SOME NONSTABLE HOMOTOPY GROUPS OF LIE GROUPS

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The main result of [6], stating that S^{4n-1} is not parallelizable except for $n = 1$ and 2, can be reformulated in terms of homotopy groups of the rotation group $SO(4n-1)$ as follows: For $n \geqq 3$, $\pi_{4n-2}(SO(4n-1))$ is not zero; or equivalently, for $n \geqq 3$, $\pi_{4n-2}(SO(4n-2))$ is not zero. (Compare [6], Lemma 2.)

In the present paper, the results of R. Bott [2] on the stable homotopy of the classical groups and the isomorphism $\pi_{2q}(U(q)) \cong Z/q!\,Z$ are used to derive more precise information on $\pi_{4n-2}(SO(4n-1))$, $\pi_{4n-2}(SO(4n-2))$, and further nonstable homotopy groups of the rotation group $SO(m)$ and the unitary group $U(m)$. Our results also rely essentially on the computations of G. F. Paechter [8].

As seen from the tables below, periodicity persists "for some time" in the nonstable range in the sense that $\pi_{r+m}(SO(m))$ for $r \leqq 1$ and large m depends only on the remainder class of $r + m$ modulo 8. (Periodicity breaks down for low values of m, due to the fact that S^1, S^3, S^7 are parallelizable.) Similarly, for m large enough and $r \leqq 2$, $\pi_{2m+r}(U(m))$ depends only on the parity of r.

$\pi_{2m+r}(U(m))$ is given for $r \leqq 2$ by the following table:

${}_r\backslash^m$	$2k-1$	$2k$
1	0	Z_2
2	$Z_{(2k)!/2}$	$Z_2 + Z_{(2k+1)!}$ for $k > 1$ Z_{12} for $k = 1$

$\pi_{m+r}(SO(m))$ is given by the following table, valid for $s \geqq 1$:

${}_r\backslash^m$	$8s$	$8s+1$	$8s+2$	$8s+3$	$8s+4$	$8s+5$	$8s+6$	$8s+7$
-1	$Z+Z$	Z_2+Z_2	$Z+Z_2$	Z_2	$Z+Z$	Z_2	Z	Z_2
0	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_4	Z	Z_2+Z_2	Z_2	Z_4	Z
1	$Z_2+Z_2+Z_2$	Z_8	Z	Z_2	Z_2+Z_2	Z_8	Z	Z_2+Z_2
2	$Z_{24}+Z_8$	$Z+Z_2$	Z_{12}	Z_2+Z_2	Z_4+Z_{24d}	$Z+Z_2$	$Z_{12}+Z_2$	Z_2+Z_2
3	$Z+Z_2$	0	Z_2	Z_{8d}	$Z+Z_2$	Z_2	Z_2	Z_8
4	0	Z_2	Z_{8d}	$Z+Z_2$	Z_2	Z_2	Z_8	$Z+Z_2$

In this table d is ambiguously 1 or 2.

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For low values of m, $\pi_{m+r}(SO(m))$ is mostly well known. We mention for completeness the following table:

$r \backslash m$	3	4	5	6	7
-1	0	$Z + Z$	Z_2	Z	0
0	Z	$Z_2 + Z_2$	Z_2	0	Z
1	Z_2	$Z_2 + Z_2$	0	Z	$Z_2 + Z_2$
2	Z_2	$Z_{12} + Z_{12}$	Z	Z_{24}	$Z_2 + Z_2$
3	Z_{12}	$Z_2 + Z_2$	0	Z_2	Z_8
4	Z_2	$Z_2 + Z_2$	0	$Z_{120} + Z_2$	$Z + Z_2$

The knowledge of $\pi_{m+r}(SO(m))$ provides information on the homomorphisms of the homotopy exact sequence of the fibering $SO(m)/SO(m-1) = S^{m-1}$. We obtain

THEOREM 1. *Let η_{m-1} be the generator of $\pi_m(S^{m-1}) \approx Z_2$. Then $\partial\eta_{4n-2} \neq 0$ for $n \geqq 3$, where $\partial: \pi_m(S^{m-1}) \to \pi_{m-1}(SO(m-1))$ is the boundary homomorphism.*

Remark. $\partial\eta_2 = 0$, $\partial\eta_6 = 0$ because S^3 and S^7 are parallelizable. It is well known that $\partial\eta_{4n-1} = 0$, $\partial\eta_{4n} \neq 0$, and $\partial\eta_{4n+1} \neq 0$. (Compare P. J. Hilton and J. H. C. Whitehead [4], [5].)

