Definition 3: $\lambda(y^0) = -\sup \{\lambda': u(\tau,0,u_0,y^0) \exp \lambda' \tau$ is bounded for all τ, $0 \leqq \tau < +\infty\}$.

Type numbers (or their negatives, Lyapunov numbers) are usually defined somewhat more generally than this,[2] but only the radial type numbers defined above are needed here. Now, any hypothesis concerning the real parts of the characteristic roots of A when $k = 1$ can be formulated in terms of the radial type numbers of the limit solutions when $k \geqq 1$. For it can be shown that when $k = 1$, the set of radial type numbers of the limit solutions of (3) is precisely the set of real parts of the characteristic roots of A. In particular, Lyapunov's theorem can be generalized.

THEOREM. *If $k \geqq 1$ and if the radial type number of every limit solution of (3) is negative, then the trivial solution, $x = 0$, of (1) is asymptotically stable.*

The proof consists of showing that the hypothesis implies that the radial type number of every solution of (3) is negative and that this in turn implies that the trivial solution of (2) is asymptotically stable. A theorem of Zubov[3] is used to prove this latter assertion. Finally, it is a theorem of Massera[4] that if the trivial solution of (2) is asymptotically stable, so is the trivial solution of (1).

3. The results of an earlier paper[5] and some remarks made by S. Lefschetz were the stimuli for what has been presented here. It should be noted that the main theorem of reference (5) is a special case of the theorem stated above.

* This research was supported in part by the National Science Foundation under contract number NSF G 12932.

[1] Lefschetz, S., *Differential Equations: Geometric Theory* (New York: Interscience, 1957).

[2] Cesari, L., *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations* (Berlin: Springer, 1959).

[3] Zubov, V., *Metody, A. M. Lyapunova i ih Primenenie* (Leningrad: 1957).

[4] Massera, J., "Contributions to Stability Theory," *Ann. Math.*, **64**, 182–205 (1956).

[5] Coleman, C., "Asymptotic Stability in 3-Space," in *Contributions to the Theory of Nonlinear Oscillations*, ed. S. Lefschetz (Princeton University Press: 1961), vol. 5, pp. 257–268.

ON 2-SPHERES IN 4-MANIFOLDS

BY MICHEL A. KERVAIRE AND JOHN W. MILNOR

INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, AND DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated by N. E. Steenrod, August 23, 1961

Let M^{2n} be a simply connected differentiable manifold, and let $\xi \epsilon \pi_n(M^{2n})$ be a given homotopy class of maps $S^n \rightarrow M^{2n}$. It is known that if $n > 2$, the class ξ can be represented by a differentiable imbedding f: $S^n \rightarrow M^{2n}$. This follows from a reasoning similar to the one used by H. Whitney[9] to prove that every differentiable n-manifold can be differentiably imbedded in Euclidean $2n$-space. (Compare Milnor,[6] Lemma 6.) It is also included in a more general theorem of A. Haefliger.[3] Both arguments, however, break down for $n = 2$. This leads to the following question:

Let M^4 be a simply connected differentiable manifold. Is every element of $\pi_2(M^4)$ representable by a differentiably imbedded sphere?

In 1951, Rohlin[7] announced the erroneous result that the stable 3-stem $\pi_{n+3}(S^n)$, $n \geq 5$, was cyclic of order 12. (This result was corrected in a subsequent paper.[8]) It was pointed out to us by Rohlin that the only mistake in reference 7 was to take for granted an affirmative answer to the above problem. Indeed, a specific counter-example can be extracted from Rohlin's paper, using the fact that $\pi_{n+3}(S^n) \cong Z_{24}$ for $n \geq 5$.

In the following, we present a generalized version of Rohlin's counter-example and also study the corresponding combinatorial problem.

Let M be a 4-dimensional closed connected, oriented, differentiable manifold. The *signature* $\sigma(M)$ *of* M is defined to be the signature of the symmetric bilinear form

$$H_2(M; Q) \otimes H_2(M; Q) \rightarrow Q$$

determined by the intersection number. We will say that an integral homology class $\xi \in H_2(M; Z)$ is *dual* to the Stiefel-Whitney class $w_2(M)$ if the natural homomorphisms

$$H_2(M; Z) \rightarrow H_2(M; Z_2) \overset{\cong}{\rightarrow} H^2(M; Z_2)$$

(reduction mod 2 followed by Poincaré duality) carry ξ into $w_2(M)$.

