ON THE PONTRYAGIN CLASSES OF CERTAIN $SO(n)$-BUNDLES OVER MANIFOLDS.*

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One of the main steps in the proof of the well known

Theorem (Rochlin [4]). *If* $w_2(M_4) = 0$, *then* $p_1(M_4)$ *is divisible by* 48, is the following lemma due to Pontryagin who stated it without proof in the Doklady Akademii Nauk S.S.S.R., XLVII, N. 5, pp. 327-330. (In Rochlin's theorem, $w_2(M_4)$ is the 2-dimensional Stiefel-Whitney class of the 4-dimensional, orientable, closed, differentiable manifold M_4 and $p_1(M_4)$ the first (4-dimensional) Pontryagin class of M_4.)

Lemma. *Let the closed differentiable manifold* M_4 *of dimension* 4 *be the base space of a principal* $\boldsymbol{SO}(n)$*-bundle* $\mathfrak{N}$. *Assume that* $\mathfrak{N}$ *admits a cross-section over the subset* $M - U$ *of* M, *where* U *is a spherical neighborhood of some point in* M_4. *Let* $\lambda \in \pi_3(\boldsymbol{SO}(n)) \cong \boldsymbol{Z}$ *be the integer representing, up to sign, the obstruction to the extension over* M *of the cross-section given over* $M - U$. *Then,* $p_1(M_4) = 2\lambda$, *up to sign.*

A tentative method toward generalizing Rochlin's theorem starts with generalizing this lemma. This is done below. I do not know whether this will actually lead to a reasonable generalization of Rochlin's theorem. However, the generalized lemma might have some interest in itself.

I am indebted to A. Borel for valuable discussions during the preparation of the present paper.

1. Let M_{4s} be a closed manifold of dimension $4s$. Let $\mathfrak{N}$ be a principal $\boldsymbol{SO}(n)$-bundle over M_{4s} which admits a cross-section over $M - U$. We shall say for brevity that the bundle $\mathfrak{N}$ is "*almost trivial.*" Assume that $4s < n$ and let λ denote the integer representing the homotopy class in $\pi_{4s-1}(\boldsymbol{SO}(n)) \cong \boldsymbol{Z}$ (see [1]) of the obstruction to the extension over M_{4s} of the cross-section given over $M - U$. All manifolds considered in this paper are assumed to be orientable.

* Received January 10, 1958.

LEMMA (1.1). *The value on M_{4s} of the 4s-dimensional Pontryagin class p_s of the bundle $\mathfrak{N}$ is given by the formula: $p_s[M_{4s}] = a_s \cdot \lambda \cdot (2s-1)!$, up to sign, where a_s is equal to 1 or 2 according to whether s is even or odd respectively.*

Results concerning the same question in the non-stable range $(n < 4s+1)$ can be deduced (using Lemma (3.1) below) from results in a forthcoming paper by A. Borel and F. Hirzebruch: "On the characteristic classes of homogeneous spaces."

Before we proceed to the proof of the above lemma, let us recall briefly the argument by which it implies Rochlin's theorem:

2. Let the orientable, closed, differentiable manifold M_4 be imbedded in euclidean $(n+4)$-space E_{n+4} in such a way that U is the hemisphere $x_0 \leqq 0$ of the unit sphere $S_4 \subset E_5 \subset E_{n+4}$. Assume that $M - U$ lies in $\{x_0 \geqq 0\}$ and that the imbedding of M_{4s} in E_{n+4} is differentiable of class C^2. The assumption $w_2(M_4) = 0$ is equivalent to the existence of a field of n-frames $\boldsymbol{F}_n$ defined over $M - U$ and orthogonal to M_4 in E_{n+4} (because M_4 was assumed to be orientable and $\pi_2(\boldsymbol{SO}(n)) = 0$). By Lemma (1.1), $\bar{p}_1[M_4] = 2\lambda$, where $\bar{p}_1$ is the 4-dimensional Pontryagin class of the normal bundle over M_4 in E_{n+4}. The integer λ represents the homotopy class of the mapping $\mu\colon S_3 \to \boldsymbol{SO}(n)$ which sends a point $x \in U^{\cdot}$ (identified with S_3) into the matrix of components of the vectors of $\boldsymbol{F}_n$ at x relative to the vectors at x of some field of normal n-frames over S_4 in E_{n+4} (Take for instance the field consisting of the normal to S_4 in E_5 followed by the vectors of an orthonormal basis of the orthogonal complement of E_5 in E_{n+4}). By the interpretation of J (the Hopf's homomorphism) given in [2] and the fact that $\boldsymbol{F}_n \mid U^{\cdot}$ can be extended over $M - U$, it follows that the element $J\mu \in \pi_{n+3}(S_n)$ is zero. Since $J\pi_3(\boldsymbol{SO}(n)) = \pi_{n+3}(S_n) \approx \boldsymbol{Z}_{24}$, it follows that λ is divisible by 24. Consequently $\bar{p}_1[M_4]$ is divisible by 48. By Whitney duality (which holds over the integers in this special case), we have $p_1[M_4] = -\bar{p}_1[M_4]$. Hence the theorem.

