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GROUPS OF HOMOTOPY SPHERES: I

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1. Introduction

All manifolds, with or without boundary, are to be compact, oriented, and differentiable of class C^∞. The boundary of M will be denoted by bM. The manifold M with orientation reversed is denoted by $-M$.

DEFINITION. The manifold M is a *homotopy n-sphere* if M is closed (that is, $bM = \emptyset$) and has the homotopy type of the sphere S^n.

DEFINITION. Two closed n-manifolds M_1 and M_2 are *h-cobordant*[1] if the disjoint sum $M_1 + (-M_2)$ is the boundary of some manifold W, where both M_1 and $(-M_2)$ are deformation retracts of W. It is clear that this is an equivalence relation.

The *connected sum* of two connected n-manifolds is obtained by removing a small n-cell from each, and then pasting together the resulting boundaries. Details will be given in § 2.

THEOREM 1.1. *The h-cobordism classes of homotopy n-spheres form an abelian group under the connected sum operation.*

This group will be denoted by Θ_n, and called the n^{th} *homotopy sphere cobordism group*. It is the object of this paper (which is divided into 2 parts) to investigate the structure of Θ_n.

It is clear that $\Theta_1 = \Theta_2 = 0$. On the other hand these groups are not all zero. For example, it follows easily from Milnor [14] that $\Theta_7 \neq 0$.

The main result of the present Part I will be:

THEOREM 1.2. *For $n \neq 3$ the group Θ_n is finite.*

(Our methods break down for the case $n = 3$. However, if one assumes the Poincaré hypothesis, then it can be shown that $\Theta_3 = 0$.)

More detailed information about these groups will be given in Part II. For example, for $n = 1, 2, 3, \cdots, 18$, it will be shown that the order of the group Θ_n is respectively:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[\Theta_n]$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16.

Partial summaries of results are given also at the end of § 4 and of § 7.

[1] The term "J-equivalent" has previously been used for this relation. Compare [15], [16], [17].

REMARK. S. Smale [25] and J. Stallings [27], C. Zeeman [33] have proved that every homotopy n-sphere, $n \neq 3, 4$, is actually homeomorphic to the standard sphere S^n. Furthermore, Smale has proved [26] that two homotopy n-spheres, $n \neq 3, 4$, are h-cobordant if and only if they are diffeomorphic. Thus for $n \neq 3, 4$ (and possibly for all n) the group Θ_n can be described as the set of all diffeomorphism classes of differentiable structures on the topological n-sphere. These facts will not be used in the present paper.

2. Construction of the group Θ_n

First we give a precise definition of the connected sum $M_1 \# M_2$ of two connected n-manifolds M_1 and M_2. (Compare Seifert [22] and Milnor [15], [16].) The notation D^n will be used for the unit disk in euclidean n-space. Choose imbeddings

$$i_1: D^n \to M_1\ , \qquad i_2: D^n \to M_2$$

so that i_1 preserves orientation and i_2 reverses orientation. Now obtain $M_1 \# M_2$ from the disjoint sum

$$(M_1 - i_1(0)) + (M_2 - i_2(0))$$

by identifying $i_1(tu)$ with $i_2((1-t)u)$ for each unit vector $u \in S^{n-1}$ and each $0 < t < 1$. Choose the orientation for $M_1 \# M_2$ which is compatible with that of M_1 and M_2. (This makes sense since the correspondence $i_1(tu) \to i_2((1-t)u)$ preserves orientation.)

It is clear that the sum of two homotopy n-spheres is a homotopy n-sphere.

LEMMA 2.1. *The connected sum operation is well defined, associative, and commutative up to orientation preserving diffeomorphism. The sphere S^n serves as identity element.*

PROOF. The first assertions follow easily from the lemma of Palais [20] and Cerf [5] which asserts that any two orientation preserving imbeddings $i, i': D^n \to M$ are related by the equation $i' = f \circ i$, for some diffeomorphism $f: M \to M$. The proof that $M \# S^n$ is diffeomorphic to M will be left to the reader.

LEMMA 2.2. *Let M_1, M_1' and M_2 be closed and simply connected.*[2] *If M_1 is h-cobordant to M_1' then $M_1 \# M_2$ is h-cobordant to $M_1' \# M_2$.*

PROOF. We may assume that the dimension n is $\geqq 3$. Let $M_1 + (-M_1') = bW_1$, where M_1 and $-M_1'$ are deformation retracts of W_1. Choose a differentiable arc A from a point $p \in M_1$ to a point $p' \in -M_1'$ within W_1 so that

[2] This hypothesis is imposed in order to simplify the proof. It could easily be eliminated.

a tubular neighborhood of this arc is diffeomorphic to $R^n \times [0, 1]$. Thus we obtain an imbedding

$$i\colon R^n \times [0, 1] \to W_1$$

with $i(R^n \times 0) \subset M_1$, $i(R^n \times 1) \subset M_1'$, and $i(0 \times [0, 1]) = A$. Now form a manifold W from the disjoint sum

$$(W_1 - A) + \big(M_2 - i_2(0)\big) \times [0, 1]$$

by identifying $i(tu, s)$ with $i_2((1 - t)u) \times s$ for each $0 < t < 1$, $0 \leqq s \leqq 1$, $u \in S^{n-1}$. Clearly W is a compact manifold bounded by the disjoint sum

$$M_1 \# M_2 + \big(-(M_1' \# M_2)\big) .$$

We must show that both boundaries are deformation retracts of W.

First it is necessary to show that the inclusion map

$$M_1 - p \xrightarrow{j} W_1 - A$$

is a homotopy equivalence. Since $n \geqq 3$, it is clear that both of these manifolds are simply connected. Mapping the homology exact sequence of the pair $(M_1, M_1 - p)$ into that of the pair $(W_1, W_1 - A)$, we see that j induces isomorphisms of homology groups, and hence is a homotopy equivalence. Now it follows easily, using a Mayer-Vietoris sequence, that the inclusion

$$M_1 \# M_2 \to W$$

is a homotopy equivalence; hence that $M_1 \# M_2$ is a deformation retract of W. Similarly $M_1' \# M_2$ is a deformation retract of W, which completes the proof of Lemma 2.2.

LEMMA 2.3. *A simply connected manifold M is h-cobordant to the sphere S^n if and only if M bounds a contractible manifold.*

(Here the hypothesis of simple connectivity cannot be eliminated.)

PROOF. If $M + (-S^n) = bW$ then filling in a disk D^{n+1} we obtain a manifold W' with $bW' = M$. If S^n is a deformation retract of W, then it clearly follows that W' is contractible.

Conversely if $M = bW'$ with W' contractible, then removing the interior of an imbedded disk we obtain a simply connected manifold W with $bW = M + (-S^n)$. Mapping the homology exact sequence of the pair (D^{n+1}, S^n) into that of the pair (W', W), we see that the inclusion $S^n \to W$ induces a homology isomorphism; hence S^n is a deformation retract of W. Now applying the Poincaré duality isomorphism

$$H_k(W, M) \simeq H^{n+1-k}(W, S^n) ,$$

we see that the inclusion $M \to W$ also induces isomorphisms of homology groups. Since M is simply connected, this completes the proof.

LEMMA 2.4. *If M is a homotopy sphere, then $M \# (-M)$ bounds a contractible manifold.*

PROOF. Let $H^2 \subset D^2$ denote the half-disk consisting of all $(t \sin \theta, t \cos \theta)$ with $0 \leqq t \leqq 1$, $0 \leqq \theta \leqq \pi$, and let $\frac{1}{2}D^n \subset D^n$ denote the disk of radius $\frac{1}{2}$. Given an imbedding $i\colon D^n \to M$, form W from the disjoint union

$$\left(M - i\left(\frac{1}{2}D^n\right)\right) \times [0, \pi] + S^{n-1} \times H^2$$

by identifying $i(tu) \times \theta$ with $u \times ((2t-1)\sin\theta, (2t-1)\cos\theta)$ for each $\frac{1}{2} < t \leqq 1$, $0 \leqq \theta \leqq \pi$. (Intuitively we are removing the interior of $i(\frac{1}{2}D^n)$ from M and then "rotating" the result through 180° around the resulting boundary.) It is easily verified that W is a differentiable manifold with $bW = M \# (-M)$. Furthermore W contains $M -$Interior $i(\frac{1}{2}D^n)$ as deformation retract, and therefore is contractible. This proves Lemma 2.4.

PROOF OF THEOREM 1.1. Let Θ_n denote the collection of all h-cobordism classes of homotopy n-spheres. By Lemmas 2.1 and 2.2 there is a well defined, associative, commutative addition operation in Θ_n. The sphere S^n serves as zero element. By Lemmas 2.3, 2.4, each element of Θ_n has an inverse. Therefore Θ_n is an additive group.

Clearly Θ_1 is zero. For $n \leqq 3$, Munkres [19] and Whitehead [31] have proved that a topological n-manifold has a differentiable structure which is unique up to diffeomorphism. It follows that $\Theta_2 = 0$. If the Poincaré hypothesis were proved, it would follow that Θ_3 is zero; but at present the structure of Θ_3 remains unknown. For $n > 3$ the structure of Θ_n will be studied in the following sections.

Addendum. There is a slight modification of the connected sum construction which is frequently useful. Let W_1 and W_2 be $(n+1)$-manifolds with connected boundary. Then the sum $bW_1 \# bW_2$ is the boundary of a manifold W constructed as follows. Let H^{n+1} denote the half-disk consisting of all $x = (x_0, x_1, \cdots, x_n)$ with $|x| \leqq 1$, $x_0 \geqq 0$ and let D^n denote the subset $x_0 = 0$. Choose imbeddings

$$i_q\colon (H^{n+1}, D^n) \to (W_q, bW_q)\ , \qquad q = 1, 2\ ,$$

so that $i_2 \circ i_1^{-1}$ reverses orientation. Now form W from

$$(W_1 - i_1(0)) + (W_2 - i_2(0))$$

by identifying $i_1(tu)$ with $i_2((1-t)u)$ for each $0 < t < 1$, $u \in S^n \cap H^{n+1}$.

It is clear that W is a differentiable manifold with $bW = bW_1 \# bW_2$.

Note that W has the homotopy type of $W_1 \vee W_2$: the union with a single point in common.

W will be called the *connected sum along the boundary* of W_1 and W_2. The notation $(W, bW) = (W_1, bW_1) \# (W_2, bW_2)$ will be used for this sum.

3. Homotopy spheres are S-parallelizable

Let M be a manifold with tangent bundle $\tau = \tau(M)$, and let ε^1 denote a trivial line bundle over M.