THEOREM 1. *Let $\xi \in H_2(M; Z)$ be dual to the Stiefel-Whitney class $w_2(M)$. If ξ is represented by a differentiably imbedded 2-sphere in M, then the self-intersection number $\xi \cdot \xi$ must be congruent* to $\sigma(M)$ modulo 16.*

Two examples will illustrate this theorem. First let M be the product $S^2 \times S^2$ and let $\alpha, \beta \in H_2(S^2 \times S^2; Z)$ be the standard generators. Thus, $\alpha + \beta$ is the diagonal class.

COROLLARY 1. *The homology class $2(\alpha + \beta)$ is not represented by any differentiably imbedded 2-sphere in $S^2 \times S^2$.*

Next, let M be the complex projective plane $PC(2)$, with generator $\gamma \in H_2(PC(2); Z)$.

COROLLARY 2. *The homology class 3γ in $PC(2)$ is not represented by any differentiably imbedded 2-sphere.*

A third example will be presented at the end of the paper. In contrast, we will prove the following.

THEOREM 2. *In either $S^2 \times S^2$ or $PC(2)$, every 2-dimensional homology class can be represented by a combinatorially imbedded 2-sphere.*

(In general, this 2-sphere will have exceptional points at which it is not locally flat.) The proof of Theorem 2 will apply to some other manifolds. (See Lemma 2 below.) However, it is not known whether every two-dimensional homology class in every simply connected 4-manifold can be represented by a combinatorially imbedded sphere.

Proof of Theorem 1: We will use the following known result (see Rohlin,[8] Milnor and Kervaire[5]):

LEMMA 1. *If the Stiefel-Whitney class $w_2(M^4)$ is zero, then† the signature $\sigma(M^4)$ is congruent to zero modulo 16.*

(Note that this lemma can be considered as a special case of Theorem 1, corresponding to the choice $\xi = 0$.)

Reversing the orientation of M if necessary, we may assume that the self-intersection number $\xi \cdot \xi$ is nonpositive. Let $P_1, \ldots, P_{s+1}$ be $s + 1$ copies of the complex projective plane $PC(2)$, where $s = -\xi \cdot \xi$.

Form the connected sum

$$M_1 = M \# P_1 \# \ldots \# P_{s+1}.$$

Using the natural isomorphism

$$j: \quad H_2(M; Z) \oplus H_2(P_1; Z) \oplus \ldots \oplus H_2(P_{s+1}; Z) \rightarrow H_2(M_1; Z),$$

let

$$\eta = j(\xi \oplus \gamma_1 \oplus \ldots \oplus \gamma_{s+1}).$$

Here γ_i denotes a generator of $H_2(P_i; Z)$. Note that the self-intersection number $\eta \cdot \eta$ is given by

$$\eta \cdot \eta = \xi \cdot \xi + \textstyle\sum_i \gamma_i \cdot \gamma_i = +1.$$

Using the hypothesis that ξ can be represented by a differentiable imbedding of S^2 in M, it follows easily that η can be represented by a differentiable imbedding

$$f: \quad S^2 \rightarrow M_1.$$

Since $\eta \cdot \eta = 1$, the Euler class of the normal bundle of $f(S^2) \subset M_1$ is the standard generator of $H^2(S^2; Z)$. (Compare the argument on page 51 of ref. 6.) Hence, the normal circle-bundle of $f(S^2) \subset M_1$ is the Hopf fibration $S^3 \rightarrow S^2$.

Let M_2 be the new differentiable manifold obtained from M_1 by removing a tubular neighborhood of $f(S^2)$ and replacing it by a 4-cell, matching the 3-sphere boundaries. Evidently, the original M_1 is diffeomorphic to the connected sum $M_2 \# PC(2)$. Hence, we have

$$\sigma(M_1) = \sigma(M_2) + 1.$$

We claim that the Stiefel-Whitney class $w_2(M_2)$ is zero. Clearly, η is dual to the Stiefel-Whitney class $w_2(M_1)$. This implies that the class $w_2(M_1 - f(S^2))$ is zero; hence $w_2(M_2) = 0$. Therefore, we have $\sigma(M_2) \equiv 0$ modulo 16; hence,

$$\sigma(M_1) \equiv 1 \pmod{16}.$$

On the other hand,

$$\sigma(M_1) = \sigma(M) + \sigma(P_1) + \ldots + \sigma(P_{s+1}) = \sigma(M) + s + 1.$$

Thus,

$$\sigma(M) + s \equiv 0 \pmod{16}$$

which completes the proof of Theorem 1.