Similarly, we obtain

THEOREM 2. *Let Θ_{m-1} be the generator of the group $\pi_{m+1}(S^{m-1}) \approx Z_2$. ($\Theta_{m-1} = \eta_{m-1} \circ \eta_m$.) Then $\partial\Theta_{4n+1} \neq 0$ for $s \geqq 2$.*

Remark. $\partial\Theta_5 = 0$, $\partial\Theta_{4n-2} = 0$, $\partial\Theta_{4n-1} = 0$, $\partial\Theta_{4n} \neq 0$ are well known. (Compare [4], [5].)

THEOREM 3. *Let ν_{m-1} be the generator of the stable group $\pi_{m+2}(S^{m-1})$ $(m \geqq 6)$, and $\partial: \pi_{m+2}(S^{m-1}) \to \pi_{m+1}(SO(m-1))$ the boundary operator of the homotopy sequence of the fibering $SO(m)/SO(m-1)$. We have*

(i) $\partial\nu_{8s-3} \neq 0$, $2\partial\nu_{8s-3} = 0$, *for* $s \geqq 2$; $\partial\nu_5 = 0$,
(ii) *the kernel of* $\partial: \pi_{8s+1}(S^{8s-2}) \to \pi_{8s}(SO(8s-2))$ *contains* 0, *and* $12\nu_{8s-2} = \eta_{8s-2} \circ \eta_{8s-1} \circ \eta_{8s}$,
(iii) $\partial\nu_{8s-1} = 0$,
(iv) $\partial: \pi_{8s+3}(S^{8s}) \to \pi_{8s+2}(SO(8s))$ *is injective,*
(v) $\partial\nu_{8s+1} \neq 0$, $2\partial\nu_{8s+1} = 0$,
(vi) *the kernel of* $\partial: \pi_{8s+5}(S^{8s+2}) \to \pi_{8s+4}(SO(8s+2))$ *is cyclic of order* 2 *generated by* $12\nu_{8s+2} = \eta_{8s+2} \circ \eta_{8s+3} \circ \eta_{8s+4}$,
(vii) $\partial\nu_{8s+3} \neq 0$, $2\partial\nu_{8s+3} = 0$,
(viii) *the kernel of* $\partial: \pi_{8s+7}(S^{8s+4}) \to \pi_{8s+6}(SO(8s+4))$ *is at most* Z_2.

Some lemmas

The following preliminary lemma is a generalization of a lemma of B. Eckmann (compare [3]).

Let ξ be a fibre space with projection p, and let

$$\pi_{i+1}(E) \xrightarrow{p} \pi_{i+1}(X) \xrightarrow{\partial} \pi_i(F) \to \pi_i(E)$$

be the homotopy sequence of ξ.

LEMMA 1. *If $\alpha \in \pi_{i+1}(X)$ has the form $\alpha = \alpha' \circ E\beta$, where $\beta \in \pi_i(S^m)$ and $\alpha' \in \pi_{m+1}(X)$, then $\partial\alpha = (\partial\alpha') \circ \beta$.*

Proof. Let $f': (B^{m+1}, S^m) \to (E, F)$ be such that $p \circ f$ represents α'. Then $f' \mid S^m$ represents $\partial\alpha'$. Let $C\beta: (B^{i+1}, S^i) \to (B^{m+1}, S^m)$ be the mapping induced by β, and define $f: (B^{i+1}, S^i) \to (E, F)$ to be $f = f' \circ C\beta$. Clearly $p \circ f$ represents $\alpha = \alpha' \circ E\beta$. Hence $\partial(\alpha' \circ E\beta) = \partial(p \circ f) = f \mid S^i = (f' \mid S^m) \circ \beta = \partial\alpha' \circ \beta$.

LEMMA 2. *Let ε_i be a generator of the stable group $\pi_i(SO(m))$ (whenever nonzero). We have the relations*

$$\varepsilon_{8s-1} \circ \eta_{8s-1} = \varepsilon_{8s}, \qquad \varepsilon_{8s-1} \circ \Theta_{8s-1} = \varepsilon_{8s+1}$$

for all $s \geqq 1$.

