A NOTE ON OBSTRUCTIONS AND CHARACTERISTIC CLASSES.*

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The present paper is a generalization of [7]. Relations will be established between the obstructions associated with cross-sections in a stable $\boldsymbol{U}(n)$, $\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$-bundle over a *complex* K and the characteristic classes of such bundles.

In the $\boldsymbol{U}(n)$-case, we obtain as a corollary a theorem of F. Peterson [10] stating that a $\boldsymbol{U}(n)$-bundle over a torsion free complex K of dimension $\leqq 2n$ is trivial if and only if the Chern classes of the bundle vanish.

A similar statement in the $\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$ case, involving the Pontryagin, resp. symplectic Pontryagin classes, would be wrong. In case of an $\boldsymbol{SO}(n)$ [resp. $\boldsymbol{Sp}(n)$] bundle there are obstructions in $H^{8s+1}(K;\boldsymbol{Z}_2)$ and $H^{8s+2}(K;\boldsymbol{Z}_2)$ [resp. $H^{8s+5}(K;\boldsymbol{Z}_2)$ and $H^{8s+6}(K;\boldsymbol{Z}_2)$] which are not expressible in terms of characteristic classes of the bundle (see Lemma 4.3 for a precise statement). The information about these obstructions is still very poor.

In [10], F. Peterson deduces his theorem from a computation of the Postnikov decomposition of $B_{\boldsymbol{U}(n)}$. We proceed the other way around and obtain the Postnikov decomposition of $B_{\boldsymbol{U}(n)}$, $B_{\boldsymbol{SO}(n)}$ and $B_{\boldsymbol{Sp}(n)}$ in the stable range from the main lemma (Lemma 1.1).

I am indebted to J. Milnor, B. Eckmann and A. Borel for their suggestions during the preparation of this paper.

1. Let $\boldsymbol{G}$ be one of the groups $\boldsymbol{U}(n)$, $\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$. Let ξ be a stable principal $\boldsymbol{G}$-bundle over a CW-complex K (stability means that the homotopy groups $\pi_{q-1}(\boldsymbol{G})$ are stable for $q \leqq \dim K$). Assume that ξ admits a cross-section f over the $(q-1)$-skeleton $K^{(q-1)}$ of K. Take q to be even $=2r$ if $\boldsymbol{G}=\boldsymbol{U}(n)$ and q divisible by 4, $q=4k$, if $\boldsymbol{G}=\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$. Then, by [4], $\pi_{q-1}(\boldsymbol{G}) \approx \boldsymbol{Z}$ in all cases. The obstruction class $\mathfrak{o}(\xi,f) \in H^q(K;\pi_{q-1}(\boldsymbol{G}))$ to extending f over the q-skeleton can be regarded up to sign as an integer class. Denote by $c_r(\xi)$, $p_k(\xi)$, $e_k(\xi)$ the Chern class, the Pontryagin or the symplectic Pontryagin class[1] of ξ in dimension q according as $\boldsymbol{G}=\boldsymbol{U}(n)$, $\boldsymbol{SO}(n)$ or $\boldsymbol{Sp}(n)$ respectively.

* Received October 5, 1958.

[1] See [3], 9.6, for the definition.

LEMMA 1.1. *The characteristic classes $c_r(\xi)$, $p_k(\xi)$, $e_k(\xi)$ are given by the formulae*

$$\text{(i)} \qquad c_r(\xi) = (r-1)!\,\mathfrak{o}(\xi,f) \qquad \text{if } \boldsymbol{G}=\boldsymbol{U}(n),$$

$$\text{(ii)} \qquad p_k(\xi) = (2k-1)!\,a_k\mathfrak{o}(\xi,f) \qquad \text{if } \boldsymbol{G}=\boldsymbol{SO}(n),$$

$$\text{(iii)} \qquad e_k(\xi) = (2k-1)!\,b_k\mathfrak{o}(\xi,f) \qquad \text{if } \boldsymbol{G}=\boldsymbol{Sp}(n),$$

where, as in [6], $a_k\cdot b_k = 2$ *and* a_k *is equal to* 1 *for* k *even and to* 2 *for* k *odd.*

Proof. Let $\boldsymbol{G}=\boldsymbol{U}(n)$. Denote by ξ' the associated bundle with fibre $W_{n,n-r+1}=\boldsymbol{U}(n)/\boldsymbol{U}(r-1)$. Let $q\colon \boldsymbol{U}(n)\to\boldsymbol{U}(n)/\boldsymbol{U}(r-1)$ be the natural projection and q' the induced map of the total space of ξ into the total space of ξ'. The map $f'=q'\circ f$ is a cross-section of ξ' restricted to the $(q-1)$-skeleton. Denote by $q_*\colon \pi_{2r-1}(\boldsymbol{U}(n))\to\pi_{2r-1}(W_{n,n-r+1})$ and

$$q_{**}\colon H^{2r}(K;\pi_{2r-1}(\boldsymbol{U}(n)))\to H^{2r}(K;\pi_{2r-1}(W_{n,n-r+1}))$$

the homomorphisms induced by q. Clearly, $q_{**}\mathfrak{o}(\xi,f)=\mathfrak{o}(\xi',f')=$ the obstruction to extending f' over the $2r$-skeleton. In other words $q_{**}\mathfrak{o}(\xi,f)=c_r(\xi)$. Identifying $\pi_{2r-1}(\boldsymbol{U}(n))$ and $\pi_{2r-1}(W_{n,n-r+1})$ with $\boldsymbol{Z}$ (disregarding signs), we have $q_{**}u=(r-1)!\,u$ for any $u\in H^*(K;\boldsymbol{Z})$ because q_* maps a generator of $\pi_{2r-1}(\boldsymbol{U}(n))$ onto $(r-1)!$ times a generator of $\pi_{2r-1}(W_{n,n-r+1})$ according to [5]. Thus $c_r(\xi)=q_{**}\mathfrak{o}(\xi,f)=\pm(r-1)!\,\mathfrak{o}(\xi,f)$.