DEFINITION. M will be called S-*parallelizable* if the Whitney sum $\tau \oplus \varepsilon^1$ is a trivial bundle.[3] The bundle $\tau \oplus \varepsilon^1$ will be called the *stable tangent bundle* of M. It is a stable bundle in the sense of [10]. (The expression S-parallelizable stands for stably parallelizable.)

THEOREM 3.1. *Every homotopy sphere is* S-*parallelizable.*

In the proof, we will use recent results of J. F. Adams [1], [2].

PROOF. Let Σ be a homotopy n-sphere. Then the only obstruction to the triviality of $\tau \oplus \varepsilon^1$ is a well defined cohomology class

$$\mathfrak{o}_n(\Sigma) \in H^n(\Sigma; \pi_{n-1}(\mathrm{SO}_{n+1})) = \pi_{n-1}(\mathrm{SO}_{n+1}) .$$

The coefficient group may be identified with the stable group $\pi_{n-1}(\mathrm{SO})$. But these stable groups have been computed by Bott [4], as follows, for $n \geqq 2$:

residue class of n mod 8:	0	1	2	3	4	5	6	7
$\pi_{n-1}(\mathrm{SO})$	Z	Z_2	Z_2	0	Z	0	0	0 .

(Here Z, Z_2, 0 denote the cyclic groups of order ∞, 2, 1 respectively.)

Case 1. $n \equiv 3, 5, 6$, or 7 (modulo 8). Then $\pi_{n-1}(\mathrm{SO}) = 0$, so that $\mathfrak{o}_n(\Sigma)$ is trivially zero.

Case 2. $n \equiv 0$ or 4 (modulo 8). Say that $n = 4k$. According to [18], [10], some non-zero multiple of the obstruction class $\mathfrak{o}_n(\Sigma)$ can be identified with the Pontrjagin class $p_k(\tau \oplus \varepsilon^1) = p_k(\tau)$. But the Hirzebruch signature[4] theorem implies that $p_k[\Sigma]$ is a multiple of the signature $\sigma(\Sigma)$ which is zero since $H^{2k}(\Sigma) = 0$. Therefore every homotopy $4k$-sphere is S-parallelizable.

Case 3. $n \equiv 1$ or 2 (modulo 8), so that $\pi_{n-1}(\mathrm{SO})$ is cyclic of order 2. For

[3] The authors have previously used the term "π-manifold" for an S-parallelizable manifold.

[4] We will substitute the word "signature" for "index" as used in [7; 14; 17; 18; 28] since this is more in accord with the usage in other parts of mathematics. The *signature* of the form $x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{k+l}^2$ is defined to be $\sigma = k - l$.

each homotopy sphere Σ the residue modulo 2

$$\mathfrak{o}_n[\Sigma] \in \pi_{n-1}(\mathrm{SO}) \simeq Z_2$$

is well defined. It follows from an argument of Rohlin that

$$J_{n-1}(\mathfrak{o}_n) = 0 \ ,$$

where J_{n-1} denotes the Hopf-Whitehead homomorphism

$$J_{n-1}\colon \pi_{n-1}(\mathrm{SO}_k) \longrightarrow \pi_{n+k-1}(S^k)$$

in the stable range $k > n$. (Compare [18, Lemma 1].) But J_{n-1} is a monomorphism for $n \equiv 1$ or 2 (modulo 8). For the case $n = 2$ this fact is well known, and for $n = 9, 10$ it has been proved by Kervaire [11]. For $n =$ 17, 18, it has been verified by Kervaire and by Toda in unpublished computations. A proof that J_{n-1} is injective for all $n \equiv 1$ or 2 (modulo 8) has recently been given by J.F. Adams [1], [2]. Now the relation $J_{n-1}(\mathfrak{o}_n) = 0$ together with the information that J_{n-1} is a monomorphism implies that $\mathfrak{o}_n = 0$. This finishes the proof of Theorem 3.1.

In conclusion, here are two lemmas which clarify the concept of s-parallelizability. The first is essentially due to J. H. C. Whitehead [32].

LEMMA 3.3. *Let M be an n-dimensional submanifold of S^{n+k}, $n < k$. Then M is s-parallelizable if and only if its normal bundle is trivial.*

LEMMA 3.4. *A connected manifold with non-vacuous boundary is s-parallelizable if and only if it is parallelizable.*

The proofs will be based on the following lemma. (Compare Milnor [17, Lemma 4].)

Let ξ be a k-dimensional vector space bundle over an n-dimensional complex, $k > n$.

LEMMA 3.5. *If the Whitney sum of ξ with a trivial bundle ε^r is trivial then ξ itself is trivial.*

PROOF. We may assume that $r = 1$, and that ξ is oriented. An isomorphism $\xi \oplus \varepsilon^1 \approx \varepsilon^{k+1}$ gives rise to a bundle map f from ξ to the bundle γ^k of oriented k-planes in $(k + 1)$-space. Since the base space of ξ has dimension n, and since the base space of γ^k is the sphere S^k, $k > n$, it follows that f is null-homotopic; and hence that ξ is trivial.

PROOF OF LEMMA 3.3. Let τ, ν denote the tangent and normal bundles of M. Then $\tau \oplus \nu$ is trivial hence $(\tau \oplus \varepsilon^1) \oplus \nu$ is trivial. Applying Lemma 3.5 the conclusion follows.

PROOF OF LEMMA 3.4. This follows by a similar argument. The hypothesis on the manifold guarantees that every map into a sphere of the same dimension is null-homotopic.

4. Which homotopy spheres bound parallelizable manifolds?

Define a subgroup $bP_{n+1} \subset \Theta_n$ as follows. A homotopy n-sphere M represents an element of bP_{n+1} if and only if M is the boundary of a parallelizable manifold. We will see that this condition depends only on the h-cobordism class of M, and that bP_{n+1} does form a subgroup. The object of this section will be to prove the following

THEOREM 4.1. *The quotient group Θ_n/bP_{n+1} is finite.*

PROOF. Given an s-parallelizable closed manifold M of dimension n, choose an imbedding

$$i: M \to S^{n+k}$$

with $k > n + 1$. Such an imbedding exists and is unique up to differentiable isotopy. By Lemma 3.3 the normal bundle of M is trivial. Now choose a specific field φ of normal k-frames. Then the Pontrjagin-Thom construction yields a map

$$p(M, \varphi): S^{n+k} \to S^k .$$

(See Pontrjagin [21, pp. 41–57], Thom [28].) The homotopy class of $p(M, \varphi)$ is a well defined element of the stable homotopy group

$$\Pi_n = \pi_{n+k}(S^k) .$$

Allowing the normal frame field φ to vary, we obtain a set of elements

$$p(M) = \{p(M, \varphi)\} \subset \Pi_n .$$

LEMMA 4.2. *The subset $p(M) \subset \Pi_n$ contains the zero element of Π_n if and only if M bounds a parallelizable manifold.*

PROOF. If $M = bW$ with W parallelizable then the imbedding $i: M \to S^{n+k}$ can be extended to an imbedding $W \to D^{n+k+1}$, and W has a field ψ of normal k-frames. We set $\varphi = \psi \mid M$. Now the Pontrjagin-Thom map $p(M, \varphi): S^{n+k} \to S^k$ extends over D^{n+k+1}, hence is null-homotopic.

Conversely if $p(M, \varphi) \simeq 0$, then M bounds a manifold $W \subset D^{n+k+1}$, where φ extends to a field ψ of normal frames over W. It follows from Lemmas 3.3 and 3.4 that W is parallelizable. This completes the proof of Lemma 4.2.

LEMMA 4.3. *If M_0 is h-cobordant to M_1, then $p(M_0) = p(M_1)$.*

PROOF. If $M_0 + (-M_1) = bW$, we choose an imbedding of W in $S^{n+k} \times [0, 1]$ so that $M_q \to S^{n+k} \times (q)$ for $q = 0, 1$. Then a normal frame field φ_q on M_q extends to a normal frame field ψ on W which restricts to some normal frame field φ_{1-q} on M_{1-q}. Clearly (W, ψ) gives rise to a homotopy between $p(M_0, \varphi_0)$ and $p(M_1, \varphi_1)$.

LEMMA 4.4. *If M and M' are* s-*parallelizable then*

$$p(M) + p(M') \subset p(M \# M') \subset \Pi_n .$$

PROOF. Start with the disjoint sum

$$M \times [0, 1] + M' \times [0, 1] ,$$

and join the boundary components $M \times 1$ and $M' \times 1$ together, as described in the addendum at the end of § 2. Thus we obtain a manifold W bounded by the disjoint sum

$$(M \# M') + (-M) + (-M') .$$

Note that W has the homotopy type of $M \vee M$, the union with a single point in common.

Choose an imbedding of W in $S^{n+k} \times [0, 1]$ so that $(-M)$ and $(-M')$ go into well separated submanifolds of $S^{n+k} \times 0$, and so that $M \# M'$ goes into $S^{n+k} \times 1$. Given fields φ and φ' of normal k-frames on $(-M)$ and $(-M')$, it is not hard to see that there exists an extension defined throughout W. Let ψ denote the restriction of this field to $M \# M'$. Then clearly $p(M, \varphi) + p(M', \varphi')$ is homotopic to $p(M \# M', \psi)$. This completes the proof.

LEMMA 4.5. *The set $p(S^n) \subset \Pi_n$ is a subgroup of the stable homotopy group Π_n. For any homotopy sphere Σ the set $p(\Sigma)$ is a coset of this subgroup $p(S^n)$. Thus the correspondence $\Sigma \to p(\Sigma)$ defines a homomorphism p' from Θ_n to the quotient group $\Pi_n/p(S^n)$.*

PROOF. Combining Lemma 4.4 with the identities

(1) $S^n \# S^n = S^n$,

(2) $S^n \# \Sigma = \Sigma$,

(3) $\Sigma \# (-\Sigma) \sim S^n$

we obtain

$$(1) \qquad p(S^n) + p(S^n) \subset p(S^n) ,$$

which shows that $p(S^n)$ is a subgroup of Π_n;

$$(2) \qquad p(S^n) + p(\Sigma) \subset p(\Sigma) ,$$

which shows that $p(\Sigma)$ is a union of cosets of this subgroup; and

$$(3) \qquad p(\Sigma) + p(-\Sigma) \subset p(S^n) ,$$

which shows that $p(\Sigma)$ must be a single coset. This completes the proof of Lemma 4.5.

