefoil T_+. (Compare § 3.)

There are 8 states associated to the standard knot diagram. One readily sees that

$$<T_+> = A^3 d + 3A^2 B d^0 + 3AB^2 d + B^3 d^2 .$$

Substituting $d = -(A^2 + A^{-2})$ and $B = A^{-1}$ one gets

$$<T_+> = -A^5 - A^{-3} + A^{-7} .$$

As $w(T_+) = 3$, one gets

$$f_{T_+}(A) = (-A)^{-9} <T_+> = A^{-4} + A^{-12} - A^{-16} .$$

Substituting $t = A^{1/4}$ one finally obtains

$$V_{T_+}(t) = t^{-1} + t^{-3} - t^{-4} = t^{-4}(-1 + t + t^3) .$$

Now, if one uses our computation in § 3

$$P(T_+) = -2a_-a_+^{-1} - a_-^2a_+^{-2} + a_+^{-2}a_0^2$$

and substitutes $a_+ = l, a_- = l^{-1}, a_0 = m$ one gets

$$P_{T_+}(l, m) = (-2l^{-2}-l^{-4})m^0 + l^{-2}m^2 .$$

The last substitution $l = it$; $m = i(t^{1/2}-t^{-1/2})$ gives (with relief!) the same result for Jones one variable polynomial. (Bulletin AMS definition.)

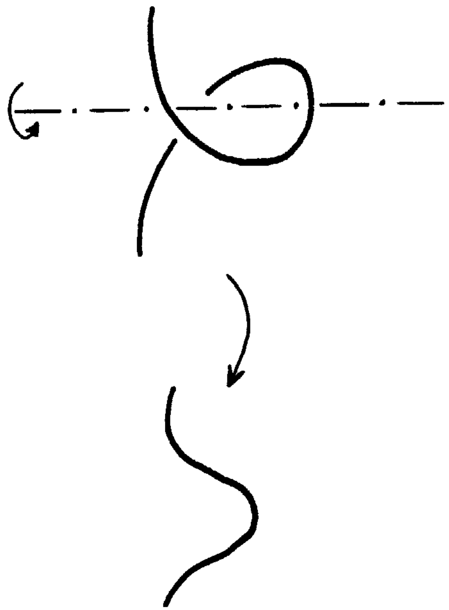
§ 9. Tait conjectures

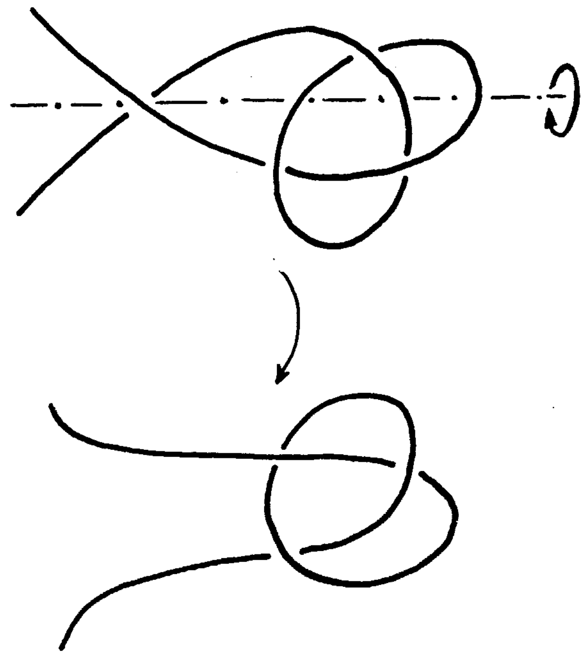
Tait was primarily interested in the classification of knots (i.e. one component links). He organized the job in two steps.

Step 1. Classify generic immersions of the circle in S^2 (not $\mathbf{R}^2$!) modulo homeomorphisms (possibly orientation reversing) of S^2. This was mostly done by the Rev. T. P. Kirkman (around 1880).

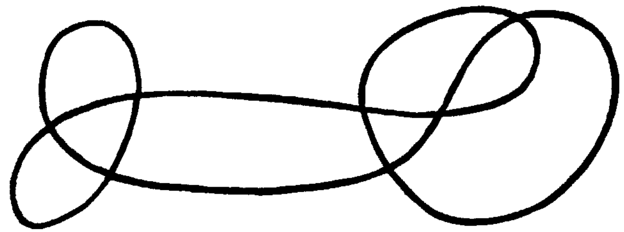
In this process, one has to remember that one is looking at knots in $\mathbf{R}^3$ and that one is trying to list knots according to their "knottiness", i.e. their minimal crossing number. So, Tait first reduced the number of double points of a generic immersion by making one "local 180° rotation".

Examples.





So, really, the problem was to list *reduced* generic immersion of S^1. Tait also recognised that is was sufficient to classify "prime" immersions, i.e. immersions indecomposables with respect to connected sum.
Example of a connected sum:



Step 2. Find how many knot types correspond to the same generic immersion. Tait's first observation was:

PROPOSITION 9.1. *A link projection being given, one can always choose the heights at the double points in order that the corresponding link diagram be alternating.*

By definition, a link diagram is alternating if, when one follows any string, the crossings are alternatively over and under.

We now reproduce Tait's proof, because it will play its part in § 11.

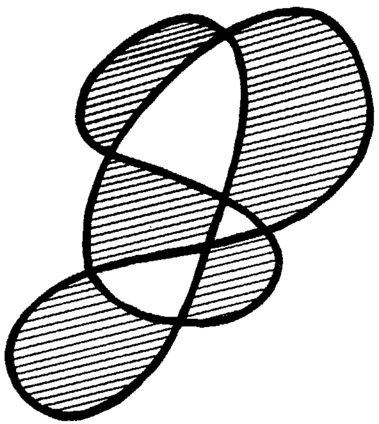
Proof of proposition 9.1. Let L be a link projection in S^2, not passing through the north pole N.

Call "region" a connected component of $S^2 - L$.

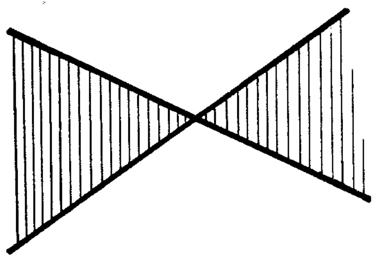
If $P \in S^2 - L$, let $I(P)$ be the intersection number mod 2 of L and a generic 1-chain joining P to N.

Shade the regions for which $I \equiv 1 \bmod 2$. S^2 is thus painted like a chessboard, the region containing N being unshaded.

Example.



Let X be a double point of L. Near X, two opposite regions are shaded and two aren't.



Choose a thread and travel along this thread toward the crossing point and a little further. Call this thread "rl" if the shaded region is first on your right and then on your left, while you travel. Notice that this does not depend on the orientation you choose on the thread.

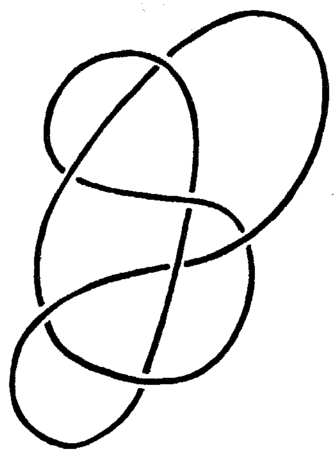
At each double point, one thread will be "rl" and the other will be "lr".

To construct an alternating link diagram from the link projection L we make the following *convention*: A "rl" thread always passes over a "lr" thread.

