Relative Characteristic Classes
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RELATIVE CHARACTERISTIC CLASSES.*

By Michel A. Kervaire.

1. Introduction. The main purpose of this paper [1] is to prove a lemma (Lemma (1.2) below) conjectured in [8].

In the proof, we shall make use of a not quite classical form of Whitney duality, involving Stiefel-Whitney characteristic classes which have to be considered as *relative* cohomology classes. Since these slightly generalized characteristic classes may have some interest for themselves, the present paper is divided into two parts as follows.

In Part I, an attempt is made to give a systematic treatment of relative characteristic classes. Beside Stiefel-Whitney classes, relative Chern and Pontryagin characteristic classes will also be considered. It will be seen that most of the properties of the usual characteristic classes may be adapted to hold for the relative classes. In particular, the relative classes satisfy a generalized Whitney duality theorem and Wu's theorem [16] remains true if suitably stated. The fact that Wu's theorem may be extended to the case of a manifold with boundary was communicated to me by R. Thom and was the starting point of the proof of Lemma (1.2). According to R. Thom, this extension of Wu's theorem was first known to H. Cartan, who proved it using (Φ)-cohomology (unpublished). For our purpose, it will be sufficient to reduce (by Lemma (6.1)) the extended Wu's theorem to the ordinary one, thus avoiding (Φ)-cohomology. The proof of the generalized Whitney duality will be based on the interpretation of the relative characteristic classes as symmetric functions. The original author's proof was very cumbersome and will be omitted. The proof given here is due to A. Borel and is reproduced with his permission.

Part II will be concerned with the following situation considered in [8]: Let M_d be a differentiable closed manifold imbedded [2] into some euclidean

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[2] All manifolds considered are of class C^∞. Imbedding will mean regular imbedding (and similarly for immersion).

space E_{d+n}, and assume that there exists a continuous field $\boldsymbol{F}_n$ of n-frames (n mutually orthogonal unit vectors) *normal* to M_d in E_{d+n}. To such a pair $(M_d; \boldsymbol{F}_n)$ we attach

(a) a map $\omega\colon M_d \to V_{d+n,n}$ of M_d into the Stiefel manifold of n-frames with base point at the origin in E_{d+n}: the map ω is defined by

$$\omega(x) = \{\boldsymbol{v}_1(x), \boldsymbol{v}_2(x), \cdots, \boldsymbol{v}_n(x)\},$$

where $\boldsymbol{v}_1(x), \cdots, \boldsymbol{v}_n(x)$, are the vectors based at the origin in E_{d+n} and parallel to the vectors of $\boldsymbol{F}_n$ at x; and we also attach

(b) a map $f\colon S_{d+n} \to S_n$ of the $(d+n)$-dimensional sphere into the n-sphere defined as follows: Take a tubular neighborhood U of M_d in E_{d+n}. Any point $u \in U$ lies in a uniquely determined n-plane N_x normal to M_d at a (uniquely defined) point $x \in M_d$. Using the coordinate system in N_x which is defined by the mutually orthogonal unit vectors of $\boldsymbol{F}_n$ at x, we attach coordinates $y_1, \cdots, y_n$ to the point u. It may be assumed that $\sum_i (y_i)^2 = 1$, if and only if u lies on the boundary of U. The definition of f will involve the mapping

$$r\colon (B_n, \partial B_n) \to (S_n, q^*)$$

of the n-ball B_n onto the n-sphere S_n given (for instance) by the formula

$$(1.1)\quad r(y_1, \cdots, y_n) = (1 - 2y^2, 2y_1(1-y^2)^{\frac{1}{2}}, 2y_2(1-y^2)^{\frac{1}{2}}, \cdots, 2y_n(1-y^2)^{\frac{1}{2}}),$$

where $y^2 = \sum_k (y_k)^2$ and $q^* \in S_n$ is the point with coordinates $(-1, 0, \cdots, 0)$. Identifying now S_{d+n} with $E_{d+n} + \infty$, the desired mapping $f\colon S_{d+n} \to S_n$ is given by

$$f(u) = r(y_1, \cdots, y_n) \text{ for } u \in U, \qquad f(x) = q^* \text{ for } x \in S - U.$$

The homotopy class of the map ω (defined under (a) above) is determined[3] by the generalized *curvatura integra* c which represents the homology class of the cycle $\omega(M_d)$ in $H_d(V_{d+n,n}; \boldsymbol{Z})$. The number c is an integer for d even or $n = 1$ and a remainder mod 2 for d odd $(n > 1)$. Define γ by $\gamma = c - \chi^*(M_d)$, where $\chi^*(M_d)$, the semi-characteristic of M_d, is equal to $\frac{1}{2}\chi(M_d)$ for d even and to $\sum_{i=0}^{s-1} (-1)^i p_i(M_d)$ for d odd, $p_i(M_d)$ being then the rank of $H_i(M_d; \boldsymbol{Z}_2)$, and $2s = d + 1$.

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[3] **Recall that the Stiefel manifold of n-frames in $(d+n)$-space is $(d-1)$-connected and its d-dimensional integer homology group is infinite cyclic, or cyclic of order 2, depending on whether d is even or $n = 1$, or d is odd and $n > 1$ respectively.**

It has been proved in [8] (Théorème II) that if a manifold M_d' with normal n-frame field F_n' in E_{d+n} leads (by procedure (b)) to a map f'; $S_{d+n} \to S_n$ homotopic to f, then $\gamma = \gamma'$, i. e., $c - \chi^*(M_d) = c' - \chi^*(M_d')$, this formula being valid only mod 2 for d odd.

Actually each homotopy class in $\pi_{d+n}(S_n)$ contains maps $f\colon S_{d+n} \to S_n$ obtained by the procedure described under (b) and γ is a homomorphism of $\pi_{d+n}(S_n)$ into $\boldsymbol{Z}$ or $\boldsymbol{Z}_2$ according as d is even or odd. It has been proved in [8] (Théorème IV) that, for d even, γ is always zero.

Another homotopy invariant associated to f which will play a role in the present paper is Hopf's invariant as generalized by Steenrod [11]: Consider the cell complex $K = S_n \cup e_{d+n+1}$ obtained by attaching the cell e_{d+n+1} to S_n by the given map f, then Steenrod's generalization of Hopf's invariant, which will be denoted by $h(f)$, is the remainder mod 2 defined by

$$Sq^{d+1}(u) = h(f) \cdot v,$$

where u and v are the generators of $H^n(K;\boldsymbol{Z}_2)$ and $H^{n+d+1}(K;\boldsymbol{Z}_2)$ respectively and Sq^{q+1} is the Steenrod square which raises the degree by $d+1$.

Each of the invariants $h(f)$ and $\gamma(f)$ has the following properties which can be verified easily (see [8]) for $\gamma(f)$ and were proved by Steenrod ([11], §18) for $h(f)$:

(1) It is a homomorphism of $\pi_{d+n}(S_n)$ into $\boldsymbol{Z}_2$ defined for every $d \geqq 1$ and $n \geqq 1$.

(2) It vanishes for d even.

(3) It takes the same value on a homotopy class and on its Freudenthal suspension.

(4) It is zero for every composition map $g \circ f\colon S_p \to S_r$, where $f\colon S_p \to S_q$, $g\colon S_q \to S_r$, provided that $p > q > r$.[4]

(5) If S_d is parallelizable, it takes the value 1 on the standard Hopf's map $s\colon S_{2d+1} \to S_{d+1}$ with Hopf's invariant 1.

It is not unlikely that properties (1)-(5) should characterize completely $h(f)$. This proves to be true for $d \leqq 7$, even with property (4) omitted.[5] I have no proof of this conjecture for general d. However, using the explicit definitions of $h(f)$ and $\gamma(f)$, we shall prove below the following lemma, which was conjectured in [8]:

LEMMA (1.2). *Using the above notations,* $\gamma(f) = h(f)$.

[4] This fact has not been proved for γ in [8] but is easily seen.

[5] See [8], page 242.

The use which can be made of relative characteristic classes to prove this lemma will become apparent during the proof in Section 8.

I am indebted and wish to express my gratitude to A. Borel and R. Thom for many fruitful discussions during the preparation of the present paper.

PART I. Relative characteristic classes.

2. Definition. We shall treat first Stiefel-Whitney classes. The relative Chern and Pontryagin classes may be obtained similarly and will be discussed briefly in an Appendix (§ 11). The coefficients will be the remainder mod 2, except in § 11 (and § 6).

Let $\mathfrak{B} = (B, p, K, S_{n-1}, \boldsymbol{O}(n))$ be a sphere-bundle over the simplicial complex K with the orthogonal group of n variables $\boldsymbol{O}(n)$ as structural group. Denote by $\mathfrak{B}^q = (B^q, p^q, K, V_{n,n-q}, \boldsymbol{O}(n))$ the bundle associated to $\mathfrak{B}$ with fibre the Stiefel manifold $V_{n,n-q}$ of $(n-q)$-frames (based at a fixed point) in euclidean n-space.

Let L be a subcomplex of K and assume that a cross-section θ^r over L is given in the associated bundle $\mathfrak{B}^r$. This section induces (by the projection $B^r \to B^q$) cross-sections θ^q over L in the associated bundle $\mathfrak{B}^q$ for $q \geqq r$. Roughly speaking, θ^r defines an $(n-r)$-vectorfield $\boldsymbol{F}_{n-r}$ over L, and θ^q is given by the $(n-q)$-vectorfield consisting of the first $(n-q)$ vectors of $\boldsymbol{F}_{n-r}$.

Let W^{q+1} be the $(q+1)$-dimensional Stiefel-Whitney class of the bundle $\mathfrak{B}$. Suppose $q \geqq r$. A representative w^{q+1} of W^{q+1} may be obtained by the usual stepwise extension process over $K^0 \cup L, K^1 \cup L, K^2 \cup L, \cdots, K^q \cup L$ of the cross-section θ^q in $\mathfrak{B}^q$ induced by θ^r over L. The *requirement* that the cross-section over K^q in $\mathfrak{B}^q$ should coincide over L with the given section θ^q leads to a representative w^{q+1} of W^{q+1} which takes the value zero on every $(q+1)$-simplex of L. The cocycle w^{q+1} is thus a representative of a *relative* cohomology class $W_R^{q+1} \in H^{q+1}(K, L; \boldsymbol{Z}_2)$ defined for $q \geqq r$, which will be called the $(q+1)$-dimensional *relative characteristic* Stiefel-Whitney class mod L corresponding to θ^r.

W_R^{q+1} does not depend on the choice of the extension of θ^q over the 0-, 1-, $\cdots$, q-simplexes of K-L (see [12], 33.5). It does, however, depend, in general, on the choice of the given cross-section over L. Let θ^0 and θ^1 be two cross-sections over L in $\mathfrak{B}^r$ and let θ^{0q}, θ^{1q} be their projections in $\mathfrak{B}^q$ $(q \geqq r)$. The stepwise attempt to make the cross-sections θ^{0q} and θ^{1q} coincide meets with an obstruction in dimension q on L. Let $b^q \in H^q(L)$ be the obstruction cohomology class. It is easily seen, that δb^q is the difference of the $(q+1)$-dimensional Stiefel-Whitney classes corresponding to θ^0 and θ^1, where δ is the coboundary operator of the cohomology sequence:

$$\cdots \leftarrow H^{q+1}(K) \leftarrow H^{q+1}(K,L) \xleftarrow{\delta} H^q(L) \leftarrow H^q(K) \leftarrow \cdots.$$

3. Naturality. Suppose we are given two principal $\boldsymbol{O}(n)$-bundles $\mathfrak{B}$ and $\mathfrak{B}'$ over simplicial complexes K, K' respectively, such that $\mathfrak{B}$ is induced from $\mathfrak{B}'$ by a map $f\colon K \to K'$ which we assume to be simplicial.

Let L and L' be subcomplexes of K and K' respectively, such that $f(L)$ is a subcomplex of L'. Suppose that cross-sections θ^r and θ'^r over L and L' are given in the associated bundles $\mathfrak{B}^r$ and $\mathfrak{B}'^r$ with fibre $V_{n,n-r}$, such that $h\theta^r(x) = \theta'^r f(x)$ for every $x \in L$ (h is the bundle map covering f).

Then, for $q \geqq r$, the relative characteristic Stiefel-Whitney classes $W_R{}^{q+1}$ and $W'_R{}^{q+1}$ of the two bundles are both defined.

LEMMA (3.1). *We have $W_R{}^{q+1} = f^*(W'_R{}^{q+1})$, where f^* is the dual homomorphism $f^*\colon H(K',L';\boldsymbol{Z}_2) \to H(K,L;\boldsymbol{Z}_2)$ induced by $f\colon (K,L) \to (K',L')$.* (See [12], 32.7).

Choose an extension over $K'^q \cup L'$ of the cross-section θ'^q in $\mathfrak{B}'^q$ given over L'. Define $W'_R{}^{q+1}$ using this extension. Define θ^q over $K^q \cup L$ as the reverse image by h of the cross-section θ'^q. By assumption, this definition is consistent with the given cross-section over L in $\mathfrak{B}^q$ and the extended θ^q may be used to define $W_R{}^{q+1}$. Let s be a $(q+1)$-simplex of K. If $f(s) = 0$, $|f(s)|$ is a subset of the q-dimensional skeleton of K', thus θ^q may be defined over s and $w^{q+1}(s) = w'^{q+1}(fs) = 0$. If $f(s) \neq 0$, the restriction of f on s is a homeomorphism, and $w^{q+1}(s) = w'^{q+1}(fs)$ follows from $h\theta^q \mid \dot{s} = \theta'^q f \mid \dot{s}$, together with the fact that h induces the identity $h_*\colon \boldsymbol{Z}_2 \to \boldsymbol{Z}_2$ (identifying $H_q(V_{n,n-q};\boldsymbol{Z}_2)$ with $\boldsymbol{Z}_2$). Thus, for the cohomology classes of w^{q+1} and w'^{q+1}, $W_R{}^{q+1} = f^*(W'_R{}^{q+1})$.

4. The Whitney duality. Let $\mathfrak{B}_i$ be two principal $\boldsymbol{O}(n_i)$-bundles $(i = 1, 2)$ over the same simplicial complex K, and suppose that cross-sections $\theta_1{}^{r_1}$ and $\theta_2{}^{r_2}$ over subcomplexes L_1, L_2 of K are given in the corresponding associated bundles $\mathfrak{B}_1{}^{r_1}$, $\mathfrak{B}_2{}^{r_2}$.

Let $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$ be the Whitney sum of $\mathfrak{B}_1$ and $\mathfrak{B}_2$. The bundle $\mathfrak{B}$ is an $\boldsymbol{O}(n_1 + n_2)$-bundle. The cross-sections $\theta_1{}^{r_1}$, $\theta_2{}^{r_2}$ induce over $L = L_1 \cap L_2$ (where they are both defined) a cross-section θ^r in the associated bundle $\mathfrak{B}^r$

with fibre $V_{n,n-r}$ $(n = n_1 + n_2, r = r_1 + r_2)$. We shall refer to θ^r as the *sum* of the given cross-sections $\theta_1^{r_1}$ and $\theta_2^{r_2}$.