By Lemma 4.2 the kernel of $p'\colon \Theta_n \to \Pi_n/p(S^n)$ consists exactly of all h-cobordism classes of homotopy n-spheres which bound parallelizable manifolds. Thus these elements form a group which we will denote by $bP_{n+1} \subset \Theta_n$. It follows that Θ_n/bP_{n+1} is isomorphic to a subgroup of $\Pi_n/p(S^n)$. Since Π_n is finite (Serre [24]), this completes the proof of Theorem 4.1.

REMARKS. The subgroup $p(S^n) \subset \Pi_n$ can be described in more familiar terms as the image of the Hopf-Whitehead homomorphism

$$J_n\colon \pi_n(\mathrm{SO}_k) \to \pi_{n+k}(S^k) .$$

(See Kervaire [9, p. 349].) Hence $\Pi_n/p(S^n)$ is the cokernel of J_n. The actual structure of these groups for $n \leqq 8$ is given in the following table. For details, and for higher values of n, the reader is referred to Part II of this paper.

n	1	2	3	4	5	6	7	8
Π_n	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{240}	$Z_2 + Z_2$
$\Pi_n/p(S^n)$	0	Z_2	0	0	0	Z_2	0	Z_2
Θ_n/bP_{n+1}	0	0	0	0	0	0	0	Z_2

The prime $q \geqq 3$ first divides the order of Θ_n/bP_{n+1} for $n = 2q(q-1)-2$.

Using Theorem 4.1, the proof of the main theorem (Theorem 1.2), stating that Θ_n is finite for $n \neq 3$, reduces now to proving that bP_{n+1} is finite for $n \neq 3$.

We will prove that the group bP_{n+1} is zero for n even (§§ 5, 6), and is finite cyclic for n odd, $n \neq 3$, (see §§ 7, 8). The first few groups can be given as follows:

n	1	3	5	7	9	11	13	15	17	19
order of bP_{n+1}	1	?	1	28	2	992	1	8128	2	130,816.

(Again see Part II for details.) The cyclic group bP_{n+1} has order 1 or 2 for $n \equiv 1 \pmod 4$, but the order grows more than exponentially for $n \equiv 3 \pmod 4$.

5. Spherical modifications

This section, and § 6 which follows, will prove that the groups bP_{2k+1} are zero.[5] That is:

THEOREM 5.1. *If a homotopy sphere of dimension $2k$ bounds an s-parallelizable manifold M, then it bounds a contractible manifold M_1.*

[5] An independent proof of this theorem has been given by C.T.C. Wall [29].

For the case $k = 1$, this assertion is clear since every homotopy 2-sphere is actually diffeomorphic to S^2. The proof for $k > 1$ will be based on the technique of "spherical modifications." (See Wallace [30], Milnor [15; 17][6].)

DEFINITION. Let M be a differentiable manifold of dimension $n = p + q + 1$ and let

$$\varphi: S^p \times D^{q+1} \to M$$

be a differentiable imbedding. Then a new differentiable manifold $M' = \chi(M, \varphi)$ is formed from the disjoint sum

$$\bigl(M - \varphi(S^p \times 0)\bigr) + D^{p+1} \times S^q$$

by identifying $\varphi(u, tv)$ with (tu, v) for each $u \in S^p$, $v \in S^q$, $0 < t \leqq 1$. We will say that M' is obtained from M by the *spherical modification* $\chi(\varphi)$. Note that the boundary of M' is equal to the boundary of M.

In order to prove Theorem 5.1 we will show that the homotopy groups of M can be completely killed by a sequence of such spherical modifications. The effect of a single modification $\chi(\varphi)$ on the homotopy groups of M can be described as follows.

Let $\lambda \in \pi_p M$ denote the homotopy class of the map $\varphi \mid S^p \times 0$ from $S^p \times 0$ to M.

LEMMA 5.2. *The homotopy groups of M' are given by*

$$\pi_i M' \simeq \pi_i M \qquad \textit{for } i < \operatorname{Min}(p, q)\ ,$$

and

$$\pi_p M' \simeq \pi_p M / \Lambda\ ,$$

provided that $p < q$; where Λ denotes a certain subgroup of $\pi_p M$ containing λ.

The proof is straightforward. (Compare [17, Lemma 2].)

Thus if $p < q$ (that is, if $p \leqq n/2 - 1$), the effect of the modification $\chi(\varphi)$ is to kill the homotopy class λ.

Now suppose that some homotopy class $\lambda \in \pi_p M$ is given.

LEMMA 5.3. *If M^n is* s*-parallelizable and if $p < n/2$, then the class λ is represented by some imbedding $\varphi: S^p \times D^{n-p} \to M$.*

PROOF. (Compare [17, Lemma 3].) Since $n \geqq 2p + 1$ it follows from a well known theorem of Whitney that λ can be represented by an imbedding

$$\varphi_0: S^p \to M\ .$$

[6] The term "surgery" is used for this concept in [15; 17].

It follows from Lemma 3.5 that the normal bundle of $\varphi_0 S^p$ in M is trivial. Hence φ_0 can be extended to the required imbedding $S^p \times D^{n-p} \to M$.

Thus Lemmas 5.2 and 5.3 assert that spherical modifications can be used to kill any required element $\lambda \in \pi_p M^n$ provided that $p \leqq n/2 - 1$. There is one danger however. If the imbedding φ is chosen badly then the modified manifold $M' = \chi(M, \varphi)$ may no longer be s-parallelizable. However the following was proven in [17]. Again let $n \geqq 2p + 1$.

LEMMA 5.4. *The imbedding φ: $S^p \times D^{n-p} \to M$ can be chosen within its homotopy class so that the modified manifold $\chi(M, \varphi)$ will also be* s-*parallelizable.*

For the proof, the reader may either refer to [17, Theorem 2], or make use of the sharper Lemma 6.2 which will be proved below.

Now combining Lemmas 5.2, 5.3, 5.4, one obtains the following. (Compare [17, p. 46].)

THEOREM 5.5. *Let M be a compact, connected* s-*parallelizable manifold of dimension $n \geqq 2k$. By a sequence of spherical modifications on M one can obtain an* s-*parallelizable manifold M_1 which is $(k-1)$-connected.*

Recall that $bM_1 = bM$.

PROOF. Choosing a suitable imbedding φ: $S^1 \times D^{n-1} \to M$, one can obtain an s-parallelizable manifold $M' = \chi(M, \varphi)$ such that $\pi_1 M'$ is generated by fewer elements than $\pi_1 M$. Thus after a finite number of steps, one can obtain a manifold M'' which is 1-connected. Now, after a finite number of steps, one can obtain an s-parallelizable manifold M''' which is 2-connected, and so on until we obtain a $(k-1)$-connected manifold. This proves Theorem 5.5.

In order to prove 5.1, where $\dim M = 2k + 1$, we must carry this argument one step further obtaining a manifold M_1 which is k-connected. It will then follow from the Poincaré duality theorem that M_1 is contractible.

The difficulty in carrying out this program is that Lemma 5.2 is no longer available. Thus if $M' = \chi(M, \varphi)$ where φ imbeds $S^k \times D^{k+1}$ in M, the group $\pi_k M'$ may actually be larger than $\pi_k M$. It is first necessary to describe in detail what happens to $\pi_k M$ under such a modification. Since we may assume that M is $(k-1)$-connected with $k > 1$, the homotopy group $\pi_k M$ may be replaced by the homology group $H_k M = H_k(M; Z)$.

LEMMA 5.6. *Let $M' = \chi(M, \varphi)$ where φ imbeds $S^k \times D^{k+1}$ in M, and let*

$$M_0 = M - \left(\text{interior } \varphi(S^k \times D^{k+1})\right).$$

Then there is a commutative diagram

$$\begin{tikzcd}
 & & H_{k+1}M' \arrow[d, "\cdot\lambda'"] & & \\
 & & Z \arrow[d, "\varepsilon"] \arrow[dr, "\lambda"] & & \\
H_{k+1}M \arrow[r, "\cdot\lambda"] & Z \arrow[r, "\varepsilon'"] \arrow[dr, "\lambda'"'] & H_kM_0 \arrow[r, "i"] \arrow[d, "i'"] & H_kM \arrow[r] & 0 \\
 & & H_kM' \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

such that the horizontal and vertical sequences are exact. It follows that the quotient group $H_kM/\lambda(Z)$ is isomorphic to $H_kM'/\lambda'(Z)$.

Here the following notations are to be understood. The symbol λ denotes the element of H_kM which corresponds to the homotopy class $\varphi \mid S^k \times 0$, and λ also denotes the homomorphism $Z \to H_kM$ which carries 1 into λ. On the other hand $\cdot\lambda$: $H_{k+1}M \to Z$ denotes the homomorphism which carries each $\mu \in H_{k+1}M$ into the intersection number $\mu \cdot \lambda$. The symbols λ' and $\cdot\lambda'$ are to be interpreted similarly. The element $\lambda' \in H_kM'$ corresponds to the homotopy class $\varphi' \mid 0 \times S^k$ where

$$\varphi' \colon D^{k+1} \times S^k \to M'$$

denotes the canonical imbedding.

PROOF OF LEMMA 5.6. As horizontal sequences take the exact sequence

$$H_{k+1}M \longrightarrow H_{k+1}(M, M_0) \xrightarrow{\varepsilon'} H_kM_0 \xrightarrow{i} H_kM \longrightarrow H_k(M, M_0)$$

of the pair (M, M_0). By excision, the group $H_j(M, M_0)$ is isomorphic to

$$H_j(S^k \times D^{k+1}, S^k \times S^k) \simeq \begin{cases} Z & \text{for } j = k+1 \\ 0 & \text{for } j < k+1 \,. \end{cases}$$

Thus we obtain

$$H_{k+1}M \longrightarrow Z \xrightarrow{\varepsilon'} H_kM_0 \xrightarrow{i} H_kM \longrightarrow 0 \,,$$

as asserted. Since a generator of $H_{k+1}(M, M_0)$ clearly has intersection number ± 1 with the cycle $\varphi(S^k \times 0)$ which represents λ, it follows that the homomorphism $H_{k+1}M \to Z$ can be described as the homomorphism $\mu \to \mu \cdot \lambda$. The element $\varepsilon' = \varepsilon'(1) \in H_kM_0$ can clearly be described as the homology class corresponding to the "meridian" $\varphi(x_0 \times S^k)$ of the torus

$\varphi(S^k \times S^k)$, where x_0 denotes a base point in S^k.

The vertical exact sequence is obtained in a similar way. Thus $\varepsilon = \varepsilon(1) \in H_k M_0$ is the homology class of the "parallel" $\varphi(S^k \times x_0)$ of the torus. Clearly $i(\varepsilon) \in H_k M$ is equal to the homology class λ of $\varphi(S^k \times 0)$. Similarly $i'(\varepsilon') = \lambda'$.