The proofs of (ii) and (iii) are entirely similar and are left to the reader (compare also [9]).

2. As a corollary we obtain the

THEOREM 2.1 (F. Peterson). *Let ξ be a $\boldsymbol{U}(n)$-bundle over a complex K with* $\dim K\leqq 2n$ *and assume that $H^{2r}(K;\boldsymbol{Z})$ has no torsion except possibly prime to $(r-1)!$ for $r=1,2,\cdots$. Then ξ is trivial if and only if the Chern classes $c_1,\cdots,c_n$ vanish.*

Proof. Half of the statement is trivial. We prove that ξ is the product bundle provided $c_1(\xi)=0$, $c_2(\xi)=0,\cdots,c_n(\xi)=0$ by stepwise extension of a cross-section in the associated principal bundle ξ_P.

If f is a cross-section in ξ_P restricted to $K^{(q-1)}$ and q is odd, there is no obstruction to extending f to $K^{(q)}$ since $\pi_{2i}(\boldsymbol{U}(n))=0$ for $i<n$ by [4]. Let q be even: $q=2r$. Then by Lemma 1.1 the obstruction class $\mathfrak{o}(\xi,f)$ satisfies the identity $c_r=\pm(r-1)!\,\mathfrak{o}(\xi,f)$. Under the assumptions of the theorem this implies $\mathfrak{o}(\xi,f)=0$. It follows (see [11], 34.2) that $f\,|\,K^{(q-2)}$ is extendable over $K^{(q)}$. This proves the theorem by induction on q.

Some information on the obstructions arising in the $\boldsymbol{SO}(n)$ and $\boldsymbol{Sp}(n)$ cases is given in Lemmas 4.1 and 4.2 below. We need a preliminary lemma.

3. Let $\boldsymbol{G}$ be any Lie group and $\boldsymbol{H}$ a closed subgroup of $\boldsymbol{G}$ such that the sequence

$$(3.1) \qquad 0 \to \pi_q(\boldsymbol{G}/\boldsymbol{H}) \xrightarrow{\partial} \pi_{q-1}(\boldsymbol{H}) \xrightarrow{i_*} \pi_{q-1}(\boldsymbol{G}) \to 0$$

is exact for some q (here i_* is induced by the inclusion $i\colon \boldsymbol{H} \to \boldsymbol{G}$). Assume that the $\boldsymbol{G}$-bundle ξ over the complex K admits a cross-section f over the $(q-1)$-skeleton. Let $\mathfrak{o}(\xi, f) \in H^q(K; \pi_{q-1}(\boldsymbol{G}))$ be the obstruction class to extending f over $K^{(q)}$. We want to compute $\delta^* \mathfrak{o}(\xi, f)$, where δ^* is the boundary homomorphism $\delta^*\colon H^q(K; \pi_{q-1}(\boldsymbol{G})) \to H^{q+1}(K; \pi_q(\boldsymbol{G}/\boldsymbol{H}))$ of the cohomology exact sequence associated with the coefficient sequence (3.1). (Compare Steenrod [11], 38.5.)

Let ξ' be the associated bundle with fibre $\boldsymbol{G}/\boldsymbol{H}$. The cross-section f induces a cross-section f' of ξ' restricted to the $(q-1)$-skeleton.

LEMMA 3.2. *Under the above exactness assumption of* (3.1), *the cross-section f' is always extendable to a cross-section F' of ξ' restricted to $K^{(q)}$. Let $\mathfrak{o}(\xi', F') \in H^{q+1}(K; \pi_q(\boldsymbol{G}/\boldsymbol{H}))$ be the obstruction class to extending F' over $K^{(q+1)}$. Then $\delta^* \mathfrak{o}(\xi, f) = \mathfrak{o}(\xi', F')$.*

Proof. Let $p\colon \pi_{q-1}(\boldsymbol{G}) \to \pi_{q-1}(\boldsymbol{G}/\boldsymbol{H})$ be induced by the projection $\boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{H}$, and $p_*\colon Z^q(K; \pi_{q-1}(\boldsymbol{G})) \to Z^q(K; \pi_{q-1}(\boldsymbol{G}/\boldsymbol{H}))$ be the homomorphism induced by the coefficient homomorphism p. Let $z \in \mathfrak{o}(\xi, f)$ be the obstruction *cocycle* to extending f over $K^{(q)}$ and z' be the obstruction cocycle to extending f' over $K^{(q)}$. We have $p_* z = z'$ and since p is zero, it follows $z' = 0$. In other words, f' can be extended to a cross-section F' of ξ' restricted to $K^{(q)}$. The map $F'\colon K^{(q)} \to E'$, where E' is the total space of ξ' induces over $K^{(q)}$ an $\boldsymbol{H}$-bundle η (E' is the quotient of the total space E of ξ by the action of $\boldsymbol{H}$ as a subgroup of $\boldsymbol{G}$). $f_\eta(x) = (x, f(x))$ for $x \in K^{(q-1)}$ defines a cross-section of η restricted to $K^{(q-1)}$ (compare Steenrod [11], 10.2). Let $u_\eta \in Z^q(K^{(q)}; \pi_{q-1}(\boldsymbol{H}))$ be the obstruction cocycle to extending f_η over $K^{(q)}$ and let $u \in C^q(K; \pi_{q-1}(\boldsymbol{H}))$ be defined by $u[\tau] = u_\eta[\tau]$ for every q-cell $\tau \in K^{(q)} \subset K$. Clearly, $i_{**} u = z$, where $i_{**}\colon C^q(K; \pi_{q-1}(\boldsymbol{H})) \to C^q(K; \pi_{q-1}(\boldsymbol{G}))$ is induced by $i_*\colon \pi_{q-1}(\boldsymbol{H}) \to \pi_{q-1}(\boldsymbol{G})$. The assertion of the lemma can be stated as

$$(3.3) \qquad \delta u = \partial_* z',$$

where $\partial_*\colon Z^{q+1}(K; \pi_q(\boldsymbol{G}/\boldsymbol{H})) \to Z^{q+1}(K; \pi_{q-1}(\boldsymbol{H}))$ is induced by $\partial\colon \pi_q(\boldsymbol{G}/\boldsymbol{H})$

$\to \pi_{q-1}(\boldsymbol{H})$, and z' is the obstruction to extending F' over $K^{(q+1)}$. (Compare Steenrod [11], 38.5 formula (8).)