Example 1: Let $M = S^2 \times S^2$ and let $\xi = p\alpha + q\beta$ be an arbitrary element of $H_2(M; Z)$. Then $\xi \cdot \xi = 2pq$. Since $w_2(M) = 0$, the class ξ is dual to $w_2(M)$ if and only if p and q are both even. If $p \equiv q \equiv 2 \pmod 4$, then $\xi \cdot \xi \equiv 8 \pmod{16}$. Hence, such a ξ cannot be represented by a differentiably imbedded 2-sphere in $S^2 \times S^2$. This proves Corollary 1. On the other hand, if $|p| < 2$, then the class $p\alpha + q\beta$ *is* representable by a differentiably imbedded sphere. In general, for higher values of p and q, the problem remains unsolved.

Example 2: Let M be the complex projective plane $PC(2)$, and let $\xi = r\gamma$. Then, $\xi \cdot \xi = r^2$. The class ξ is dual to $w_2(M)$ if and only if r is odd. If $r \equiv \pm 3 \pmod 8$, then

$$\xi \cdot \xi = r^2 \equiv 9 \pmod{16}.$$

Since $\sigma(M) = 1$, this shows that such a ξ is not representable by a differentiably imbedded sphere. This proves Corollary 2.

On the other hand, for $|r| < 3$, the class $\xi = r\gamma$ is representable. The problem remains unsolved for other $r \not\equiv \pm 3 \pmod 8$.

The proof of Theorem 2 is elementary. We will first illustrate the proof idea in the case of the projective plane $PC(2)$. We will then prove a more general statement (Lemma 2 below) which includes Theorem 2 as a special case.

$PC(2)$ can be regarded as the union $U \cup V$ of a tubular neighborhood U of the projective line $PC(1) = S^2 \subset PC(2)$ and a 4-disk V. Let p: $U \rightarrow S^2$ be the normal projection. The inverse image $p^{-1}(x)$ of a point $x \in S^2$ is a 2-disk. The boundary $U^{\cdot}$ of U is diffeomorphic to S^3, and p restricted to $U^{\cdot}$ is the Hopf fibration $S^3 \rightarrow S^2$. Let m be a positive integer, and let D_i, $i = 1,\ldots,m$, denote the 2-disk $p^{-1}(x_i)$, where $x_1,\ldots,x_m$ are distinct points on S^2. The disks $D_1,\ldots,D_m$ are oriented so that their intersection numbers with $PC(1)$ are equal to $+1$. Let f_j: $I \times I \rightarrow U^{\cdot}$, $j = 1,\ldots,m-1$, be imbeddings such that

(1) $f_j(I \times 0) \subset D_j^{\cdot}$ and $f_j(I \times 1) \subset D_{j+1}^{\cdot}$, where $f_j|I \times 1$ is orientation-preserving and $f_j|I \times 0$ orientation-reversing;

(2) The images E_j of f_j, $j = 1,\ldots,m-1$, are disjoint, and intersect $D_k^{\cdot}$ only as specified in (1). (See Figs. 1 and 2.)

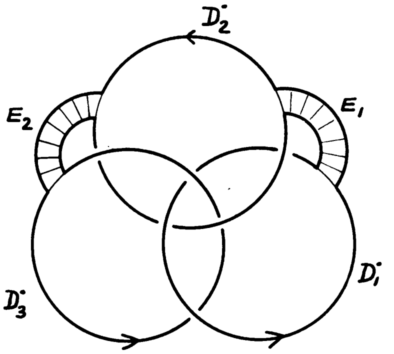

FIG. 1.—$F^{\cdot}$ is a right-handed trefoil knot.

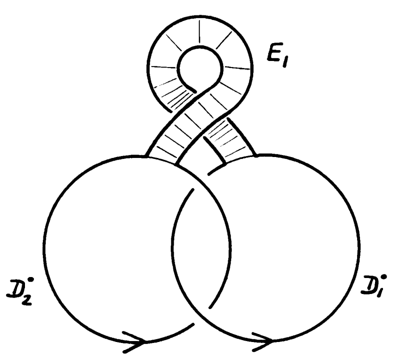

FIG. 2.