Proof. Let $b: SO(n) \to \Omega^8 SO(16n)$ be the Bott map.[1]

Since $b\varepsilon_i = \pm\varepsilon_{i+8}$, the above relations hold if they do for $s = 1$. Thus we have only to verify that $\varepsilon_7 \circ \eta_7 \neq 0$, $\varepsilon_7 \circ \Theta_7 \neq 0$. In fact, $J(\varepsilon_7 \circ \eta_7) \neq 0$ and $J(\varepsilon_7 \circ \Theta_7) \neq 0$.

Notice that $J(\varepsilon_7) = E^m\gamma$, where $\gamma: S^{15} \to S^8$ is the Hopf map. (See Milnor-Kervaire [7].)

Now $\gamma \circ \eta$ and $\gamma \circ \Theta$ are known to be nonzero (see Adams [1]). (Recall that $J(\alpha \circ \beta) = \pm J\alpha \circ E^m\beta$, where $\alpha \in \pi_k(SO(m))$, $\beta \in \pi_j(S^k)$.)

I. The unitary groups

LEMMA I.1. *Let $q: U(n) \to S^{2n-1}$ be the natural projection. Then $q_*: \pi_{2n}(U(n)) \to \pi_{2n}(S^{2n-1})$ is given by*

$$q_* \alpha_n = 0 \qquad \text{for } n \text{ odd},$$

$$q_* \alpha_n = \eta_{2n-1} \quad \text{for } n \text{ even},$$

where α_n is a generator of $\pi_{2n}(U(n))$.

Specifically, we shall take α_n to be $\alpha_n = \partial i_{2n+1}$, where ∂ is the boundary homomorphism in

$$\pi_{2n+1}(U(n+1)) \to \pi_{2n+1}(S^{2n+1}) \xrightarrow{\partial} \pi_{2n}(U(n)) \to \pi_{2n}(U(n+1)) = 0.$$

Proof. Let $n = 2k$. Consider the homotopy sequence of $W_{2k+1,2}/S^{4k-1} = S^{4k+1}$:

I.2. $$\pi_{4k+1}(W_{2k+1,2}) \to \pi_{4k+1}(S^{4k+1}) \xrightarrow{\Delta} \pi_{4k}(S^{4k-1}) \to \pi_{4k}(W_{2k+1,2}).$$

[1] Added in proof. See R. BOTT, *The stable homotopy of the classical groups*, Ann. of Math.(2), vol. 70 (1959), pp. 313–337.

Since S^{4k+1} does not admit a 3-field, $\Delta i_{4k+1} \neq 0$. Hence

I.3. $$\Delta i_{4k+1} = \eta_{4k-1} .$$

Since $\Delta = q_* \partial$, it follows that $q_* \alpha_{2k} = q_* \partial i_{4k+1} = \Delta i_{4k+1} = \eta_{4k-1}$. Let $n = 2k - 1$. Since $W_{2k,2}/W_{2k-1,1} = S^{4k-1}$ has a cross section, it follows that $\Delta i_{4k-1} = 0$. Hence $q_* \alpha_{2k-1} = q_* \partial i_{4k-1} = \Delta i_{4k-1} = 0$.

LEMMA I.4. *$\pi_{4k-1}(U(2k - 1)) = 0$, $\pi_{4k+1}(U(2k)) = Z_2$ ($k \geqq 1$), generated by $\partial \eta_{4k+1}$.*

This is an immediate consequence of Lemma I.1, by using the exactness of the sequences

$$\pi_{4k}(U(2k)) \xrightarrow{q_*} \pi_{4k}(S^{4k-1}) \xrightarrow{\partial} \pi_{4k-1}(U(2k-1)) \rightarrow 0,$$

$$\pi_{4k+2}(U(2k+1)) \xrightarrow{q_*} \pi_{4k+2}(S^{4k+1}) \xrightarrow{\partial} \pi_{4k+1}(U(2k)) \rightarrow 0.$$

LEMMA I.5. *$q_* \partial \eta_{4k+1} = \Theta_{4k-1}$, for $k \geqq 1$, where*

$$q_* : \pi_{4k+1}(U(2k)) \rightarrow \pi_{4k+1}(S^{4k-1}).$$

Proof. $q_* \partial \eta_{4k+1} = \Delta \eta_{4k+1}$, where Δ is the boundary operator of the fibering $W_{2k+1,2}/W_{2k,1} = S^{4k+1}$. By Lemma 1, $\Delta \eta_{4k+1} = \Delta i_{4k+1} \circ \eta_{4k} = \eta_{4k-1} \circ \eta_{4k} = \Theta_{4k-1}$. (Compare I.3.)