3. Using the following lemma, we reduce the study of almost trivial bundles over a closed manifold M_d to the study of bundles over the sphere S_d:

LEMMA (3.1). *Let $f\colon M_d \to B_G$ be a mapping of M_d into the classifying space for G and assume that the induced principal $\boldsymbol{G}$-bundle over M_d is almost trivial* (i.e., *admits a cross-section over $M - U$*). *Then f is homo-*

topic to a composite map $f' \circ \alpha$, where $\alpha: M_d \to S_d$ has degree 1 and f' maps S_d into B_G.

Proof of lemma (3.1). Denote by $\pi: E_G \to B_G$ the projection in the classifying fibration. Since the bundle over M_d induced by f is almost trivial, there exists a mapping $F: M - U \to E_G$, such that $\pi \circ F = f \mid M - U$. We may assume that f is constant on $U: f(U) = b \in B_G$. The map F is homotopic to zero (E_G is contractible). Let F_t be a homotopy: $F_0 = F$, F_1 maps $M - U$ into some points of E_G. It is convenient for later purposes to assume that $F_t = F$ for $0 \leqq t \leqq \frac{1}{2}$. Introduce generalized polar coordinates (x, r) in U ($x \in U^{\cdot}$, $r =$ distance from (x, r) to the center of U). The required homotopy between f and a map of the form $f' \circ \alpha$ is given by

$$f_t(z) = \pi F_t(z) \qquad \text{for } z \in M - U,$$

$$f_t(x, r) = \pi F_{tr}(x, 1) \quad \text{for } (x, r) \in U$$

(We take the radius of U to be 1).

Clearly, $f_t(z)$ is continuous in both z and t. We verify that $f_0 = f$: for

$$z \in M - U,\ f_0(z) = \pi F_0(z) = \pi F(z) = f(z);\ \text{for}$$

$$(x, r) \in U,\ f_0(x, r) = \pi F_0(x, 1) = \pi F(x, 1) = f(x, 1) = f(x, r) = b.$$

For $t = 1$, we have

$$f_1(z) = \pi F_1(z) = b' \text{ a constant point } (z \in M - U),$$

$$f_1(x, r) = \pi F_r(x, 1).$$

Clearly, f_1 has the form $f' \circ \alpha$, where $\alpha: M_d \to S_d$ shrinks $M - U$ into a point and maps the interior of U homeomorphically into S_d (thus α has degree 1), and f' sends $\alpha(x, r)$ into $\pi F_r(x, 1)$.

4. Let $f: M_{4s} \to B_{\boldsymbol{SO}(n)}$ induce an almost trivial bundle over M_{4s} ($4s + 1 < n$). We may assume, without loss of generality, that the section over $M - U$ is given by a mapping $F: M - U \to E_{\boldsymbol{SO}(n)}$ (such that $\pi F = f \mid M - U$). Assuming for convenience as in § 3 that $f(U) = b \in B_{\boldsymbol{SO}(n)}$, $F \mid U^{\cdot}$ defines a mapping of S_{4s-1} into $\boldsymbol{SO}(n)$. Since we assumed $4s + 1 < n$, one has by [1], $\pi_{4s-1}(\boldsymbol{SO}(n)) \cong \boldsymbol{Z}$. Let λ be the integer (determined up to sign) representing the homotopy class of $F \mid U^{\cdot}$ in $\boldsymbol{SO}(n)$. Denote by

$p_s \in H^{4s}(M_{4s}; \boldsymbol{Z})$ the Pontryagin class of dimension $4s$ of the $\boldsymbol{SO}(n)$-bundle over M_{4s} induced by f. We have to prove that $p_s[M_{4s}] = a_s \cdot \lambda \cdot (2s-1)!$.