In other words, we connect the disks $D_1,\ldots,D_m$ with each other using $m - 1$ strips $E_1,\ldots,E_{m-1}$ imbedded in $U^{\cdot}$ so that the resulting surface with boundary $F = D_1 \cup E_1 \cup D_2 \cup \ldots \cup D_m$ has an orientation compatible with the orientations of $D_1,\ldots,D_m$. The Euler characteristic of F is 1, hence F is a 2-element in U. The boundary $F^{\cdot}$ of F is a circle imbedded in the 3-sphere $U^{\cdot} = V^{\cdot}$. Since V is a 4-disk, we can extend the imbedding $F^{\cdot} \subset V^{\cdot}$ to a (combinatorial) imbedding $CF^{\cdot} \rightarrow V$ of the cone over the boundary of F into V. [NOTE: In general, $F^{\cdot} \subset V^{\cdot}$ will be a knotted circle. Hence, the imbedded disk $CF^{\cdot} \subset V$ will have a "singular point" (Cf. ref. 2) at the vertex of the cone.] The surface $F \cup CF^{\cdot}$ is a 2-sphere Σ^2 combinatorially imbedded in $PC(2)$. Clearly Σ^2 represents the homology class $m\gamma$ providing that its orientation is compatible with the orientations of the disks D_i. Otherwise, it represents the class $(-m\gamma)$.

Remark.—It is not hard to see that the strips connecting the disks $D_1, \ldots, D_m$ can be chosen so that $F^\cdot$ is a torus knot of type $(m - 1, - m)$ in S^3. For example, for $m = 2$, the strip E_1 can be chosen so that $F^\cdot$ is unknotted. For $m = 3$, we obtain the (right-handed) trefoil knot as illustrated by Figure 1. It follows from the statement in Example 1 that for $m \equiv 3$ or 5 modulo 8, such a knot does not belong to the trivial knot cobordism class. (Compare R. Fox and J. Milnor.[2] Actually, Fox and Milnor have proved that no two distinct torus knots belong to the same knot cobordism class.) Notice, however, that the knot cobordism class of $F^\cdot$ is not uniquely determined by the homology class of $F \cup CF^\cdot$. For instance, in the case $m = 2$ we can add a twist to the strip E_1 so as to obtain the (right-handed) trefoil knot (compare Fig. 2).

We now come back to the proof of Theorem 2. We will prove the following statement.

LEMMA 2. *Let M be a closed combinatorial 4-manifold containing a 2-dimensional subcomplex K such that*

(1) *$M - K$ is acyclic, and*

(2) *The boundary of a regular neighborhood‡ of K is a 3-sphere.*

Then, every homology class $\xi \in H_2(M; Z)$ can be represented by a combinatorially imbedded sphere.

Clearly, both $S^2 \times S^2$ and $PC(2)$ satisfy the hypothesis of Lemma 2. It is not known whether every simply connected 4-manifold satisfies this hypothesis.

Proof of Lemma 2: Start with a cohomology class c in $H^2(K; Z)$ and choose an iterated barycentric subdivision $K^{(i)}$ of K $(i \geq 1)$ such that this cohomology class is represented by a cocycle c_0 with $|c_0(\Delta)| \leq 1$ for every 2-simplex $\Delta \in K^{(i)}$. Let U be the star neighborhood of K in $M^{(i+1)}$. Then U is a regular neighborhood of K (see ref. 10, Theorem 22); hence, the boundary $U^\cdot$ is a 3-sphere. Each 2-simplex $\Delta \in K^{(i)}$ has a dual 2-cell $\Delta^* \subset U \subset M^{(i+1)}$ whose boundary is contained in the boundary of U. Let $\{D_1, \ldots, D_m\}$ be the set of those dual cells Δ^* for which $c(\Delta) \neq 0$. Orient each D_i so that it has intersection number $c(\Delta)$ with the corresponding simplex Δ. The boundaries $D_1^\cdot, \ldots, D_m^\cdot$ are m disjoint oriented circles in the 3-sphere $U^\cdot$, where $m = \sum |c_0(\Delta)|$. Now just as in the preceding proof, choose $m - 1$ strips $E_1, \ldots, E_{m-1}$ in $U^\cdot$ which connect these m circles. Again, the orientations are to be chosen so that the 2-element