LEMMA I.6. *$\pi_{4k}(U(2k - 1)) = Z_{(2k)!/2}$, $\pi_{4k+2}(U(2k)) = Z_2 + Z_{(2k+1)!}$ for $k > 1$. For $k = 1$, $\pi_{4k+2}(U(2k)) = \pi_6(U(2)) = \pi_6(S^1 \times S^3) = Z_{12}$, as is well known.*

Proof. Consider the homotopy sequence of the fibering

$$U(2k)/U(2k-1) = S^{4k-1} : \pi_{4k+1}(U(2k)) \xrightarrow{q_*} \pi_{4k+1}(S^{4k-1})$$
$$\xrightarrow{\partial} \pi_{4k}(U(2k-1)) \xrightarrow{i_*} \pi_{4k}(U(2k)) \xrightarrow{q_*} \pi_{4k}(S^{4k-1}).$$

The above results show that the sequence

$$0 \rightarrow \pi_{4k}(U(2k-1)) \xrightarrow{i_*} Z_{(2k)!} \rightarrow Z_2 \rightarrow 0$$

is exact. It follows that $\pi_{4k}(U(2k - 1)) \cong Z_{(2k)!/2}$.

Similarly, the homotopy sequence of $U(2k + 1)/U(2k) = S^{4k+1}$, i.e.,

$$\pi_{4k+3}(S^{4k+1}) \xrightarrow{\partial} \pi_{4k+2}(U(2k)) \xrightarrow{i_*} \pi_{4k+2}(U(2k+1)) \xrightarrow{q_*} \pi_{4k+2}(S^{4k+1})$$

shows that the sequence

$$0 \rightarrow Z_2 \rightarrow \pi_{4k+2}(U(2k)) \rightarrow Z_{(2k+1)!} \rightarrow 0$$

is exact. ($\partial \Theta_{4k+1} = \partial E \Theta_{4k} = \alpha_{2k} \circ \Theta_{4k} \neq 0$, since $q_*(\alpha_{2k} \circ \Theta_{4k}) = \eta_{4k-1} \circ \Theta_{4k} = 12\nu_{4k-1} \neq 0$.) If $\pi_{4k+2}(U(2k))$ is cyclic, then $\partial \Theta_{4k+1}$ is divisible by $(2k + 1)!$.

Hence, $12\nu_{4k-1} = q_* \partial\Theta_{4k+1}$ is also divisible by $(2k+1)!$. This implies $k = 1$. In other words, for $k > 1$, $\pi_{4k+2}(U(2k))$ is the trivial extension: $Z_2 + Z_{(2k+1)!}$.

II. The rotation groups

We shall need the following information about homotopy groups of *complex* Stiefel manifolds.

The following isomorphisms hold for $n \geqq 3$:

LEMMA II.1. (i) $\pi_{4n-1}(W_{2n-1,2}) = 0$, (ii) $\pi_{4n-2}(W_{2n-1,2}) = Z_{12}$, (iii) $\pi_{4n-1}(W_{2n,3}) = Z$.

Proof. The first assertion follows from the exact homotopy sequence of $W_{2n-1,2}/W_{2n-2,1} = S^{4n-3}$:

$$\pi_{4n-1}(S^{4n-5}) \to \pi_{4n-1}(W_{2n-1,2}) \to \pi_{4n-1}(S^{4n-3}) \xrightarrow{\Delta} \pi_{4n-2}(S^{4n-5}).$$

For $n \geqq 3$, the groups $\pi_{4n-1}(S^{4n-5})$ are zero (compare Serre [9]); $\pi_{4n-1}(S^{4n-3})$ is cyclic of order 2, generated by $\Theta_{4n-3} = \eta_{4n-3} \circ \eta_{4n-2}$. We have

$$\Delta\Theta_{4n-3} = \Delta E\Theta_{4n-4} = \Delta i_{4n-3} \circ \Theta_{4n-4} = q_* \partial i_{4n-3} \circ \Theta_{4n-4} = q_* \alpha_{2n-2} \circ \Theta_{4n-4}.$$