Let $\Phi: B^{q+1} \to K$ be the attaching map of a $(q+1)$-cell σ of K, and denote by ϕ the restriction of Φ to the boundary S^q of B^{q+1}. Then $F' \circ \phi: S^q \to E'$ induces over S^q an $\boldsymbol{H}$-bundle whose characteristic map $\chi = \Delta\alpha$ is the image of the element $\alpha \in \pi_q(E')$ represented by $F' \circ \phi$ under the boundary operator $\Delta: \pi_q(E') \to \pi_{q-1}(H)$ of the homotopy sequence of $H \to E \xrightarrow{\pi} E'$. Since $F' \circ \phi(S^q) \subset \rho^{-1}(\Phi B)$, where ρ is the projection $\rho: E' \to K$ of the bundle ξ', $F' \circ \phi$ as a map in $\rho^{-1}(\Phi B)$ represents an element of $\pi_q(G/H)$ which is equal to $z'[\sigma]$. Thus $j'_*(z'[\sigma]) = \alpha$, where $j': \boldsymbol{G}/\boldsymbol{H} \to E'$ is the inclusion of the fibre. On the other hand χ is equal to $u[\phi S^q]$. Consider the commutative diagram

$$\begin{array}{ccc} \pi_q(\boldsymbol{G}/\boldsymbol{H}) & \xrightarrow{\partial} & \pi_{q-1}(\boldsymbol{H}) \\ \downarrow{\scriptstyle j'_*} & & \| \\ \pi_q(E') & \xrightarrow{\Delta} & \pi_{q-1}(\boldsymbol{H}) \end{array}$$

induced by the bundle map

$$\begin{array}{ccc} \boldsymbol{G} & \xrightarrow{j} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ \boldsymbol{G}/\boldsymbol{H} & \xrightarrow{j'} & E'. \end{array}$$

We have

$$\delta u[\sigma] = u[\phi S^q] = \Delta\alpha = \Delta j'_*(z'[\sigma]) = \partial_*(z'[\sigma]) = (\partial_* z')[\sigma].$$

Since this is true for any $(q+1)$-cell σ, the proof of (3.3) is complete.

4. We apply this to the cases $\boldsymbol{G} = \boldsymbol{SO}(2n)$ and $\boldsymbol{G} = \boldsymbol{Sp}(n)$, $\boldsymbol{H} = \boldsymbol{U}(n)$.

LEMMA 4.1. *Let the stable principal $\boldsymbol{SO}(2n)$-bundle ξ admit a cross-section f over the $(8s+1)$-skeleton $K^{(8s+1)}$ of the base complex K. Let $\mathfrak{o}(\xi, f) \in H^{8s+2}(K; \boldsymbol{Z}_2)$ be the obstruction class* (compare Bott [4]). *The induced cross-section f' of the associated bundle ξ' with fibre $\boldsymbol{SO}(2n)/\boldsymbol{U}(n)$ restricted to $K^{(8s+1)}$ is extendable over $K^{(8s+2)}$ to a cross-section F'. One has $\beta\mathfrak{o}(\xi, f) = \pm\, \mathfrak{o}(\xi', F')$, where β is the Bockstein operation and $\mathfrak{o}(\xi', F')$ is the obstruction class to extending F' over $K^{(8s+3)}$.*

Proof. Take $\boldsymbol{G} = \boldsymbol{SO}(2n)$ and $\boldsymbol{H} = \boldsymbol{U}(n)$ in Lemma 3.2. The sequence

$$0 \to \pi_{8s+2}(\boldsymbol{SO}(2n)/\boldsymbol{U}(n)) \to \pi_{8s+1}(\boldsymbol{U}(n)) \to \pi_{8s+1}(\boldsymbol{SO}(2n)) \to 0$$

is exact and reads $0 \to \boldsymbol{Z} \to \boldsymbol{Z} \to \boldsymbol{Z}_2 \to 0$. (Compare Bott [4].) The associated coboundary homomorphism δ^* is β by definition. Similarly,

LEMMA 4.2. *Let the stable principal* $\boldsymbol{Sp}(n)$*-bundle* ξ *admit a cross-section* f *over the* $(8s+5)$*-skeleton of the base complex* K. *Denote by* $\mathfrak{o}(\xi, f) \in H^{8s+6}(K; \boldsymbol{Z}_2)$ *the obstruction class.* f *induces a cross-section* f' *of the associated bundle* ξ' *with fibre* $\boldsymbol{Sp}(n)/\boldsymbol{U}(n)$ *restricted to* $K^{(8s+5)}$. *Then* f' *is extendable over the* $(8s+6)$*-skeleton. Let* F' *be an extension and* $\mathfrak{o}(\xi', F') \in H^{8s+7}(K; \boldsymbol{Z})$ *the obstruction class (up to sign). We have* $\beta\mathfrak{o}(\xi, f) = \pm\, \mathfrak{o}(\xi', F')$.