From this diagram the isomorphisms

$$H_k M/\lambda(Z) \simeq H_k M_0/\varepsilon(Z) + \varepsilon'(Z) \simeq H_k M'/\lambda'(Z)$$

are apparent. This completes the proof of Lemma 5.6.

As an application, suppose that one chooses an element $\lambda \in H_k M$ which is *primitive* in the sense that $\mu \cdot \lambda = 1$ for some $\mu \in H_{k+1} M$. It follows that

$$i \colon H_k M_0 \to H_k M$$

is an isomorphism, and hence that

$$H_k M' \simeq H_k M/\lambda(Z) .$$

Thus:

ASSERTION. *Any primitive element of $H_k M$ can be killed by a spherical modification.*

In order to apply this assertion we assume the following:

Hypothesis. M is a compact, s-parallelizable manifold of dimension $2k+1$, $k>1$, and is $(k-1)$-connected. The boundary bM is either vacuous or a homology sphere.

This hypothesis will be assumed for the rest of § 5 and for § 6.

LEMMA 5.7. *Subject to this hypothesis, the homology group $H_k M$ can be reduced to its torsion subgroup by a sequence of spherical modifications. The modified manifold M_1 will still satisfy the hypothesis.*

PROOF. Suppose that $H_k M \simeq Z \oplus \cdots \oplus Z \oplus T$ where T is the torsion subgroup. Let λ generate one of the infinite cyclic summands. Using the Poincaré duality theorem one sees that $\mu_1 \cdot \lambda = 1$ for some element $\mu_1 \in H_{k+1}(M, bM)$. But the exact sequence

$$H_{k+1} M \to H_{k+1}(M, bM) \to H_k(bM) = 0$$

shows that μ_1 can be lifted back to $H_{k+1} M$. Therefore λ is primitive, and can be killed by a modification. After finitely many such modifications, one obtains a manifold M_1 with $H_k M_1 \simeq T \subset H_k M$. This completes the proof of Lemma 5.7.

Let us specialize to the case k even. Let M be as above, and let $\varphi \colon S^k \times D^{k+1} \to M$ be any imbedding.

LEMMA 5.8. *If k is even then the modification $\chi(\varphi)$ necessarily changes* *the k^{th} Betti number of M.*

The proof will be based on the following lemma. (See Kervaire [8, Formula (8.8)].)

Let F be a fixed field and let W be an orientable homology manifold of dimension $2r$. Define the *semi-characteristic* $e^*(bW; F)$ to be the following residue class modulo 2:

$$e^*(bW; F) \equiv \sum_{i=0}^{r-1} \operatorname{rank} H_i(bW; F) \pmod 2 .$$

LEMMA 5.9. *The rank of the bilinear pairing*

$$H_r(W; F) \otimes H_r(W; F) \to F ,$$

given by the intersection number, is congruent modulo 2 to $e^*(bW; F)$ plus the Euler characteristic $e(W)$.

[For the convenience of the reader, here is a proof. Consider the exact sequence

$$H_r W \xrightarrow{h} H_r(W, bW) \longrightarrow H_{r-1}(bW) \longrightarrow \cdots \longrightarrow H_0(W, bW) \longrightarrow 0 ,$$

where the coefficient group F is to be understood. A counting argument shows that the rank of the indicated homomorphism h is equal to the alternating sum of the ranks of the vector spaces to the right of h in this sequence. Reducing modulo 2 and using the identity

$$\operatorname{rank} H_i(W, bW) = \operatorname{rank} H_{2r-i} W ,$$

this gives

$$\begin{aligned} \operatorname{rank} h &\equiv \sum_{i=0}^{r-1} \operatorname{rank} H_i(bW) + \sum_{i=0}^{2r} \operatorname{rank} H_i W \\ &\equiv e^*(bW; F) + e(W) \pmod 2 . \end{aligned}$$

But the rank of

$$h: H_r W \to H_r(W, bW) \simeq \operatorname{Hom}_F(H_r W, F)$$

is just the rank of the intersection pairing. This completes the proof.]

PROOF OF LEMMA 5.8. First suppose that M has no boundary. As shown in [15] or [17] the manifolds M and $M' = \chi(M, \varphi)$, suitably oriented, together bound a manifold $W = W(M, \varphi)$ of dimension $2k + 2$. For the moment, since no differentiable structure on W is needed, we can simply define W to be the union

$$(M \times [0, 1]) \cup (D^{k+1} \times D^{k+1}) ,$$

where it is understood that $S^k \times D^{k+1}$ is to be pasted onto $M \times 1$ by the imbedding φ. Clearly W is a topological manifold with

$$bW = M \times 0 + M' \times 1 .$$

Note that W has the homotopy type of M with a $(k+1)$-cell attached. Since the dimension $2k+1$ of M is odd, this means that the Euler characteristic

$$e(W) = e(M) + (-1)^{k+1} = (-1)^{k+1} .$$

Since k is even, the intersection pairing

$$H_{k+1}(W; Q) \otimes H_{k+1}(W; Q) \to Q$$

is skew symmetric, hence has even rank. Therefore Lemma 5.9 (with rational coefficients) asserts that

$$e^*(M + M'; Q) + (-1)^{k+1} \equiv 0 \pmod 2 ,$$

and hence that

$$e^*(M; Q) \not\equiv e^*(M'; Q) .$$

But $H_iM \simeq H_iM' \simeq 0$ for $0 < i < k$, so this implies that

$$\operatorname{rank} H_k(M; Q) \not\equiv \operatorname{rank} H_k(M'; Q) .$$

This proves Lemma 5.8 provided that M has no boundary.

If M is bounded by a homology sphere, then attaching a cone over bM, one obtains a homology manifold M_* without boundary. The above argument now shows that

$$\operatorname{rank} H_k(M_*; Q) \not\equiv \operatorname{rank} H_k(M'_*; Q) .$$

Therefore the modification $\chi(\varphi)$ changes the rank of $H_k(M; Q)$ in this case also. This completes the proof of Lemma 5.8.

It is convenient at this point to insert an analogue of 5.8 which will only be used later. (See the end of § 6.) Let M be as above, with k even or odd, and let $W = W(M, \varphi)$.

LEMMA 5.10. *Suppose that every* mod 2 *homology class*

$$\xi \in H_{k+1}(W; Z_2)$$

has self-intersection number $\xi \cdot \xi = 0$. *Then the modification* $\chi(\varphi)$ *necessarily changes the rank of the* mod 2 *homology group* $H_k(M; Z_2)$.

The proof is completely analogous to that of 5.8. The hypothesis, $\xi \cdot \xi = 0$ for all ξ, guarantees that the intersection pairing

$$H_{k+1}(W; Z_2) \otimes H_{k+1}(W; Z_2) \to Z_2$$

will have even rank.

We now return to the case k even.

PROOF OF THEOREM 5.1, *for k even.* According to 5.6, we can assume that H_kM is a torsion group. Choose

$$\varphi: S^k \times D^{k+1} \to M$$

as in 5.4 so as to represent a non-trivial $\lambda \in H_k M$. According to 5.6 we have

$$H_k M/\lambda(Z) \simeq H_k M'/\lambda'(Z) .$$

Since the group $\lambda(Z)$ is finite, it follows from 5.8 that $\lambda'(Z)$ must be infinite. Thus the sequence

$$0 \longrightarrow Z \xrightarrow{\lambda'} H_k M' \longrightarrow H_k M'/\lambda'(Z) \longrightarrow 0$$

is exact. It follows that the torsion subgroup of $H_k M'$ maps monomorphically into $H_k M'/\lambda'(Z)$; and hence is definitely smaller than $H_k M$. Now according to 5.7, we can perform a modification on M' so as to obtain a new manifold M'' with

$$H_k M'' \simeq \text{Torsion subgroup of } H_k M' < H_k M .$$

Thus in two steps one can replace $H_k M$ by a smaller group. Iterating this construction a finite number of times, the group $H_k M$ can be killed completely. This completes the proof of Theorem 5.1 for k even.

6. Framed spherical modifications

This section will complete the proof of Theorem 5.1 by taking care of the case k odd. This case is somewhat more difficult than the case k even (which was handled in § 5), since it is necessary to choose the imbeddings φ more carefully, taking particular care not to lose s-parallelizability in the process. Before starting on the proof, it is convenient to sharpen the concepts of s-parallelizable manifold, and of spherical modification.

DEFINITION. A *framed manifold* (M, f) will mean a differentiable manifold M together with a fixed trivialization f of the stable tangent bundle $\tau_M \oplus \varepsilon_M$.

Now consider a spherical modification $\chi(\varphi)$ of M. Recall that M and $M' = \chi(M, \varphi)$ together bound a manifold

$$W = (M \times [0, 1]) \cup (D^{p+1} \times D^{q+1})$$

where the subset $S^p \times D^{q+1}$ of $D^{p+1} \times D^{q+1}$ is pasted onto $M \times 1$ by the imbedding φ. (Compare Milnor [17].) It is easy to give W a differentiable structure, except along the "corner" $S^p \times S^q$. A neighborhood of this corner will be "diffeomorphic" with $S^p \times S^q \times Q$ where

$$Q \subset R^2$$

denotes the three-quarter disk consisting of all $(r\cos\theta, r\sin\theta)$ with $0 \leqq r < 1$, $0 \leqq \theta \leqq 3\pi/2$. In order to "straighten" this corner, map Q onto the half-disk H, consisting of all $(r\cos\theta', r\sin\theta')$ with $0 \leqq r < 1$, $0 \leqq \theta' \leqq \pi$; by setting $\theta' = 2\theta/3$. Now carrying the differentiable structure of H back to Q, this makes Q into a differentiable manifold. Carrying out the same transformation on the neighborhood of $S^p \times S^q$, this makes $W = W(M, \varphi)$ into the required differentiable manifold. Note that both boundaries of W get the correct differential structures.

Now identify M with $M \times 0 \subset W$, and identify the stable tangent bundle $\tau_M \oplus \varepsilon_M$ with the restriction $\tau_W \mid M$. Thus a framing f of M determines a trivialization of $\tau_W \mid M$.