Using the sum cross-section θ^r over L, we define the relative Stiefel-Whitney classes W_R^{q+1} of $\mathfrak{B} \bmod L$ for $q \geqq r$. Denote by X_R^{k+1} and Y_R^{l+1} the relative Stiefel-Whitney classes of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ corresponding to the cross-sections $\theta_1^{r_1}$, $\theta_2^{r_2}$. They are defined for $k \geqq r_1$ and $l \geqq r_2$ respectively.

In the Theorem (4.1) below, expressing Whitney duality for relative classes, the classes X_R^{k+1}, Y_R^{l+1} which are relative classes $\bmod L_1$ and $\bmod L_2$ respectively are regarded as relative classes $\bmod L$. Precisely, we write X_R^{k+1}. meaning the image of X_R^{k+1} by the homomorphism $H^*(K, L_1) \to H^*(K, L)$ induced by the inclusion $(K, L) \to (K, L_1)$, and similarly for Y_R^{l+1}.

THEOREM (4.1). *The Whitney duality holds for relative characteristic Stiefel-Whitney classes in the following form: For $q \geqq r$, we have*

$$W_R^{q+1} = X_R^{q+1} + X_R^{q} \cdot Y^1 + \cdots + X_R^{r_1+1} \cdot Y^{q-r_1} + \\ + X^{r_1} \cdot Y_R^{q-r_1+1} + \cdots + Y_R^{q+1}.$$

Notice that some absolute (usual) characteristic classes occur in the right hand side of the above formula. However, in each cup-product $X^a \cdot Y^b$ with $a + b - 1 = q \geqq r_1 + r_2$, either $a - 1 \geqq r_1$ or $b - 1 \geqq r_2$ (or both), because $a \leqq r_1$ and $b \leqq r_2$ would imply $a + b \leqq r_1 + r_2$. Therefore, in each product $X^a \cdot Y^b$, at least one of the classes X^a or Y^b is a relative class and so is every product in the right hand side of (4.1).

The proof of the above theorem will be carried out by showing that the relative characteristic classes may be equivalently defined as symmetric functions (Theorem (5.1)). This alternative definition in turn implies immediately the above duality theorem.

5. Relative characteristic classes as symmetric functions. Proof of Whitney duality. Let us recall Borel's definition of the (usual) characteristic classes [4]. Let $\mathfrak{B} = (E, p, K, \boldsymbol{O}(n))$ be a principal $\boldsymbol{O}(n)$-bundle over K. Let $\boldsymbol{Q}(n)$ denote the subgroup of $\boldsymbol{O}(n)$ consisting of the diagonal matrices $(e_i\delta_{ij})$, with $e_i = +1$ or -1. The space E is also the bundle space of a fibre bundle with fibre $\boldsymbol{Q}(n) = \boldsymbol{O}(1) \times \boldsymbol{O}(1) \times \cdots \times \boldsymbol{O}(1)$ (n factors), which is a covering space since $\boldsymbol{Q}(n)$ is discrete. The base space $\bar{K} = E/\boldsymbol{Q}(n)$ of this bundle is called the space of flags over K. The space $\bar{K}$ is the bundle space of a bundle with fibre $\boldsymbol{F}(n) = \boldsymbol{O}(n)/\boldsymbol{Q}(n)$ and base space K. Let $\rho\colon \bar{K} \to K$ be the projection.

The bundle $\mathfrak{E} = (E, \pi, \bar{K}, \boldsymbol{Q}(n))$ is the Whitney sum of n bundles $\mathfrak{E} = \mathfrak{E}^1 \oplus \mathfrak{E}^2 \oplus \cdots \oplus \mathfrak{E}^n$, where $\mathfrak{E}^i$ is a principal bundle over $\bar{K}$ with structural group $\boldsymbol{O}(1) \cong \boldsymbol{Z}_2$, thus a two-fold covering. Let $x_i \in H^1(\bar{K}; \boldsymbol{Z}_2)$ be the 1-dimensional (only non-zero positive dimensional) Stiefel-Whitney classes of $\mathfrak{E}^i$, $i = 1, 2, \cdots, n$, defined, for instance, as obstruction to the construction of a cross-section in $\mathfrak{E}^i$ over the 1-dimensional skeleton of $\bar{K}$. We have the

THEOREM (A. Borel). *The dual homomorphism $\rho^*: H^*(K) \to H^*(\bar{K})$ is a monomorphism and the ρ^*-image of $H^*(K)$ contains the symmetric functions of the variables $x_1, x_2, \cdots, x_n$.*

It is then legitimate to define the characteristic class W^q of the bundle $\mathfrak{B}$ by the formula $\rho^*(W^q) = S^q(x_1, \cdots, x_n)$, where $S^q(x_1, \cdots, x_n)$ denotes the elementary symmetric function of degree q in the variables $x_1, \cdots, x_n$.

A. Borel has proved in [4] (Théorème 5.1) that the class W^q defined in this way coincides with the characteristic class defined as obstruction.

We come back to relative classes: Suppose a cross-section θ^r is given in the bundle $\mathfrak{B}^r = (E^r, K, V_{n,n-r}, \boldsymbol{O}(n))$ associated to $\mathfrak{B}$ over a subcomplex L of K. Consider the space of flags $\bar{K} = E/\boldsymbol{Q}(n)$ over K. The section θ^r determines a subspace $\bar{L}$ of $\bar{K}$ as follows: Regarding $\mathfrak{B}$ as induced by a map $f: K \to B_{\boldsymbol{O}(n)}$, $f(x)$, $x \in K$, is a non-oriented n-plane in euclidean space of large dimension. A point of E consists of a point $x \in K$, together with an n-frame $v_1, v_2, \cdots, v_n$ in $f(x)$ (see [12], 10.2). Thus a point of $\bar{K}$ may be represented by $\{x; d_1, d_2, \cdots, d_n\}$, where $x \in K$ and $d_1, d_2, \cdots, d_n$ is an ordered set of mutually orthogonal straight lines in $f(x)$. The set $\bar{L}$ consists of those points $\{x; d_1, \cdots, d_n\}$ of $\bar{K}$ such that $x \in L$ and $d_{r+1}, \cdots, d_n$ carry the $(n-r)$ vectors of θ^r.

Let $x_1, \cdots, x_n \in H^1(\bar{K}; \boldsymbol{Z}_2)$ be, as before, the 1-dimensional characteristic classes of the two-fold coverings $\mathfrak{E}^1, \mathfrak{E}^2, \cdots, \mathfrak{E}^n$ over $\bar{K}$, the Whitney sum of which is the bundle $(E, \pi, \bar{K}, \boldsymbol{Q}(n))$. Because $\mathfrak{E}^{r+1}, \cdots, \mathfrak{E}^n$ admit cross-sections over $\bar{L}$ given by θ^r, the $(n-r)$ last x_i, $i = r+1, r+2, \cdots, n$, may be defined as *relative* characteristics classes $\operatorname{mod} \bar{L}$ (obstruction to extending over the 1-dimensional skeleton of $\bar{K}$ the cross-section in $\mathfrak{E}^i$, $r+1 \leqq i \leqq n$, already given over $\bar{L}$). Because in any product of $(q+1)$ distinct factors from $x_1, \cdots, x_n$ with $q \geqq r$, at least one of the last $n-r$ must occur, it follows that, for $q \geqq r$, the elementary symmetric function $S^{q+1}(x_1, \cdots, x_n)$ may be (and will be) defined as a relative cohomology class, which we shall denote by $S_R{}^{q+1}(x_1, \cdots, x_n)$, using the relative $x_{r+1}, \cdots, x_n$.

THEOREM (5.1). (a) *The homomorphism* $\rho_R^*: H^*(K, L) \to H^*(\bar{K}, \bar{L})$ *is a monomorphism and* (b) $\rho_R^*(W_R^{q+1}) = S_R^{q+1}(x_1, \cdots, x_n)$ *for* $q \geqq r$, *where* W_R^{q+1} *is the relative characteristic class* $\bmod L$ *corresponding to the cross-section* θ^r.

The proof of this theorem which will be given below is due to A. Borel. In the first formulation of this paper, the point (b) was proved using an inductive argument (on n) and the special case of relative Whitney duality in which one of the bundles involved in the Whitney sum is a two-fold covering. This special case of Whitney duality was proved directly using the definition of characteristic classes as obstruction, which is rather cumbersome. Futhermore, the point (a) of Theorem (5.1) could not be obtained by this method. The Lemma (5.2) below is also due to A. Borel and was unknown to the author.

The Whitney duality formula (4.1) is an immediate consequence of Theorem (5.1) and of the identity

$$S_R^{q+1}(x_1, \cdots, x_m, y_1, \cdots, y_n) = \sum_{a+b=q+1} S^a(x_1, \cdots, x_m) \cdot S^b(y_1, \cdots, y_n),$$

where each product in the right hand sum is a relative class (at least one factor in each product is a relative class if the left hand side is).

Before we proceed to the proof of Theorem (5.1), we give a property of the relative Stiefel-Whitney classes, which will be needed for this proof.

Let L be a subcomplex of $B_{\boldsymbol{O}(n)}$ and θ^r a cross-section over L in the associated bundle with fibre $V_{n,n-r}$. The cross-section θ^r induces cross-sections θ^q over L in every associated bundle with fibre $V_{n,n-r}$ for $q \geqq r$. The bundle space of the associated bundle with fibre $V_{n,n-r}$ is classifying space for $\boldsymbol{O}(q)$ and will consequently be denoted by $B_{\boldsymbol{O}(q)}$. The projection map $B_{\boldsymbol{O}(q)} \to B_{\boldsymbol{O}(n)}$ is the Borel map $\rho(\boldsymbol{O}(q), \boldsymbol{O}(n))$ corresponding to the inclusion $\boldsymbol{O}(q) \to \boldsymbol{O}(n)$. We write $\rho_{q,n}$, meaning $\rho(\boldsymbol{O}(q), \boldsymbol{O}(n))$, for notational convenience.

LEMMA (5.2). *The relative Stiefel-Whitney characteristic class* W_R^{q+1} *is the only non-zero element in* $H^{q+1}(B_{\boldsymbol{O}(n)}, L; \boldsymbol{Z}_2)$ *belonging to the kernel of* $\rho_{q,n}^*: H^*(B_{\boldsymbol{O}(n)}, L) \to H^*(B_{\boldsymbol{O}(q)}, \theta^q L)$.

We first prove that W_R^{q+1} belongs to the kernel of $\rho_{q,n}^*$: The map $\rho_{q,n}: B_{\boldsymbol{O}(q)} \to B_{\boldsymbol{O}(n)}$ induces over $B_{\boldsymbol{O}(q)}$ as base space an $\boldsymbol{O}(q)$-bundle with fibre $V_{n,n-q}$ and the induced cross-section over $\theta^q L$ may be trivially extended all over $B_{\boldsymbol{O}(q)}$. By naturality the Stiefel-Whitney class corresponding to this cross-section is $\rho_{q,n}^*(W_R^{q+1})$. Since the cross-section may be extended, $\rho_{q,n}^*(W_R^{q+1}) = 0$.

In order to prove that W_R^{q+1} is the *only* non-zero element of dimension $q+1$ in the kernel of $\rho_{q,n}^*$, let us consider the following diagram, where the rows are the cohomology sequences of the pairs $(B_{O(q)}, \theta^q L)$ and $(B_{O(n)}, L)$ respectively:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(B_{O(q)}) \arrow[r, "j^*"] & H^i(\theta^q L) \arrow[r, "\delta"] \arrow[d, "\theta^*"] & H^{i+1}(B_{O(q)}, \theta^q L) \arrow[r] & H^{i+1}(B_{O(q)}) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^i(B_{O(n)}) \arrow[r, "i^*"] \arrow[u, "\rho^*"] & H^i(L) \arrow[r, "\delta"] & H^{i+1}(B_{O(n)}, L) \arrow[r, "a^*"] \arrow[u, "\rho_{q,n}^*"] & H^{i+1}(B_{O(n)}) \arrow[r] \arrow[u, "\rho^*"] & \cdots
\end{tikzcd}$$

(coefficients in $\boldsymbol{Z}_2$). One has $H^*(B_{O(n)}; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[W^1, W^2, \cdots, W^n]$, $H^*(B_{O(q)}; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[W^1, W^2, \cdots, W^q]$ and the W^i for $i \leqq q$ correspond to each other by ρ^*. Thus, for $i \leqq q$, the homomorphism ρ^* is an isomorphism.

By the Five Lemma, it follows that for $i \leqq q$, the homomorphism $\rho_{q,n}^*$ is also an isomorphism (θ^* is an isomorphism in every dimension).

Let x be an element of the kernel of $\rho_{q,n}^*$ in $H^{q+1}(B_{O(n)}, L)$. By commutativity in the above diagram, a^*x is an element of the kernel of ρ^*. Therefore, by [4], Lemma 5.1, $a^*x = c \cdot W^{q+1}$ with $c = 0$ or 1. Thus $y = x + c \cdot W_R^{q+1}$ belongs to the kernels of both $\rho_{q,n}^*$ and a^*. By exactness, there exist elements z and t in $H^q(L)$ and $H^q(B_{O(q)})$ respectively, such that $\delta z = y$ and $j^* t = \theta^{*-1} z$. Since ρ^* is an epimorphism in every dimension, there exists a class $w \in H^q(B_{O(n)})$, such that $\rho^* w = t$. It follows that $y = \delta i^* w = 0$, by exactness.

This completes the proof of Lemma (5.2).

Notice that Lemma (5.2) suggests a new (more general) definition of the relative Stiefel-Whitney classes: Let A be a closed subset of $B_{O(n)}$ and suppose a cross-section θ^r over A is given in the bundle with fibre $V_{n,n-r}$ associated to the universal bundle over $B_{O(n)}$. The $(q+1)$-dimension universal relative Stiefel-Whitney class mod A corresponding to θ^r $(r \geqq q)$, may be defined as being the only non-zero element in the kernel of $\rho_{q,n}^*$: $H^*(B_{O(n)}, A) \to H^*(B_{O(q)}, \theta^q A)$. The proof of uniqueness runs as in Lemma (5.2), replacing L by A. To complete the definition, one has to show the existence of an element in kernel $\rho_{q,n}^*$, the image of which by a^* is the ordinary universal Stiefel-Whitney class and which is thus different from zero. Extension of this definition to the characteristic classes of a principal $\boldsymbol{O}(n)$-bundle over any compact finite dimensional space X is immediate. However, the naturality is less easy to prove in this general case. We omit the details here and shall treat relative Chern classes by this method (see Appendix).

Proof of Theorem (5.1). *Proof of* (a). This part is not concerned with characteristic classes. Let X be the base space (assumed to be a compact finite dimensional topological space) of an $\boldsymbol{O}(n)$-bundle induced by some map $X \to B_{\boldsymbol{O}(n)}$ and let $\bar{X}$ be the space of flags over X (denote by $\rho_X\colon \bar{X} \to X$ the projection). According to [4], Théorème 5.1, the fibre $\boldsymbol{F}(n) = \boldsymbol{O}(n)/\boldsymbol{Q}(n)$ is totally non-homologous to zero in $\bar{X}$ and thus $\rho_X{}^*\colon H^*(X) \to H^*(\bar{X})$ is a monomorphism.