Proof. The sequence

$$0 \to \pi_{8s+6}(\boldsymbol{Sp}(n)/\boldsymbol{U}(n)) \to \pi_{8s+5}(\boldsymbol{U}(n)) \to \pi_{8s+5}(\boldsymbol{Sp}(n)) \to 0$$

is exact and reads $0 \to \boldsymbol{Z} \to \boldsymbol{Z} \to \boldsymbol{Z}_2 \to 0$. (Compare Bott [4].)

Next we show that the Stiefel-Whitney class w_q of an $\boldsymbol{SO}(n)$-bundle ξ is in general independent of the obstruction to extending over $K^{(q)}$ a cross-section of ξ restricted to $K^{(q-1)}$. In fact:

LEMMA 4.3. *If there exists a cross-section of the* $\boldsymbol{SO}(n)$*-bundle* ξ *restricted to the* $(q-1)$*-skeleton of the base complex* K, $q > 0$ *and* $q \neq 2$, 4, 8, *then the Stiefel-Whitney class* w_q *of* ξ *is zero.*

Proof. Let $\mathfrak{o}(\xi, f)$ be the cohomology class of the obstruction to extending the given cross-section f over the q-skeleton. Let $p\colon \boldsymbol{SO}(n) \to V_{n,n-q+1} = \boldsymbol{SO}(n)/\boldsymbol{SO}(q-1)$ be the natural projection, and

$$p_*\colon \pi_{q-1}(\boldsymbol{SO}(n)) \to \pi_{q-1}(V_{n,n-q+1}),$$

$$p_{**}\colon H^q(K; \pi_{q-1}(\boldsymbol{SO}(n))) \to H^q(K; \pi_{q-1}(V_{n,n-q+1}))$$

the induced homomorphisms. Clearly, $p_{**}\mathfrak{o}(\xi, f) = w_q$. The lemma follows from the fact that p_* is zero provided $q \neq 2, 4, 8$. This is trivial for $q \equiv 3$, 5, 6, 7 modulo 8 ($\pi_{q-1}(\boldsymbol{SO}(n))$ is zero in these cases by [4]). It is also trivial for $q \equiv 1 \bmod 8$ since $\pi_0(\boldsymbol{SO}(n)) = 0$ and $\pi_{8s}(\boldsymbol{SO}(n)) = \boldsymbol{Z}_2$, $\pi_{8s}(V_{n,n-8s}) \approx \boldsymbol{Z}$ for $s > 0$. For $q \equiv 2 \bmod 8$, $p_* = 0$ follows easily from considering the homotopy exact sequence of $\boldsymbol{SO}(n)/\boldsymbol{SO}(q-1)$. For $q \equiv 4 \bmod 8$, $p_* = 0$ was stated and proved in [6], Lemma 3. The case $q \equiv 0 \bmod 8$ will be treated in a forth coming paper [8].[2]

Using the Lemma 1.1 above, we obtain the k-invariants of the classifying spaces $B_{\boldsymbol{U}(n)}$, $B_{\boldsymbol{SO}(n)}$, $B_{\boldsymbol{Sp}(n)}$ in the *stable range.* (For the k-invariants of

[2] (Added in proof) $p_* = 0$ also follows from comparison with $p'_*\colon \pi_{q-1}(\boldsymbol{SO}(q)) \to \pi_{q-1}(S^{q-1})$ as in the proof of Lemma 6.4 below.

$B_{U(n)}$, compare F. Peterson [10]). We need a few probably well known lemmas about k-invariants which we derive in the next section.

5. Let X be a simply connected space and $X_q \supset X$ such that

(i) (X_q, X) is a relative CW-complex (compare G. W. Whitehead [12]),

(ii) $\pi_i(X_q) = 0$ for $q < i$,

(iii) $\pi_i(X) \approx \pi_i(X_q)$ for $i \leqq q$ under inclusion.

Then, $\pi_i(X_q, X) = 0$ for $i \leqq q+1$ and since $\pi_1(X) = 0$ by assumption, it follows from the relative Hurewicz theorem that $H_i(X_q, X) = 0$ for $i \leqq q+1$, and $\pi_{q+1}(X) \approx \pi_{q+2}(X_q, X) \approx H_{q+2}(X_q, X)$. It follows that $H^{q+2}(X_q, X; \pi_{q+1}) = \mathrm{Hom}(H_{q+2}(X_q, X), \pi_{q+1})$ contains a fundamental class u (π_{q+1} denotes $\pi_{q+1}(X)$ for brevity).

LEMMA 5.1. *$a^*u = \boldsymbol{k}^{q+2} \in H^{q+2}(X_q; \pi_{q+1})$ is the $(q+2)$-dimensional k-invariant of X, where $a^*\colon H^*(X_q, X; \pi_{q+1}) \to H^*(X_q; \pi_{q+1})$ is induced by the inclusion $a\colon (X_q, 0) \to (X_q, X)$.*

In fact the lemma is a special case of the following definition of the characteristic class.

Let $p\colon E \to B$ be a fibering with q-connected fibre F, and let $i\colon F \to E$ be the inclusion of the fibre. Assume that E (and thus also B) is simply connected. By the homotopy exact sequence, $p_*\colon \pi_i(E) \to \pi_i(B)$ is an isomorphism for $i \leqq q$, and $p_*\colon \pi_{q+1}(E) \to \pi_{q+1}(B)$ is surjective. Therefore, $\pi_i(B', E) = 0$ for $i \leqq q+1$, where B' is the mapping cylinder of p (B and B' have the same homotopy type). Since $\pi_1(E) = 0$, it follows $H_i(B', E) = 0$ for $i \leqq q+1$, and $\pi_{q+2}(B', E) \approx H_{q+2}(B', E)$. Thus $H^{q+2}(B', E; \pi_{q+2}(B', E)) = \mathrm{Hom}(H_{q+2}(B', E), \pi_{q+2}(B', E))$ contains a fundamental class u.