ASSERTION. *The link diagram thus obtained is alternating.*

Proof. If one follows a string, after a double point a "rl" thread becomes a "lr" thread and conversely. Q.E.D.

Picture:



Suppose that L is a connected link projection. There are exactly two ways to obtain an alternating link diagram from it. In this setting, the question of amphicheirality is very natural: Are the two links ambient isotopic? If yes, they are amphicheiral (nowadays, one also says achiral). If not, they are now called "chiral".

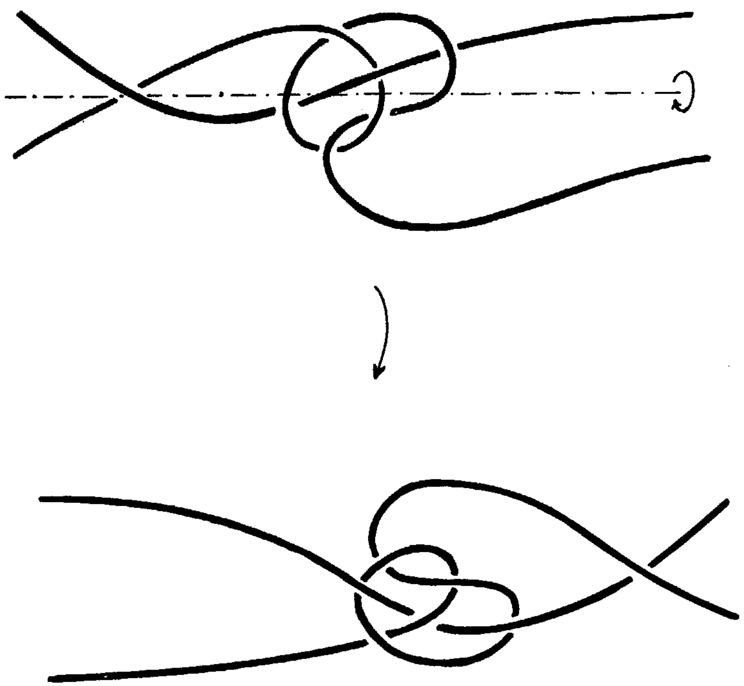
Roughly speaking the chirality question arose more or less in these terms in Tait. It is however obscured by considerations pertaining to knot projections rather than to knots in $\mathbf{R}^3$.

In order to classify alternating knots, Tait used the following principles, now called Tait conjectures:

CONJECTURE A. Two reduced alternating diagrams of the same alternating knot have the same number of crossing points. This number is minimal among all diagrams.

A stronger form of conjecture A would be: The minimal diagrams of an alternating knot are exactly the reduced alternating ones.

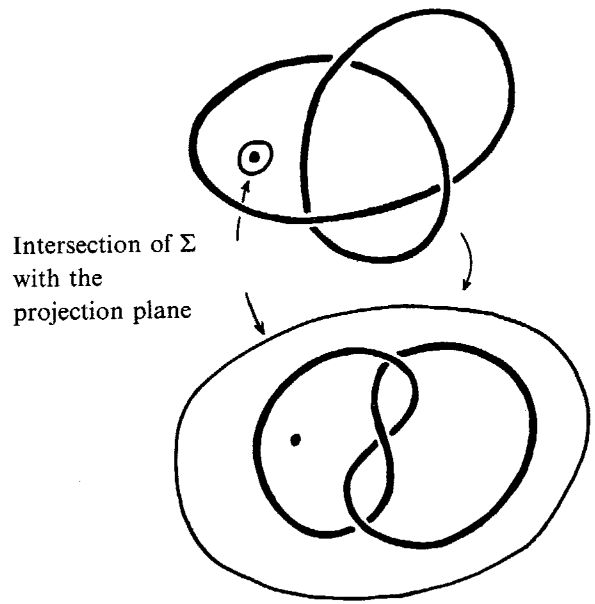
CONJECTURE B. Two reduced alternating diagrams of the same knot are "essentially unique". More precisely one can pass from one to another by a sequence of the following two operations:



(i) Another kind of "local 180° rotation" illustrated in the above picture, and called "twisting" by Tait. (An analogous operation is called by him "distortion".)

(ii) An inversion with respect to a 2-sphere Σ in S^3 intersecting the projection "plane" in a circle, followed by a mirror through the projection plane (in order that the composition be orientation preserving). For that, Tait introduced the name "flype", an old Scottish word meaning "to turn outside in".

Example.



Remarks. 1. If conjectures A and B were true, the classification of alternating knots would mainly rely on listing generic immersions of S^1 in S^2.

2. If conjecture A is true, then an alternating reduced knot diagram with at least one crossing point represents a non trivial knot. This was first proved by C. Bankwitz, with a mistake corrected by R. Crowell. See [Ba], and [Cr].

3. Tait noticed that, from eight crossings on, there exist non alternating knots. No actual proof was given. Tait had no "principles" to classify non alternating knots.

4. Conjecture B is still open.

Let us now come back to the notion of writhe number of a knot diagram L defined in § 8. Recall that, by definition, $w(L)$ is the sum of the signs of the crossing points.

A topological interpretation of $w(L)$ is the following: take a small tubular neighborhood of L and restrict the projection onto $\mathbf{R}^2$ to the boundary of this neighborhood. This restriction will have two curves of singularities: the "contour apparent". Choose one of them; it is a parallel of the knot. The linking coefficient of this parallel with the knot is precisely $w(L)$. Notice that this parallel is defined only when a projection is chosen.

A careful reader of Tait [Tai] on p. 308 will remark that Tait knew that. The Gaussian integral, interpreted via Maxwell theory, takes place of the linking coefficient. In Tait's point of view the parallel is turned 90° downward on each fiber of the regular neighborhood of the knot.

C. N. Little also introduced the number $w(L)$. He used it to classify knots by making the following statement:

Little principle: Any two minimal diagrams of the same knot have the same writhe number. (See [Li].)

This principle is known to be false; a counter-example is given by Little's duplication: the knot diagrams listed in Rolfsen's book as 10_{161} and 10_{162} have distinct writhe number, but represent the same knot as discovered by K. Perko [Pe].

However, the following is still open:

CONJECTURE C. Any two reduced and alternating diagrams of an (alternating) knot have the same writhe number.

If L is a knot diagram, let $L^{\times}$ denote the mirror image of L. Clearly: $w(L) = -w(L^{\times})$. So, if one believes some of the above conjectures, one is ready to make the following conjecture, used by Tait as a fact:

CONJECTURE D. If K is an alternating and amphicheiral knot, then any minimal projection of K has an even number of double points.

More daring people would conjecture that minimal diagrams of an amphicheiral knot have Tait number zero (i.e. writhe number zero).

Helped by these statements, Tait gave a list of twenty knots up to ten crossings which are amphicheiral and believed that the list was complete (which it is!).

We conclude this paragraph by recalling a few dates:

a. First proof that knots do exist: H. Tietze in 1908 [Ti] proved that the trefoil is knotted.

b. First proof that non amphicheiral knots do exist: M. Dehn in 1914 [De] proved that the left handed trefoil is not ambient isotopic to the right handed trefoil.

c. First proof that non alternating knots do exist: R. Crowell [Cr] and K. Murasugi [Mu_1] proved in 1957 that the (3, 4) torus knot is non alternating. This result was already stated by C. Bankwitz.

§ 10. L. Kauffman's and K. Murasugi's results

Definition. Let $g(t) \in \mathbf{Z}[t^{\pm 1/2}]$ be a non-zero element:

$$g(t) = \sum_{i=n}^{m} a_i t^i, \quad i \in \frac{1}{2}\mathbf{Z}, \quad a_n \neq 0, \quad a_m \neq 0.$$

Define span $g(t) = m - n$.