DEFINITION. A *framed spherical modification* $\chi(\varphi, F)$ of the framed manifold (M, f) will mean a spherical modification $\chi(\varphi)$ of M together with a trivialization F of the tangent bundle of W, satisfying the condition

$$F \mid M = f \, .$$

Note that the modified manifold $M' = \chi(M, \varphi)$ automatically acquires a framing

$$f' = F \mid M' \, .$$

It is only necessary to identify $\tau_W \mid M'$ with the stable tangent bundle $\tau_{M'} \oplus \varepsilon_{M'}$. To do this, we identify the positive direction in $\varepsilon_{M'}$ with the outward normal direction in $\tau_W \mid M'$.

The following question evidently arises. Given a modification $\chi(\varphi)$ of M and a framing f of M, does f extend to a trivialization F of τ_W? The obstructions to such an extension lie in the cohomology groups

$$H^{r+1}(W, M; \pi_r(\mathrm{SO}_{n+1})) \simeq \begin{cases} \pi_p(\mathrm{SO}_{n+1}) & \text{for } r = p \\ 0 & \text{for } r \neq p \, . \end{cases}$$

Thus the only obstruction to extending f is a well defined class

$$\gamma(\varphi) \in \pi_p(\mathrm{SO}_{n+1}) \, .$$

The modification $\chi(\varphi)$ can be framed if and only if this obstruction $\gamma(\varphi)$ is zero.

Now consider the following alteration of the imbedding φ. Let

$$\alpha\colon S^p \longrightarrow \mathrm{SO}_{q+1}$$

be a differentiable map, and define

$$\varphi_\alpha\colon S^p \times D^{q+1} \longrightarrow M$$

by

$$\varphi_\alpha(u, v) = \varphi(u, v\cdot\alpha(u)) \, ,$$

where the dot denotes the usual action of SO_{q+1} on D^{q+1}. Clearly φ_α is an imbedding which represents the same homotopy class $\lambda \in \pi_p M$ as φ.

LEMMA 6.1. *The obstruction $\gamma(\varphi_\alpha)$ depends only on $\gamma(\varphi)$ and on the homotopy class (α) of α. In fact*

$$\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$$

where $s_*: \pi_p(SO_{q+1}) \to \pi_p(SO_{n+1})$ is induced by the inclusion $s: SO_{q+1} \to SO_{n+1}$.

PROOF. (Compare [17], proof of Theorem 2.) Let W_α be the manifold constructed as W above, now using φ_α. There is a natural differentiable imbedding

$$i_\alpha: D^{p+1} \times \operatorname{int} D^{q+1} \longrightarrow W_\alpha ,$$

and $i_\alpha \mid S^p \times D^{q+1}$ coincides with $\varphi_\alpha: S^p \times D^{q+1} \longrightarrow M$ followed by the inclusion $M \longrightarrow M \times 1 \subset W_\alpha$.

$\gamma(\varphi_\alpha)$ is the obstruction to extending $f \mid \varphi_\alpha(S^p \times 0)$ to a trivialization of $\tau(W_\alpha)$ restricted to $i_\alpha(D^{p+1} \times 0)$ Let $t^{n+1} = e^{p+1} \times e^{q+1}$ be the standard framing on $D^{p+1} \times D^{q+1}$. Then $i'_\alpha(t^{n+1})$ is a trivialization of the tangent bundle of W_α restricted to $i_\alpha(D^{p+1} \times D^{q+1})$, and $\gamma(\varphi_\alpha)$ is the homotopy class of the map $g: S^p \longrightarrow SO_{n+1}$, where $g(u)$ is the matrix $\langle f^{n+1}, i'_\alpha(t^{n+1})\rangle$ at $\varphi_\alpha(u, 0)$.

Since $i_\alpha \mid D^{p+1} \times 0$ is independent of α, and $i_\alpha \mid S^p \times D^{q+1} = \varphi_\alpha$, we have

$$i'_\alpha(t^{n+1}) = i'(e^{p+1}) \times \varphi'_\alpha(e^{q+1})$$

at every point $(u, 0) \in S^p \times D^{q+1}$.

Since

$$\varphi'_\alpha(e^{q+1}) = \varphi'(e^{q+1}) \cdot \alpha(u)$$

at $(u, 0)$, it follows that

$$i'_\alpha(t^{n+1}) = i'(t^{n+1}) \cdot s(\alpha) .$$

Hence

$$\langle f^{n+1}, i'_\alpha(t^{n+1})\rangle = \langle f^{n+1}, i'(t^{n+1})\rangle \cdot s(\alpha)$$

and the lemma follows.

Now suppose (as usual) that $p \leqq q$. Then the homomorphism

$$s_*: \pi_p(SO_{q+1}) \longrightarrow \pi_p S(O_{n+1})$$

is onto. Hence α can be chosen so that

$$\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$$

is zero. Thus we obtain:

LEMMA 6.2. *Given $\varphi: S^p \times D^{q+1} \to M$ with $p \leqq q$, a map α can be chosen* *so that the modification* $\chi(\varphi_a)$ *can be framed.*

In particular, it follow that the manifold $\chi(M, \varphi_\alpha)$ will be s-parallelizable. Thus we have proved Lemma 5.4 in a sharpened form.

We note however that α is not always uniquely determined. In the case $p = q = k$ odd, the homomorphism

$$s_*: \pi_k(\mathrm{SO}_{k+1}) \to \pi_k(\mathrm{SO}_{n+1})$$

has an infinite cyclic kernel. This freedom in the choice of α will be the basis of the proof of 5.1 for k odd.

Let us study the homology of the manifold

$$M'_\alpha = \chi(M, \varphi_\alpha) ,$$

where φ is now chosen, by Lemma 6.1, so that the spherical modification $\chi(\varphi)$ can be framed. Clearly the deleted manifold

$$M_0 = M - \big(\text{interior } \varphi_\alpha(S^k \times D^{k+1})\big)$$

does not depend on the choice of α. Furthermore the meridian $\varphi_\alpha(x_0 \times S^k)$ of the torus $\varphi_\alpha(S^k \times S^k) \subset M_0$ does not depend on the choice of α; hence the homology class

$$\varepsilon' \in H_k M_0$$

does not depend on α. On the other hand the parallel $\varphi_\alpha(S^k \times x_0)$ does depend on α. In fact it is clear that the homology class $\varepsilon_\alpha \in H_k M_0$ of this parallel is given by

$$\varepsilon_\alpha = \varepsilon + j(\alpha)\varepsilon'$$

where the homomorphism

$$j_*: \pi_k(\mathrm{SO}_{k+1}) \to Z \simeq \pi_k(S^k)$$

is induced by the canonical map

$$\rho \xrightarrow{j} x_0 \cdot \rho$$

from SO_{k+1} to S^k.

The spherical modification $\chi(\varphi_\alpha)$ can still be framed provided α is an element of the kernel of

$$s_*: \pi_k(\mathrm{SO}_{k+1}) \to \pi_k(\mathrm{SO}_{n+1}) .$$

Identifying the stable group $\pi_k(\mathrm{SO}_{n+1})$ with the stable group $\pi_k(\mathrm{SO}_{k+2})$, there is an exact sequence

$$\pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(\mathrm{SO}_{k+1}) \xrightarrow{s_*} \pi_k(\mathrm{SO}_{k+2})$$

associated with the fibration $\mathrm{SO}_{k+2}/\mathrm{SO}_{k+1} = S^{k+1}$. It is well known that the composition

$$\pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(\mathrm{SO}_{k+1}) \xrightarrow{j_*} \pi_k(S^k)$$

carries a generator of $\pi_{k+1}(S^{k+1})$ onto twice a generator of $\pi_k(S^k)$, provided that k is odd. Therefore the integer $j_*(\alpha)$ can be any multiple of 2.

Let us study the effect of replacing ε by $\varepsilon_\alpha = \varepsilon + j(\alpha)\varepsilon'$ on the homology of the modified manifold. Consider the exact sequence

$$0 \longrightarrow Z \xrightarrow{\varepsilon'} H_k M_0 \xrightarrow{i} H_k M \longrightarrow 0$$

of 5.6, where i carries ε into an element λ of order $l > 1$. Evidently $l\varepsilon$ must be a multiple of ε', say:

$$l\varepsilon + l'\varepsilon' = 0 \ .$$

Since ε' is not a torsion element, these two elements can satisfy no other relation. Since $\varepsilon_\alpha = \varepsilon + j_*(\alpha)\varepsilon'$ it follows that

$$l\varepsilon_\alpha + \big(l' - lj(\alpha)\big)\varepsilon' = 0 \ .$$

Now using the sequence

$$Z \xrightarrow{\varepsilon_\alpha} H_k M_0 \xrightarrow{i'_\alpha} H_k M'_\alpha \longrightarrow 0 \,,$$

we see that the inclusion homomorphism i'_α carries ε' into an element

$$\lambda'_\alpha \in H_k M'_\alpha$$

of order $| l' - lj(\alpha) |$. Since $H_k M'_\alpha/\lambda'_\alpha(Z)$ is isomorphic to $H_k M/\lambda(Z)$, we see that the group $H_k M'_\alpha$ is smaller than $H_k M_\alpha$ if and only if

$$0 < | l' - lj(\alpha) | < l \ .$$

But $j(\alpha)$ can be any even integer. Thus $j(\alpha)$ can be chosen so that

$$-l < l' - lj(\alpha) \leqq l \ .$$

This choice of $j(\alpha)$ will guarantee an improvement except in the special case where l' happens to be divisible by l.

Our progress so far can be summarized as follows.

LEMMA 6.3. *Let M be a framed $(k-1)$-connected manifold of dimension $2k+1$ with k odd, $k > 1$, such that $H_k M$ is finite. Let $\chi(\varphi, F)$ be a framed modification of M which replaces the element $\lambda \in H_k M$ of order $l > 1$ by an element $\lambda' \in H_k M'$ of order $\pm l'$. If $l' \not\equiv 0 \bmod l$ then it is possible to choose $(\alpha) \in \pi_k(\mathrm{SO}_{k+1})$ so that the modification $\chi(\varphi_\alpha)$ can still be framed, and so that the group $H_k M'_\alpha$ is definitely smaller than $H_k M$.*

Thus one must study the residue class of l' modulo l. Recall the definition of linking numbers. (Compare Seifert-Threlfall [23, § 77].) Let

$\lambda \in H_pM$, $\mu \in H_qM$ be homology classes of finite order, with $\dim M = p + q + 1$. Consider the homology sequence

$$\cdots \longrightarrow H_{p+1}(M; Q/Z) \xrightarrow{\beta} H_pM \xrightarrow{i_*} H_p(M; Q) \longrightarrow \cdots$$

associated with the coefficient sequence

$$0 \longrightarrow Z \xrightarrow{i} Q \longrightarrow Q/Z \longrightarrow 0 .$$

Since λ is of finite order, $i_*\lambda = 0$ and $\lambda = \beta(\nu)$ for some $\nu \in H_{p+1}(M; Q/Z)$. The pairing

$$Q/Z \otimes Z \to Q/Z$$

defined by multiplication induces a pairing

$$H_{p+1}(M; Q/Z) \otimes H_qM \to Q/Z$$

defined by the intersection of homology classes. We denote this pairing by a dot.