By Lemma I.1, $q_* \alpha_{2n-2} = \eta_{4n-5}$. Thus $\Delta\Theta_{4n-3} = 12\nu_{4n-5} \neq 0$. Extending the above sequence

$$\pi_{4n-1}(S^{4n-3}) \xrightarrow{\Delta} \pi_{4n-2}(S^{4n-5}) \to \pi_{4n-2}(W_{2n-1,2}) \to \pi_{4n-2}(S^{4n-3}) \xrightarrow{\Delta},$$

and using $\Delta\eta_{4n-3} = \eta_{4n-5} \circ \eta_{4n-4} = \Theta_{4n-5} \neq 0$, we obtain $\pi_{4n-2}(W_{2n-1,2}) = Z_{12}$. Now, the last assertion of the lemma follows from exactness of the sequence

$$\pi_{4n-1}(W_{2n-1,2}) \to \pi_{4n-1}(W_{2n,3}) \xrightarrow{q''} \pi_{4n-1}(S^{4n-1}) \xrightarrow{\Delta} \pi_{4n-2}(W_{2n-1,2}) = Z_{12}.$$

Incidentally we see that q'' maps a generator onto a times a generator, where a is a divisor of 12.

Consider now the commutative diagram

$$\begin{tikzcd}
\pi_{4n-1}(SO(2m)) \arrow[r, "p'"] & \pi_{4n-1}(V_{2m,2m-4n+6}) \arrow[r] & \pi_{4n-2}(SO(4n-6)) \arrow[r] & 0 \\
\pi_{4n-1}(U(m)) \arrow[r, "q'"] \arrow[u, "\beta"] & \pi_{4n-1}(W_{m,m-2n+3}) \arrow[r] \arrow[u, "\beta'"] & \pi_{4n-2}(U(2n-3)) \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

where m is to be large $(2n < m)$. $\pi_{4n-1}(W_{m,m-2n+3})$ is independent of m for $m \geqq 2n$, and the projection $\pi_{4n-1}(W_{m,m-2n+3}) \to \pi_{4n-1}(W_{m,m-2n+1})$ can be identified with $q'': \pi_{4n-1}(W_{2n,3}) \to \pi_{4n-1}(S^{4n-1})$ considered above. Since $q = q'' \circ q': \pi_{4n-1}(U(m)) \to \pi_{4n-1}(W_{m,m-2n+1})$ maps a generator onto $(2n-1)!$ times a generator, it follows that q' multiplies by $(2n-1)!/a$ (a, divisor of 12).

The map β' is also imbedded in the following diagram

$$\begin{tikzcd}
\pi_{4n-1}(V_{2m,2m-4n+6}) \arrow[r, "p''"] & \pi_{4n-1}(V_{2m,2m-4n+2}) \arrow[r] & \pi_{4n-2}(V_{4n-2,4}) \arrow[r] & Z_{b_n} \arrow[r] & 0 \\
\pi_{4n-1}(W_{m,m-2n+3}) \arrow[r, "q''"] \arrow[u, "\beta'"] & \pi_{4n-1}(W_{m,m-2n+1}) \arrow[r] \arrow[u, "\beta''"] & \pi_{4n-2}(W_{2n-1,2}) \arrow[r] \arrow[u] & \cdots ,
\end{tikzcd}$$

where b_n is equal to 1 for n odd, 2 for n even. Since $\pi_{4n-1}(V_{2m,2m-4n+2}) \cong Z_4$, and $\pi_{4n-2}(V_{4n-2,4}) \cong Z_2$ for $n > 1$, it follows that $\operatorname{Im} p'' = 2\cdot\pi_{4n-1}(V_{2m,2m-4n+2})$ for n odd, and p'' is surjective for n even. It is easily seen that β'' is surjective.

Let n be odd: $n = 2s + 1$. We have $\pi_{8s+3}(V_{2m,2m-8s+2}) \cong Z_8$ (see [8]). $\beta'q'$ is divisible by $(2n-1)!/2$. Consequently, $\beta'q'$ is zero for $n \geqq 5$. By commutativity, $p'\beta = 0$ for $n \geqq 5$, i.e., $s \geqq 2$. Since

$$\beta: \pi_{8s+3}(U(m)) \to \pi_{8s+3}(SO(2m))$$

is surjective ($\pi_{8s+3}(SO(2m)/U(m)) = \pi_{8s+4}(SO(2m)) = 0$, by [2]), it follows that $p': \pi_{8s+3}(SO(2m)) \to \pi_{8s+3}(V_{2m,2m-8s+2})$, and hence

$$\Phi_{8s+3}^{8s-i}: \pi_{8s+3}(SO(2m)) \to \pi_{8s+3}(V_{2m,2m-8s+i})$$

is zero for $s \geqq 2$, $i \leqq 2$. Therefore,

II.2. $\quad \pi_{8s+2}(SO(8s-i)) \cong \pi_{8s+3}(V_{2m,2m-8s+i}) \qquad$ for $i \leqq 2$ and $s \geqq 2$.