In principle span $g(t) \in \frac{1}{2}\mathbf{Z}$. But, if $g(t)$ is the one variable Jones polynomial of an oriented link in S^3, the span of $g(t)$ will actually be an integer. To see that, use induction on complexity, like in § 3.

Definition. Let K be a link in S^3.

K is said to be *splittable* if there exists a 2-sphere $\Sigma \subset S^3$ such that:

1. $\Sigma \cap K = \emptyset$.
2. There is at least one component of K in each connected component of $S^3 - \Sigma$.

THEOREM 10.1. *Let* $K \subset S^3$ *be an oriented unsplittable link. Then:*

$$\text{span } V_K(t) \leqslant c(K).$$

Comments. (i) One can define the number $s(K)$ of split components of K. Then, theorem 10.1 generalizes to:

$$\text{span } V_K(t) \leqslant c(K) + s(K) - 1 .$$

See [Mu_2].

(ii) At first sight, there is something disturbing in this inequality: the polynomial $V_K(t)$ depends on the orientation of K, while the minimal crossing number $c(K)$ does not. But, in fact, span $V_K(t)$ does not depend on orientations, thanks mainly to Jones reversing result.

THEOREM 10.2. *Let L be a connected and oriented link diagram. Suppose L alternating and reduced. Then:*

$$\text{span } V_L(t) = c(L) .$$

Recall that a link is prime if it cannot be decomposed (non trivially) in a connected sum.

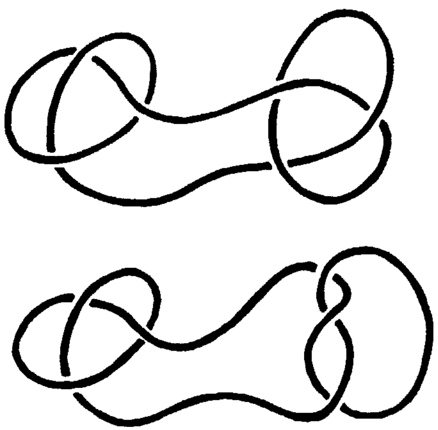
THEOREM 10.3. *Let K be a prime oriented link. Then, for any non alternating diagram L of K one has:*

$$\text{span } V_K(t) < c(L) .$$

Comments. (i) We emphazise that the inequality is strict.

(ii) Primeness is necessary, as the following example shows:

Let K be the connected sum of a left-handed and a right-handed trefoil. (This is the so called "square knot".) It is easily proved, for instance by using results of this paper, that $c(K) = 6$. Here are one alternating, and one non-alternating minimal diagrams of K:



As consequences one obtains:

THEOREM 10.4. *Tait conjecture A is true for unsplittable links. (Not only for knots.) The stronger form of conjecture A is true for unsplittable prime links. (For instance for prime knots.)*

This has the following extraordinary consequence concerning knot tabulations, which we illustrate on an example: Suppose you want to prove that the knots 8_{19}, 8_{20} and 8_{21} are non alternating. You may proceed like this:

1. Make the list of knot diagrams with at most 7 crossings (prime or not). Prove the list is exhaustive. (This has already been done by Tait!)
2. Prove that the knots 8_{19}, 8_{20} and 8_{21} are distinct from the preceding ones. Alexander and Jones polynomials may help. Note that the spans of the Jones polynomials for these three knots are strictly smaller than 8.
3. Observe that the knot diagrams 8_{19}, 8_{20} and 8_{21} are non alternating.

Then you know that the knots 8_{19}, 8_{20} and 8_{21} are genuine non-alternating knots!

Proceeding like this step by step (7 crossings, then 8 crossings, etc.), and using computers, M. B. Thistlethwaite can go up to 13 crossings. See [Thi].

By inspection among the 12 695 prime knots with at most 13 crossings, he proves that 6 236 of them are non-alternating. This is a striking example (among others) of the effectiveness of Jones polynomial for proving concrete facts.

THEOREM 10.5. *Conjecture D is true.*

Proof. We know that, for a knot,

$$V_K(t) \in \mathbf{Z}[t^{\pm 1}] . \qquad \text{(i.e. no "halves")} .$$

Moreover $V_K(t) = V_{K^*}(t^{-1})$.

So, if K is amphicheiral, the span of V_K must be even.

But, for an alternating knot, the span is equal to the minimal crossing number. Q.E.D.

Note. The two references for L. Kauffman and K. Murasugi's results are [Ka_3] and [Mu_2].

§ 11. Proof of the theorems of L. Kauffman and K. Murasugi

Let Γ be an unoriented link projection in S^2. We shall always suppose that the image is connected, to avoid unnecessary complications. Observe that all projections of an unsplittable link have this property.

We consider the chessboard associated to Γ. To the shaded regions we associate a graph $\Sigma \subset S^2$ in the following way: In each shaded region we select a point which will be a vertex of Σ. If two shaded regions meet at a double point of Γ, we draw an edge joining the two vertices through the double point. (If the two regions are not distinct, we will get a loop.)

We proceed in the same way with the unshaded (lightened) regions, to obtain another graph $\Lambda \subset S^2$.

Notice that, if c is the number of double points of Γ and if R is the number of regions determined by Γ, one has $R = c + 2$. This is an immediate consequence of Euler formula and the fact that the image of Γ is a quadrivalent graph.

Now, let L be an unoriented link diagram and write Γ for the underlying link projection.

Let S be a state of L. We shall associate to S a subgraph Σ_S of Σ and a subgraph Λ_S of Λ in the following way:

(i) Σ_S contains all the vertices of Σ.

(ii) Λ_S contains all the vertices of Λ.

(iii) At each double point of Γ, one edge of Λ and one edge of Σ cross each other. We keep the edge which joins the two regions which are connected by the choice (marker) of S at the crossing point and we discard the other edge.

Lemma 11.1. *Σ_S is a deformation retract of $S^2 - \Lambda_S$ and Λ_S is a deformation retract of $S^2 - \Sigma_S$. In other words, Σ_S and Λ_S are duals in S^2 in the sense of J. H. C. Whitehead.*

Let Γ_S be the configuration of disjoint simple closed curves in S^2 obtained by cutting and glueing Γ at each crossing point according to the indication given by S. By definition, $|S|$ is the number of connected components of Γ_S.

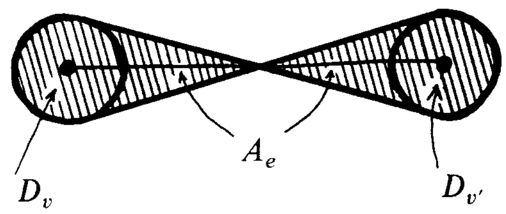
Lemma 11.2. *Γ_S is the boundary of a regular neighborhood of Σ_S in S^2.*

As Σ_S and Λ_S are Whitehead duals, we can replace Σ_S by Λ_S if we wish.

Proof of lemmas 11.1 and 11.2. Let us observe that we can recapture from Σ the union of the shaded regions in the chessboard by the following procedure:

1) Choose a small disc D_v around each vertex v of Σ.

2) For each edge e in Σ, choose a double apex A_e like in the picture:



The union $\bigcup_v D_v \cup \bigcup_e A_e$ is equal, up to an homeomorphism of S^2, to the union of the shaded regions of the chessboard. Its boundary (frontier) is the link projection Γ.