$$F = D_1 \cup E_1 \cup D_2 \cup \ldots \cup D_m \subset U$$

has an orientation compatible with the orientations of the D_i. The boundary $F^\cdot$ is now a (knotted) circle in $U^\cdot$. But $F^\cdot$ also bounds a 2-element F_1 in the closure V of $M - U$. Indeed, a regular neighborhood of $U^\cdot = V^\cdot$ in V is combinatorially equivalent to $S^3 \times I$, and therefore the imbedding $F^\cdot \to V^\cdot$ can be extended to an imbedding of the cone over $F^\cdot$ into V.

The union $F \cup F_1$ is a combinatorial 2-sphere in M which represents a homology class $\xi \in H_2(M; Z)$. Clearly, ξ corresponds to the cohomology class $c \in H^2(K; Z)$ under the isomorphisms

$$H^2(K) \xleftarrow{i^*} H^2(U) \xrightarrow{p} H_2(U, U^\cdot) \xrightarrow{j_*} H_2(M, V) \xleftarrow{k_*} H_2(M).$$

Here, i, j, and k denote inclusion maps and p denotes the Poincaré duality isomorphism. (k_* is an isomorphism since V is acyclic.) Since c was an arbitrary cohomology class, this completes the proof of Lemma 2.

Example 3: In conclusion, we will consider the connected sum $P_1 \# P_2$ of two copies of the complex projective plane. Let $\gamma_1, \gamma_2 \in H_2(P_1 \# P_2; Z)$ be the standard generators.

LEMMA 3. *The homology class $3\gamma_1$ can be represented by a differentiably imbedded 2-sphere $f: S^2 \to P_1 \# P_2$.*

This is rather surprising in view of Corollary 2.

The homology class γ_2 can also be represented by an imbedded sphere $g: S^2 \to P_1 \# P_2$. It is interesting to note that the two images $f(S^2)$, $g(S^2)$ must necessarily intersect, even though the intersection number $3\gamma_1 \cdot \gamma_2$ is zero. For otherwise it would be possible to choose an imbedded sphere representing the homology class $3\gamma_1 + \gamma_2$ of the sum. This is impossible by Theorem 1, since

$$(3\gamma_1 + \gamma_2)\cdot(3\gamma_1 + \gamma_2) = 10 \not\equiv \sigma(P_1 \# P_2) = 2 \pmod{16}.$$

Proof of Lemma 3: Let U denote a tubular neighborhood of $PC(1)$ in $PC(2)$, as in the proof of Theorem 2. Then $P_1 \# P_2$ can be considered as the union $U_1 \cup U_2$ of two copies of U, with the 3-sphere boundaries matched under an orientation *reversing* homeomorphism $h: U_1^{\cdot} \to U_2^{\cdot}$. As illustrated in Figure 1, it is possible to imbed a 2-disk F_1 in U_1 so as to represent the homology class $3\gamma_1$, and so that the boundary $F_1^{\cdot}$ is a right-handed trefoil in $U_1^{\cdot}$. On the other hand, it is possible to imbed a 2-disk F_2 in U_2 representing the trivial homology class, and so that the boundary $F_2^{\cdot}$ is a left-handed trefoil in $U_2^{\cdot}$. The construction is illustrated in Figure 3. (This is similar to Figure 2 except that the orientation of D_2 has been

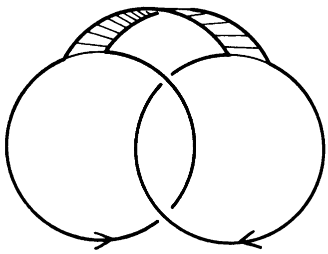

FIG. 3.—$F^{\cdot}$ is a left-handed trefoil.

reversed, in order to obtain the zero homology class in $H_2(U_2, U_2^{\cdot}; Z)$.)

Deforming this left-handed trefoil so that it matches the image of the right-handed trefoil under h, we obtain a 2-sphere $F_1 \cup F_2 \subset P_1 \# P_2$ representing the required homology class $3\gamma_1$. It is clear that the "angles" can be rounded off so as to make $F_1 \cup F_2$ into a differentiably imbedded 2-sphere. This completes the proof of Lemma 3.