Define a homomorphism $\phi\colon \pi_i(F) \to \pi_{i+1}(B', E)$ for every i as follows: If $f\colon S^i \to F$ represents a class $\alpha \in \pi_i(F)$, the formula

$$f'(x, t) = (f(x), t),$$

$0 \leqq t \leqq 1$ defines a mapping of the cone over S^i into B'. The boundary of this cone is mapped into $F \subset E$. Let $\phi\alpha$ be the class of f' in $\pi_{i+1}(B', E)$.

LEMMA 5.1′. *ϕ is an isomorphism, and $\phi_* c = a^* u$, where ϕ_* is induced by the coefficient homomorphism $\phi\colon \pi_{q+1}(F) \to \pi_{q+2}(B', E)$; c is the characteristic class and a^* is induced by the inclusion $(B', 0) \subset (B', E)$.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
\pi_{i+1}(E) & \longrightarrow & \pi_{i+1}(E,F) & \xrightarrow{\partial} & \pi_i(F) & \xrightarrow{i_*} & \pi_i(E) & \longrightarrow & \pi_i(E,F) \\
\downarrow \text{id.} & & \downarrow p_* & & \downarrow \phi & & \downarrow \text{id.} & & \downarrow p_* \\
\pi_{i+1}(E) & \xrightarrow{p_*} & \pi_{i+1}(B) & \xrightarrow{a_*} & \pi_{i+1}(B',E) & \xrightarrow{\partial'} & \pi_i(E) & \xrightarrow{p_*} & \pi_i(B).
\end{array}$$

It is easily seen from the definition of ϕ that commutativity holds in each square. Since p_* is an isomorphism, ϕ is an isomorphism by the 5-Lemma.

Let $\psi\colon H_i(F) \to H_{i+1}(B',E)$ be defined similarly to ϕ. The cycle z being a representative of a class $\zeta \in H_i(F)$, $\psi\zeta$ is the class in $H_{i+1}(B',E)$ of the cone over z in B', regarded as a cycle modulo E. Clearly, the following diagram is commutative:

$$\begin{array}{ccccccccc}
H_{i+1}(E) & \longrightarrow & H_{i+1}(E,F) & \xrightarrow{\partial} & H_i(F) & \xrightarrow{i_*} & H_i(E) & \longrightarrow & H_i(E,F) \\
\downarrow \text{id.} & & \downarrow p_* & & \downarrow \psi & & \downarrow \text{id.} & & \downarrow p_* \\
H_{i+1}(E) & \longrightarrow & H_{i+1}(B) & \xrightarrow{a_*} & H_{i+1}(B',E) & \xrightarrow{\partial'} & H_i(E) & \longrightarrow & H_i(B).
\end{array}$$

It follows by a standard argument that the dual diagram with coefficient group $\pi_j(F) = \pi_{j+1}(B',E)$, identified by $\phi\colon \pi_j(F) \to \pi_{j+1}(B',E)$, is also commutative. In particular,

$$\begin{array}{ccc}
H^{q+2}(E,F;\pi_{q+1}) & \xleftarrow{\delta} & H^{q+1}(F;\pi_{q+1}) \\
\uparrow p^* & & \uparrow \psi^* \\
H^{q+2}(B;\pi_{q+1}) & \xleftarrow{a^*} & H^{q+2}(B',E;\pi_{q+1})
\end{array}$$

is commutative (π_{q+1} denotes $\pi_{q+1}(F)$ identified with $\pi_{q+2}(B',E)$ by ϕ). Since c is characterized by $p^*c = \delta v$, where v is the fundamental class in $H^{q+1}(F;\pi_{q+1})$, it remains only to prove that $\psi^*u = v$. This is obvious, and the proof of Lemma 5.1′ is complete.

Consider the cohomology exact sequence

$$\cdots \to H^{q+1}(X;\pi_{q+1}) \xrightarrow{\delta} H^{q+2}(X_q, X;\pi_{q+1}) \xrightarrow{a^*} H^{q+2}(X_q;\pi_{q+1}) \to \cdots$$

Clearly, δ annihilates every decomposable element of $H^{q+1}(X;\pi_{q+1})$.

In fact, if $c \in H^{q+1}(X;\pi_{q+1})$ is any class, δc as a homomorphism of π_{q+1} into π_{q+1} is given by

(5.2) $$\delta c[\alpha] = \alpha^* c, \text{ for every } \alpha \in \pi_{q+1},$$

where $\alpha^*\colon H^{q+1}(X;\pi_{q+1}) \to H^{q+1}(S^{q+1};\pi_{q+1})$ is induced by any map $S^{q+1} \to X$ representing α, and $H^{q+1}(S^{q+1};\pi_{q+1})$ is identified with π_{q+1}.

6. As an application of Lemma 1.1 and the preceding remarks, we obtain:

THEOREM 6.1. *Let* $X = B_{U(n)}$ *and* $\pi_{q+1} = \pi_{q+1}(B_{U(n)}) \approx \pi_q(\boldsymbol{U}(n))$. *Then, for* $i < n$, *we have* $\boldsymbol{k}^{2i+2} = 0$, *and* X_{2i+1} *may be taken equal to* X_{2i}. $H^{2i+3}(X_{2i+1}; \pi_{2i+2}) = H^{2i+3}(X_{2i+1}; \boldsymbol{Z})$ *is a finite cyclic group of order* $i!$ *generated by the k-invariant* $\boldsymbol{k}^{2i+3}$.