Let A be a closed subspace of X and suppose that a cross-section θ^r over A is given in the associated bundle over X with fibre $V_{n,n-r}$. Let $\bar{A}$ be the subset of $\bar{X}$ the points of which are the sets $\{a; d_1, \cdots, d_n\}$, such that $a \in A$, and $d_{r+1}, \cdots, d_n$ carry the $(n-r)$ vectors given over A by θ^r. Notice that $\bar{A}$ is homeomorphic to the space of the bundle $(\bar{A}, \rho_A, \boldsymbol{F}(r), \boldsymbol{O}(r))$ defined as follows: E being the space of the $\boldsymbol{O}(n)$-bundle over X, consider the fibering $(E/\boldsymbol{Q}(r), \rho_E, E/\boldsymbol{O}(r), \boldsymbol{F}(r), \boldsymbol{O}(r))$, where $\boldsymbol{Q}(r) \subset \boldsymbol{O}(r) \subset \boldsymbol{O}(n)$. The bundle $(\bar{A}, \rho_A, A, \boldsymbol{F}(r), \boldsymbol{O}(r))$ is induced by the cross-section $\theta^r\colon A \to E/\boldsymbol{O}(r)$. Therefore, by Borel's theorem, $\rho_A{}^*$ is also a monomorphism.

Consider the following commutative diagram in which the rows are the cohomology sequences of $(\bar{X}, \bar{A})$ and (X, A) respectively:

$$
(5.3) \qquad
\begin{tikzcd}
\cdots & H^k(\bar{X}) \arrow[l] & H^k(\bar{X}, \bar{A}) \arrow[l, "\bar{j}^*"'] & H^{k-1}(\bar{A}) \arrow[l, "\delta"'] & H^{k-1}(\bar{X}) \arrow[l, "\bar{i}^*"'] & \cdots \arrow[l] \\
\cdots & H^k(X) \arrow[l] \arrow[u, "\rho_X{}^*"] & H^k(X, A) \arrow[l, "j^*"'] \arrow[u, "\rho_R{}^*"'] & H^{k-1}(A) \arrow[l, "\delta"'] \arrow[u, "\rho_A{}^*"'] & H^{k-1}(X) \arrow[l, "i^*"'] \arrow[u, "\rho_X{}^*"'] & \cdots \arrow[l]
\end{tikzcd}
.
$$

We have to prove that $\rho_R{}^*$ is a monomorphism. Let $a \in H^k(X, A)$ be a cohomology class, such that $\rho_R{}^* a = 0$. Since $\bar{j}^* \rho_R{}^* a = \rho_X{}^* j^* a = 0$ and $\rho_X{}^*$ is a monomorphism, it follows that $j^* a = 0$. Thus by exactness, there exists a class $b \in H^{k-1}(A)$, such that $\delta b = a$. Because $\delta \rho_A{}^* b = \rho_R{}^* \delta b = \rho_R{}^* a = 0$, there exists (by exactness) an element $w \in H^{k-1}(\bar{X})$, such that $\bar{i}^* w = \rho_A{}^* b$. The desired conclusion $a = 0$ would be granted if we knew the existence of an element $x \in H^{k-1}(X)$, such that $i^* v = b$. Indeed, from the existence of v follows, by exactness, $a = \delta b = \delta i^* v = 0$.

It remains to prove the

LEMMA. *Let $b \in H^*(A)$ and $w \in H^*(\bar{X})$ be cohomology classes, such that $\rho_A{}^* b = \bar{i}^* w$, then there exists a cohomology class $v \in H^*(X)$, such that $i^* v = b$* (Notations as in diagram (5.3)).

Proof. Let $x_1, \cdots, x_n$ denote, as before, the 1-dimensional characteristic classes of the two-fold coverings $\mathfrak{E}^1, \mathfrak{E}^2, \cdots, \mathfrak{E}^n$ over $\bar{X}$. By definition of $\bar{A}$, we have $\bar{\imath}^*x_a = y_a$ for $a = 1, 2, \cdots, r$ and $\bar{\imath}^*x_{r+b} = 0$ for $b = 1, \cdots, n - r$, where $y_1, \cdots, y_r$ denote the characteristic classes of the restriction over $\bar{A}$ of the two-fold coverings $\mathfrak{C}^1, \mathfrak{C}^2, \cdots, \mathfrak{C}^r$. The map $\bar{\imath}: \bar{A} \to \bar{X}$ restricted on a fibre (we denote this restriction again by $\bar{\imath}$) induces the inclusion $\boldsymbol{F}(r) \to \boldsymbol{F}(n)$ corresponding to the inclusion $(\boldsymbol{O}(r), \boldsymbol{Q}(r)) \to (\boldsymbol{O}(n), \boldsymbol{Q}(n))$. According to the results of A. Borel in [4] (Théorème 11.1), one has

$$H^*(\boldsymbol{F}(n); \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2[x_1, \cdots, x_n]/(S^+(x_1, \cdots, x_n))$$

and similarly for $H^*(\boldsymbol{F}(r); \boldsymbol{Z}_2)$, where $(S^+(x_1, \cdots, x_n))$ denotes the ideal (in $\boldsymbol{Z}_2[x_1, \cdots, x_n]$) generated by the symmetric functions of positive degrees in the variables $x_1, \cdots, x_n$.

It is easily seen that there exists a basis $h_1, h_2, \cdots, h_t$ of $H^*(\boldsymbol{F}(n))$ over $\boldsymbol{Z}_2$ with the following properties:

(1) $h_1 = 1$, the h_i are monomials in the $x_1, \cdots, x_n$;

(2) $\bar{\imath}^*h_1, \bar{\imath}^*h_2, \cdots, \bar{\imath}^*h_s$ form a basis of $H^*(\boldsymbol{F}(r); \boldsymbol{Z}_2)$;

(3) $\bar{\imath}^*h_{s+1} = \bar{\imath}^*h_{s+2} = \cdots = \bar{\imath}^*h_t = 0$.

Such a basis may be obtained by writing down in some order, beginning with 1, all monomials in the variables $x_1, \cdots, x_r$ the degree of which does not exceed $\dim \boldsymbol{F}(r)$, followed by the other monomials in $x_1, \cdots, x_n$. By omitting in this list the monomials which are linearly dependent (modulo the ideal generated by $S^+(x_1, \cdots, x_n)$) of preceding ones, one obtains the desired basis $h_1, h_2, \cdots, h_t$.

By Borel's results, the spectral sequence of the fibering $\rho_X: \bar{X} \to X$ is trivial ($E_2 \cong E_\infty$). Furthermore, since we have a coefficient *field*, $\boldsymbol{Z}_2$, the term E_∞ is additively isomorphic to $H^*(\bar{X})$. Therefore,

$$H^*(\bar{X}) \cong H^*(X) \otimes H^*(\boldsymbol{F}(n))$$

is a module over $\rho_X^*H^*(X)$ with the basis $h_1, h_2, \cdots, h_t$. Similarly, $H^*(\bar{A})$ is a module over $\rho_A^*H^*(A)$ with the basis $\bar{\imath}^*h_1, \bar{\imath}^*h_2, \cdots, \bar{\imath}^*h_s$.

Any element $w \in H^*(\bar{X})$ admits a unique decomposition in the form

$$w = \sum_{1 \leqq a \leqq t} \rho_X^*(v_a) \cdot h_a, \text{ where } v_i \in H^*(X).$$

We have $\bar{\imath}^*w = \sum_{1 \leqq a \leqq s} \bar{\imath}^*\rho_X^*(v_a) \cdot \bar{\imath}^*h_a$, because $\bar{\imath}^*h_a = 0$ for $a = s + 1, \cdots, t$.

If $\bar{\imath}^*w = \rho_A^*b$, as we have assumed in the lemma we are proving, then

$$\rho_A^*b = \sum_{1 \leqq a \leqq s} \rho_A^*i^*(v_a) \cdot \bar{\imath}^*h_a.$$

This is (by uniqueness of the representation) only possible if $i^*v_2 = i^*v_3 = \cdots = i^*v_s = 0$. Thus $\rho_A{}^*b = \rho_A{}^*i^*(v_1)$ $(h_1 = 1)$. Therefore, $b = i^*v_1$, since $\rho_A{}^*$ is a monomorphism. This completes the proof of the lemma.

Proof of (b). By naturality, it is sufficient to prove that the relation $\rho_R{}^*(W_R{}^{q+1}) = S_R{}^{q+1}(x_1, \cdots, x_n)$, $q \geqq r$, holds for the bundle

$$(B_{\boldsymbol{Q}(n)}, \rho(\boldsymbol{Q}(n), \boldsymbol{O}(n)), B_{\boldsymbol{O}(n)}, \boldsymbol{F}(n), \boldsymbol{O}(n)),$$

where ρ_R is written for $\rho_R(\boldsymbol{Q}(n), \boldsymbol{O}(n))$. This is easily seen using the fact that the map $f\colon K \to B_{\boldsymbol{O}(n)}$ inducing the given $\boldsymbol{O}(n)$-bundle over K may be approached by a *simplicial injective* map g as soon as $\dim B_{\boldsymbol{O}(n)} \geqq 2 \cdot \dim K + 1$ (for the existence of g, see Theorem 5 in S. Eilenberg, On spherical cycles, Bulletin of the American Mathematical Society, vol. 47 (1941), pp. 432-434).

Let $(B_{\boldsymbol{O}(r)}, \rho_{r,n}, B_{\boldsymbol{O}(n)}, V_{n,n-r}, \boldsymbol{O}(n))$ be the bundle associated to the classifying bundle for $\boldsymbol{O}(n)$, with fibre $V_{n,n-r}$. The bundle space of this bundle is classifying space for $\boldsymbol{O}(r)$ and is consequently denoted by $B_{\boldsymbol{O}(r)}$. The projection $\rho_{r,n}$ is the Borel map $\rho(\boldsymbol{O}(r), \boldsymbol{O}(n))$.

Consider the following commutative diagram

$$(5.4) \qquad \begin{array}{ccc} B_{\boldsymbol{Q}(r)} & \xrightarrow{\mu} & B_{\boldsymbol{Q}(n)} \\ {\scriptstyle \mu^1}\downarrow & & \downarrow \\ B_{\boldsymbol{O}(r)} & \xrightarrow{\rho_{r,n}} & B_{\boldsymbol{O}(n)} \end{array}$$

and let A be a closed subset of $B_{\boldsymbol{O}(n)}$, such that a cross-section θ^r over A is given in the bundle $(B_{\boldsymbol{O}(r)}, \rho_{r,n}, B_{\boldsymbol{O}(n)}, V_{n,n-r}, \boldsymbol{O}(n))$. Let $\bar{A}$ be defined as at the beginning of this section. As noticed previously, $\bar{A}$ is homeomorphic to a subset $\bar{A}^1$ of $B_{\boldsymbol{Q}(r)}$ and $\mu(\bar{A}^1) = \bar{A}$. We have to prove that the element $y = \rho_R{}^*(W_R{}^{q+1}) - S_R{}^{q+1}(x_1, \cdots, x_n) \in H^*(B_{\boldsymbol{Q}(n)}, \bar{A})$ is zero.

Let us consider the following commutative diagram

$$\begin{array}{ccccccccc} \cdots \to & H^q(B_{\boldsymbol{Q}(r)}) & \xrightarrow{\bar{i}^*} & H^q(\bar{A}^1) & \xrightarrow{\delta} & H^{q+1}(B_{\boldsymbol{Q}(r)}, \bar{A}^1) & \longrightarrow & H^{q+1}(B_{\boldsymbol{Q}(r)}) & \to \cdots \\ & \uparrow{\scriptstyle \mu^*} & & \uparrow{\scriptstyle \mu_A{}^*} & & \uparrow{\scriptstyle \mu_R{}^*} & & \uparrow{\scriptstyle \mu^*} & \\ \cdots \to & H^q(B_{\boldsymbol{Q}(n)}) & \xrightarrow{i^*} & H^q(\bar{A}) & \xrightarrow{\delta} & H^{q+1}(B_{\boldsymbol{Q}(n)}, \bar{A}) & \xrightarrow{a^*} & H^{q+1}(B_{\boldsymbol{Q}(n)}) & \to \cdots \end{array}$$

in which the rows are the cohomology sequences of the pairs $(B_{\boldsymbol{Q}(r)}, \bar{A}^1)$ and $(B_{\boldsymbol{Q}(n)}, \bar{A})$ respectively.

In order to prove $y=0$, it is sufficient to prove $\mu_R^*(y)=0$ and $a^*(y)=0$. Indeed, if these two equalities hold, there exist elements $z\in H^q(\bar{A})$ and $t\in H^q(B_{Q(r)})$, such that $\bar{\imath}^*t=\mu_A^*z$ and $\delta z=y$. Now,

$$H^*(B_{Q(n)};\boldsymbol{Z}_2)=\boldsymbol{Z}_2[x_1,\cdots,x_n], H^*(B_{Q(r)};\boldsymbol{Z}_2)=\boldsymbol{Z}_2[y_1,\cdots,y_r]$$

and $\mu^*x_i=y_i$ for $i=1,2,\cdots,r$, $\mu^*x_{r+j}=0$ for $j=1,\cdots,n-r$. Thus, μ^* is an epimorphism in every dimension. Therefore, there exists an element $w\in H^q(B_{Q(n)})$, such that $\mu^*w=t$. It follows that $y=\delta z=\delta\mu_A^{*-1}\bar{\imath}^*\mu^*w$ $=\delta i^*w=0$.

It remains to prove that $\mu_R^*y=0$ and $a^*y=0$.

Proof of $a^*y=0$. This is obvious with regard to the corresponding result of Borel on absolute characteristic classes:

$$a^*y=a^*\rho_R^*W_R^{q+1}-a^*S_R^{q+1}(x_1,\cdots,x_n)=\rho^*W^{q+1}-S^{q+1}(x_1,\cdots,x_n)=0$$

Proof of $\mu_R^*y=0$. We prove separately that $\mu_R^*\rho_R^*W_R^{q+1}=0$ and $\mu_R^*S_R^{q+1}(x_1,\cdots,x_n)=0$.

The first assertion follows from $\mu_R^*\rho_R^*W_R^{q+1}=\mu_1^*\rho_{r,n}^*W_R^{q+1}$ (see diagram (5.4)), and $\rho_{r,n}^*W_R^{q+1}=0$ proved in Lemma (5.2).

The second assertion, $\mu_R^*S_R^{q+1}(x_1,\cdots,x_n)=0$ follows from $\mu^*x_{r+j}=0$ for $j=1,2,\cdots,n-r$ (and thus $\mu_R^*x_{r+j}=0$, since $a^*\colon H^1(B_{Q(r)},\bar{A}^1)\to H^1(B_{Q(r)})$ is a monomorphism, A being assumed to be non-void) and the fact that for $q\geqq r$, each product of $(q+1)$ distinct factors from $x_1,\cdots,x_n$ must contain at least one x_{r+j} with $1\leqq j\leqq n-r$. Thus each product in $S_R^{q+1}(x_1,\cdots,x_n)$ is mapped into zero by μ_R^*.

This completes the proof of Theorem (5.1).

6. A lemma on Lefschetz-Poincaré duality. Let $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{G}$ be coefficient groups (abelian) with a pairing $\boldsymbol{G}_1\times\boldsymbol{G}_2\to\boldsymbol{G}$ of the two first groups to the third.