DEFINITION. The *linking number* $L(\lambda, \mu)$ is the rational number modulo 1 defined by

$$L(\lambda, \mu) = \nu \cdot \mu .$$

This linking number is well defined, and satisfies the symmetry relation

$$L(\mu, \lambda) + (-1)^{pq} L(\lambda, \mu) = 0 .$$

(Compare Seifert and Threlfall.)

LEMMA 6.4. *The ratio l'/l modulo 1 is, up to sign, equal to the self-linking number $L(\lambda, \lambda)$.*

PROOF. Since

$$l\varepsilon + l'\varepsilon' = 0$$

in H_kM_0, we see that the cycle $l\varepsilon + l'\varepsilon'$ on bM_0 bounds a chain c in M_0. Let $c_1 = \varphi(x_0 \times D^{k+1})$ denote the cycle in $\varphi(S^k \times D^{k+1}) \subset M$ with boundary ε'. Then the chain $c - l'c_1$ has boundary $l\varepsilon$; hence $(c - l'c_1)/l$ has boundary ε, representing the homology class λ in H_kM. Taking the intersection of this chain with $\varphi(S^k \times 0)$, representing λ, we obtain $\pm l'/l$, since c is disjoint and c_1 has intersection number ∓ 1. Thus $L(\lambda, \lambda) = \pm l'/l \bmod 1$.

Now if $L(\lambda, \lambda) \neq 0$, then $l' \not\equiv 0 \pmod{l}$, hence the class λ can be replaced by an element of smaller order under a spherical modification. Hence, unless $L(\lambda, \lambda) = 0$ for all $\lambda \in H_kM$, this group can be simplified.

LEMMA 6.5. *If H_kM is a torsion group, with $L(\lambda, \lambda) = 0$ for every*

$\lambda \in H_k M$, *and if k is odd, then this group $H_k M$ must be a direct sum of cyclic groups of order* 2.

PROOF. The relation

$$L(\eta, \xi) + (-1)^{pq} L(\xi, \eta) = 0$$

with $p = q \equiv 1 \pmod 2$ implies that

$$L(\eta, \xi) = L(\xi, \eta) .$$

Now if self-linking numbers are all zero, the identity

$$L(\xi + \eta, \xi + \eta) = L(\xi, \xi) + L(\eta, \eta) + L(\xi, \eta) + L(\eta, \xi)$$

implies that

$$2L(\xi, \eta) = 0$$

for all ξ and η. But, according to the Poincaré duality theorem for torsion groups (see [23, p. 245]), L defines a completely orthogonal pairing

$$T_p M \otimes T_q M \to Q/Z .$$

Hence the identity $L(2\xi, \eta) = 0$ for all η implies that $2\xi = 0$. This proves Lemma 6.5.

It follows that, by a sequence of modifications, one can reduce $H_k M$ to a group of the form $Z_2 \oplus \cdots \oplus Z_2 = sZ_2$.

Now let us apply Lemma 5.10. Since the modification $\chi(\varphi_\alpha)$ is framed, the corresponding manifold $W = W(M, \varphi_\alpha)$ is parallelizable. It follows from the formulas of Wu that the Steenrod operation

$$\mathrm{Sq}^{k+1}\colon H^{k+1}(W, bW; Z_2) \to H^{2k+2}(W, bW; Z_2)$$

is zero. (See Kervaire [8, Lemma (7.9)].) Hence every $\xi \in H_{k+1}(W; Z_2)$ has self-intersection number $\xi \cdot \xi = 0$. Thus, according to 5.10, the modification $\chi(\varphi_\alpha)$ changes the rank of $H_k(M; Z_2)$.

But the effect of $\chi(\varphi_\alpha)$ on $H_k(M; Z)$, provided that α is chosen properly, will be to replace the element λ of order $l = 2$ by an element λ'_α of order l'_α where

$$-2 < l'_\alpha \leqq 2 , \qquad l'_\alpha \equiv 0 \qquad \pmod 2 .$$

Thus l'_α must be 0 or 2. Now using the sequence

$$0 \to Z_{l'_\alpha} \to H_k M'_\alpha \to H_k M_\alpha / \lambda_\alpha(Z) \to 0 ,$$

where the right hand group is isomorphic to $(s-1)Z_2$, we see that $H_k M'_\alpha$ is given by one of the following:

$$H_kM'_\alpha \simeq \begin{cases} Z + (s-1)Z_2 , \\ Z_2 + (s-1)Z_2 , \\ Z + (s-2)Z_2 , \text{ or} \\ Z_4 + (s-2)Z_2 . \end{cases}$$

But the first two possibilities cannot occur, since they do not change the rank of $H_k(M; Z_2)$. In the remaining two cases, a further modification will replace $H_kM'_\alpha$ by a group which is definitely smaller than H_kM. Thus in all cases H_kM can be replaced by a smaller group by a sequence of framed modifications.

This completes the proof of Theorem 5.1. Actually we have proved the following result which is slightly sharper.

THEOREM 6.6. *Let M be a compact, framed manifold of dimension $2k+1$, $k>1$, such that bM is either vacuous or a homology sphere. By a sequence of framed modifications, M can be reduced to a k-connected manifold M_1.*

If bM is vacuous then the Poincaré duality theorem implies that M_1 is a homotopy sphere. If bM is a homology sphere, then M_1 is contractible.

The proof of 6.6 is contained in the above discussion, provided that M is connected. But using [17, Lemma 2′] it is easily seen that a disconnected manifold can be connected by framed modifications. This completes the proof.

7. The groups bP_{2k}

The next two sections will prove that the groups bP_{2k} are finite cyclic for $k \neq 2$. In fact for k odd, the group bP_{2k} has at most two elements. For $k = 2m \neq 2$ we will see in Part II that bP_{4m} is a cyclic group of order[7]

$$\varepsilon_m 2^{2m-2}(2^{2m-1}-1) \text{ numerator } (4B_m/m) ,$$

where B_m denotes the m^{th} Bernoulli number, and ε_m equals 1 or 2.

The proofs will be based on the following.

LEMMA 7.1. *Let M be a $(k-1)$-connected manifold of dimension $2k$, $k \geqq 3$, and suppose that H_kM is free abelian with basis $\{\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_r\}$ where*

$$\lambda_i \cdot \lambda_j = 0 , \qquad \lambda_i \cdot \mu_j = \delta_{ij}$$

for all i, j (where δ_{ij} denotes a Kronecker delta). Suppose further that every imbedded sphere in M which represents a homology class in the subgroup generated by $\lambda_1, \cdots, \lambda_r$ has trivial normal bundle. Then H_kM

[7] This expression for the order of bP_{4m} relies on recent results of J. F. Adams [1].

can be killed by a sequence of spherical modifications.

PROOF. According to [17, Lemma 6] or Haefliger [6] any homology class in $H_k M$ can be represented by a differentiably imbedded sphere.

REMARK. It is at this point that the hypothesis $k \geqq 3$ is necessary. Our methods break down completely for the case $k = 2$ since a homology class in $H_2(M^4)$ need not be representable by a differentiably imbedded sphere. (Compare Kervaire and Milnor [13].)

Choose an imbedding $\varphi_0 \colon S^k \to M$ so as to represent the homology class λ_r. Since the normal bundle is trivial, φ_0 can be extended to an imbedding $\varphi \colon S^k \times D^k \to M$. Let $M' = \chi(M, \varphi)$ denote the modified manifold, and let

$$M_0 = M - \text{Interior } \varphi(S^k \times D^k) = M' - \text{Interior } \varphi'(D^{k+1} \times S^{k-1}) .$$

The argument now proceeds just as in [17, p. 54]. There is a diagram

$$\begin{tikzcd} Z \arrow[d] \arrow[dr, "\lambda_r"] & & & & \\ 0 \arrow[r] & H_k M_0 \arrow[r] \arrow[d] & H_k M \arrow[r, "\cdot \lambda_r"] & Z \arrow[r] & H_{k-1} M_0 \arrow[r] & 0 \\ & H_k M' \arrow[d] & & & & \\ & 0 & & & & \end{tikzcd}$$

where the notation and the proof is similar to that of Lemma 5.6. Since $\mu_r \cdot \lambda_r = 1$ it follows that $H_{k-1} M_0 = 0$. From this fact one easily proves that M_0 and M' are $(k-1)$-connected. The group $H_k M_0$ is isomorphic to the subgroup of $H_k M$ generated by $\{\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_{r-1}\}$. The group $H_k M'$ is isomorphic to a quotient group of $H_k M_0$. It has basis $\{\lambda'_1, \cdots, \lambda'_{r-1}, \mu'_1, \cdots, \mu'_{r-1}\}$ where each λ'_i corresponds to a coset

$$\lambda_i + \lambda_r Z \subset H_k M ,$$

and each μ'_j corresponds to a coset $\mu_j + \lambda_r Z$.

The manifold M' also satisfies the hypothesis of 7.1. In order to verify that

$$\lambda'_i \cdot \lambda'_j = 0 , \qquad \lambda'_i \cdot \mu'_j = \delta_{ij} ,$$

note that each λ'_i or μ'_j can be represented by a sphere imbedded in M_0 and representing the homology class λ_i or μ_j of M. Thus the intersection numbers in M' are the same as those in M. In order to verify that any imbedded sphere with homology class $n_1 \lambda'_1 + \cdots + n_{r-1} \lambda'_{r-1}$ has trivial normal bundle, note that any such sphere can be pushed off $\varphi'(0\times S^{k-1})$, and hence can be deformed into M_0. It will then represent a homology class

$$(n_1\lambda_1 + \cdots + n_{r-1}\lambda_{r-1}) + n_r\lambda_r \in H_kM ,$$

and thus will have trivial normal bundle.

Iterating this construction r times, the result will be a k-connected manifold. This completes the proof of Lemma 7.1.

Now consider an s-parallelizable manifold M of dimension $2k$, bounded by a homology sphere. By Theorem 5.5, we can assume that M is $(k-1)$-connected. Using the Poincaré duality theorem it follows that H_kM is free abelian, and that the intersection number pairing

$$H_kM \otimes H_kM \to Z$$

has determinant ± 1. The argument now splits up into three cases.