Let $f' \circ \alpha \simeq f$ be the factorization given by Lemma (3.1). Denote by χ the characteristic map of the bundle over S_{4s} induced by f'. We shall prove (up to sign):

(1) $\chi = \lambda$, (we use also χ to denote the integer representing the homotopy class of χ),

(2) $p_s[M_{4s}] = p'[S_{4s}]$, where p'_s is the $4s$-dimensional Pontryagin class of the bundle over S_{4s} induced by f'.

(3) $p'_s[S_{4s}] = a_s \cdot \chi \cdot (2s-1)!$.[1]

Proof of (1). Define F': $S_{4s} - V \to E_{\boldsymbol{SO}(n)}$ by $F'(\alpha(x,r)) = F_r(x,1)$, where V is the image by α of the set of those $(x,r) \in U$ with $0 \leqq r \leqq \frac{1}{2}$. Since we assumed $F_t = F$ for $0 \leqq t \leqq \frac{1}{2}$ (see § 3), $\pi F'(V^{\cdot}) = \pi F(U_{\frac{1}{2}}^{\cdot}) = f(U_{\frac{1}{2}}^{\cdot}) = b$, where $U_{\frac{1}{2}}^{\cdot}$ is the set of points $(x, \frac{1}{2}) \in U$. On the other hand, $\pi F' = f' \mid S_{4s} - V$. In other words, $F' \mid V^{\cdot}$ defines a mapping $S_{4s-1} \to \boldsymbol{SO}(n)$ which is precisely the characteristic map χ of the bundle over S_{4s} induced by f' (we use χ for the map and its homotopy class). But $F'(x, \frac{1}{2}) = F_{\frac{1}{2}}(x,1) = F(x,1)$. This means that χ is also represented by $F \mid U^{\cdot}$. Hence $\chi = \lambda$.

Proof of (2). Trivial by naturality of the Pontryagin classes (and the fact that α has degree 1).

Proof of (3). We are left with the following situation: Let $(E, \boldsymbol{SO}(n), S_{4s})$ be a principal $\boldsymbol{SO}(n)$-bundle over S_{4s} and χ its characteristic map. $\chi = \partial i_{4s}$, where ∂ is the boundary operator in the homotopy sequence of π: $E \to S_{4s}$. Let p_s be the $4s$-dimensional Pontryagin class of $(E, \boldsymbol{SO}(n), S_{4s})$. We have to prove:

$$p_s[S_{4s}] = a_s \cdot \chi \cdot (2s-1)!$$

(up to sign).

Let α: $\boldsymbol{SO}(n) \to \boldsymbol{U}(n)$ be the standard injection and denote by c_{2s} the $4s$-dimensional Chern class of the extended bundle $(E', \boldsymbol{U}(n), S_{4s})$. By definition $p_s[S_{4s}] = c_{2s}[S_{4s}]$ up to sign. It is well known that $\pi_{4s-1}(W_{n,n-2s+1}) \approx \boldsymbol{Z}$. We have $\partial_A i_{4s} = c_{2s}[S_{4s}] \cdot \epsilon_W$, where $\partial_A : \pi_{4s}(S_{4s}) \pi_{4s-1}(W_{n,n-2s+1})$ is the boundary operator of the homotopy sequence of the fibration with fibre $W_{n,n-2s+1}$ associated with $(E', \boldsymbol{U}(n), S_{4s})$ and ϵ_W is a generator of $\pi_{4s-1}(W_{n,n-2s+1})$.