Let (X,A) be an admissible pair for cohomology theory. Assume X to be connected and A to be a neighborhood retract in X (hence the excision $e\colon(X,A)\to(Y,X')$ induces an isomorphism $H^*(Y,X')\to H^*(X,A)$).

Let Y be the space obtained by matching together two copies of X along the copies of A (i.e., $Y=X+X'$, with A and A' pointwise identified).

Let n be some positive integer. Denote by ${}_1H$, ${}_2H$, H cohomology groups with coefficients in $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{G}$ respectively.

LEMMA (6.1). *Assume $H^n(X)=0$, then the pairings of ${}_1H^q(X)$ with ${}_2H^{n-q}(X,A)$ to $H^n(X,A)$ and of ${}_1H^q(X,A)$ with ${}_2H^{n-q}(X)$ to $H^n(X,A)$*

given by the cup-product are completely orthogonal if and only if the pairing of ${}_1H^q(Y)$ *with* ${}_2H^{n-q}(Y)$ *to* $H^n(Y)$ *is completely orthogonal* (q, n fixed, $0 \leqq q \leqq n$).

Recall that a pairing is said to be completely orthogonal if either of the first two groups involved is the group of all homomorphisms of the other into the third.

Proof. Consider for $a = 1, 2$ the cohomology sequence of the pair (Y, X)

$$\cdots \to {}_aH^{q-1}(X) \xrightarrow{\delta} {}_aH^q(Y, X) \xrightarrow{h^*} {}_aH^q(Y) \xrightarrow{i^*} {}_aH^q(X) \to \cdots.$$

Since (by excision) the inclusion map $e\colon (X', A') \to (Y, X)$ induces isomorphisms $e^*\colon {}_aH^q(Y, X) \to {}_aH^q(X', A')$, we may substitute ${}_aH^q(X', A')$ for ${}_aH^q(Y, X)$ in the above sequence. Specifically, we consider the exact sequence

$$\cdots \to {}_aH^{q-1}(X) \xrightarrow{\delta'} {}_aH^q(X', A') \xrightarrow{j'^*} {}_aH^q(Y) \xrightarrow{i^*} {}_aH^q(X) \to \cdots,$$

where $\delta' = e^*\delta$ and $j'^* = h^*e^{*-1}$.

Let $k\colon Y \to X$ be the map defined by $k(x) = x$, $k(x') = x$, where x' corresponds to x in the copy X' of X. One has $ki = \text{id.}$, and therefore $i^*k^* = \text{id.}$. Thus i^* is an epimorphism, k^* a monomorphism and δ' is trivial. The sequence

$$0 \to {}_aH^q(X', A') \xrightarrow{j'^*} {}_aH^q(Y) \xrightarrow{i^*} {}_aH^q(X) \to 0$$

is exact.

Moreover, ${}_aH^q(Y)$ is the direct sum

$$(6.2) \qquad {}_aH^q(Y) = j'({}_aH^q(X', A')) + k({}_aH^q(X))$$

(we drop the stars by j'^* and k^* for notational convenience), or alternatively, interchanging X and X':

$$(6.3) \qquad {}_aH^{n-q}(Y) = k'({}_aH^{n-q}(X')) + j({}_aH^{n-q}(X, A)).$$

Let us denote by $l'\colon {}_aH^q(Y) \to {}_aH^q(X', A')$, respectively $l\colon {}_aH^q(Y) \to {}_aH^q(X, A)$ the homomorphisms, such that $l'j' = id.$, and $lj = id.$.

We have, for every $u_1 \in {}_1H^q(Y)$ and $x \in {}_2H^{n-q}(X, A)$,

$$(6.4) \qquad j(iu_1 \cdot x) = u_1 \cdot jx,$$

which may be proved using 3.4 of [11]. Setting $u_1 = ka$ and $x = ly$, one obtains

(6.5) $$l(ka \cdot y) = a \cdot ly.$$

Similarly,

(6.5′) $$l'(b \cdot k'x') = l'b \cdot x'.$$

1. *Assume the pairings of* ${}_1H^q(X)$ *with* ${}_2H^{n-q}(X,A)$ *to* $H^n(X,A)$ *and of* ${}_1H^q(X,A)$ *with* ${}_2H^{n-q}(X)$ *to* $H^n(X,A)$ *to be completely orthogonal.*

(1a) Let $h\colon {}_2H^{n-q}(Y) \to H^n(Y)$ be a homomorphism. Define the homomorphisms $g\colon {}_2H^{n-q}(X,A) \to H^n(X,A)$ and $g'\colon {}_2H^{n-q}(X') \to H^n(X',A')$ by $g(x) = lhjx$ and $g'(x') = l'hk'x'$ respectively. By assumption, there exist elements $a \in {}_1H^q(X)$ and $a' \in {}_1H^q(X',A')$, such that $g(x) = a \cdot x$ and $g'(x') = a' \cdot x'$ for every $x \in {}_2H^{n-q}(X,A)$ and $x' \in {}_2H^{n-q}(X')$. Set $b = ka + j'a'$ (thus $ib = a$, $l'b = a'$). We have $h(y) = b \cdot y$ for every $y \in {}_2H^{n-q}(Y)$. Indeed, each y admits a decomposition $y = k'x' + jx$ and $lhjx = g(x) = a \cdot x$, $l'hk'x' = g'(x') = a' \cdot x'$.

Therefore, $h(y) = hjx + hk'x' = j(a \cdot x) + j'(a' \cdot x')$. Notice that

$$l'(b \cdot k'x') = l'b \cdot x' = a' \cdot x'$$

(by (6.5′)). We obtain

$$h(y) = j(ib \cdot x) + j'l'(b \cdot k'x') = b \cdot jx + b \cdot k'x' = b \cdot (jx + k'x') = b \cdot y$$

($j'l' = \mathrm{id.}$ in dimension n follows from $H^n(X) = 0$, because then $l'\colon H^n(Y) \to H^n(X',A')$ is an isomorphism).

(1b) Suppose that $b \cdot y = 0$ for some $b \in {}_1H^q(Y)$ and every $y \in {}_2H^{n-q}(Y)$. We have to prove $b = 0$. By (6.2), b admits a decomposition $b = j'a' + ka$. We have $j(a \cdot x) = j(ib \cdot x) = b \cdot jx = 0$, furthermore, $a' \cdot x' = l'(b \cdot k'x') = l'(0) = 0$ for every $x \in {}_2H^{n-q}(X,A)$ and $x' \in {}_2H^{n-q}(X')$. By assumption, it follows that $a = a' = 0$. Thus, $b = 0$.

(1a) and (1b) prove that the pairing of ${}_1H^q(Y)$ with ${}_2H^{n-q}(Y)$ to $H^n(Y)$ is completely orthogonal.

2. *Assume now the pairing of* ${}_1H^q(Y)$ *with* ${}_2H^{n-q}(Y)$ *to* $H^n(Y)$ *to be completely orthogonal.*

(2a) Take a homomorphism $h\colon {}_2H^{n-q}(X,A) \to H^n(X,A)$. By (6.3), each $y \in {}_2H^{n-q}(Y)$ may be written uniquely in the form $y = k'x' + jx$. The map $g\colon {}_2H^{n-q}(Y) \to H^n(Y)$ defined by $g(y) = jh(x)$ is a homomorphism and thus, by assumption, there exists an element $u_1 \in {}_1H^q(Y)$, such that $u_1 \cdot y = jh(x)$ for every $y = k'x' + jx \in {}_2H^{n-q}(Y)$. One has (with $u = iu_1$),

$j(u\cdot x)=j(iu_1\cdot x)=u_1\cdot jx=jh(x)$. Since j is a monomorphism, $u\cdot x=h(x)$ for every $x\in {}_2H^{n-q}(X,A)$.

(2b) Let $a\cdot x=0$ for some $a\in {}_1H^q(X)$ and every $x\in {}_2H^{n-q}(X,A)$. Then $a=0$. Indeed, consider $ka=b\in {}_1H^q(Y)$. We have $b\cdot y=0$ for every $y\in {}_2H^{n-q}(Y)$, because $l(b\cdot y)=l(ka\cdot y)=a\cdot ly=0$. In dimension n, l is a monomorphism, because $H^n(X)=0$. The pairing of ${}_1H^q(Y)$ with ${}_2H^{n-q}(Y)$ to $H^n(Y)$ being assumed to be completely orthogonal, it follows that $b=0$. Thus, $a=ika=ib=0$.

The proofs of (2a′): every homomorphism ${}_2H^{n-q}(X)\to H^n(X,A)$ may be "realized" by cup-product with an element of ${}_1H^q(X,A)$ and of (2b′): if $a\cdot x=0$ for some $a\in {}_1H^q(X,A)$ and every $x\in {}_2H^{n-q}(X)$, then $a=0$, are mechanical and similar to (2a) and (2b) and will be omitted.

Remark. The assumption $H^n(Y)=0$ is actually needed in the proofs of both (1) and (2) as is shown by the following examples.

(1) Let $X=P_5$ be the 5-dimensional real projective space and A a point of P_5. Let $\boldsymbol{G}_1=\boldsymbol{G}_2=\boldsymbol{G}=\boldsymbol{Z}_2$ and take $n=5$, $q=2$. Then ${}_1H^2(X)\cong {}_2H^3(X,A)\cong H^5(X,A)\cong \boldsymbol{Z}_2$ and the pairing of ${}_1H^2(X)$ with ${}_2H^3(X,A)$ to $H^5(X,A)$ by cup-product is completely orthogonal. Similarly, the pairing of ${}_1H^2(X,A)$ with ${}_2H^3(X,A)$ is also completely orthogonal.

Now $Y=P_5\vee P_5$ and ${}_1H^2(Y)\cong {}_2H^3(Y)\cong H^5(Y)\cong \boldsymbol{Z}_2+\boldsymbol{Z}_2$. However, $\mathrm{Hom}(\boldsymbol{Z}_2+\boldsymbol{Z}_2,\boldsymbol{Z}_2+\boldsymbol{Z}_2)\cong \boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$. One has $H^5(X)\cong \boldsymbol{Z}_2$.

(2) Let $X=A=S_1$. Then $Y=X=S_1$. Take $n=1$, $q=0$ and $\boldsymbol{G}_1=\boldsymbol{G}_2=\boldsymbol{G}=\boldsymbol{Z}$. The pairing of ${}_1H^0(Y)=\boldsymbol{Z}$ with ${}_2H^1(Y)=\boldsymbol{Z}$ to $H^1(Y)=\boldsymbol{Z}$ is completely orthogonal, but since ${}_2H^1(X,A)=0$, $H^1(X,A)=0$, the pairing of ${}_1H^0(X)=\boldsymbol{Z}$ with ${}_2H^1(X,A)$ to $H^1(X,A)$ is not. One has $H^1(X)=\boldsymbol{Z}$.

7. The relative Wu classes. We come back to coefficients in the field $\boldsymbol{Z}_2$.

Let K be a complex of dimension n and L a non-void subcomplex of K. Suppose that relative Lefschetz-Poincaré duality holds in $K \bmod L$. In other words, for every $q=0,1,\cdots,n$, the pairing of $H^q(K;\boldsymbol{Z}_2)$ with $H^{n-q}(K,L;\boldsymbol{Z}_2)$ to $H^n(K,L;\boldsymbol{Z}_2)$ is completely orthogonal. Then we may define Wu classes $U^q\in H^q(K;\boldsymbol{Z}_2)$ by the requirement that for every relative class $X_R{}^{n-q}\in H^{n-q}(K,L)$

$$Sq^q(X_R{}^{n-q})=U^q\cdot X_R{}^{n-q} \tag{7.1}$$

should hold.

Because L is non-empty, $H^0(K,L)=0$, and thus, by duality, $H^n(K)=0$. Consequently, according to Lemma (6.1), absolute Poincaré duality holds in $M=K+K'$ (L and L' identified). Let S^q be the q-dimensional Wu class of M (in the ordinary sense, see [16]), i.e.,

$$Sq^q(X^{n-q})=S^q\cdot X^{n-q} \tag{7.2}$$

for every class $X^{n-q}\in H^{n-q}(M)$.

LEMMA (7.3). *Let $i\colon K\to M$ be the inclusion map and i^* the dual homomorphism induced by i, then $U^q=i^*S^q$.*

Proof. Let $j^*\colon H^*(K,L)\to H^*(M)$ be as in (6.3). We have, using (6.4), $j^*(i^*S^q\cdot X_R{}^{n-q})=S^q\cdot j^*X_R{}^{n-q}=Sq^q(j^*X_R{}^{n-q})=j^*(Sq^q(X_R{}^{n-q}))$. Since j^* is a monomorphism and the class U^q is determined uniquely by (7.1), it follows that $U^q=i^*S^q$.

Suppose now a sphere-bundle is given over M such that its Stiefel-Whitney characteristic classes $W_M{}^q$ be connected to the Wu classes by the relation

$$W_M{}^q=\sum_{0\leqq p\leqq q} Sq^{q-p}(S^p), \tag{7.4}$$

which we may write more conveniently as $W_M=Sq(S)$, denoting by W_M and S the "total" classes, i.e.,

$$W_M=1+W_M{}^1+\cdots+W_M{}^n,\qquad S=1+S^1+\cdots+S^n$$

and where the operator Sq stands for $Sq=Sq^0+Sq^1+\cdots+Sq^k+\cdots$.

According to a theorem of Wu in [16], the situation described by formula (7.4) arises in particular if M is a closed differentiable manifold and the sphere-bundle considered is its tangent bundle. From the relative Lefschetz-Poincaré duality for manifolds with regular boundary ([11], 7), it follows that (7.4) *holds in particular with $M=K+K'$ if K is a manifold with regular boundary L.*

Since the characteristic classes of the restricted bundle over K are the i^*-images of the characteristic classes of the bundle over M (by naturality), we have $W=Sq(U)$, where W denotes the total Stiefel-Whitney class of the bundle over K.

Suppose now that

(7.5) *a cross-section over the subcomplex L is given in the associated principal bundle.*

Then, for every q, W^q admits a representative cocycle which vanishes on L and thus defines a relative class $W_R{}^q$.

Since $H^0(K,L)=0$, the algebra $H^*(K,L)$ has no unit element. It is convenient to consider, rather than $H^*(K,L)$, the direct sum $H_1^*=\mathbf{Z}_2+H^*(K,L)$. In other words, we introduce formally a unit into the algebra $H^*(K,L)$. The requirement $1\cdot x=x\cdot 1=x$ for every $x\in H_1^*$ gives to H_1^* a ring structure. We shall furthermore allow the Steenrod squares to operate (as homomorphisms again) in H_1^* by setting $Sq^0(1)=1$, $Sq^i(1)=0$ for $i>0$.

We may then use the classes W_R^q to define a total relative class by $W_R=1+W_R^1+\cdots+W_R^n\in H_1^*$. Since the endomorphism $Sq\colon H_1^*\to H_1^*$ defined for each $X\in H_1$ by $Sq(X)=Sq^0(X)+Sq^1(X)+\cdots$ is a monomorphism and therefore maps H_1^*, as a finite dimensional vector space, *onto* itself, we may define *relative Wu classes* U_R^q by

$$W_R=Sq(U_R), \tag{7.6}$$

U_R being the total class $U_R=1+U_R^1+U_R^2+\cdots\in H_1^*$.