THEOREM 6.2. *Let* $X = B_{SO(n)}$ *and* $\pi_{q+1} = \pi_{q+1}(B_{SO(n)}) \approx \pi_q(\boldsymbol{SO}(n))$. *Then, in the stable range,* $\boldsymbol{k}^{4j} = 0$, $\boldsymbol{k}^{8i-2} = 0$, $\boldsymbol{k}^{8i-1} = 0$. *One can take* $X_{4j-1} = X_{4j-2}$, $X_{8i-2} = X_{8i-3} = X_{8i-4}$. *The k-invariants* $\boldsymbol{k}^{4j+1}$, $\boldsymbol{k}^{8i+2}$, $\boldsymbol{k}^{8i+3}$ *are different from zero. Specifically, we have exact sequences*

$$(6.2') \qquad 0 \to \boldsymbol{Z}_{(2j-1)!a_j} \xrightarrow{\bar{a}^*} H^{4j+1}(X_{4j-1}; \pi_{4j}) \to H^{4j+1}(X; \pi_{4j}) \to 0$$

which split for $j \geqq 3$. *The k-invariant* $\boldsymbol{k}^{4j+1}$ *is the image under* $\bar{a}^*$ *of a generator of* $\boldsymbol{Z}_{(2j-1)!a_j}$. *For* $j = 1$ *or* 2, $\boldsymbol{k}^{4j+1}$ *can be halved and* $\frac{1}{2}\boldsymbol{k}^{4j+1}$ *can be chosen so as to project onto* $W^{4j+1} \in H^{4j+1}(X; \pi_{4j})$. *Similarly, the sequences*

$$(6.2'') \qquad \begin{aligned} &0 \to \boldsymbol{Z}_2 \xrightarrow{a^*} H^{8i+2}(X_{8i}; \pi_{8i+1}) \to H^{8i+2}(X; \pi_{8i+1}) \to 0, \\ &0 \to \boldsymbol{Z}_2 \xrightarrow{a^*} H^{8i+3}(X_{8i+1}; \pi_{8i+2}) \to H^{8i+3}(X; \pi_{8i+2}) \to 0, \end{aligned}$$

are exact and split. $\boldsymbol{k}^{8i+2}$, *resp.* $\boldsymbol{k}^{8i+3}$ *are the images under* a^* *of the generator of* $\boldsymbol{Z}_2$. ($\pi_{4j} \approx \boldsymbol{Z}$, $\pi_{8i+1} \approx \pi_{8i+2} \approx \boldsymbol{Z}_2$, by [4].)

THEOREM 6.3. *Let* $X = B_{Sp(n)}$ *and* $\pi_{q+1} = \pi_{q+1}(B_{Sp(n)}) \approx \pi_q(\boldsymbol{Sp}(n))$. *Then, in the stable range,* $\boldsymbol{k}^{4j} = 0$, $\boldsymbol{k}^{8i+2} = 0$, $\boldsymbol{k}^{8i+3} = 0$. *One can take* $X_{4j-1} = X_{4j-2}$, $X_{8i+2} = X_{8i+1} = X_{8i}$. *The other k-invariants are non-zero. Precisely,*

$$H^{4j+1}(X_{4j-1}; \boldsymbol{Z}) \approx \boldsymbol{Z}_{(2j-1)!b_j} \text{ generated by } \boldsymbol{k}^{4j+1},$$

$$H^{8i-2}(X_{8i-4}; \boldsymbol{Z}_2) \approx \boldsymbol{Z}_2 \text{ generated by } \boldsymbol{k}^{8i-2},$$

$$H^{8i-1}(X_{8i-3}; \boldsymbol{Z}_2) \approx \boldsymbol{Z}_2 \text{ generated by } \boldsymbol{k}^{8i-1}.$$

Proof of Theorem 6.1. The first assertion is trivial since $\pi_{2i+1}(B_{U(n)}) = 0$ for $i < n$. Consider the cohomology exact sequence

$$H^{2i+2}(X) \xrightarrow{\delta} H^{2i+3}(X_{2i+1}, X) \xrightarrow{a^*} H^{2i+3}(X_{2i+1}) \to H^{2i+3}(X)$$

with coefficients in $\pi_{2i+2}(X) \approx \boldsymbol{Z}$. Since $H^*(X) \approx \boldsymbol{Z}[c_1, \cdots, c_n]$, $\deg c_j = 2j$ (see [2], Theorem 21.3), we have $H^{2i+3}(X) = 0$ and $H^{2i+2}(X)$ is the direct sum of $\pi(i+1)$ copies of $\boldsymbol{Z}$, where $\pi(i+1)$ is the number of partitions of $i+1$. By (5.2) and Lemma (1.1), δ is zero on the decomposable elements and $\delta c_{i+1} = \pm i! \cdot u$, where u is the fundamental class in $H^{2i+3}(X_{2i+1}, X) \approx \boldsymbol{Z}$. It follows that $H^{2i+3}(X_{2i+1})$ is cyclic of order $i!$, generated by $a^* u = \boldsymbol{k}^{2i+3}$.

Proof of Theorem 6.2. The first assertions are trivial since $\pi_{4j-2}(\boldsymbol{SO}(n))$, $\pi_{8i-4}(\boldsymbol{SO}(n))$ and $\pi_{8i-3}(\boldsymbol{SO}(n))$ are zero in the stable range (see [4]). Consider the cohomology exact sequence.