Let $n = 3$, or $s = 1$. We use the diagram

$$\begin{tikzcd}
\pi_{11}(SO(2m)) \arrow[r, "p'"] & \pi_{11}(V_{2m,2m-8}) \arrow[r] & \pi_{10}(SO(8)) \arrow[r] & \pi_{10}(SO(2m)) = 0 \\
\pi_{11}(U(m)) \arrow[r, "q'"] \arrow[u, "\beta"] & \pi_{11}(W_{m,m-4}) \arrow[r] \arrow[u, "\beta'"] & \pi_{10}(U(4)) \arrow[r] \arrow[u, "\beta_0"] & \pi_{10}(U(m)) = 0.
\end{tikzcd}$$

By Lemma I.6, $\pi_{10}(U(4)) \cong Z_2 + Z_{120}$. Hence q' is divisible by 120. Since $\pi_{11}(V_{2m,2m-8}) \cong Z_{24} + Z_8$ (see [8]), it follows that $\beta'q' = 0$. Since β is an isomorphism, p' is zero, and a fortiori $\Phi_{11}^{i}: \pi_{11}(SO(2m)) \to \pi_{11}(V_{2m,2m-i})$ is zero for $i \geqq 8$. This gives $\pi_{11}(V_{2m,2m-i}) = \pi_{10}(SO(i))$ for $i \geqq 8$. From G. F. Paechter's table, we obtain

II.3.
$$\pi_{10}(SO(8)) \cong Z_{24} + Z_8, \qquad \pi_{10}(SO(9)) \cong Z_8,$$
$$\pi_{10}(SO(10)) \cong Z_4, \qquad \pi_{10}(SO(11)) \cong Z_2.$$

Since $\pi_{11}(V_{2m,2m-7}) \to \pi_{11}(V_{2m,2m-8})$ is injective ($\pi_{11}(S^7) = 0$), it follows that also $\pi_{11}(V_{2m,2m-7}) \cong \pi_{10}(SO(7))$. This gives

II.4. $$\pi_{10}(SO(7)) \cong Z_8.$$

Let n be even: $n = 2s$. We have $\pi_{8s-1}(V_{2m,2m-8s+6}) \cong Z_{16}$ for $s \geqq 2$. Now $\beta' q'$ is divisible by $(2n-1)!$. Hence $\beta' q' = 0$ for $n \geqq 4$, i.e., $s \geqq 2$. Since $\beta : \pi_{8s-1}(U(m)) \to \pi_{8s-1}(SO(2m))$ maps a generator onto 2 times a generator, and $p'\beta = \beta' q' = 0$ for $s \geqq 2$, it follows that in the sequence

II.5. $$\pi_{8s-1}(SO(m)) \xrightarrow{p'} \pi_{8s-1}(V_{m,m-8s+6}) \to \pi_{8s-2}(SO(8s-6)) \to \pi_{8s-2}(SO(m)) = 0,$$

p' is divisible by 8 for $s \geqq 2$. Now $\pi_{8s-1}(V_{2m,2m-8s+3}) \cong Z_8$. Therefore $\Phi^{8s-i}_{8s-1} : \pi_{8s-1}(SO(2m)) \to \pi_{8s-1}(V_{2m,2m-8s+i})$ is zero for $i \leqq 3$, $s = 2$. Hence

II.6. $$\pi_{8s-2}(SO(8s-i)) \cong \pi_{8s-1}(V_{2m,2m-8s+i}) \qquad \text{for } i \leqq 3 \text{ and } s \geqq 2.$$

The groups $\pi_{8s-2}(SO(8s-6))$, $\pi_{8s-2}(SO(8s-5))$, $\pi_{8s-2}(SO(8s-4))$ are either Z_8, Z_8, $Z_{24} + Z_4$, respectively, or Z_{16}, Z_{16}, $Z_{48} + Z_4$, respectively. I do not know whether the decision of this alternative depends on s or not.