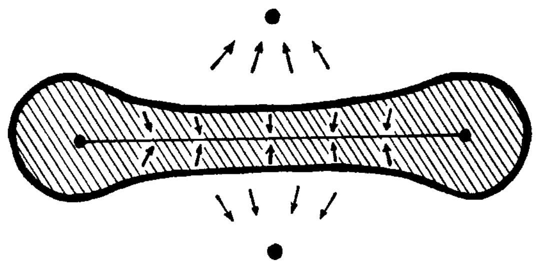
Of course, we could have replaced everywhere in the construction "shaded" by "lightened".

Now, let S be a state for L. Let P be a double point of Γ. The cutting and glueing operation associated to S at P will remove the double point P.

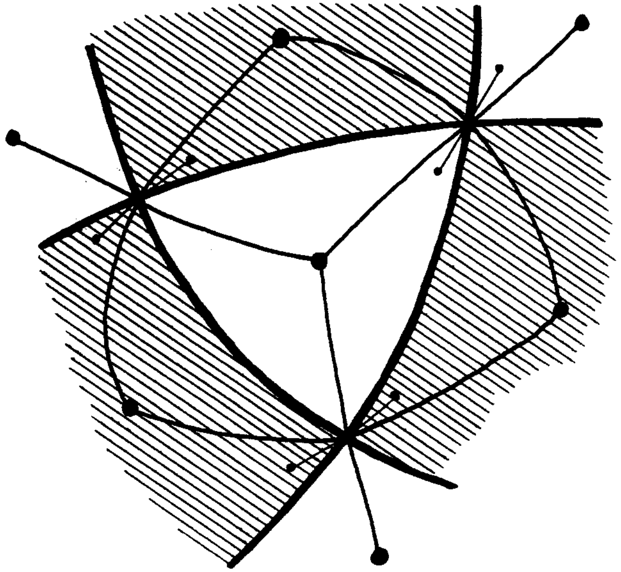
Near P, Γ_S will be the boundary of the shaded surface newly obtained. (And also the boundary of the lightened surface newly obtained.) Suppose, for instance, that the state S chooses at P the marker corresponding to the shaded regions. Then, it is easy to see that, locally around P, the new shaded surface deformation retracts to the edge of Σ_S going through P.

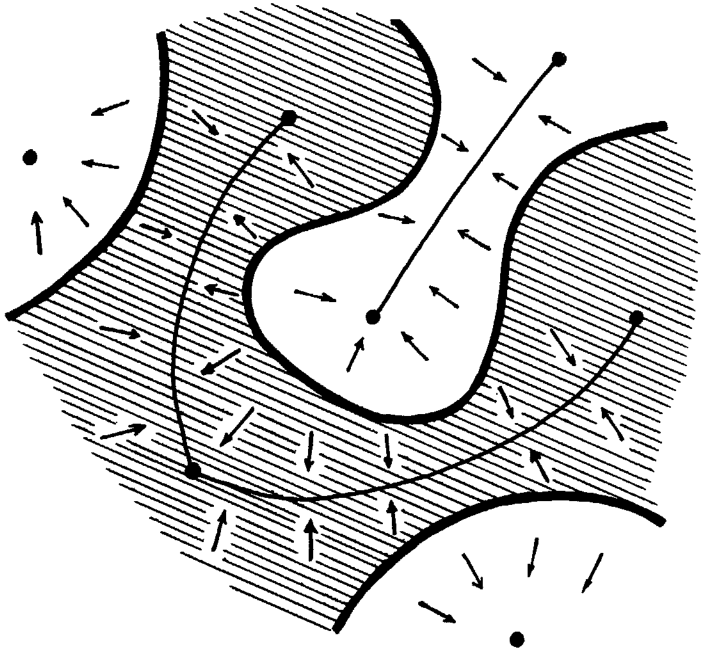
It is also easy to see that, locally around P, the new lightened region deformation retracts on the two vertices of the edge of Λ which has been deleted to obtain Λ_S.

Picture:



The following pictures should help to see what happens locally:

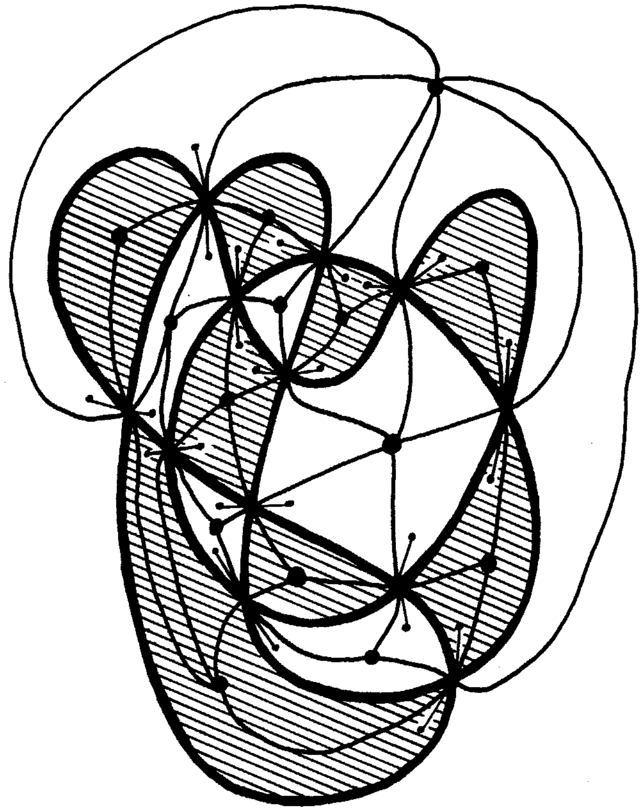


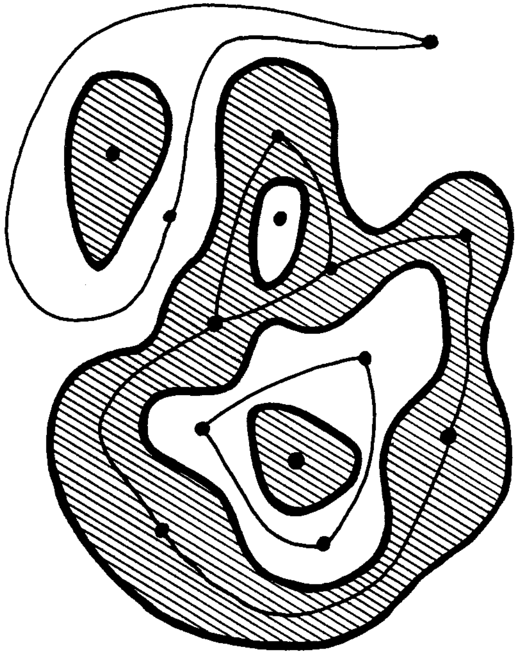


These small deformation retractions can be pieced together in order that globally the newly shaded surface is a regular neighborhood $N(\Sigma_S)$ of Σ_S. In the same way, the newly lightened surface is a regular neighborhood $N(\Lambda_S)$ of Λ_S. The common boundary of $N(\Sigma_S)$ and $N(\Lambda_S)$ is Γ_S.

These constructions are illustrated in the next two pictures. In the first one, a knot projection is shown, with its chessboard, its graphs Σ and Λ. A state S is indicated. The second picture shows Γ_S, Σ_S, Λ_S.

This ends the proofs of lemmas 11.1 and 11.2.





LEMMA 11.3. *Let G be a graph in S^2 and let N be a regular neighborhood of G. Then the number of connected components of ∂N is equal to $b_0(G) + b_1(G)$.*

Notation. $b_i(G)$ denotes the i-th Betti number.