$$H^{4j}(X) \xrightarrow{\delta} H^{4j+1}(X_{4j-1}, X) \xrightarrow{a^*} H^{4j+1}(X_{4j-1}) \to H^{4j+1}(X) \xrightarrow{\delta'}$$

with coefficients in $\pi_{4j}(B_{SO(n)}) \approx \boldsymbol{Z}$. Since $H^*(B_{SO}; \boldsymbol{R}) = \boldsymbol{R}[p_1, \cdots, p_r, \cdots]$, $H^*(B_{SO(n)}; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_2, \cdots, w_n]$, and $B_{SO(n)}$ has no other torsion than 2-torsion (see [3], Appendix II), every class $c \in H^*(B_{SO(n)})$ with $\deg c < n$ can be written in the form $c = P(p_1, \cdots, p_r) + \beta Q(w_2, \cdots, w_s)$, where β is the Bockstein homomorphism and P, Q are polynomials. Hence, $\delta \colon H^{4j}(X) \to H^{4j+1}(X_{4j-1}, X)$ kills every element except possibly p_j. It follows from (5.2) and Lemma (1.1) that $\delta p_j = \pm(2j-1)!\, a_j \cdot u$, where $u \in H^{4j+1}(X_{4j-1}, X)$ is the fundamental class. If $c \in H^{4j+1}(X)$, it has the form $\beta c'$, with $c' \in H^{4j}(X; \boldsymbol{Z}_2)$. Thus $\delta' c = 0$, except possibly if $c = \beta w_{4j} = W_{4j+1}$, and then $\delta' W_{4j+1} = \beta \delta w_{4j}$. Since by (5.2) δw_{4j} is the $4j$-dimensional Stiefel-Whitney class of the $\boldsymbol{SO}(n)$-bundle induced over S^{4j} by a map $S^{4j} \to B_{SO(n)}$ representing a generator of $\pi_{4j}(B_{SO(n)}) \approx \boldsymbol{Z}$, we obtain information about δw_{4j} from the following lemma:

LEMMA 6.4. *Let ξ be the stable $\boldsymbol{SO}(n)$-bundle induced over S^{4j} by a map representing a generator of $\pi_{4j}(B_{SO(n)}) \approx \boldsymbol{Z}$. The Stiefel-Whitney class $w_{4j}(\xi) \in H^{4j}(S^{4j}; Z_2) \approx Z_2$ is different from zero if and only if S^{4j-1} is parallelizable.* (Compare Bott and Milnor [5].)

Since S^{4j-1} is parallelizable only for $j = 1, 2$ (see [5] or [6]), it follows that for $j \geqq 3$, the sequence

$$0 \to \boldsymbol{Z}_{(2j-1)!a_j} \xrightarrow{\bar{a}^*} H^{4j+1}(X_{4j-1}) \to H^{4j+1}(X) \to 0,$$

is exact. In order to show that the sequence splits, consider the diagram

$$\begin{array}{ccccccc} H^{4j}(X) & \xrightarrow{\delta} & H^{4j+1}(X_{4j-1}, X) & \xrightarrow{a^*} & H^{4j+1}(X_{4j-1}) & \to & H^{4j+1}(X) \\ \downarrow \phi_* & & \downarrow \phi_* & & \downarrow \phi_* & & \downarrow \phi_* \\ H_2^{4j}(X) & \xrightarrow{\delta} & H_2^{4j+1}(X_{4j-1}, X) & \xrightarrow{a^*} & H_2^{4j+1}(X_{4j-1}) & \to & H_2^{4j+1}(X) \end{array}$$

where $H_2{}^*$ is the cohomology with coefficients in $\boldsymbol{Z}_2$ and ϕ_* is induced by the coefficient epimorphism $\phi\colon \boldsymbol{Z}\to\boldsymbol{Z}_2$. Since $\phi_* u\neq 0$ and δ kills decomposable elements in $H_2{}^{4j}(X)$, it follows that $\phi_*\boldsymbol{k}^{4j+1}=0$ if and only if $\delta w_{4j}\neq 0$. Hence, by Lemma 6.4 above, $\boldsymbol{k}^{4j+1}$ cannot be halved unless $j=1$ or 2. Since $H^{4j+1}(X;\boldsymbol{Z})$ is the direct sum of copies of $\boldsymbol{Z}_2$, it follows that every element $\neq 0$ in $H^{4j+1}(X;\boldsymbol{Z})$ is the image of an element *of order two* in $H^{4j+1}(X_{4j-1};\boldsymbol{Z})$. Thus (6.2′) splits for $j\geqq 3$.

Now, let $j=1$ or 2. Since S^3 and S^7 are parallelizable, it follows that $\delta w_4\neq 0$, $\delta w_8\neq 0$. Thus $\boldsymbol{k}^5$ and $\boldsymbol{k}^9$ can be halved. Consider first the case $j=1$: The sequence we are interested in reads

$$H^4(X)\to H^5(X_3,X)\xrightarrow{a^*}H^5(X_3)\to H^5(X)\xrightarrow{\delta'}\cdots.$$

Since $H^5(B_{SO(n)})\approx\boldsymbol{Z}_2$ generated by W_5, the only possible value of the projection in $H^5(X)$ of $\frac{1}{2}\boldsymbol{k}^5$ is W_5. Therefore, δ' is trivial also in this case and exactness of (6.2′) holds for $j=1$.

Let $j=2$. Consider the sequence

$$H^8(X)\xrightarrow{\delta}H^9(X_7,X)\xrightarrow{a^*}H^9(X_7)\xrightarrow{p^*}H^9(X)\xrightarrow{\delta'}\cdots.$$

$H^9(B_{SO(n)})\approx\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$ generated by p_1W_5, $(W_3)^3$ and W_9. Let h be an element in $H^9(X_7)$ such that $2h=\boldsymbol{k}^9$. Claim: $p^*h=W_9+\alpha p_1W_5+\beta(W_3)^3$, where α and β are remainders mod 2 depending on the choice of h. This is equivalent to proving $p^*h\neq\alpha p_1W_5+\beta(W_3)^3$. Since p^* is an isomorphism in dimensions <8, there exist classes p'_1, W'_5, W'_3 whose projection under p^* are p_1, W_5, W_3. Thus $\alpha p'_1W'_5+\beta(W'_3)^3\in H^9(X_7)$ is an element of order 2 whose image by p^* is $\alpha p_1W_5+\beta(W_3)^3$. If p^*h were equal to $\alpha p_1W_5+\beta(W_3)^3$, we would get an element $h'=h-\alpha p'_1W'_5-\beta(W'_3)^3$ with the properties $p^*h'=0$ and $2h'=\boldsymbol{k}^9$. Such an element however does not exist. It follows from $p^*h=W_9+$ decomposable elements, that $\delta W_9=0$. Hence (6.2′) is seen to be exact in any case.