The groups $\pi_{8s+1}(SO(8s-i))$ *for* $-2 \leqq i \leqq 3$ are obtained from the sequence

$$\pi_{8s+2}(SO(m)) \to \pi_{8s+2}(V_{m,m-8s+i}) \to \pi_{8s+1}(SO(8s-i)) \to \pi_{8s+1}(SO(m)) \to \pi_{8s+1}(V_{m,m-8s+i}),$$

where $\pi_{8s+2}(SO(m)) = 0$.

Since $\pi_{8s+1}(V_{m,m-8s+4}) = 0$ for $s \geqq 2$, it follows that

II.7. $$\Phi^{8s+1}_{8s-i} : \pi_{8s+1}(SO(m)) \to \pi_{8s+1}(V_{m,m-8s+i})$$

is zero for $i \leqq 4$ and $s \geqq 2$, and the sequence

$$0 \to \pi_{8s+2}(V_{m,m-8s+i}) \to \pi_{8s+1}(SO(8s-i)) \to \pi_{8s+1}(SO(m)) \to 0$$

is exact for $i \leqq 4$, $s \geqq 2$. Because of commutativity in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_{8s+2}(V_{m,m-8s+i-1}) \arrow[r] & \pi_{8s+1}(SO(8s-i+1)) \arrow[r] & \pi_{8s+1}(SO(m)) \arrow[r] & 0 \\
0 \arrow[r] & \pi_{8s+2}(V_{m,m-8s+i}) \arrow[r] \arrow[u] & \pi_{8s+1}(SO(8s-i)) \arrow[r] \arrow[u, "i_*"] & \pi_{8s+1}(SO(m)) \arrow[r] \arrow[u, "\mathrm{id}"] & 0,
\end{tikzcd}$$

it follows that the upper sequence is a split extension if the lower is. The sequence splits trivially for $i = 3$, since $\pi_{8s+2}(V_{m,m-8s+3}) = 0$, for $s \geqq 1$. Thus

II.8. $$\pi_{8s+1}(SO(8s-i)) \approx Z_2 + \pi_{8s+2}(V_{m,m-8s+i}) \qquad \text{for } i \leqq 3,\ s \geqq 2.$$

For $s = 1$, we have to study $\Phi^9_5 : \pi_9(SO(m)) \to \pi_9(V_{m,m-5}) = Z_2$. Φ^9_5 is an epimorphism, since $\pi_8(SO(5)) = 0$ (see Serre [10]). Therefore, $\pi_9(SO(5)) = \pi_{10}(V_{m,m-5}) = 0$. Now the sequence

$$\pi_9(SO(m)) \to \pi_9(V_{m,m-6}) \to \pi_8(SO(6)) \to \pi_8(SO(m)),$$

which reads $Z_2 \to Z_{12} \to Z_{24} \to Z_2$ (compare Serre [10]), shows that Φ_6^9 is zero. Therefore, the sequence

$$0 \to \pi_{10}(V_{m,m-8+i}) \to \pi_9(SO(8-i)) \to \pi_9(SO(m)) \to 0$$

is exact for $i \leqq 2$. This sequence splits, because it splits trivially for $i = 2$ ($\pi_{10}(V_{m,m-6}) = 0$, according to Paechter [8]). We obtain

II.9.
$$\pi_9(SO(6)) = Z_2, \qquad \pi_9(SO(7)) = Z_2 + Z_2,$$
$$\pi_9(SO(8)) = Z_2 + Z_2 + Z_2,$$
$$\pi_9(SO(9)) = Z_2 + Z_2, \qquad \pi_9(SO(10)) = Z + Z_2.$$

The groups $\pi_{8s}(SO(8s-i))$ *for* $1 \leqq i \leqq 4$. Consider the sequence

$$\pi_{8s+1}(SO(m)) \to \pi_{8s+1}(V_{m,m-8s+i}) \to \pi_{8s}(SO(8s-i)) \to \pi_{8s}(SO(m)) \to \pi_{8s}(V_{m,m-8s+i}),$$

where the first homomorphism (Φ_{8s-i}^{8s+1}) is zero for $i \leqq 4$, $s \geqq 2$, bijective for $i = 3$, $s = 1$, and zero for $i \leqq 2$, $s = 1$.