Proof of Lemma 11.3. By Alexander duality:

$$b_0(\partial N) = b_0(N) + b_0(S^2 - N) - 1$$

and

$$b_1(N) = b_0(S^2 - N) - 1 .$$

As N deformation retracts onto G, the result follows.

Recall that the number $|S|$ of connected components of Γ_S is an important ingredient in Kauffman's polynomial.

PROPOSITION 11.4. $|S| = b_1(\Sigma_S) + b_1(\Lambda_S) + 1$.

Note. This proposition is the generalization to any state S of lemma 2 of K. Murasugi's paper [Mu$_2$].

Proof of proposition 11.4. We know that $|S| = b_0(\Gamma_S)$. Now $\Gamma_S = \partial N(\Sigma_S)$. So, if we apply lemma 11.3 to $G = \Sigma_S$, we get

$$b_0(\Gamma_S) = b_0(\Sigma_S) + b_1(\Sigma_S) .$$

As Σ_S and Λ_S are S-duals, Alexander duality implies that

$$b_0(\Sigma_S) = b_1(\Lambda_S) + 1 .$$

We substitute and the proof is finished.

LEMMA 11.5. *Let G be a connected graph. Let G_1 and G_2 be two subgraphs of G such that (1) $G = G_1 \cup G_2$. Let $G_0 = G_1 \cap G_2$ and suppose that (2) G_0 contains no edge. Then*

$$b_1(G_1) + b_1(G_2) \leqslant b_1(G) .$$

Suppose moreover that (3) G_1 and G_2 have no isolated vertices. Then, one has $b_1(G_1) + b_1(G_2) = b_1(G)$ if and only if each vertex of G_0 is a cut vertex (for the partition associated to G_1 and G_2).

Consequence: Suppose that G_1 and G_2 have no isolated vertices and that G has no cut vertex at all. Then, if $b_1(G_1) + b_1(G_2) = b_1(G)$ one has that G_1 or G_2 is empty (and $G_2 = G$ or $G_1 = G$).

Before proving lemma 11.5, we make some comments on the notion of cut vertex.

Let v be a vertex of a graph H. Let E_v be the set of edges of H which have v in their boundary. Suppose given a partition of E_v into two non empty classes E_1 and E_2. Then the chopping of H at v is constructed in the following way:

Replace v by two vertices v_1 and v_2 and declare that the edges in E_i will have v_i in their boundary instead of v $(i = 1, 2)$.

Definition. v is a *cut vertex* for the partition $E_1 \amalg E_2$ if the chopping of H we just described produces a graph with one more connected component. v is a cut vertex if there exists a partition such that... etc., etc.

Proof of lemma 11.5. The inequality is an immediate consequence of Mayer-Vietoris, using that $b_1(G_0) = 0$.

Now observe that conditions (1) and (2) amount to say that G_1 and G_2 produce a (global) partition of the edges of G in two classes.

Suppose that moreover condition (3) is also satisfied. Let v be a vertex of G_0. Then G_1 and G_2 induce a partition of the set E in two non-empty classes. Hence, the chopping of G at v is well defined.

Write $\hat{G}$ for the graph obtained by chopping G at all the vertices of G_0. Remark that G_1 and G_2 naturally embed in $\hat{G}$. Their union is $\hat{G}$ and their intersection is empty. So

$$b_1(G_1) + b_1(G_2) = b_1(\hat{G}).$$

Now, let $\pi: \hat{G} \to G$ be the natural projection which identifies the pairs of vertices created by the chopping. Remark that identifying two vertices has homologically the same effect as adding a new edge between the two vertices. This replaces π by an inclusion. If we write the end of the homology exact sequence of this inclusion, we see immediately that π induces a monomorphism

$$H_1(\hat{G}) \hookrightarrow H_1(G).$$

The same exact sequence shows that the monomorphism is an isomorphism if and only if each vertex of G_0 is a cut vertex for the partition induced by G_1 and G_2.

End of proof of lemma 11.5.

Notation. Let σ_S be the subgraph of Σ_S obtained by removing the isolated vertices of Σ_S. Let λ_S be the subgraph of Λ_S obtained in the same way.

Of course $b_1(\Sigma_S) = b_1(\sigma_S)$ and $b_1(\Lambda_S) = b_1(\lambda_S)$. So, proposition 11.4 gives $|S| = b_1(\sigma_S) + b_1(\lambda_S) + 1$.

Definition. If S is a state, L. Kauffman calls $\check{S}$ the dual state of S if, at every double point of Γ, the choice opposite to S is made.

It is obvious from the definitions that:

(1) $\sigma_S \cup \sigma_{\check{S}} =$

(2) $\sigma_S \cap \sigma_{\check{S}}$ contains no edge.

(3) σ_S and $\sigma_{\check{S}}$ have no isolated vertices.

The same holds for λ_S and $\lambda_{\check{S}}$ in Λ.

LEMMA 11.6. *$b_1(\Sigma) + 1 = l$ = number of lightened region of the chessboard. $b_1 |\Lambda| + 1 = s$ = number of shaded region in the chessboard.*

Proof. Obvious.

PROPOSITION 11.7. $|S| + |\check{S}| \leqslant l+s = R = c+2$.

Comment. This inequality is the "dual state lemma" of L. Kauffman.

Proof of proposition 11.7.

$$\begin{aligned} |S| + |\check{S}| &\leqslant b_1(\sigma_S) + b_1(\lambda_S) + 1 + b_1(\sigma_{\check{S}}) + b_1(\lambda_{\check{S}}) + 1 \\ &\leqslant b_1(\Sigma) + b_1(\Lambda) + 2 = l+s. \end{aligned}$$

Q.E.D.

Recall that L is an unoriented link diagram and that Γ is the underlying link projection. Write A for the state defined by choosing "A" at every double point of L. Write B for the state defined by choosing "B" everywhere. Of course, A and B are dual states.

Notation. If S is a state of L, write $\varphi_S(A)$ for the contribution of the state S to the polynomial $<L>$. $\varphi_S(A)$ is an element of $\mathbf{Z}[A^{\pm 1}]$.

Write D_S for the maximal degree of the monomials in $\varphi_S(A)$ and write d_S for the minimal degree.

LEMMA 11.8. *For any state S one has:*

$$D_S \leqslant D_{\mathrm{A}} \quad \text{and} \quad d_{\mathrm{B}} \leqslant d_S.$$

Proof of lemma 11.8. We prove $D_S \leqslant D_{\mathrm{A}}$, the proof of $d_{\mathrm{B}} \leqslant d_S$ being analogous. Write $b = b(S)$ for the number of times "B" has been chosen in the state S. There is a sequence of states:

$\mathrm{A} = S_0, S_1, \ldots, S_b = S$ where S_i differs from S_{i-1} in one double point of L where the "A" has been replaced by a "B".

CLAIM: $D_{S_i} \leqslant D_{S_{i-1}}$.

Obviously the claim implies that $D_S \leqslant D_{\mathrm{A}}$. Come back to the definition of $<L>$. The contribution of S_i is

$$A^{a(S_i)} B^{b(S_i)} d^{|S|-1},$$

where $B = A^{-1}$ and $d = -(A^2+A^{-2})$. The degree of $A^{a(S_i)} B^{b(S_i)}$ is then

$$a(S_i) - b(S_i).$$

So (*) $a(S_i) - b(S_i) = a(S_{i-1}) - b(S_{i-1}) - 2$.