Notice furthermore, that according to the above conventions, we have $Sq(1)=1$, and therefore the product formula of H. Cartan

$$Sq(X\cdot Y)=Sq(X)\cdot Sq(Y) \tag{7.7}$$

holds also in H_1^*.

We state now some properties of the relative Wu class U_R.

LEMMA (7.8). *Let $h\colon(K,0)\to(K,L)$ be the inclusion map and h^* the induced homomorphism, then $U^q=h^*U_R^q$ for every $q>0$.*

The proof is immediate: Apply h^* to both sides of the equation (7.6) $W_R=Sq(U_R)$, with the convention $h^*(1)=1$. We obtain $W=Sq(h^*U_R)$. Since the class U is uniquely determined by $W=Sq(U)$, Sq being an automorphism of $H^*(K)$, it follows that $U=h^*U_R$.

LEMMA (7.9). *For every relative class $X_R^{n-q}\in H^{n-q}(K,L)$, we have $Sq^q(X_R^{n-q})=U_R^q\cdot X_R^{n-q}$.*

This is again obvious, according to the preceding lemma.

The following is a discrepancy between properties of absolute and relative Wu classes: Absolute classes the dimension of which exceed $\frac{1}{2}n$ vanish (if $q>\frac{1}{2}n$, then $q>n-q$; thus, in formula (7.1), the square is zero for every X_R^{n-q} and, by duality U^q, must be zero, too). This need *not* be the case for relative Wu classes.

We shall be mainly interested in the sequel in the case *n even.* We state some lemmas in this case:

LEMMA (7.10). *Let n be even, $n=2s$. Under assumptions* (7.4) *and* (7.5), *we have $W_R{}^n = U_R{}^s \cdot U_R{}^s + U_R{}^{2s}$.*

Proof.

$$\begin{aligned} W_R{}^n &= Sq^s(U_R{}^s) + Sq^{s-1}(U_R{}^{s+1}) + \cdots + Sp^{s-i}(U_R{}^{s+i}) + \cdots + U_R{}^{2s} \\ &= U_R{}^s \cdot U_R{}^s + U_R{}^{s-1} \cdot U_R{}^{s+1} + \cdots + U_R{}^{s-i} \cdot U_R{}^{s+i} + \cdots + U_R{}^{2s} \\ &= U_R{}^s \cdot U_R{}^s + Sq^{s+1}(U_R{}^{s-1}) + \cdots + Sq^{s+i}(U_R{}^{s-i}) + \cdots + U_R{}^{2s} \\ &= U_R{}^s \cdot U_R{}^s + U_R{}^{2s}. \end{aligned}$$

LEMMA (7.11). *Suppose K has even dimension $n=2s$. Let r be the rank of the bilinear form $f(X,Y)$ over $\mathbf{Z}_2$ defined for $X, Y \in H^s(K,L;\mathbf{Z}_2)$ by $X \cdot Y = f(X,Y)A$, where A denotes the generator of $H^n(K,L)$. Then, under assumptions* (7.4) *and* (7.5) $r \cdot A = U_R{}^s \cdot U_R{}^s$.

Proof. Introduce in $H^s(K,L)$ a basis $Z_1, \cdots, Z_r, Z_{r+1}, \cdots, Z_p$, such that $Z_i \cdot Z_j = \delta_{ij} A$ for $1 \leqq i, j \leqq r$ and $Z_i \cdot Z_j = 0$ if $r < i$ or $r < j$.

With respect to such a basis, $U_R{}^s$ must have the form

$$U_R{}^s = Z_1 + Z_2 + \cdots + Z_r + c_{r+1} Z_{r+1} + \cdots + c_p Z_p.$$

Indeed, let $U_R{}^s = \sum_{1 \leqq i \leqq p} c_i Z_i$, with $c_i \in \mathbf{Z}_2$, and let $X = \sum_{1 \leqq i \leqq p} x_i Z_i$ be any class in $H^s(K,L)$. One has

$$Sq^s(X) = X \cdot X = x_1{}^2 + x_2{}^2 + \cdots + x_r{}^2 = x_1 + x_2 + \cdots + x_r \pmod 2,$$
$$U_R{}^s \cdot X = c_1 x_1 + c_2 x_2 + \cdots + c_r x_r.$$

Thus, for any choice of $x_1, \cdots, x_r$, Lemma (7.9) implies

$$x_1 + x_2 + \cdots + x_r = c_1 x_1 + c_2 x_2 + \cdots + c_r x_r.$$

This is only possible if $c_1 = c_2 = \cdots = c_r = 1$.

Now from $U_R{}^s = Z_1 + Z_2 + \cdots + Z_r + c_{r+1} Z_{r+1} + \cdots + c_p Z_p$, it follows that $U_R{}^s \cdot U_R{}^s = Z_1 \cdot Z_1 + Z_2 \cdot Z_2 + \cdots + Z_r \cdot Z_r = rA$, and the proof of (7.11) is complete.

We consider now the dual classes $\bar{W}_R$ and $\bar{U}_R$ which are defined by

$$(7.12) \qquad W_R \cdot \bar{W}_R = 1, \qquad U_R \cdot \bar{U}_R = 1$$

respectively. $\bar{W}_R$ and $\bar{U}_R$ are uniquely defined (as relative classes) because the cup-product with a total class U_R or W_R defines an *isomorphism* $H_1{}^* \to H_1{}^*$

of $H_1{}^*$ onto itself. Of course, $\bar{W}=h^*\bar{W}_R$ and similarly with U substituted for W.

We have

$$\bar{W}_R=Sq(\bar{U}_R). \tag{7.13}$$

Proof. Denoting $Sq(\bar{U}_R)$ by $\tilde{W}_R=Sq(\bar{U}_R)$, we have

$$W_R\cdot\tilde{W}_R=Sq(U_R)\cdot Sq(\bar{U}_R)=Sq(U_R\cdot\bar{U}_R)=Sq(1)=1.$$

Since $W_R\cdot\bar{W}_R=1$ determines $\bar{W}_R$ uniquely, $\tilde{W}_R=\bar{W}_R$.

LEMMA (7.14). *We have (with $n=2s$ and under assumptions (7.4), (7.5)) $rA^n=W^{n-1}\cdot\bar{W}_R{}^1+W^{n-2}\cdot\bar{W}_R{}^2+\cdots+W^1\cdot\bar{W}_R{}^{n-1}$, where $W^q=h^*W_R{}^q$ as in (7.8); r and A^n were defined in (7.11).*

Proof. We prove first $\bar{W}_R{}^n=U_R{}^n$, as follows:

$$\begin{aligned}\bar{W}_R{}^n&=Sq^0(\bar{U}_R{}^n)+Sq^1(\bar{U}_R{}^{n-1})+\cdots+Sq^s(\bar{U}_R{}^s)\\&=\bar{U}_R{}^n+U_R{}^1\cdot\bar{U}_R{}^{n-1}+\cdots+U_R{}^s\cdot\bar{U}_R{}^s\\&=U_R{}^{s+1}\cdot\bar{U}_R{}^{s-1}+\cdots+U_R{}^{n-1}\cdot\bar{U}_R{}^1+U_R{}^n\ \text{(by (7.12))}\\&=Sq^{s+1}(\bar{U}_R{}^{s-1})+\cdots+Sq^{n-1}(\bar{U}_R\ )+U_R{}^n=U_R{}^n.\end{aligned}$$

From $W_R{}^n+W_R{}^{n-1}\cdot\bar{W}_R{}^1+\cdots+W_R{}^1\cdot\bar{W}_R{}^{n-1}+\bar{W}_R{}^n=0$, we obtain

$$W_R{}^n=W_R{}^{n-1}\cdot\bar{W}_R{}^1+\cdots+W_R{}^1\cdot\bar{W}_R{}^{n-1}+U_R{}^n.$$

Comparing this formula with (7.10) and (7.11), we see that

$$rA^n=W_R{}^{n-1}\cdot\bar{W}_R{}^1+\cdots+W_R{}^1\cdot\bar{W}_R{}^{n-1},$$

hence $rA^n=W^{n-1}\cdot\bar{W}_R{}^1+\cdots+W^1\cdot\bar{W}_R{}^{n-1}$. This completes the proof of Lemma (7.14).

PART II. Proof and Consequences of Lemma (1.2).

8. The proof. All homology and cohomology groups occurring in this section will be based on remainders mod 2 as coefficients (we shall therefore omit to mention the coefficient field explicitly).

Consider the situation described in Section 1: a C^∞-d-manifold M_d regularly imbedded in euclidean $(d+n)$-space E_{d+n} with a continuous field of normal n-frames $\boldsymbol{F}_n$. In order to prove the Lemma (1.2), i.e., $\gamma=h$, it is sufficient to consider the special case $n=d+1$. Indeed, if $\gamma(f)=h(f)$ has been proved for every $f\in\pi_{2d+1}(S_{d+1})$, the general assertion follows from the fact that γ and h are both "stable" by suspension (Section 1, property (3)): If $f \in \pi_{d+n}(S_n)$, with $n \leqq d+1$, then $\gamma(f) = \gamma(E^{d+1-n}f) = h(E^{d+1-n}f) = h(f)$. If $f \in \pi_{d+n}(S_n)$, with $n \geqq d+1$, then there exists by Freudenthal's theorems a map $g \in \pi_{2d+1}(S_{d+1})$, such that $E^{n-d-1}g = f$, and for the same reason $\gamma(f) = \gamma(g) = h(g) = h(f)$.

In the sequel $n = d+1$. Let $f\colon S_{2d+1} \rightarrow S_{d+1}$ be the sphere map corresponding to the given manifold M_d in E_{2d+1}, together with the n-field $\boldsymbol{F}_n$ $(n = d+1)$ of mutually orthogonal unit vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ normal to M_d in E_{d+n} (see Section 1, (b)). It is easily seen from the definition of f (see (1.1)) that $M_d = f^{-1}(q)$, where q is the point of S_n with the coordinates $q = (1, 0, \cdots, 0)$. Furthermore, the manifold M'_d, which is the locus of the endpoint of the vector $\epsilon\boldsymbol{v}_1(x)$ as x runs over M_d (ϵ fixed, a small positive real number) is also the reverse image $M'_d = f^{-1}(q')$ of some point $q' \in S_n$ (precisely, q' is the point $q' = (1-2\epsilon^2, 2\epsilon(1-\epsilon^2)^{\frac{1}{2}}, 0, \cdots, 0)$, if the mapping $r\colon B_n \rightarrow S_n$ is indeed chosen as in (1.1)).

By the original definition of Hopf's invariant, one has

$$L(M_d, M'_d) = h(f) \mod 2, \tag{8.1}$$

where $L(\ ,\ )$ denotes the looping coefficient in E_{2d+1}.

Considering E_{2n-1} $(= E_{2d+1})$ as the linear subspace of E_{2n} defined by $y_{2n} = 0$; $y_1, y_2, \cdots, y_{2n-1}$ being coordinates in E_{2n-1}, we show first that there exists in E_{2n} an immersed (not necessarily orientable) manifold X_n, the regular boundary (mod 2) of which is the given manifold M_d imbedded in E_{2n-1}.

The existence of an abstract C^∞-manifold V_n with regular boundary (mod 2) diffeomorphic to M_d follows, by a theorem of R. Thom (see [14], Théorème IV. 10), from the fact that all Stiefel-Whitney numbers of M_d vanish mod 2 (this because the normal bundle over M_d is trivial). We want to prove the existence of an immersion $i\colon V_n \rightarrow E_{2n}$, such that $i|\partial V_n$ be the given imbedding $f\colon M_d \rightarrow E_{2n-1}$.

By Theorem 1 in Whitney's paper [15], there exists an analytic manifold A_n in euclidean $(2n+1)$-space E, which is C^2-homeomorphic to V_n. Map A_n into E_{2n} by F_0 defined as follows: $F_0|\partial A_n$ is the given imbedding f of $M = \partial A_n$ into E_{2n-1}. Consider a neighborhood $N \approx \partial A_n \times I$ of ∂A_n in A_n and represent points $u \in N$ by pairs (x, t), where $x \in \partial A_n$ and $0 \leqq t \leqq 1$. Define $F_0(u) = (fx, t) =$ the point of E_{2n} with $(2n-1)$ first coordinates coinciding with those of fx and the $2n$-th coordinate of which is t. Extend F_0 over A_n, such that $F_0(A - N) \subset \{y_{2n} > 1\}$. Let N_1 be the subset of N characterized by $0 \leqq t \leqq \frac{1}{3}$. Extending again F_0 over E now, we use Weierstrass' approximation theorem to get a C^2 map $F_1\colon A_n \to E_{2n}$, such that $F_1\overline{(A - N_1)} \cap E_{2n-1} = 0$. Let ω^0 be a real valued C^2 function on A_n, such that $\omega^0 = 1$ in $A - N$, $\omega^0 = 1$ in N for $t \geqq \frac{2}{3}$, $0 \leqq \omega^0 \leqq 1$ for $\frac{1}{3} \leqq t \leqq \frac{2}{3}$, $\omega^0 = 0$ for $t \leqq \frac{1}{3}$. Take, for instance, $\omega^0(x,t) = \frac{1}{4}(1 + \cos(3\pi t))^2$ for (x,t) in N with $\frac{1}{3} \leqq t \leqq \frac{2}{3}$. For $u \in N$, $F_1(u)$ has the form $F_1(u) = F_0(u) + \xi(u)$. Define $F\colon A_n \to E_{2n}$ by $F|A - N = F_1|A - N$ and $F(u) = F_0(u) + \omega^0(u)\xi(u)$. Then F is C^2 and its restriction over ∂A_n is f. Moreover, $F|N_1 = F_0|N_1$, and thus F is completely regular in N_1 since F_0 is.

Now, by a theorem of H. Whitney (see [15], Theorem 2, assertions (a) and (b)), we can approximate F, together with its first derivatives by a completely regular immersion $j\colon A_n \to E_n$. Let N_2 be the subset of N_1 defined by $0 \leqq t \leqq \frac{1}{9}$ and define the C^2 real-valued function ω^1 over A_n by $\omega^1(x) = 1$ for $x \in A - N_1$, $\omega^1(u) = 1$ for $\frac{2}{9} \leqq t \leqq \frac{1}{3}$, $\omega^1(u) = \frac{1}{4}\cdot(1 + \cos(9\pi t))^2$ for $\frac{1}{9} \leqq t \leqq \frac{2}{9}$ and $\omega^1(u) = 0$ for $u \in N_2$. Substitute for j the immersion $i\colon A_n \to E_{2n}$ defined by $i|A - N_1 = j|A - N_1$. In N_1, j takes the form $j(u) = F(u) + \eta(u)$. Define $i(u)$ by $i(u) = F(u) + \omega^1(u)\eta(u)$. Since we may take $\eta(u)$ together with its first derivatives arbitrarily small and the derivatives of $\omega^1(u)$ are all zero except the derivative with respect to t which is $\leqq 20$, it follows that, for suitably chosen j, the map i will be completely regular ($F|N_1 = F_0|N_1$ is completely regular). Moreover, $i|N_2 = F_0|N_2$, and therefore *the normal vector to M_d and tangent to $i(A_n)$ is the constant vector $\boldsymbol{v}_0$, normal to E_{2n-1} in E_{2n}*. Finally, we can obviously manage that $i(A - \partial A)$ has no common point with E_{2n-1}. Denote $i(A_n)$ by X_n.