Case 1. Let $k = 3$ or 7. (Compare [17, Theorem 4′].) Since k is odd the intersection pairing is skew symmetric. Hence there exists a "symplectic" basis for H_kM; that is, a basis $\{\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_r\}$ with

$$\lambda_i \cdot \lambda_j = \mu_i \cdot \mu_j = 0 , \qquad \lambda_i \cdot \mu_j = \delta_{ij} .$$

Since $\pi_{k-1}(\mathrm{SO}_k) = 0$ for $k = 3, 7$, any imbedded k-sphere will have trivial normal bundle. Thus Lemma 7.1 implies that H_kM can be killed. Since an analogous result for $k = 1$ is easily obtained, this proves:

LEMMA 7.2. *The groups bP_2, bP_6, and bP_{14} are zero.*

Case 2. k is odd, but $k \neq 1, 3, 7$. Again one has a symplectic basis; but the normal bundle of an imbedded sphere is not necessarily trivial. This case will be studied in § 8.

Case 3. k is even, say $k = 2m$. Then the following is true. (Compare [17, Theorem 4].)

LEMMA 7.3. *Let M be a framed manifold of dimension $4m > 4$, bounded by a homology sphere.*[8] *The homotopy groups of M can be killed by a sequence of framed spherical modifications if and only if the signature $\sigma(M)$ is zero.*

Since a proof of 7.3 is essentially given in [17] we will only give an outline here.

In one direction the lemma follows from the assertion that $\sigma(M)$ is invariant under spherical modifications. (See [17, p. 41]. The fact that M has a boundary does not matter here, since we can adjoin a cone over

[8] This lemma is of course also true if bM is vacuous. In this case the signature $\sigma(M)$ is necessarily zero, by Hirzebruch's signature theorem.

the boundary, thus obtaining a closed homology manifold with the same signature.)

Conversely suppose that $\sigma(M) = 0$. We may assume that M is $(k-1)$-connected. Since the quadratic form $\lambda \to \lambda \cdot \lambda$ has determinant ± 1 and signature zero, it is possible to choose a basis $\{\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_r\}$ for $H_k M$ so that $\lambda_i \cdot \lambda_j = 0$, $\lambda_i \cdot \mu_j = \delta_{ij}$. The proof is analogous to that of [17, Lemma 9], but somewhat simpler since we do not put any restriction on $\mu_i \cdot \mu_j$. For any imbedded sphere with homology class $\lambda = n_1\lambda_1 + \cdots + n_r\lambda_r$, the self-intersection number $\lambda \cdot \lambda$ is zero. Therefore, according to [17, Lemma 7], the normal bundle is trivial.

Thus M satisfies the hypothesis of 7.1. It follows that $H_k M$ can be killed by spherical modifications. Since the homomorphism

$$\pi_k(\mathrm{SO}_k) \to \pi_k(\mathrm{SO}_{2k+1})$$

is onto for k even, it follows from Lemma 6.2 that we need use only framed spherical modifications. This completes the proof of Lemma 7.3.

LEMMA 7.4. *For each $k = 2m$ there exists a parallelizable manifold M_0 whose boundary bM_0 is the ordinary $(4m-1)$-sphere, such that the signature $\sigma(M_0)$ is non-zero.*

PROOF. According to Milnor and Kervaire [18, p. 457] there exists a closed "almost parallelizable" $4m$-manifold whose signature is non-zero. Removing the interior of an imbedded $4m$-disk from this manifold, we obtain the required parallelizable manifold M_0.

Now consider the collection of all $4m$-manifolds M_0 which are s-parallelizable, and are bounded by the $(4m-1)$-sphere. Clearly the corresponding signatures $\sigma(M_0) \in Z$ form a group under addition. Let $\sigma_m > 0$ denote the generator of this group.

THEOREM 7.5. *Let Σ_1 and Σ_2 be homotopy spheres of dimension $4m-1$, $m > 1$, which bound s-parallelizable manifolds M_1 and M_2 respectively. Then Σ_1 is h-cobordant to Σ_2 if and only if*

$$\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_m .$$

PROOF. First suppose that

$$\sigma(M_1) = \sigma(M_2) + \sigma(M_0) .$$

Form the connected sum along the boundary

$$(M, bM) = (-M_1, -bM_1) \# (M_2, bM_2) \# (M_0, bM_0)$$

as in § 2; with boundary

$$bM = -\Sigma_1 \# \Sigma_2 \# S^{4m-1} \approx -\Sigma_1 \# \Sigma_2 .$$

Since

$$\sigma(M) = -\sigma(M_1) + \sigma(M_2) + \sigma(M_0) = 0$$

it follows from 7.3 that $bM = -\Sigma_1 \# \Sigma_2$ belongs to the trivial h-cobordism class. Therefore Σ_1 is h-cobordant to Σ_2.

Conversely let W be an h-cobordism between $-\Sigma_1 \# \Sigma_2$ and the sphere S^{4m-1}. Pasting W onto $(-M_1, -bM_1)\#(M_2, bM_2)$ along the common boundary $-\Sigma_1 \# \Sigma_2$, we obtain a differentiable manifold M bounded by the sphere S^{4m-1}. Since M is clearly S-parallelizable, we have

$$\sigma(M) \equiv 0 \qquad (\mathrm{mod}\ \sigma_m) .$$

But

$$\sigma(M) = -\sigma(M_1) + \sigma(M_2) .$$

Therefore

$$\sigma(M_1) \equiv \sigma(M_2) \qquad (\mathrm{mod}\ \sigma_m) ,$$

which completes the proof.

COROLLARY 7.6. *The group bP_{4m}, $m > 1$, is isomorphic to a subgroup of the cyclic group of order σ_m. Hence bP_{4m} is finite cyclic.*

The proof is evident.

Discussion and computations. In Part II we will see that bP_{4m} is cyclic of order precisely $\sigma_m/8$. In fact a given integer σ occurs as $\sigma(M)$ for some S-parallelizable M bounded by a homotopy sphere if and only if

$$\sigma \equiv 0 \qquad (\text{modulo } 8) .$$

The following equality is proved in [18, p. 457]:

$$\sigma_m = 2^{2m-1}(2^{2m-1} - 1)B_m j_m a_m/m ,$$

where B_m denotes the m^{th} Bernoulli number, j_m denotes the order of the cyclic group

$$J(\pi_{4m-1}(\mathrm{SO})) \subset \Pi_{4m-1} ,$$

and a_m equals 1 or 2 according as m is even or odd. Thus bP_{4m} is cyclic of order

$$(1) \qquad \sigma_m/8 = 2^{2m-4}(2^{2m-1} - 1)B_m j_m a_m/m .$$

According to recent work of J. F. Adams [1], the integer j_m is precisely equal to the denominator of $B_m/4m$, at least when m is odd. (Compare [18, Theorem 4].) Therefore

$$B_m j_m a_m/4m = a_m \text{ numerator } (B_m/4m) = \text{numerator } (4B_m/m)$$

where the last equality holds since the denominator of B_m is divisible by 2 but not 4. Thus bP_{4m} is cyclic of order

$$(2)\qquad \sigma_m/8 = 2^{2m-2}(2^{2m-1}-1) \text{ numerator } (4B_m/m)\ ,$$

when m is odd.

One can also give a formula for the order of the full group Θ_{4m-1}. In Part II we will see that Θ_{4m-1}/bP_{4m} is isomorphic to $\Pi_{4m-1}/J(\pi_{4m-1}(\mathrm{SO}))$. (Compare § 4.) Together with formula (1) above this implies that:

$$\text{order } \Theta_{4m-1} = (\text{order } \Pi_{4m-1})2^{2m-4}(2^{2m-1}-1)B_m a_m/m\ .$$

8. A cohomology operation

Let $2 \leqq k \leqq n-2$ be integers and let (K, L) be a CW-pair which satisfies the following:

Hypothesis. The cohomology groups $H^i(K, L; G)$ vanish for $k < i < n$ and for all coefficient groups G.

Then a cohomology operation

$$\psi\colon H^k(K, L; Z) \to H^n\big(K, L; \pi_{n-1}(S^k)\big)$$

is defined as follows[9]. Let $e^0 \in S^k$ denote a base point and let

$$s \in H^k(S^k, e^0; Z)$$

denote a generator. Then $\psi(c)$ will denote the first obstruction to the existence of a map

$$f\colon (K, L) \to (S^k, e^0)$$

satisfying the condition $f^*(s) = c$.

To be more precise let K^r denote the r-skeleton of K. Then given any class

$$x \in H^k(K, L; Z) \simeq H^k(K^{n-1} \cup L, L; Z)\ ,$$

it follows from standard obstruction theory that there exists a map

$$f_x\colon (K^{n-1} \cup L, L) \to (S^k, e^0)$$

with $f_x^* s = x$; and that the restriction

$$f_x \mid (K^{n-2} \cup L, L)$$

is well defined up to homotopy. The obstruction to extending f_x over $K^n \cup L$ is the required class

$$\psi(x) \in H^n\big(K, L; \pi_{n-1}(S^k)\big)\ .$$

[9] A closely related operation φ_0 has been studied by Kervaire [12]. The operation φ_0 would serve equally well for our purposes.

LEMMA 8.1. *The function*

$$\psi\colon H^k(K, L; Z) \to H^n(K, L; \pi_{n-1}(S^k))$$

is well defined, and is natural in the following sense. If the CW-*pair* (K', L') *also satisfies the hypothesis above, then for any map*

$$g\colon (K', L') \to (K, L) ,$$

and any $x \in H^k(K, L; Z)$ *the identity*

$$g^* \psi(x) = \psi g^*(x)$$

is satisfied.

The proof is straightforward. It follows that ψ does not depend on the particular cell structure of the pair (K, L).

Now let us specialize to the case $n = 2k$.

LEMMA 8.2. *The operator* ψ *satisfies the identity*

$$\psi(x + y) = \psi(x) + \psi(y) + [i, i](x \smile y) ,$$

where the last term stands for the image of the class $x \smile y \in H^{2k}(K, L; Z)$ *under that coefficient homomorphism*

$$Z \to \pi_{2k-1}(S^k)$$

which carries 1 *into the Whitehead product class* $[i, i]$.

PROOF. Let $U = e^0 \cup e^k \cup \{e_i^{2k}\} \cup \{e_j^{2k+1}\} \cup \cdots$ denote a complex formed from the sphere S^k by adjoining cells of dimensions $\geqq 2k$ so as to kill the homotopy groups in dimensions $\geqq 2k - 1$. Let

$$u \in H^k(U, e^0; Z)$$

be a standard generator. Evidently the functions

$$\psi\colon H^k U \to H^{2k}(U; \pi_{2k-1}(S^k))$$

and

$$\psi\colon H^k(U \times U) \to H^{2k}(U \times U; \pi_{2k-1}(S^k))$$

are defined. We will first evaluate $\psi(u \times 1 + 1 \times u)$.