To evaluate $c_{2s}[S_{4s}]$, consider the commutative diagram

[1] The divisibility of $p'_s[S_{4s}]$ by $(2s-1)!$ is also known to R. Bott.

$$\begin{array}{ccccc}
\pi_{4s-1}(\boldsymbol{SO}(n)) & \xrightarrow{\alpha_*} & \pi_{4s-1}(\boldsymbol{U}(n)) & \xrightarrow{q_*} & \pi_{4s-1}(W_{n,n-2s+1}) \\
\uparrow \partial & & \uparrow & & \uparrow \partial_A \\
\pi_{4s}(S_{4s}) & \xrightarrow{\text{identity}} & \pi_{4s}(S_{4s}) & \xrightarrow{\text{identity}} & \pi_{4s}(S_{4s}),
\end{array}$$

where $q\colon \boldsymbol{U}(n) \to W_{n,n-2s+1}$ is the natural projection. Using the definition of $c_{2s}[S_{4s}]$, we get

$$c_{2s}[S_{4s}] \cdot \epsilon_W = \partial_A(i_{4s}) = q_*\alpha_*\partial(i_{4s}) = \chi q_*\alpha_*\epsilon_{SO} = a_s \cdot \chi \cdot q_*\epsilon_U.$$

where χ denotes also the integer representing the homotopy class of χ.

Consider the homotopy sequence of the fibration $\boldsymbol{U}(n)/\boldsymbol{U}(2s-1) = W_{n,n-2w+1}$ with projection q:

$$\cdots \to \pi_{4s-1}(\boldsymbol{U}(n)) \longrightarrow \pi_{4s-1}(W_{n,n-2s+1}) \to \pi_{4s-2}(\boldsymbol{U}(2s-1)) \to \pi_{4s-2}(\boldsymbol{U}(n)).$$

By R. Bott, $\pi_{4s-2}(\boldsymbol{U}(n)) = 0$ and $\pi_{4s-2}(\boldsymbol{U}(2s-1)) = \boldsymbol{Z}/(2s-1)!\boldsymbol{Z}$. Therefore, q_* is the multiplication by $(2s-1)!$ (precisely, $q_*\epsilon_U = (2s-1)!\epsilon_W$). It follows that

$$p_s[S_{4s}] = a_s \cdot \chi \cdot (2s-1)!,$$

up to sign. It was proved in [3], Lemma 3, that $a_{2k+1} = 2$ and $a_{2k} = 1$. This completes the proof of Lemma (1.1).

5. *Remarks.* The same method gives the Chern class c_s of an almost trivial $\boldsymbol{U}(n)$-bundle over a $2s$-dimensional manifold ($s \leqq n$) and similarly the symplectic Pontryagin class e_s of an almost trivial $\boldsymbol{Sp}(n)$-bundle over a $4s$-dimensional manifold (for the "symplectic" Pontryagin classes, see A. Borel and F. Hirzebruch, loc. cit., Chapter I, 9.6).

LEMMA (5.1). *Let M_{2s} be the base space of an almost trivial principal $\boldsymbol{U}(n)$-bundle $\mathfrak{N}$. Let $\lambda(F)$ be the obstruction to the extension M_{2s} of a cross-section F in $\mathfrak{N}$ given over $M - U$ ($\lambda(F) \in \pi_{2s-1}(\boldsymbol{U}(n))$). Assume $s < n$, thus $\pi_{2s-1}(\boldsymbol{U}(n)) \approx \boldsymbol{Z}$. Let c_s denote the $2s$-dimensional Chern class of $\mathfrak{N}$. We have $c_s[M_{2s}] = (s-1)!\lambda(F)$.*

Similarly, let M_{4s} be the base space of an almost trivial $\boldsymbol{Sp}(n)$-bundle $\mathfrak{N}$. Assume $s < n$, then the obstruction to the extension over M of a cross-section defined over $M - U$ (U some spherical neighborhood in M) can be represented by an integer $\lambda(\pi_{4s-1}(\boldsymbol{Sp}) \approx \boldsymbol{Z}$ by R. Bott [1]). Let σ be the standard inclusion $\sigma\colon \boldsymbol{Sp}(n)\to\boldsymbol{U}(2n)$ and $\sigma^*\colon H^*(B_{\boldsymbol{U}(2n)})\to H^*(B_{\boldsymbol{Sp}(n)})$ the induced homomorphism in integral cohomology of the classifying spaces. The symplectic Pontryagin classes of an $\boldsymbol{Sp}(n)$-bundle induced by a mapping $f\colon M\to B_{\boldsymbol{Sp}(n)}$ are, up to sign, the images by $f^*\circ\sigma^*$ of the universal Chern classes $C_{2i}\in H^{4i}(B_{\boldsymbol{U}(2n)})$.