Moreover: $|S_{i-1}| - 1 \leqslant |S_i| \leqslant |S_{i-1}| + 1$.

So (**) the maximal degree in A of $(-A^2-A^{-2})^{|S_i|-1}$ is at most two more than the one of $(-A^2-A^{-2})^{|S_{i-1}|-1}$.

Putting together (*) and (**) finishes the proof of lemma 11.8.

An easy computation shows that:

$$D_A = c + 2(|A|-1),$$
$$d_B = -[c+2(|B|-1)].$$

Proof of theorem 10.1. Let L be any projection of an unsplittable link K in $\mathbf{R}^3$. Then

$$\text{Span } f_L = \text{span } <L> \leqslant D_A - d_B$$

and

$$D_A - d_B = c + c + 2|A| + 2|B| - 4 \leqslant 2c + 2R-4$$
$$= 2c + 2c + 4-4 = 4c.$$

As $V_K(t) = f_L(t^{1/4})$, this gives at once a proof of theorem 10.1.

We now proceed towards the proof of theorems 10.2 and 10.3.

LEMMA 11.9. *Let L be a link diagram. Then L is alternating if and only if either all the "A" are shaded or all the "B" are shaded.*

Recall that we suppose that the image of the projection is connected. Recall also that our convention to make a projection alternating was that the "A" should be shaded.

This lemma is essentially Tait's theorem of § 9.

LEMMA 11.10. *Let L be a link diagram, alternating according to the convention. Suppose L without nugatory crossing, i.e. L reduced. Let S be any state, distinct from* A *and* B. *Then*

$$D_S < D_A \quad \text{and} \quad d_B < d_S.$$

Proof of lemma 11.10. The proof begins like the proof of lemma 11.8. We assert that, because the link diagram is reduced, one has

$$D_{S_1} < D_{S_0} = D_A.$$

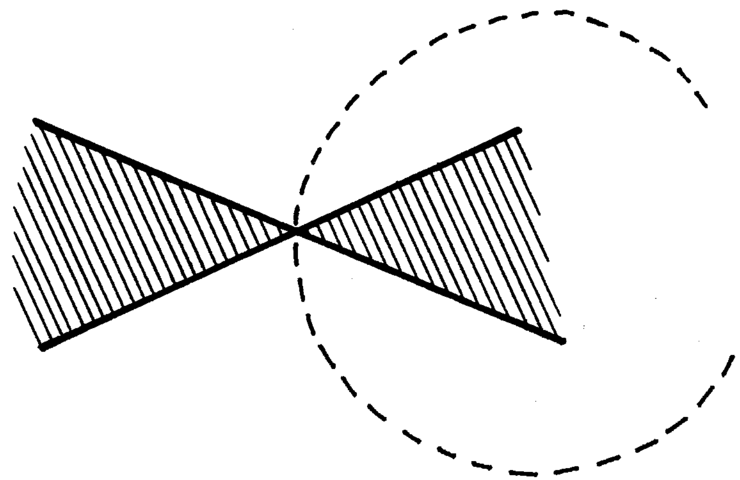
If the reader goes back to lemma 11.8, he will see that the assertion is all that is needed to get lemma 11.10.

We prove the assertion:

As the link diagram alternates, according the convention the "A" are shaded. So $|A| = l$ = number of lightened regions.

We claim that $|S_1| = l-1$, the reason being the following: At exactly one double point P of Γ, the marker has passed from A = shade to B = light. By this operation, two *different* lightened regions have been connected, and the newly shaded surface is still connected. (This immediately implies $|S_1| = l-1$.)

If not, the lightened spots in the neighborhood of P would belong to the same lightened region. One could thus draw a circle entirely in the light, joining the two spots:



This means that L would not be reduced, contrary to the hypotheses. The same kind of argument proves $d_B < d_S$.

This finishes the proof of lemma 11.10.

Notation. Let S be the state obtained by choosing "shade" at every double point and let L be the state obtained by choosing "light" at every double point. Of course, S and L are dual states.

LEMMA 11.11. $|S| + |L| = R$.

Proof of lemma 11.11. One has

$$\sigma_S = \Sigma \qquad \lambda_S = \emptyset$$

and

$$\sigma_L = \emptyset \qquad \lambda_L = \Lambda.$$

Then apply the proof of proposition 11.7. Q.E.D.

Proof of theorem 10.2. First of all, we do not restrict the generality by supposing that the diagram alternates according to the convention.

Now lemma 11.10 implies that the highest degree of the monomials in $<L>$ is D_A and that the lowest degree is d_B. The coefficients of these monomials are different from zero.

Moreover A = S and B = L.

So $|A| + |B| = R$ by lemma 11.11.

Hence:

$$\text{Span} <L> = D_A - d_B = 2c + 2|A| + 2|B| - 4 = 2c + 2R - 4$$
$$= 2c + 2(c+2) - 4 = 4c .$$

As span $V_K(t) = \frac{1}{4}$ span $<L>$, this finishes the proof.

PROPOSITION 11.12. *Suppose that the graphs* Σ *and* Λ *have no cut vertex. Suppose that for a state* S *we have*

$$|S| + |\check{S}| = R .$$

Then S = S or S = L.

Remark. Σ and Λ have no cut vertex if and only if Γ is not a non-trivial connected sum. See also proof of prop. 11.7.

The proof of proposition 11.12 follows immediately from the consequence of lemma 11.5.

Remark. There is an obvious generalisation of proposition 11.12 to the case of a connected sum. Use the full lemma 11.5 instead of its consequence.

We now state an equivalent form of theorem 10.3.

THEOREM 10.3′. *Let* L *be a link diagram such that* Σ *and* Λ *have no cut vertex. (This will be fulfilled if the link is prime.) Suppose that* span $V_K(t) = c(L)$. *Then* L *is reduced and alternating.*

Remark. There is a generalisation of theorem 10.3′ to the case of a connected sum: the only possible counter-examples to non-alternativity are non-alternating connected sums of alternating links, as in the square knot. We leave this to the reader. (Use generalisation of proposition 11.12.)

Proof of theorem 10.3′. If L were not reduced, we could reduce it. But this would contradict theorem 10.1.

Now, the computation of $D_A - d_B$ in the proof of theorem 10.1 shows that, if span $<L> = 4c$, one has $D_A - d_B = 4c$ and so $|A| + |B| = R$.

As Σ and Λ have no cut vertex, the proposition 11.12 implies that A = S or A = L.

By lemma 11.9, this means that L is alternating. Q.E.D.

§ 12. THE PATH FROM VON NEUMANN ALGEBRAS TO KNOT POLYNOMIALS

The discovery of the knot polynomials discussed here is due to Jones' investigations on von Neumann algebras, and not to the flourishing activity in low dimensional topology. In the light of previous work by J. Conway on Alexander's polynomial and of subsequent work by L. Kauffman (among others) on Jones' polynomial, such a genesis may seem unexpected. However this cannot be challenged, and should indeed appear rather as a delight of the subject than as any unpleasant awkwardness. With this point of view, we offer some guidelines for (some of) the surprising relationships put into light by V. Jones' work.

FACTORS OF TYPE II_1

An involution on a complex algebra M is a conjugate linear transformation $x \mapsto x^*$ of M such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in M$. The algebra $L(H)$ of all continuous operators on a Hilbert space H has a canonical involution, with x^* the adjoint of x, defined by $<x^*\xi|\eta> = <\xi|x\eta>$ for all $\xi, \eta \in H$. A representation of an involutive algebra M on H is a morphism of algebras $\pi: M \to L(H)$ with $\pi(x^*) = (\pi(x))^*$ for all $x \in M$. The algebra $L(H)$ carries several useful topologies, and in particular the weak topology, for which a sequence $(x_i)_{i \in I}$ of operators converges to 0 iff the numerical sequences $(<x_i\xi|\eta>)_{i \in I}$ converge to 0 for all pairs (ξ, η) of vectors in H.