We replace now the given field $\boldsymbol{F}_n$ of n-frames $\boldsymbol{v}_1(x), \cdots, \boldsymbol{v}_n(x)$ normal to M_d in E_{2n-1} by the $(n+1)$-field $\boldsymbol{F}_{n+1}$ of $(n+1)$-frames $\boldsymbol{v}_0(x), \boldsymbol{v}_1(x), \cdots, \boldsymbol{v}_n(x)$ consisting of the (constant) vector $\boldsymbol{v}_0(x)$ tangent to X_n and normal to M_d at $x \in M_d$, followed by the vectors of the field $\boldsymbol{F}_n$.

The homotopy class of the map $\omega'\colon M_d \to V_{2n,n+1}$ induced by the $(n+1)$-field $\boldsymbol{F}_{n+1}$ is represented by the same remainder mod 2, i.e., c, as the homotopy class of the map $\omega\colon M_d \to V_{2n-1,n}$ induced by the given field $\boldsymbol{F}_n$ (*we have assumed* d *odd*, which is no loss of generality in view of $\gamma = h = 0$ for d even by [8], Théorème IV. *Thus $n = d + 1$ is even, $n = 2s$*).

On the other hand, using certain vectors of $\boldsymbol{F}_{n+1}$ as a cross-section over $M_d \subset X_n$, we can define relative characteristic Stiefel-Whitney classes as follows: the n last vectors of $\boldsymbol{F}_{n+1}$ provide a cross-section over M_d in the principal normal bundle over X_n and lead in every positive dimension to a class $\bar{W}_R{}^i \bmod M_d$ $(1 \leqq i \leqq n)$. The vector field v_0 leads to the characteristic class $W_R{}^n$ of the tangent bundle to X_n. Using finally the vectors of $\boldsymbol{F}_{n+1}$ altogether, we can define the relative n-dimensional class $S_R{}^n$ of the Whitney sum: tangent $\oplus$ normal bundles over X_n. According to Whitney duality for relative characteristic classes (Theorem (4.1)), we have [6]

$$(8.2) \qquad S_R{}^n = W_R{}^n + W^{n-1}\cdot \bar{W}_R{}^1 + \cdots + W^1 \cdot \bar{W}_R{}^{n-1} + \bar{W}_R{}^n.$$

The remaining part of the proof of Lemma (1.2) consists in interpreting the several terms occurring in formula (8.2).

First,

$$(8.3) \qquad S_R{}^n = c\cdot A_R{}^n,$$

where $A_R{}^n$ denotes the generator of $H^n(X,M;\boldsymbol{Z}_2)$. The class $S_R{}^n$ is indeed the obstruction class to the extension of $\boldsymbol{F}_{n+1}$ over X_n under the only condition that it keeps being an $(n+1)$-framefield in E_{2n}. The formula (8.3) follows then from the fact that $V_{2n,n+1}$ is $(n-2)$-connected.

Since $W_R{}^n$ is the obstruction cohomology class to the extension of the vector $\boldsymbol{v}_0(x)$ as a *tangent* vector field over X, we have by [2] (Satz I, page 549)

$$(8.4) \qquad W_R{}^n = \chi(X)\cdot A_R{}^n.$$

It has been proved in Lemma (7.14) that

$$(8.5) \qquad rA_R{}^n = W^{n-1}\cdot \bar{W}_R{}^1 + \cdots + W^1\cdot \bar{W}_R{}^{n-1}.$$

Let us prove now that

$$(8.6) \qquad \bar{W}_R{}^n = h(f)A_R{}^n.$$

Indeed, $\bar{W}_R{}^n$ is the obstruction class to the extension over X_n of the vector $\boldsymbol{v}_1(x)$ of the field $\boldsymbol{F}_{n+1}$ as a *unit* vector field *normal* to X_n in E_{2n}. The extension is possible over the $(n-1)$-skeleton (a triangulation of X is taken, such that M is a subcomplex). Suppose this extension has been constructed. Extend then $\boldsymbol{v}_1(x)$ in the interior of the n-simplexes of X_n as a *normal*

[6] Notice that we could prove (8.2) as follows: The restriction over M of the normal principal bundle $\mathfrak{N}$ over X is trivial. Let $K = X \cup C$ be obtained by attaching to X the cone over M. Then $\mathfrak{N}$ can be extended over K (i.e., the map $N\colon X \to B_{SO(n)}$ inducing $\mathfrak{N}$ can be extended to a map $K \to B_{SO(n)}$). K being realized in E_{2n}, define $T\colon K \to B_{SO(n)}$ by $T(x) = n$-plane orthogonal to $N(x)$ for every $x \in K$. This defines a bundle $\mathfrak{T}$ over K. Let w be the total Stiefel-Whitney class of $\mathfrak{T}$, $\bar{w}$ the total class of $\mathfrak{N}$. By excision, one has $H^+(K) \cong H^+(K,C) \cong H^+(X,M)$, where H^+ is the ring consisting of the elements of positive dimensions in the cohomology ring. It is not difficult to see that these isomorphisms send w^n into $S_R{}^n + W_R{}^n$ and w^i, $\bar{w}^j$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, into $W_R{}^i$, $\bar{W}_R{}^j$ respectively. Ordinary Whitney duality $w\cdot\bar{w} = 0$ goes thus over into formula (8.2) in particular. However, this raisonning does not apply in the more general situation considered in Part I.

vector of length $\leqq 1$. This is always possible if we allow $\boldsymbol{v}_1(x)$ to have length 0 in some interior points of some n-simplexes of X_n. Consider the locus X' of the endpoint of the vector $\epsilon\boldsymbol{v}_1(x)$ so extended (where $\epsilon > 0$ is smaller than the radius of a tubular neighborhood of X_n in E_{2n}; see [14], page 27). Denote by $V = X + \bar{X}$ the closed manifold obtained by adding to X its mirror image $\bar{X}$ with respect to the hyperplane E_{2n-1} in E_{2n}. We may choose the extension $\boldsymbol{v}_1(x)$, $x \in X_n$, such that X' and V are in general position in E_{2n}, so that we may determine the intersection coefficient mod 2 of V and X' by simply counting the intersection points. We assume all intersection points to be simple. We have then two kinds of intersection points of X' with V: Those arising from the impossibility of extending $\boldsymbol{v}_1(x)$ in the interior of an n-simplex of X_n as a normal unit vector (let their number be I), and those which arise from self-intersection points of X. It is clear that intersection points of the last category can occur only in *pairs,* thus their number is $2N$, where N is some integer.

By definition of $\bar{W}_R{}^n$, we have $\bar{W}_R{}^n = IA_R{}^n$ modulo 2. Furthermore, $I = I + 2N = S(V, X') = L'(V, M')$ modulo 2, where $L'(\ ,\ )$ denotes the looping coefficient in E_{2n} and M', as before, the locus of the endpoint of $\epsilon\boldsymbol{v}_1(x)$ as x runs over M_d $(\partial X' = M' \bmod 2)$. Since $V \cap E_{2n-1} = M_d$, we have $L'(V, M') = L(M, M')$, where $L(\ ,\ )$ is again the looping coefficient in E_{2n-1}. Therefore, $I = L(M, M') = h(f) \bmod 2$. In other words, $\bar{W}_R{}^n = h(f) A_R{}^n$.

Using the formulae (8.3), (8.4), (8.5) and (8.6), the formula (8.2) translates into

$$c = \chi(X) + r + h(f) \mod 2. \tag{8.7}$$

It has been proved in [8], that

$$\chi^*(M) = \chi(X) + \rho \mod 2, \tag{8.8}$$

where ρ is the rank of the bilinear form $S(x, y)$ defined by the intersection coefficient in $H_s(X_n; \mathbf{Z}_2)$, $n = 2s$ (i.e. $S(x, y)$ is the intersection coefficient of the classes $x, y \in H_s(X_n; \mathbf{Z}_2)$). It is not difficult to see that $\rho = r$. We prove, however,

$$\chi^*(M) = \chi(X) + r \mod 2 \tag{8.9}$$

more simply by considering the exact cohomology sequence of the pair (X, M), i.e.,

$$H^s(X) \xleftarrow{h^*} H^s(X, M) \xleftarrow{\delta} H^{s-1}(M) \leftarrow \cdots — H^0(M)$$
$$\xleftarrow{i^*} H^0(X) \xleftarrow{h^*} H^0(X, M) \leftarrow 0.$$

Using the completely orthogonal pairing of $H^s(X)$ with $H^s(X,M)$ to $H^n(X,M)$, it follows from $h^*C\cdot Z = C\cdot Z$, $C,Z\in H^s(X,M)$ that the kernel of h^* in $H^s(X,M)$ consists exactly of those elements C for which $C\cdot Z=0$ for every $Z\in H^s(X,M)$. Thus, we have $r=$ (rank of $H^s(X,M)$) — (rank kernel h^*). Using the exactness of the above sequence, we obtain

$$r = p_s(X,M) - p_{s-1}(M) + p_{s-1}(X) - p_{s-1}(X,M) + \cdots$$
$$+ (-1)^s p_0(M) + (-1)^{s-1} p_0(X) + (-1)^s p_0(X,M),$$

where $p_q(X,M)$, $p_q(M)$, $p_q(X)$ denote the ranks of $H^q(X,M)$, $H^q(M)$, $H^q(X)$ respectively.

Replacing in this formula $p_q(X,M)$ by $p_{n-q}(X)$ according to relative Lefschetz-Poincaré duality in $X \bmod M$, we obtain

$$r = (-1)^s\chi^*(M) + (-1)^{s-1}\sum_{0\leqq i\leqq s-1}(-1)^i p_i(X) + (-1)^s\sum_{s\leqq i\leqq n}(-1)^i p_i(X).$$

In other words,

$$\chi^*(M) = \{\sum_{0\leqq i\leqq s-1}(-1)^i p_i(X) - \sum_{s\leqq i\leqq n}(-1)^i p_i(X)\} + (-1)^s\cdot r, \tag{8.10}$$

from which formula (8.9) follows by reduction modulo 2.

Now, since by definition $\gamma(f) = c - \chi^*(M)$, formulae (8.7) and (8.9) complete the proof of the Lemma (1.2): $\gamma(f) = h(f)$.

9. Consequences of the Lemma (1.2). Using the Lemma (1.2) we may improve the generalized Curvatura Integra theorem. We obtain first from $h(f)=0$ if $n\leqq d$ (because then $Sq^{d+1}(u^n)=0$) the

THEOREM (9.1). *Let the closed differentiable manifold M_d be regularly imbedded in E_{d+n} with a field of normal n-frames $\mathbf{F}_n$ and $n\leqq d$, then the corresponding curvatura integra c does not depend on the imbedding nor on the n-field and is given by $c=\chi^*(M_d)$.*

Indeed, $c-\chi^*(M)=\gamma(f)$, where f is a map $S_{d+n}\to S_n$ as constructed in Section 1, (b). Since $\gamma(f)=h(f)=0$ because $n\leqq d$, the theorem follows.

Suppose that the integer d is such that every element in $\pi_{2d+n}(S_{d+1})$ has even (or zero) Hopf's invariant. Then every element in $\pi_{d+n}(S_n)$ with arbitrary n has zero generalized Hopf's invariant h. In this case we may therefore omit the restriction on n in the above theorem and obtain the

THEOREM (9.2). *Let the closed differentiable manifold M_d be regularly imbedded in any euclidean space E_{d+n} with a field of normal n-frames*

$\boldsymbol{F}_n$; *if d is such that there is no element of odd Hopf's invariant in $\pi_{2d+1}(S_{d+1})$, then the curvatura integra c corresponding to the imbedding and to $\boldsymbol{F}_n$ depends in fact only on M_d and is given by $c=\chi^*(M_d)$.*

We may replace in this theorem the assumption of an *imbedding* of M_d by the weaker one of an *immersion* (with self-intersections allowed). Indeed, if M_d is immersed in E_{d+n} with a field of normal n-frames $\boldsymbol{F}_n$ defining a curvatura integra c, we may imbed E_{d+n} as a linear subspace in E_{d+N} $(n \leqq N)$, with a field of normal N-frames $\boldsymbol{F}_N$ which consists of the n vectors of $\boldsymbol{F}_n$ followed by $(N-n)$ constant mutually orthogonal unit vectors normal to E_{d+n} in E_{d+N}. The curvatura integra corresponding to the new field $\boldsymbol{F}_N$ is again c and if N has been chosen sufficiently large $(d+1 \leqq N)$, a slight deformation in E_{d+N} of the immersion $M_d \rightarrow E_{d+N}$ will provide an *imbedding* of M_d into E_{d+N} with a field of normal N-frames obtained by continuous deformation from $\boldsymbol{F}_N$ (apply Theorem 2, case (c) of H. Whitney's paper [15] to obtain the imbedding and the covering homotopy theorem to obtain the desired field). The curvatura integra c still belongs to the new situation to which now Theorem (9.2) applies (d has not been changed). Thus we obtain the

THEOREM (9.2*). *The thorem (9.2) is still valid replacing the assumption of an "imbedding" of M_d by the assumption of an "immersion."*

Of special interest may be the case $n=1$, originally considered by Hopf for d even. If d is odd and the manifold M_d assumed to be imbedded in E_{d+1}, then the curvatura integra c (the degree of the Gauss mapping $M_d \rightarrow S_d$, in this case) is modulo 2 equal to $\chi^*(M_d)$. This was proved in [8] and also by J. Milnor, using a simpler method especially adapted to this special case, in [9]. As a corollary to Theorem (9.2*), we obtain the following improvement:

COROLLARY. *Let d be an integer such that there is no element of $\pi_{2d+1}(S_{d+1})$ with odd Hopf's invariant. Let M_d be a closed orientable hypersurface in E_{d+1} with self-intersections allowed, and let c be the degree of the Gauss mapping $M_d \rightarrow S_d$. Then $c=\chi^*(M)$ modulo 2.*

Let us remark that the condition on d in the Theorem (9.2) (no map with odd Hopf's invariant in $\pi_{2d+1}(S_{d+1})$) is known to be satisfied at least for $d \neq 2^a-1$, according to J. Adem [1] (explicit proofs in H. Cartan [7]).*

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* Added in proof: The details of Adem's proof have been recently published in *Algebraic Geometry and Topology*, Princeton University Press, 1957.

According to H. Toda it is satisfied for $d = 15$ too (As is well known, H. Hopf has proved that the condition is not satisfied for $d = 1, 3$ and 7).

Along the same lines, we obtain an extension of a part of N. Steenrod-J. H. C. Whitehead's theorem [13] according to which the d-sphere S_d cannot be parallelizable if $d \neq 2^a - 1$.

THEOREM (9.3). *Any manifold M_d with odd semi-characteristic $\chi^*(M)$ is not parallelizable if $d \neq 2^a - 1$, (in fact, if d is such that every element in $\pi_{2d+1}(S_{d+1})$ has even Hopf's invariant).*

Proof. Suppose M_d is parallelizable. It is known that any regular imbedding of M_d in E_{2d+1} admits in this case a field of normal $(d+1)$-frames $\boldsymbol{F}_{d+1}$, the corresponding curvatura integra c being zero (see [8], Section 8). Therefore, the semi-characteristic being odd, the corresponding value of γ is $\gamma = c - \chi^*(M) = 1$. Since by Lemma (1.2), γ is the Hopf's invariant mod 2 of some map $S_{2d+1} \to S_{d+1}$, the dimension d must be of the form $d = 2^a - 1$.