LEMMA (5.2). *Let e_s denote the 4s-dimensional symplectic Pontryagin class of $\mathfrak{N}$. We have $e_s[M_{4s}]=b_s\cdot\lambda\cdot(2s-1)!$, where b_s is equal to 1 if s is odd, to 2 if s is even.*

The proofs are entirely similar to the one of Lemma (1.1) and are left to the reader. For the proof of Lemma (5.2), one has to use that $\sigma_*\colon \pi_{4s-1}(\boldsymbol{Sp}(n))\to\pi_{4s-1}(\boldsymbol{U}(2n))$ maps a generator into b_s-times a generator, a fact which follows from [1], formula (3.6) and the knowledge of the stable homotopy groups of $\boldsymbol{Sp}(n)$.

We have two more remarks: in the situations of Lemmas (1.1), (5.1) and (5.2), it follows that $\lambda(F)$, the obstruction to the extension of the partial cross-section F, does not depend on F, a fact which could also be proved directly (the Hurewicz homomorphisms

$$\pi_{2s-1}(\boldsymbol{U}(n))\to H_{2s-1}(\boldsymbol{U}(n)),\qquad \pi_{4s-1}(\boldsymbol{SO}(n))\to H_{4s-1}(\boldsymbol{SO}(n)),$$
$$\pi_{4s-1}(\boldsymbol{Sp}(n))\to H_{4s-1}(\boldsymbol{Sp}(n))$$

are monomorphisms[2] for large n).

Finally, we notice that R. Bott's result: $\pi_{2n}(\boldsymbol{U}(n))\approx \boldsymbol{Z}/n!\boldsymbol{Z}$ which was used in both Lemmas (1.1) and (5.2) implies

(5.3) $\qquad \pi_{4s-2}(\boldsymbol{Sp}(s-1))\approx \boldsymbol{Z}/b_s(2s-1)!\boldsymbol{Z}.\quad$ (b_s as in Lemma (5.2)).

Indeed, the projection k_* in the homotopy sequence

$$\cdots\to\pi_{4s-1}(\boldsymbol{Sp}(s))\xrightarrow{k_*}\pi_{4s-1}(S_{4s-1})\to\pi_{4s-2}(\boldsymbol{Sp}(s-1))\to\pi_{4s-2}(\boldsymbol{Sp}(s))$$

is the composition $k_*=q_*\circ\sigma_*$, where $\sigma_*\colon\pi_{4s-1}(\boldsymbol{Sp}(s))\to\pi_{4s-1}(\boldsymbol{U}(2s))$ is induced by the standard inclusion $\sigma\colon\boldsymbol{Sp}(s)\to\boldsymbol{U}(2s)$ and $q_*\colon\pi_{4s-1}(\boldsymbol{U}(2s))\to\pi_{4s-1}(S_{4s-1})$ is induced by the projection $q\colon\boldsymbol{U}(2s)\to S_{4s-1}$. Now, σ_* sends generator into b_s-times generator and q_* sends generator into $(2s-1)!$-times generator. Therefore, k_* sends a generator of $\pi_{4s-1}(\boldsymbol{Sp}(s))\approx\boldsymbol{Z}$ into $b_s\cdot(2s-1)!$ times a generator of $\pi_{4s-1}(S_{4s-1})$. Since $\pi_{4s-2}(\boldsymbol{Sp}(s))=0$, we have $\pi_{4s-2}(\boldsymbol{Sp}(s-1))\approx\pi_{4s-1}(S_{4s-1})/k_*\pi_{4s-1}(\boldsymbol{Sp}(s))$. Hence (5.3).

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