A von Neumann algebra is an involutive algebra M with unit which has a faithful representation π on H with $\pi(1) = \text{id}$ and with $\pi(M)$ a weakly closed self-adjoint subalgebra of $L(H)$. (There are several equivalent definitions: see any textbook on the subject, for example one of [Di], [SZ], [Tak].) A von Neumann algebra is defined to be a *factor of type* II_1 if

(1) The center of M is reduced to scalar multiples of 1.

(2) There exists a normalized finite trace, namely a linear form $\text{tr}: M \to \mathbf{C}$ with $\text{tr}(1) = 1$ and $\text{tr}(xy) = \text{tr}(yx)$ for all $x, y \in M$.

(3) The dimension of M over $\mathbf{C}$ is infinite.

Moreover, if M is a factor of type II_1:

(4) There exists a *unique* normalized finite trace.

(5) For any real number $d \in [0, 1]$, there exists a self-adjoint idempotent $e \in M$ with $\mathrm{tr}(e) = d$.

(6) The trace is positive and faithful: $\mathrm{tr}(x^*x) \geqslant 0$ for all $x \in M$, with equality for $x = 0$ only.

(7) The algebra M is simple. In particular, any representation of M is faithful.

Let us add three comments. The notion of trace used in (2) may seem slightly unusual in the context of operator algebras, but is the same as the standard notion because we consider factors of type II_1 only; see [FH]. Because of (5), factors of type II_1 are also called finite and continuous. Concerning (7), the following may be added under suitable separability assumptions: Murray and von Neumann have defined for any representation of M a multiplicity, which is a positive number (possibly infinite), and two representations of M are unitarily equivalent iff they have the same multiplicity.

A factor M of type II_1 is said to be *hyperfinite* if it has the following property: for any integer $n \geqslant 1$, for any sequence $x_1, \ldots, x_n \in M$ and for any $\varepsilon > 0$, there exists a finite dimensional self-adjoint subalgebra K of M such that

$$d_2(x_j, K) < \varepsilon, \qquad j = 1, \ldots, n$$

where d_2 is the distance associated to the norm $x \mapsto \mathrm{tr}(x^*x)^{1/2}$ on M. Murray and von Neumann showed that two hyperfinite factors of type II_1 which can be represented on a separable Hilbert space are *-isomorphic; the standard notation for "the" hyperfinite factor of type II_1 is R. Moreover, they showed that any factor of type II_1 contains a copy of R [MN]. Instead of "hyperfinite", the factor R is also called "approximately finite dimensional", "injective", "semi-discrete" or "amenable", and there is a good reason for each of these words. A sub-factor of R is either finite dimensional or isomorphic to R itself [Co_1]. The importance of R in the theory cannot be overemphasized.

Consider for example a countable group Γ, the Hilbert space $l^2(\Gamma)$ of complex functions $\xi : \Gamma \to \mathbf{C}$ with $\sum_{g \in \Gamma} |\xi(g)|^2 < \infty$, the right regular representation $\rho : \Gamma \to L(l^2(\Gamma))$ defined by $(\rho(g)\xi)(h) = \xi(hg)$, and the algebra $W^*(\Gamma)$ of operators x on $l^2(\Gamma)$ such that $x\rho(g) = \rho(g)x$ for all $g \in \Gamma$. It can be shown that $W^*(\Gamma)$ is the von Neumann algebra generated by $\lambda(g)$ for $g \in \Gamma$, where $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$. If all conjugacy classes (other than $\{1\}$) in Γ are infinite, then $W^*(\Gamma)$ is a factor of type II_1; moreover it makes sense to write any element in $W^*(\Gamma)$ as a (usually infinite) sum $\sum_g z_g \lambda(g)$, and the normalised trace of such an element is z_1. Assuming that Γ has infinite conjugacy classes and moreover that Γ contains an element a of infinite order, we may formulate a nice exercise to illustrate property (5) above: for any $d \in [0, 1]$, show that the infinite sum

$$d + \sum_{\substack{n \in \mathbf{Z} \\ n \neq 0}} \frac{\sin(dn\pi)}{n\pi} \lambda(a^n)$$

defines in $W^*(\Gamma)$ a self-adjoint idempotent of normalized trace d (solution in [Au]).

If Γ has infinite conjugacy classes and is moreover amenable, then $W^*(\Gamma)$ is a model for the hyperfinite factor R, by [Co_1]. Examples of amenable groups: the group of permutations with finite supports of a countable set, or any solvable group.

To cut a long story short, Murray and von Neumann knew of two non isomorphic factors of type II_1, namely R and $W^*(\Gamma)$ for Γ the non abelian free group on two generators [MN]. J. Schwartz established the existence of a third one twenty years later [Sc], and D. McDuff showed there are uncountably many [McD]. During the 1970's, A. Connes made several break-throughs in the knowledge of factors; for a review of the subject before 1980, see [Co_2]. By then, it was reasonable for V. Jones to embark in the study of *relative* problems: understand subfactors (of type II_1) in a given factor of type II_1.

The index

Let $M_0 \subset M_1$ be a pair of factors of type II_1. It is natural to look for invariants of these data, with respect to conjugacy of M_0 by (possibly inner) automorphisms of M_1. For the present discussion, the most successful invariant is the *index* $[M_1 : M_0] \in [1, \infty]$. Its definition appears in [Jo_1] and [Jo_2]; see also below.

Once the index is defined, the most obvious problem is to compute exactly its possible values. If M_1 is the hyperfinite factor of type II_1, then the set of possible values $[M_1 : M_0]$ consists of a continuous spectrum $[4, \infty]$,

a discrete spectrum $\{4\cos^2(\pi/n)\}_{n=3,4,5,\ldots}$.

This was quite a surprise at the time, as continuity is so often the rule for objects defined by M_1. (If the factor M_1 is not hyperfinite, our knowledge is fragmentary and the possible values for $[M_1 : M_0]$ may constitute a proper subset of the spectrum just described. See [PP].)

Let us now define the index and indicate some steps in the proof of Jones' result about its spectrum. Given a pair $M_0 \subset M_1$, there exists a *conditional expectation* $e_1 : M_1 \to M_0$ which is a projection such that $e_1(axb) = ae_1(x)b$ and $\mathrm{tr}(e_1(x)) = \mathrm{tr}(x)$ for $a, b \in M_0$ and $x \in M_1$. In fact both e_1 and elements in M_1 may be looked at as operators on the Hilbert space $L^2(M_1, \mathrm{tr})$ obtained by completion of M_1 for the scalar product $<x|y> = \mathrm{tr}(x^*y)$; then e_1 is the orthogonal projection of M_1 onto M_0, and $x \in M_1$ acts on $L^2(M_1, \mathrm{tr})$ as the extension of the multiplication $y \mapsto xy$.

Thus it makes sense to consider the von Neumann algebra M_2 generated by e_1 and M_1. With one exception which is precisely the case in which $[M_1 : M_0] = \infty$, the algebra M_2 is again a factor of type II_1. In the later case, the definition of the index is

$$[M_1 : M_0] = \frac{1}{\mathrm{tr}_2(e_1)}$$

where tr_2 denotes the trace on M_2.