The value of the last homomorphism is obtained using Lemma 2. Since $\varepsilon_{8s} = \varepsilon_{8s-1} \circ \eta_{8s-1}$, it follows that $\Phi_{8s-i}^{8s}(\varepsilon_{8s}) = \Phi_{8s-i}^{8s-1}(\varepsilon_{8s-1}) \circ \eta_{8s-1}$. We have seen that $\Phi_{8s-i}^{8s-1}(\varepsilon_{8s-1})$ is divisible by 8 for $i \leqq 6$, $s \geqq 2$. It follows that $\Phi_{8s-i}^{8s}(\varepsilon_{8s}) = 0$ for $i \leqq 6$, $s \geqq 2$. We also have $\Phi_{8s-i}^{8s}(\varepsilon_{8s}) = 0$ for $i \leqq 2$ and $s = 1$, for $\pi_{8s}(V_{m,m-8s+2}) = 0$ for $s \geqq 1$.

This gives an exact sequence

II.10. $$0 \to \pi_{8s+1}(V_{m,m-8s+i}) \to \pi_{8s}(SO(8s-i)) \to \pi_{8s}(SO(m)) = Z_2 \to 0,$$

valid for $i \leqq 6$, $s \geqq 2$ and $i \leqq 2$, $s = 1$. Again the sequence splits for any $i \leqq i_0$ if it does for $i = i_0$. We use this with $i_0 = 4$ for $s \geqq 2$, a case where the splitting is obvious since $\pi_{8s+1}(V_{m,m-8s+4}) = 0$. If $s = 1$, the sequence is known to split for $i \leqq 1$ (see Serre [10]). However it does *not* split for $i = 2$ ($\pi_8(SO(6)) = Z_{24}$).

The groups $\pi_{8s-1}(SO(8s-i))$ *for* $1 \leqq i \leqq 5$. Consider the sequence

$$\pi_{8s}(SO(2m)) \xrightarrow{\Phi_{8s-i}^{8s}} \pi_{8s}(V_{2m,2m-8s+i}) \to \pi_{8s-1}(SO(8s-i)) \to \pi_{8s-1}(SO(2m)),$$

for $i \leqq 5$. Since Φ_{8s-i}^{8s} is zero for $s \geqq 2$, it follows that $\pi_{8s-1}(SO(8s-i))$ is isomorphic to the direct sum of Z and $\pi_{8s}(V_{2m,2m-8s+i})$. Thus,

II.11. For $s \geqq 2$, $\pi_{8s-1}(SO(8s-i)) = Z + Z_2$ for $3 \leqq i \leqq 5$, and $\pi_{8s-1}(SO(8s-i)) = Z$ for $i = 1, 2$.

For $s = 1$, the groups are well known.

The groups $\pi_{8s+3}(SO(8s-i))$ *for* $i \leqq 1$ are obtained from the sequence

$$\pi_{8s+4}(SO(m)) = 0 \to \pi_{8s+4}(V_{m,m-8s+i}) \to \pi_{8s+3}(SO(8s-i)) \to \pi_{8s+3}(SO(m)) \to \pi_{8s+3}(V_{m,m-8s+i}).$$

We know that $\Phi_{8s-i}^{8s+3}:\pi_{8s+3}(SO(m)) \to \pi_{8s+3}(V_{m,m-8s+i})$ is zero for $s \geqq 2$, $i \leqq 2$, and $s = 1$, $i \leqq 0$.

Consequently, $\pi_{8s+3}(SO(8s - i)) \approx \pi_{8s+4}(V_{m,m-8s+i}) + Z$ for $i \leqq 2$, $s \geqq 2$, and $s = 1$, $i \leqq 0$.

The groups $\pi_{8s+4}(SO(8s + i))$ *and* $\pi_{8s+5}(SO(8s + i + 1))$ *for* $i \geqq 0$ follow from the sequences

$$0 = \pi_{8s+5}(SO(m)) \to \pi_{8s+5}(V_{m,m-8s-i}) \to \pi_{8s+4}(SO(8s + i)) \to \pi_{8s+4}(SO(m)) = 0$$

and

$$0 = \pi_{8s+6}(SO(m)) \to \pi_{8s+6}(V_{m,m-8s-i-1}) \to \pi_{8s+5}(SO(8s + i + 1)) \to \pi_{8s+5}(SO(m)) = 0,$$

where m is to be large ($m > 8s + 7$), by using G. F. Paechter's computations [8].

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