I do not know if there are examples of manifolds M_d with odd semi-characteristic, carrying 2^k fields of mutually orthogonal unit vectors (k being defined by $d + 1 = 2^k(2r+1)$).

It was proved in [8] (Corollaire au théorème VIII, § 8), that the real projective space P_d may be immersed into a euclidean space E_{d+n} $(n \geqq d+1)$ with a field of normal n-frames if and only if it is parallelizable. The Lemma (1.2) provides a similar (perhaps weaker) statement for a larger class of manifolds:

THEOREM (9.4). *Suppose that the manifold M_d admits a $2m$-fold covering manifold $\bar{M}_d$ with odd semi-characteristic: $\chi^*(\bar{M}_d) = 1 \bmod 2$. Then the manifold M_d cannot be immersed into any euclidean space E_{d+n} with a field of normal n-frames unless there is in $\pi_{2d+1}(S_{d+1})$ some element of odd Hopf's invariant.*

Proof. From an immersion $g: M_d \to E_{d+n}$ of M_d into some euclidean $(d+n)$-space E_{d+n} with a field $\boldsymbol{F}_n$ of normal n-frames, we obtain by composition with the covering map $p: \bar{M}_d \to M_d$ a regular immersion $\bar{g}: M_d \to E_{d+n}$ with a field of normal n-frames $\boldsymbol{F}_n$. Let us call $\bar{c}$ the curvatura integra of $\bar{M}_d$ corresponding to $\bar{\boldsymbol{F}}_n$. Because $\bar{g}(\bar{M}_d)$ is homologous to $2m \cdot g(M_d)$ in $V_{d+n,n}$, $\bar{g}$ denoting the composition $\bar{g} = g \circ p$, we have $\bar{c} = 2m \cdot c = 0 \pmod 2$.

Therefore, $\bar{c} - \chi^*(\bar{M}_d) \neq 0$, and because of Theorem (9.2*), $\pi_{2d+1}(S_{d+1})$ must contain some element of odd Hopf's invariant. This completes the proof of Theorem (9.4).

It is known that if S_d is parallelizable, then any manifold M_d which may be immersed in some euclidean space E_{d+n} with a normal n-frame is also parallelizable (see [8], Théorème VIII). In [9], J. Milnor formulates the conjectures that if a parallelizable manifold M_d may be immersed in E_{d+1} in such a way that the Gauss degree be 1, then S_d should be parallelizable.

From Lemma (1.2) follows the

THEOREM (9.5). *If a parallelizable manifold M_d may be immersed in some euclidean space E_{d+n} with a field of normal n-frames inducing an odd curvatura integra, then there is in $\pi_{2d+1}(S_{d+1})$ some element of odd Hopf's invariant.*

Proof. Assuming $n \geqq d+1$, and using the covering homotopy theorem, we may construct on M_d a field G_n of normal n-frames in E_{d+n}, such that the induced curvatura integra is zero (see [8], § 8). Thus the given field and G_n induce different curvatura integra. By Theorem (9.2*), this implies the existence in $\pi_{2d+1}(S_{d+1})$ of an element of odd Hopf's invariant (the question whether this is possible without S_d being parallelizable is unsolved as is well known).

10. Remark. It should be noticed that if one is not interested in the value of the curvatura integra c but only in the fact that c does not depend on the imbedding nor on the normal field (as in Theorem 9.5), then a simpler proof may be given in the case d odd. Let us sketch a "direct" proof of the following weaker form of Theorem (9.2) in this case:

THEOREM (10.1). *If the odd integer d is such that each element of $\pi_{2d+1}(S_{d+1})$ has even Hopf's invariant, then the curvatura integra of any closed differentiable manifold M_d regularly immersed into E_{d+n} with a field of normal n-frames depends only on M_d.*

Proof. Let $M_d{}^i = T^i(M_d)$, $i = 1, 2$ be two regular immersions of the given manifold M_d of dimension d into euclidean spaces E_{d+n_i}. Let $F^i{}_{n_i}$ be two fields $(i = 1, 2)$ of normal n_i-frames on $M_d{}^i$ respectively, inducing maps $\phi^i \colon M_d \to V_{d+n_i,n_i}$ the classes of which are represented by c^i. In order to prove $c^1 = c^2$, let $M_d{}^0 = T^0(M_d)$ be an arbitrary regular *imbedding* of M_d into E_{2d+1} (euclidean $(2d+1)$-space). Consider E_{d+n_1}, E_{2d+1}, E_{d+n_2} as linear subspaces of the euclidean space $E_N = E_{d+n_1} \times E_{2d+1} \times E_{d+n_2}$. Consider on $M_d{}^i$ the fields $F^i{}_{N-d}$ of $(N-d)$-frames normal to $M_d{}^i$, consisting of the vectors of $F^i{}_{n_i}$ followed by $N-d-n_i$ constant unit vectors (mutually orthogonal) normal to E_{d+n_i} in E_N. Each immersion $T^i(M_d)$, $i=1,2$, is isotopic in E_N with $T^0(M_d)$. By the covering homotopy theorem, we obtain on $M_d{}^0$ two fields of normal $(N-d)$-frames in E_N, which we denote again by $\boldsymbol{F}^i{}_{N-d}$. The curvatura integra corresponding to the mapping $M_d \to V_{N,N-d}$ induced by the new field $\boldsymbol{F}^i{}_{N-d}$ on $M_d{}^0$ is c^i.

It is easily seen that $\boldsymbol{F}^i{}_{N-d}$ may be continuously deformed (keeping the $(N-d)$-frames of $\boldsymbol{F}^i{}_{N-d}$ normal to $M_d{}^0$ during the deformation), in such a way that the first $(d+1)$ vectors become vectors in E_{2d+1} and the last $N-(2d+1)$ be constant and normal to E_{2d+1}. In other words, we have obtained two fields $\boldsymbol{F}^i{}_{d+1}$, $i=1,2$, of $(d+1)$-frames normal to $M_d{}^0$ in E_{2d+1}. Moreover, the curvatura integra corresponding to the map $M_d \to V_{2d+1,d+1}$ induced by $\boldsymbol{F}^i{}_{d+1}$ (as field over $M_d{}^0$ in E_{2d+1}) is equal to the given c^i which we started from.

Recall that $T^0(M_d)=M_d{}^0$ has been assumed to be an *imbeddding* into E_{2d+1}. According to Section 1 (b), it thus corresponds to $\boldsymbol{F}^1{}_{d+1}$ and $\boldsymbol{F}^2{}_{d+1}$ sphere maps f^1, f^2 of S_{2d+1} into S_{d+1}. By assumption, these maps have the same Hopf's invariant mod 2: $h(f^1)=h(f^2) \bmod 2$. It is not difficult to see, that we may change one of the fields, $\boldsymbol{F}^1{}_{d+1}$ say, without changing c^1 (which is only defined mod 2 because d has been assumed to be odd) in such a way that $h(f^1)=h(f^2)$ as integers. Assume that such a change has been achieved. The Hopf's invariant of f^i is the looping coefficient in E_{2d+1} of $M_d{}^0$ with the locus, $V_d{}^i$ say, of the endpoint of the first vector of the field $\boldsymbol{F}^i{}_{d+1}$. These looping coefficients being equal, it is possible using again the covering homotopy theorem, to deform $\boldsymbol{F}^1{}_{d+1}$ and $\boldsymbol{F}^2{}_{d+1}$ continuously (keeping their unit vectors mutually orthogonal and normal to $M_d{}^0$) in such a way that after the deformation their first vectors coincide. Such a deformation does not change c^1 or c^2.

Let us denote by $\{\boldsymbol{v},\boldsymbol{v}_1{}^i,\boldsymbol{v}_2{}^i,\cdots,\boldsymbol{v}_d{}^i\}$ the vectors of $\boldsymbol{F}^i{}_{d+1}$ (after deformation) and by $\theta^i\colon M_d \to V_{2d+1,d+1}$ the induced mappings, the classes of which are represented by c^i $(i=1,2)$.

In order to prove $c^1=c^2$, let us first assume that M_d is a parallelizable manifold and let $\boldsymbol{t}_1,\boldsymbol{t}_2,\cdots,\boldsymbol{t}_d$ be a d-field of (mutually orthogonal unit) tangent vectors on $M_d{}^0$. Then each θ^i is homotopic to the map $\theta\colon M_d \to V_{2d+1,d+1}$ defined by $\theta(x)=\{\boldsymbol{v}(x),\boldsymbol{t}_1(x),\boldsymbol{t}_2(x),\cdots,\boldsymbol{t}_d(x)\}$. The desired homotopy is given by

$$\theta^i{}_s(x)=\{\boldsymbol{v}(x),\boldsymbol{v}_1{}^i(x)\cos(\tfrac{1}{2}\pi s)+\boldsymbol{t}_1(x)\sin(\tfrac{1}{2}\pi s),\cdots,$$
$$\boldsymbol{v}_d{}^i(x)\cos(\tfrac{1}{2}\pi s)+\boldsymbol{t}_d\sin(\tfrac{1}{2}\pi s)\},$$

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where $0 \leqq s \leqq 1$. Therefore, if M_d is parallelizable, θ^1 and θ^2 are homotopic and $c^1 = c^2$.

In general, M_d will not be parallelizable. However, if c^1 and c^2 were different, then one of them would be zero (it follows from d being odd, that c^1 and c^2 are remainders mod 2). If $c^i = 0$, $i = 1$ or 2, the corresponding map θ^i is homotopic to zero. By a reasoning of [7] (§ 8), it follows that M_d would be parallelizable.

This completes the "direct" proof of Theorem (10.1).

Appendix.

11. Relative Chern and Pontryagin characteristic classes.

11.a. *Relative Chern classes.*

In this section the coefficients are the *integers.*

Let $\mathfrak{B} = (E_{U(n)}, p, B_{U(n)}, \boldsymbol{U}(n))$ be the classifying bundle for the unitary group of n variables $\boldsymbol{U}(n)$. Suppose that a cross-section θ^r over a closed subset A of $B_{U(n)}$ is given in the associated bundle $\mathfrak{B}^r$ with fibre $W_{n,n-r}$ (the complex Stiefel manifold of $n-r$ complex vectors in C_n).

For $q \geqq r$, the relative Chern class $C_R{}^{q+1} \in H^{2(q+1)}(B_{U(n)}, A; \boldsymbol{Z})$ corresponding to the cross-section θ^r will be defined by the properties

(11.1) $\alpha^* C_R{}^{q+1} = C^{q+1}$, the ordinary (absolute) Chern class, α^* being the homomorphism $H^*(B_{U(n)}, A, \boldsymbol{Z}) \to H^*(B_{U(n)}, \boldsymbol{Z})$ induced by the inclusion $\alpha\colon (B_{U(n)}, 0) \to (B_{U(n)}, A)$,

(11.2) $\rho^*_{q,n} C_R{}^{q+1} = 0$, where $\rho^*_{q,n}\colon H^*(B_{U(n)}, A) \to H^*(B_{U(q)}, \theta^q A)$ is induced by the Borel map $\rho(\boldsymbol{U}(q), \boldsymbol{U}(n))$.

We consider the diagram

$$\begin{array}{ccccccccc}
\cdots \to & H^q(B_{U(q)}) & \to & H^q(\theta^q A) & \xrightarrow{\delta} & H^{q+1}(B_{U(q)}, \theta^q A) & \xrightarrow{\bar{\alpha}^*} & H^{q+1}(B_{U(q)}) & \to \cdots \\
& \uparrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \theta^*} & & \uparrow{\scriptstyle \rho^*_{q,n}} & & \uparrow{\scriptstyle \alpha^*} & \\
\cdots \to & H^q(B_{U(n)}) & \to & H^q(A) & \xrightarrow{\delta} & H^{q+1}(B_{U(n)}, A) & \xrightarrow{\alpha^*} & H^{q+1}(B_{U(n)}) & \to \cdots
\end{array}$$

where $B_{U(q)}$ is the space of the bundle $\mathfrak{B}^q$ ($B_{U(q)}$ is a classifying space for $\boldsymbol{U}(q)$, thus the notation) and θ^q is the cross-section over A in $\mathfrak{B}^q$ induced by θ^r $(q \geqq r)$.

By considerations similar to those made for the orthogonal group, it is easily seen that α^* is an epimorphism in every dimension and a monomorphism in dimensions not exceeding q. It follows, using exactness and commutativity in the diagram, that if $\rho^*_{q,n}z = 0$ and $a^*z = 0$ for some $z \in H^*(B_{U(n)}, A)$ then z must be zero.

The existence of at least one cohomology class with properties (11.1) and (11.2) is seen as follows: The restriction of C^{q+1} to A is zero because of the assumed existence of a cross-section over A in $\mathfrak{B}^r$. Let $C^{q+1} = a^*x$. Because $0 = \alpha^*C^{q+1} = \alpha^*a^*x = \bar{a}^*\rho^*_{q,n}x$, we have $\rho^*_{q,n}x = \delta\theta^{*-1}y$ for some $y \in H^q(A)$. The class $x - \delta y$ has the properties (11.1) and 11.2).

This proves that properties (11.1) and (11.2) indeed define the relative Chern classes uniquely for the classifying $\boldsymbol{U}(n)$-bundle.

We consider now the more general situation of a $\boldsymbol{U}(n)$-bundle over some compact finite dimensional space X induced by some map $g: X \to B_{U(n)}$. Let $(E^r, \pi, X, W_{n,n-r})$ be the associated bundle with fibre $W_{n,n-r}$ and assume a cross-section $\theta^r: A \to E^r$ to be given over the closed subset A of X. We may assume $\dim B_{U(n)}$ arbitrarily high. Take $\dim B_{U(n)} \geqq 2 \dim X + 1$ and let $f: X \to B_{U(n)}$ be an *injective* map homotopic to g. The bundle induced by f is equivalent to the one induced by g; let us denote it again by $(E^r, \pi, X, W_{n,n-r})$.

Let S be a closed subset of $B_{U(n)}$ containing $f(A)$ and such that there exists a cross-section $\psi: S \to B_{U(r)}$ in $\mathfrak{B}^r$, with the property $\psi(a) = \bar{f}\theta(a)$ for every $a \in A$ ($\bar{f}: E^r \to B_{U(r)}$ is the bundle map covering $f: X \to B_{U(n)}$). Let c_R^{q+1} be the $(q+1)$-dimensional relative Chern class of the classifying bundle mod S obtained using the cross-section $\psi (q \geqq r)$. We shall prove the

LEMMA (11.3). *$f^*c_R^{q+1}$ depends only on the homotopy class of the map g inducing the given bundle and on the cross-section θ^r over A.*

Definition. $f^*c_R^{q+1} = C_R^{q+1} \in H^{2(q+1)}(X, A; \boldsymbol{Z})$ is the relative Chern class (of dimension $2(q+1)$, defined for $q \geqq r$) mod A of the bundle (E, π, X), corresponding to the cross-section θ^r.