As $M_1 \subset M_2$ is again a pair as above, the same construction may be iterated, and one obtains a *tower*

$$M_0 \subset M_1 \subset \ldots \subset M_n \subset M_{n+1} = <M_n, e_n> \subset \ldots$$

of factors of type II_1. A basic fact is that the e_i's satisfy three types of relations

idempotence: $e_i^2 = e_i$,

braiding: $e_i e_{i\pm 1} e_i = [M_1 : M_0]^{-1} e_i$,

commutation: $e_i e_j = e_j e_i$ if $|i-j| \geqslant 2$.

Also the traces on the M_n's induce a trace tr on the algebra generated by the e_i's with

Markov property: $\mathrm{tr}(we_i) = [M_1 : M_0]^{-1}\mathrm{tr}(w)$ for w in the algebra generated by $M_0, e_1, \ldots e_{i-1}$.

The invocation of Markov here refers to the property of the trace: its value on each step $M_{n+1} = <M_n, e_n>$ is readily computable in terms of the trace on the previous step M_n. There is moreover the crucial tool of

positivity: the algebra of operators generated by the e_i's has an involution $w \mapsto w^*$ and $\mathrm{tr}(w^*w) > 0$ for any $w \neq 0$ in this algebra.

An analysis of these properties shows that, in case the index is smaller than 4, then only the discrete spectrum.

$$[M_1 : M_0] \in \{4\cos^2(\pi/n)\}_{n \geqslant 3}$$

is permitted. (The reader will have some flavour of the analysis if he solves the following exercise: consider four unit vectors $e_1, \ldots, e_4$ in the usual 3-space such that the scalar products satisfy

$$<e_1|e_2> = <e_2|e_3> = <e_3|e_4> = \cos\varphi$$
$$<e_1|e_3> = <e_1|e_4> = <e_2|e_4> = 0$$

for some angle φ; then $\cos\varphi = 1/2(\sqrt{5}-1)$ and φ can only be one of two possible angles.)

Constructing pairs with $[M_1 : M_0] \geqslant 4$ turns out to be easy (at least when M_1 is hyperfinite). For the discrete spectrum, consider first a complex number $\beta \neq 0$, an integer $n \geqslant 1$, and the algebra $\mathscr{A}_{\beta,n}$ abstractly defined (as a complex associative algebra) by

generators: $1, \varepsilon_1, \ldots, \varepsilon_{n-1}$,

relations:
$$\begin{cases} \varepsilon_i^2 = \varepsilon_i, \\ \varepsilon_i\varepsilon_{i\pm1}\varepsilon_i = \beta^{-1}\varepsilon_i, \\ \varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i \quad \text{if} \quad |i-j| \geqslant 2. \end{cases}$$

If $\beta > 0$, the construction of a pair with $[M_1 : M_0] = \beta$ reduces to finding a representation of $\mathscr{A}_{\beta,\infty} = \lim_{n\to\infty} \mathscr{A}_{\beta,n}$ by operators on a Hilbert space with each ε_i self-adjoint. Manipulations of linear algebra show that this can be done precisely when β is in the spectrum of indices; see Jones' papers, as well as the expository [GHJ].

Note finally that the e_i's and the ε_i's should not be confused: Given some pair $M_0 \subset M_1$ of index β, it is of course obvious that $\mathscr{A}_{\beta,n}$ maps onto the algebra generated by $1, e_1, \ldots, e_{n-1}$. But for β in the discrete spectrum, this map has a non trivial kernel when n is large enough.

HECKE ALGEBRAS AND POLYNOMIALS

One of the main points to retain from above is the following: an interesting problem with a surprising solution in the theory of von Neumann algebras has motivated a serious study of the algebras $\mathscr{A}_{\beta,n}$. Now $\mathscr{A}_{\beta,n}$ appears to be in close relationship with

(a) Artin's braid group B_n with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations as in § 6.

(b) The Hecke algebra of § 4, that we denote from now on by $H_{q,n}$ to stress the dependence on q, where parameters fit well if $\beta = 2 + q + q^{-1}$.

To make this relationship transparent, we turn to another presentation of $\mathscr{A}_{\beta,n}$. Choose a complex number q with $\beta = 2 + q + q^{-1}$ (observe that $q \neq -1$ as $\beta \neq 0$) and set

$$T_i = q\varepsilon_i - (1-\varepsilon_i) \quad \text{so that} \quad \varepsilon_i = \frac{T_i + 1}{q + 1}$$

for $i = 1, \dots, n-1$. Then a straightforward computation shows that $\mathscr{A}_{\beta,n}$ has a presentation with generators $T_1, \dots, T_{n-1}$ and relations

(1) $T_i^2 = (q-1)T_i + q$,

(2) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$,

(3) $T_iT_j = T_jT_i$ if $|i-j| \geqslant 2$,

(S) $T_iT_{i+1}T_i + T_iT_{i+1} + T_{i+1}T_i + T_i + T_{i+1} + 1 = 0$.

The last relation was first pointed out by R. Steinberg. One has now more precisely:

(a) The assignment $\sigma_i \mapsto T_i$ extends to a homomorphism ρ_q from B_n to the invertible elements of $\mathscr{A}_{\beta,n}$ (compare with § 6).

(b) $\mathscr{A}_{\beta,n}$ is the quotient of the Hecke algebra $H_{q,n}$ of § 4 by the relation (S).

For infinitely many values of q (namely $q \in \mathbf{R}$ and $q \geqslant 1$, corresponding to $\beta \geqslant 4$), Jones knew from his study of factors [Jo$_2$] that $\mathscr{A}_{\beta,n}$ is given with a faithful positive Markov trace tr. For each braid $\alpha \in B_n$, he set

$$V_\alpha(q) = -\left(\frac{q+1}{q^{1/2}}\right) q^{e/2}\operatorname{tr}(\rho_q(\alpha))$$

where e is the exponent sum of α as a word on the σ_i's. The first theorem in [Jo$_3$] is that V_α depends only on the link $K(\alpha)$ obtained by closing α. Also $V_\alpha(q)$ [respectively $q^{1/2}V_\alpha(q)$] is a Laurent polynomial in q

if $K(\alpha)$ has an odd [resp. even] number of components; in particular $V_\alpha(q)$ can be defined for any $q \in \mathbf{C}$, not just for those corresponding to good traces on some $\mathscr{A}_{\beta,n}$. And, most importantly for the early growth of the subject, a computation in the summer 1984 with the trefoil knot showed that V is not a mere variant of the Alexander polynomial. In fact, during a few hours, this was thought to reveal a mistake in computations! See end of § 7 for more details on the independence of the polynomials.

One way to recover the two variable polynomial is to introduce a *family* of traces on $H_{q,\infty} = \lim_{n\to\infty} H_{q,n}$, indexed by a complex parameter z. This programme was pursued by Ocneanu, and exposed in §§ 5-6 above. Observe that

(1) Only one of Ocneanu's traces pass to the quotient $\mathscr{A}_{\beta,\infty}$, namely that corresponding to $z = q(q+1)^{-2}$.

(2) Ocneanu's traces are positive for some values of the pair (q, z) only: the picture appears in Wenzl's thesis [We] and also in [Jo$_4$].

(3) It does help to keep positivity considerations in mind when studing knot polynomials: see § 14 in [Jo$_5$].

ADDED IN PROOF

1. V. Turaev has another and simpler proof of some of the geometric arguments given in § 11. See a next issue of this journal.

2. K. Murasugi has informed us that he has now proved conjecture C.

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