The $(2k + 1)$-skeleton of $U \times U$ consists of the union

$$U^{2k+1} \times e^0 \cup e^0 \times U^{2k+1} \cup e^k \times e^k .$$

Therefore the cohomology class $\psi(u \times 1 + 1 \times u) \in H^{2k}(U \times U; \pi_{2k-1}(S^k))$ can be expressed uniquely in the form

$$a \times 1 + 1 \times b + \gamma(u \times u)$$

with $a, b \in H^{2k}(U; \pi_{2k-1}(S^k))$ and $\gamma \in \pi_{2k-1}(S^k)$. Applying 8.1 to the inclusion map

$$U \times e^0 \to U \times U$$

we see that a must be equal to $\psi(u)$. Similarly b is equal to $\psi(u)$. Applying 8.1 to the inclusion

$$S^k \times S^k \to U \times U$$

we see that $\psi(s \times 1 + 1 \times s) = \gamma(s \times s)$. But $\psi(s \times 1 + 1 \times s)$ is just the obstruction to the existence of a mapping

$$f\colon S^k \times S^k \to S^k$$

satisfying $f(e^0, x) = f(x, e^0) = x$. Therefore γ must be equal to the Whitehead product class $[i, i] \in \pi_{2k-1}(S^k)$. Thus we obtain the identity

$$\begin{aligned}\psi(u \times 1 + 1 \times u) &= \psi(u) \times 1 + 1 \times \psi(u) + [i, i](u \times u)\\ &= \psi(u \times 1) + \psi(1 \times u) + [i, i]\big((u \times 1) \smile (1 \times u)\big)\,.\end{aligned}$$

Now consider an arbitrary CW-pair (K, L), and two classes $x, y \in H^k(K, L)$. Choose a map

$$g\colon (K, L) \to (U \times U, e^0 \times e^0)$$

so that $g^*(u \times 1) = x$, $g^*(1 \times u) = y$. (Such a map can be constructed inductively over the skeletons of K since the obstruction groups $H^i(H, L; \pi_{i-1}(U \times U))$ are all zero.) Then by 8.1:

$$\begin{aligned}\psi(x + y) &= g^*\psi(u \times 1 + 1 \times u)\\ &= g^*\psi(u \times 1) + g^*\psi(1 \times u) + [i, i]g^*\big((u \times 1) + (1 \times u)\big)\\ &= \psi(x) + \psi(y) + [i, i](x \smile y)\,.\end{aligned}$$

This completes the proof of Lemma 8.2.

Now let M be a $2k$-manifold which is $(k-1)$-connected. Then

$$\psi\colon H^k(M, bM) \to H^{2k}(M, bM; \pi_{2k-1}(S^k)) \simeq \pi_{2k-1}(S^k)$$

is defined.

LEMMA 8.3. *Let k be odd*[10] *and let M be s-parallelizable. Then an imbedded k-sphere in M has trivial normal bundle if and only if its dual cohomology class $v \in H^k(M, bM)$ satisfies the condition $\psi(v) = 0$.*

PROOF. Let N be a closed tubular neighborhood of the imbedded sphere, and let

$$M_0 = M - \text{Interior } N\,.$$

Then there is a commutative diagram

[10] This lemma is actually true for k even also.

$$\begin{array}{ccc} w \in H^k(N, bN) & \xrightarrow{\psi} & H^{2k}(N, bN; \pi_{2k-1}(S^k)) \\ \uparrow \simeq & & \uparrow \simeq \\ H^k(M, M_0) & \xrightarrow{\psi} & H^{2k}(M, M_0; \pi_{2k-1}(S^k)) \\ \downarrow & & \downarrow \simeq \\ v \in H^k(M, bM) & \xrightarrow{\psi} & H^{2k}(M, bM; \pi_{2k-1}(S^k)) \ ; \end{array}$$

where a generator w of the infinite cyclic group $H^k(N, bN)$ corresponds to the cohomology class v under the left hand vertical arrows. Thus[11]

$$\psi(v)[M] = \psi(w)[N] \in \pi_{2k-1}(S^k) \ .$$

It is clear that the homotopy class $\psi(w)[N]$ depends only on the normal bundle of the imbedded sphere.

The normal bundle is determined by an element ν of the group $\pi_{k-1}(\mathrm{SO}_k)$. Since M is s-parallelizable, ν must belong to the kernel of the homomorphism

$$\pi_{k-1}(\mathrm{SO}_k) \to \pi_{k-1}(\mathrm{SO}) \ .$$

But this kernel is zero for $k = 1, 3, 7$, and is cyclic of order 2 for other odd values of k. The unique non-trivial element corresponds to the tangent bundle of S^k, or equivalently to the normal bundle of the diagonal in $S^k \times S^k$.

Thus if $\nu \neq 0$ then N can be identified with a neighborhood of the diagonal in $S^k \times S^k$. Then

$$\psi(w)[N] = \psi(s \times 1 + 1 \times s)[S^k \times S^k] = [i, i] \neq 0$$

(assuming that $k \neq 1, 3, 7$). On the other hand if $\nu = 0$ then $\psi(w)$ is clearly zero. This completes the proof of Lemma 8.3.

Henceforth we will assume that k is odd and $\neq 1, 3, 7$. The subgroup of $\pi_{2k-1}(S^k)$ generated by $[i, i]$ will be identified with the standard cyclic group Z_2. Thus a function

$$\psi_0 \colon H_k M \to Z_2$$

is defined by the formula

$$\psi_0(\lambda) = \psi(x)[M]$$

where $x \in H^k(M, bM)$ denotes the Poincaré dual of the homology class λ. Evidently:

(1) $\psi_0(\lambda + \mu) \equiv \psi_0(\lambda) + \psi_0(\mu) + \lambda \cdot \mu \pmod 2$, and

[11] The symbol $[M]$ denotes the homomorphism $H^n(M, bM; G) \to G$ determined by the orientation homology class in $H_n(M, bM; Z)$.

(2) $\psi_0(\lambda) = 0$ if and only if an imbedded sphere representing the homology class λ has trivial normal bundle.

Now assume that bM has no homology in dimensions k, $k-1$, so that the intersection pairing has determinant ± 1. Then one can choose a symplectic basis for $H_k M$: that is a basis $\{\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_r\}$ such that

$$\lambda_i \cdot \lambda_j = 0 , \qquad \mu_i \cdot \mu_j = 0 , \qquad \lambda_i \cdot \mu_j = \delta_{ij} .$$

DEFINITION. The Arf *invariant* $c(M)$ is defined to be the residue class[12]

$$\psi_0(\lambda_1)\psi_0(\mu_1) + \cdots + \psi_0(\lambda_r)\psi_0(\mu_r) \in Z_2 .$$

(Compare [3].) This residue class modulo 2 does not depend on the choice of symplectic basis.

LEMMA 8.4. *If $c(M) = 0$ then $H_k M$ can be killed by a sequence of framed spherical modifications.*

The proof will depend on Lemma 7.1. Let $\{\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_r\}$ be a symplectic basis for $H_k M$. By permuting the λ_i and μ_i we may assume that

$$\begin{aligned} \psi_0(\lambda_i) &= \psi_0(\mu_i) = 1 && \text{for } i \leqq s , \\ \psi_0(\lambda_i) &= 0 && \text{for } i > s , \end{aligned}$$

where s is an integer between 0 and r. The hypothesis

$$c(M) = \sum \psi_0(\lambda_i)\psi_0(\mu_i) = 0$$

implies that $s \equiv 0 \pmod 2$.

Construct a new basis $\{\lambda'_1, \cdots, \mu'_r\}$ for $H_k M$ by the substitutions

$$\begin{aligned} \lambda'_{2i-1} &= \lambda_{2i-1} + \lambda_{2i} , \qquad & \lambda'_{2i} &= \mu_{2i-1} - \mu_{2i} , \\ \mu'_{2i-1} &= \mu_{2i-1} , & \mu'_{2i} &= \lambda_{2i} , \end{aligned}$$

for $2i \leqq s$, with

$$\lambda'_i = \lambda_i , \qquad \mu'_i = \mu_i$$

for $i > s$. This new basis is again symplectic, and satisfies the condition:

$$\psi_0(\lambda'_1) = \cdots = \psi_0(\lambda'_r) = 0 .$$

For any sphere imbedded in M with homology class $\lambda = n_1\lambda'_1 + \cdots + n_r\lambda'_r$ the invariant $\psi_0(\lambda)$ is zero, and hence the normal bundle is trivial. Thus the basis $\{\lambda'_1, \cdots, \mu'_r\}$ satisfies the hypothesis of Lemma 7.1. Thus $H_k M$ can be killed by spherical modifications.

If M is a framed manifold then it is only necessary to use framed modifications for this construction. This follows from Lemma 6.2, since

[12] This coincides with the invariant $\Phi(M)$ as defined by Kervaire [12].

the homomorphism $\pi_k(\mathrm{SO}_k) \to \pi_k(\mathrm{SO}_{2k+1})$ is onto for $k \neq 1, 3, 7$. This completes the proof of Lemma 8.4.

THEOREM 8.5. *For k odd the group bP_{2k} is either zero or cyclic of order* 2.

According to Lemma 7.2 the groups bP_2, bP_6 and bP_{14} are zero. Thus we may assume that $k \neq 1, 3, 7$.

Let M_1 and M_2 be s-parallelizable and $(k-1)$-connected manifolds of dimension $2k$, bounded by homotopy spheres. If

$$c(M_1) = c(M_2)$$

we will prove that bM_1 is h-cobordant to bM_2. This will clearly prove 8.5.

Form the connected sum $(M, bM) = (M_1, bM_1) \# (M_2, bM_2)$ along the boundary. Clearly

$$c(M) = c(M_1) + c(M_2) = 0 .$$

Therefore, according to 8.4, it follows that the boundary

$$bM = bM_1 \# bM_2$$

bounds a contractible manifold. Hence, according to Theorem 1.1 the manifold bM_1 is h-cobordant to $-bM_2$. Since a similar argument shows that bM_2 is h-cobordant to $-bM_2$, this completes the proof.

REMARK. It seems plausible that $bP_{2k} \simeq Z_2$ for all odd k other than 1, 3, 7; but this is known to be true only for $k = 5$ (compare Kervaire [12]) and $k = 9$.

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