To obtain the exactness of (6.2″), consider the cohomology sequence

$$H^{8i+s-1}(X)\xrightarrow{\delta}H^{8i+s}(X_{8i+s-2},X)\xrightarrow{a^*}H^{8i+s}(X_{8i+s-2})\xrightarrow{p^*}H^{8i+s}(X)\xrightarrow{\delta'}$$

with coefficients in $\pi_{8i+s-2}(SO(n))\approx\boldsymbol{Z}_2$ for $s=2$ or 3.

We have to prove that δ and δ' are zero. Since δ and δ' kill decomposable elements, it suffices to prove $\delta w_{8i+1}=0$, $\delta w_{8i+2}=0$, $\delta' w_{8i+2}=0$, $\delta' w_{8i+3}=0$. The first two assertions follow from (5.2), by which

$$\delta w_{8i+s-1}\in H^{8i+s}(X_{8i+s-2},X;\boldsymbol{Z}_2)\approx\boldsymbol{Z}_2$$

is the value $\alpha^* w_{8i+s-1}[S^{8i+s-1}]$ of the $(8i+s-1)$-th Stiefel-Whitney class of the $\boldsymbol{SO}(n)$-bundle over S^{8i+s-1} induced by a map representing the generator α of $\pi_{8i+s-1}(B_{SO(n)}) \approx \boldsymbol{Z}_2$. Since such a Stiefel-Whitney class vanishes, it follows that $\delta w_{8i+s-1} = 0$. To prove $\delta' w_{8i+s} = 0$, observe that

$$w_{8i+2} = Sq^2 w_{8i} \text{ for } i \geqq 1, \text{ and } w_{8i+3} = Sq^1 w_{8i+2}$$

(see Wu Wen-Tsün [13] and Borel [1]). It follows that

$$\delta' w_{8i+2} = \delta' Sq^2 w_{8i} = Sq^2 \delta w_{8i} = 0, \text{ and similarly } \delta' w_{8i+3} = 0.$$

The splitting of (6. 2″) is trivial since every element in $H^{8i+s}(X_{8i+s-2}; \boldsymbol{Z}_2)$ has order two.

It remains to prove Lemma 6. 4. Let ξ' be the associated bundle with fibre $V_{n,n-4j+1} = \boldsymbol{SO}(n)/\boldsymbol{SO}(4j-1)$, and let

$$p_*: H^{4j}(S^{4j}; \pi_{4j-1}(\boldsymbol{SO}(n))) \to H^{4j}(S^{4j}; \pi_{4j-1}(V_{n,n-4j+1}))$$

be induced by $p: \pi_{4j-1}(\boldsymbol{SO}(n)) \to \pi_{4j-1}(V_{n,n-4j+1})$. Identifying $H^{4j}(S^{4j}; \pi_{4j-1}(\boldsymbol{SO}(n)))$ with $\pi_{4j-1}(\boldsymbol{SO}(n))$), we clearly have $p_*\alpha = w_{4j}(\xi)$, where α is a generator of $\pi_{4j-1}(\boldsymbol{SO}(n))$. Since $\pi_{4j-1}(V_{n,n-4j+1}) \approx \boldsymbol{Z}_2$, it follows that $w_{4j}(\xi)$ is different from zero if and only if $p: \pi_{4j-1}(\boldsymbol{SO}(n)) \to \pi_{4j-1}(V_{n,n-4j+1})$ is surjective. The commutative diagram

$$\begin{array}{ccc} \pi_{4j-1}(\boldsymbol{SO}(4j)) & \to & \pi_{4j-1}(\boldsymbol{SO}(n)) \\ \downarrow p' & & \downarrow p \\ \pi_{4j-1}(S^{4j-1}) & \to & \pi_{4j-1}(V_{n,n-4j+1}) \end{array}$$

shows that p is surjectice if and only if p' is; in other words, if and only if S^{4j-1} is parallelizable.

Remark. The proof in [6] can be improved by using the above diagram from which Lemma 2 of [6] follows immediately.

Proof of Theorem 6. 3. Let $X = B_{Sp(n)}$. Again the first statements of the theorem follow trivially from the results of [4].

Consider the exact sequence

$$H^{4j}(X) \xrightarrow{\delta} H^{4j+1}(X_{4j-1}, X) \xrightarrow{a^*} H^{4j+1}(X_{4j-1}) \to H^{4j+1}(X)$$

with integer coefficients ($\pi_{4j}(X) \approx \boldsymbol{Z}$ by [4]).

Since $H^*(B_{Sp(n)}; \boldsymbol{Z}) = \boldsymbol{Z}[K_1, \cdots, K_n]$, $\deg K_j = 4j$, it follows that

$H^{4j+1}(X)=0$, and the multiples of K_j are the only non-decomposable elements of $H^{4j}(X)$. By (5.2) and Lemma 1.1,

$$\delta K_j=(2j-1)!\,b_j\cdot u,$$

where u is the fundamental class in $H^{4j+1}(X_{4j-1},X;\mathbf{Z})\approx\mathbf{Z}$.

It follows that $H^{4j+1}(X_{4j-1})$ is cyclic of order $(2j-1)!\,b_j$ generated by $\boldsymbol{k}^{4j+1}$. The assertions about $\boldsymbol{k}^{8i-1}$ and $\boldsymbol{k}^{8i-2}$ are trivial.

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