* Even if ξ is not representable by an imbedded 2-sphere, the weaker congruence

$$\xi . \xi \equiv \sigma(M) \pmod 8$$

will be satisfied, assuming that M is simply connected. (Compare van der Blij, ref. 1, formula (7).)

† This lemma can also be stated in the form: If $w_2(M) = 0$, then $p_1[M] = 0 \pmod{48}$. The two formulations are equivalent in view of the Thom-Rohlin identity

$$p_1[M] = 3\sigma(M).$$

(Cf. Hirzebruch, ref. 4, Theorem 8.2.2, p. 85.) The proof of Lemma 1 is based on the fact that $\pi_{n+3}(S^n)$ is cyclic of order 24 for $n \geq 5$.

‡ In the sense of J. H. C. Whitehead.[10] Any two regular neighborhoods of K in M are combinatorially equivalent by reference 10, Theorem 23.

[1] Blij, F. van der, "An invariant of quadratic forms mod 8.," *Proc. Nederl. Akad. v. Wetenschappen, Ser. A*, **62**, 291–293 (1959).

[2] Fox, R., and J. Milnor, "Singularities of 2-spheres in 4-space and equivalence of knots," *Bull. Amer. Math. Soc.*, **63**, 406, Abstract 809t (1957).

[3] Haefliger, A., "Plongements différentiables de variétés dans varietés," *Comm. Math. Helv.* (to appear).

[4] Hirzebruch, F., *Neue Topologische Methoden in der Algebraischen Geometrie* (Berlin: Springer Verlag, 1956).

[5] Milnor, J., and M. Kervaire, "Bernoulli numbers, homotopy groups and a theorem of Rohlin," *Proceedings of the International Congress of Mathematicians*, Edinburgh, 1958.

[6] Milnor, J., "A procedure for killing homotopy groups of differentiable manifolds," in *Proceedings of Symposia in Pure Mathematics*, III (Providence: American Mathematical Society, 1961).

[7] Rohlin, V., "Classification of the mappings of the $(n + 3)$-sphere into the n-sphere," *Doklady Akad. Nauk SSSR*, **81**, 19–22 (1951).

[8] Rohlin, V., "New results in the theory of 4-dimensional manifolds," *Doklady Akad. Nauk SSSR*, **84**, 221–224 (1952).

[9] Whitney, H., "The self-intersections of a smooth n-manifold in $2n$-space," *Ann. of Math.*, **45**, 220–246 (1944).

[10] Whitehead, J. H. C., "Simplicial spaces, nuclei and m-groups, *Proc. Lond. Math. Soc.*, **45**, 243–327 (1939).

AN ALGORITHM FOR EQUILIBRIUM POINTS IN BIMATRIX GAMES

BY HAROLD W. KUHN

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated by S. Bochner, August 2, 1961

1. Recently N. N. Vorobjev[1] has presented a constructive procedure for computing all equilibrium points for the case of bimatrix (i.e., finite two-person noncooperative non-zero-sum) games. The purpose of the present note is to simplify his algorithm both in theory and application. In the terms of his paper, the classification of extreme equilibrium strategies into two types is eliminated, and the enumeration of all such strategies is reduced to a single routine.

2. For the sake of easy comparison with Vorobjev's work, his notation will be used. If M is any matrix, $M_{i\cdot}$ denotes the ith row of M, $M_{\cdot j}$ denotes the jth column of M, and M^T denotes the transpose of M. Furthermore, J_p denotes the p-dimensional vector with all components equal to one, and O_p denotes the p-dimensional vector with all components equal to zero. Inequalities between vectors are to hold in all components.

A *bimatrix game* Γ is defined by two real m by n payoff matrices, $A = (a_{ij})$ and $B = (b_{ij})$: if player 1 chooses $i \epsilon \{1, \ldots, m\}$ and player 2 chooses $j \epsilon \{1, \ldots, n\}$, then player 1 is paid a_{ij} and 2 is paid b_{ij}. Mixed strategies for 1 and 2 are probability vectors of dimension m and n and are denoted by X and Y respectively. Thus,

$$XJ_m^T = 1, X \geqq O_m \quad \text{and} \quad J_nY^T = 1, \quad Y \geqq O_n.$$