In order to prove Lemma (11.3), we first notice that $f^*c_R^{q+1}$ does not depend on S. Indeed, let $i^*: H^*(B_{U(n)}, S) \to H^*(B_{U(n)}, f(A))$ be the homomorphism dual to the inclusion $i: (B_{U(n)}, f(A)) \to (B_{U(n)}, S)$. We prove that $i^*c_R^{q+1}$ is the relative Chern class mod $f(A)$ corresponding to the restriction ψ_A of ψ over $f(A)$. Consider the diagram

$$\begin{array}{ccccc}
 & & H^{2(q+1)}(B_{U(n)}, fA) & \xleftarrow{i^*} & H^{2(q+1)}(B_{U(r)}, \psi S) \\
 & & \uparrow{\scriptstyle \rho^*_{r,n}} & & \uparrow{\scriptstyle \rho'^*_{r,n}} \\
H^{2(q+1)}(B_{U(n)}) & \xleftarrow{a^*} & H^{2(q+1)}(B_{U(r)}, \psi fA) & \xleftarrow{\bar{i}^*} & H^{2(q+1)}(B_{U(n)}, S).
\end{array}$$

By commutativity, the relations $\rho^*_{r,n}(i^*c_R^{q+1})=0$ and $a^*(i^*c_R^{q+1})=C^{q+1}$ (showing, by properties (11.1) and (11.2), that $i^*c_R^{q+1}$ is indeed the relative Chern class mod fA corresponding to ψ_A) follow from the corresponding relations $\rho'^*_{r,n}c_R^{q+1}=0$ and $a'^*c_R^{q+1}=c^{q+1}$ for c_R^{q+1} (where $a'^*=a^*i^*$). Since $i^*c_R^{q+1}$ is independent of S, so is $f^*c_R^{q+1}$, since $f^*=f'^*i^*$, where f'^* is induced by $f'\colon(X,A)\to(B_{U(n)},f(A))$.

It remains to be proved that $f^*c_R^{q+1}$ does not depend on the choice of the injective map. Let f_1, f_2 be two injective maps $X\to B_{U(n)}$ homotopic to g. We may assume that $f_1A\cap f_2A=0$. Otherwise take an injective map $f_0\colon X\to B_{U(n)}$ such that $f_0X\cap f_1X=0$ and $f_0X\cap f_2X=0$ (such a map may be obtained taking $\dim B_{U(n)}\geqq 2\dim X+3$ if necessary) and apply the following proof to f_0 and f_1 first and again to f_0 and f_2. Let S be the union $f_1A\cup f_2A$. The cross-section ψ over S is given by $\psi(a_i)=\theta f_i^{-1}(a_i)$ for $a_i\in f_i(A)$. Denoting by c_{1R} and c_{2R} the relative (universal) Chern classes mod f_1A and f_2A respectively corresponding to the restriction of ψ over f_1A and f_2A, we have to prove $f_1{}^*c_{1R}=f_2{}^*c_{2R}$. By the above remark, we have $f_1{}^*c_{1R}=f_1{}^*c_R$, $f_2{}^*c_{2R}=f_2{}^*c_R$, where c_R is the $2(q+1)$-dimensional class mod S corresponding to ψ. The equality $f_1{}^*c_R=f_2{}^*c_R$ due to $f_1\simeq f_2$ completes the proof of the Lemma (11.3).

Remark. Similarly to the definition of the Stiefel-Whitney classes, the definition of the relative Chern classes could have been introduced in terms of obstructions to the extension of cross-sections originally given over a subcomplex of the base. The two definitions coincide on their common domain: $\boldsymbol{U}(n)$-bundles $\mathfrak{B}$ over a complex K modulo some subcomplex L. (Use S. Eilenberg's approximation theorem, l. c. section 5).

Naturality property.

LEMMA (11.4). *Let (E,π,X) be a $\boldsymbol{U}(n)$-bundle and let (E',π',X') be the $\boldsymbol{U}(n)$-bundle over X' induced by some map $g\colon X'\to X$. We denote by C'_R and C_R the $2(q+1)$-dimensional relative Chern classes of the bundles (E',π',X') and (E,π,X) respectively, modulo closed subsets $A'\subset X'$ and $A\subset X$ such that $g(A')\subset A$ and corresponding to cross-sections θ and θ' in the associated bundles with fibre $W_{n,n-r}$ $(r\leqq q)$, such that $g\theta'(a')=\theta g(a')$ for every $a'\in A'$. Then $C'_R=g^*(C_R)$.*

Proof. Let $f\colon X\to B_{U(n)}$ be an injective map inducing (E,π,X) and $f'\colon X'\to B_{U(n)}$ be an injective map inducing (E',π',X'). We may assume that $fg(A')$ and $f'(A')$ have no commoon point. Let S be the union $fg(A')\cup f'(A')$ and ψ the cross-section over S in the associated bundle with fibre $W_{n,n-r}$ defined by $\psi f'(a') = \bar{f}'\theta'(a')$ and $\psi fg(a') = \bar{f}\theta g(a')$. Denoting again by c_R the $2(q+1)$-dimensional (universal) relative Chern class modulo S corresponding to ψ, we have $C'_R = f'^*(c_R) = g^*f^*(c_R) = g^*(C_R)$.

Whitney duality. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two principal bundles with bundle groups $\boldsymbol{U}(n_1)$ and $\boldsymbol{U}(n_2)$ respectively, over the same base space X and let $\theta_1{}^{r_1}$ and $\theta_2{}^{r_2}$ be cross-sections over closed subsets A_1 and A_2 (of X) in the associated bundles $\mathfrak{B}_1{}^{r_1}$ and $\mathfrak{B}_2{}^{r_2}$ with fibres W_{n_1,n_1-r_1} and W_{n_2,n_2-r_2} respectively. $\theta_1{}^{r_1}$ and $\theta_2{}^{r_2}$ determine a cross-section θ^r $(r = r_1 + r_2)$ over $A = A_1 \cap A_2$ in the bundle $\mathfrak{B}^r$, with fibre $W_{n,n-r}$ $(n = n_1 + n_2)$ associated to the Whitney sum $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$.

Denote by $C_{iR}{}^{q+1}$ the relative Chern class of $\mathfrak{B}_i$, $i = 1, 2$, defined for $q \geqq r_i$, and let $C_R{}^{q+1}$ be the relative Chern class of $\mathfrak{B}$ defined for $q \geqq r$.

For the relative Chern classes, the Whitney duality takes the form

$$C_R{}^{q+1} = C_{1_R}{}^{q+1} + C_{1_R}{}^{q} \cdot C_2{}^1 + \cdots + C_{2_R}{}^{q+1}. \tag{11.5}$$

In (11.5) some absolute Chern classes occur, however, again, since each product contains at least one relative class, it is itself a relative class.

The proof of formula (11.5) will be based on a theorem similar to Theorem (5.1) stated below (Theorem (11.6)).

Relative Chern classes as symmetric functions. Let $(E, p, X, \boldsymbol{U}(n))$ be a principal $\boldsymbol{U}(n)$-bundles over a compact finite dimensional space X. Consider the subgroup $\boldsymbol{Q}(n) = \boldsymbol{U}(1) \times \boldsymbol{U}(1) \times \cdots \times \boldsymbol{U}(1)$, n factors, of $\boldsymbol{U}(n)$ and the "space of flags" $\bar{X} = E/\boldsymbol{Q}(n)$ over X. We have a fibering $\rho\colon \bar{X} \to X$ induced by the projection $p\colon E \to X$ and the cross-section θ^r over $A \subset X$ may be used to construct a subset $\bar{A} \subset \bar{X}$ as follows. Let $E^r = E/\boldsymbol{U}(r)$ be the space of the bundle with fibre $W_{n,n-r}$ associated to $(E, p, X, \boldsymbol{U}(n))$. We have the diagram

$$\begin{tikzcd} \bar{A} \arrow[r, "\bar{\theta}"] \arrow[d, "\rho_A"] & \bar{E}^r \arrow[r, "\omega"] \arrow[d, "\rho_E"] & \bar{X} \arrow[d, "\rho_X"] \\ A \arrow[r, "\theta"'] & E^r \arrow[r, "p"'] & X \end{tikzcd}$$

where $\bar{E}^r$ is the space of flags over E^r (i. e. $E/\boldsymbol{Q}(r) = \bar{E}^r$) and $\rho_E\colon \bar{E}^r \to E^r$ is induced by $\boldsymbol{Q}(r) \subset \boldsymbol{U}(r)$. The bundle $(\bar{A}, \rho_A, A, \boldsymbol{F}(r), \boldsymbol{U}(r))$ is induced by $\theta\colon A \to E^r$. The principal bundle $\mathfrak{E} = (E, \pi, \bar{X}, \boldsymbol{Q}(n))$ is the Whitney sum $\mathfrak{E} = \mathfrak{E}^1 \oplus \mathfrak{E}^2 \oplus \cdots \oplus \mathfrak{E}^n$ of n principal bundles $\mathfrak{E}^i$ with group $\boldsymbol{U}(1)$. Let $x_i \in H^2(\bar{X}; \boldsymbol{Z})$ be the 2-dimensional (only non-zero positive dimensional)

Chern class of $\mathfrak{E}^i$. For $r+1 \leqq i \leqq n$, the relative Chern class mod $\bar{A}$ of $\mathfrak{E}^i$ may be defined (using the cross-section over $\bar{A}$ in $\mathfrak{E}^i$ induced by θ^r). Let $x_{r+1}, x_{r+2}, \cdots, x_n$ mean the *relative* classes. Then the elementary symmetric function $S^{q+1}(x_1, \cdots, x_n)$ is a relative class for $q \geqq r$ (each product of $q+1$ distinct factors from the $x_1, \cdots, x_n$ must contain at least one of the variables $x_{r+1}, \cdots, x_n$). Let us denote by $S_R^{q+1}(x_1, \cdots, x_n)$ this relative class. We have the theorem

THEOREM (11.6). *Let $\rho_R^*: H^*(X, A; \boldsymbol{Z}) \to H^*(\bar{X}, \bar{A}; \boldsymbol{Z})$ be the homomorphism induced by $\rho_R: (\bar{X}, \bar{A}) \to (X, A)$. Then,*

(a) ρ_R^* *is a monomorphism,*

(b) $\rho_R^* C_R^{q+1} = S_R^{q+1}(x_1, \cdots, x_n)$ *for* $q \geqq r$.

The proof is very similar to the proof of Theorem (5.1). We are not going to enter into the details again. Along the same lines as for the proof of (5.1), we need the result

$$H^*(\boldsymbol{F}(n); \boldsymbol{Z}) = \boldsymbol{Z}[x_1, \cdots, x_n]/(S^+(x_1, \cdots, x_n)),$$

with $\boldsymbol{F}(n) = \boldsymbol{U}(n)/\boldsymbol{Q}(n)$, where $(S^+(x_1, \cdots, x_n))$ denotes the ideal generated in $\boldsymbol{Z}[x_1, \cdots, x_n]$ by the symmetric functions of positive degree (see [3], Proposition 31.1). The only points where the method of proof of (5.1) breaks down now are those where use was made of the fact that the coefficient ring for Stiefel-Whitney classes was a *field* $(\boldsymbol{Z}_2)$.

The assertion that the term E_2 in the spectral sequence of $\rho: \bar{X} \to X$ reduces to $H^*(X) \otimes H^*(\boldsymbol{F}(n))$ is, however, still true because $\boldsymbol{F}(n) = \boldsymbol{U}(n)/\boldsymbol{Q}(n)$ has no torsion.

The only non-trivial change is that it is no longer obvious that $H^*(X; \boldsymbol{Z})$ should be (at least additively) isomorphic to E_∞. This proves, however, tc hold, as will be seen from the following argument I learned from A. Borel, similar to an argument by J. P. Serre ([10], Chap. III, 7, Prop. 9).

LEMMA (11.7). *Let $\sigma: H^*(F) \to H^*(E)$ be a right-inverse (additive) homomorphism to $i^*: H^*(E) \to H^*(F)$, where E is the space of a bundle (E, π, B, F) with fibre F and i the inclusion $i: F \to E$. Assume that either B or F has no torsion. Then $H^*(E)$ is isomorphic to $H^*(B) \otimes H^*(F)$. Moreover, the isomorphism preserves the product if σ does.* (In this lemma the domain of coefficients is any commutative ring with unit).

Proof. Let $\omega: H^*(B) \otimes H^*(F) \to H^*(E)$ be the linear map defined by $\omega(x\otimes f)=\pi^*(x)\cdot\sigma(f)$. We are going to prove that ω is an isomorphism which preserves products if σ does.

We first notice that ω preserves the filtration: $H^*(B)\otimes H^*(F)$ being filtered by the ideals $A^q=\sum_{i\geqq q}H^i(B)\otimes H^*(F)$ and $H^*(E)$ by the ideals $J^q=\pi^*(\sum_{i\geqq q}H^i(B))\cdot H^*(E)$. We have for every $q, \omega(A^q)\subset J^q$. Therefore, ω induces an additive homomorphism $\bar{\omega}$ of the corresponding graded rings $\sum A^q/A^{q+1}$ into $\sum J^q/J^{q+1}$, where $\sum$ denotes the direct sum. It is well known and easily seen that ω is an isomorphism if $\bar{\omega}$ is.

By triviality of the spectral sequence of the fibering $p\colon E\to B$ (because $i^*\colon H^*(E)\to H^*(F)$ which admits a right inverse, is an epimorphism), it follows that $E_2=H^*(B,H^*(F))$ is isomorphic to the graded ring E_∞ associated to $H^*(E)$, i.e. $\sum J^q/J^{q+1}$. Let k be this isomorphism ($k^2{}_\infty$ with the notations of [3], § 1).

Since either B or F has no torsion by assumption, $E_2=H^*(B)\otimes H^*(F)$; identifying $\sum A^q/A^{q+1}$ with $H^*(B)\otimes H^*(F)$ in the natural manner, the lemma will be proved by showing that $\bar{\omega}\colon H^*(B)\otimes H^*(F)\to E_\infty$ is identical to the map $k\colon H^*(B)\otimes H^*(F)\to E_\infty$.

Let $x\otimes f\in H^*(B)\otimes H^*(F)$, where $x\in H^p(B)$, $f\in H^q(F)$. We have $\omega(x\otimes f)=\pi^*(x)\cdot\sigma(f)\in J^{p,q}=\pi^*(\sum_{i\geqq p}H^i(B))\cdot H^q(E)$, and $\bar{\omega}(x\otimes f)$ is the image of $\omega(x\otimes f)$ in $J^{p,q}/J^{p+1,q-1}=E_\infty{}^{p,q}$. By definition of the product in E_∞, we obviously have $\bar{\omega}(x\otimes f)=\bar{\omega}(x\otimes 1)\cdot\bar{\omega}(1\otimes f)$. Since k has also this property, it is sufficient to verify $\bar{\omega}=k$ on the elements of the form $x\otimes 1$ and $1\otimes f$.

$\bar{\omega}(x\otimes 1)=\pi^*(x)\in J^{p,0}$ $(J^{p+1,-1}=0)$. We have $k(x\otimes 1)=\pi^*(x)$. See [3], § 4, (b).

$\bar{\omega}(1\otimes f)=\overline{\sigma(f)}$, where $\overline{\sigma(f)}$ represents the image of $\sigma(f)\in J^{0,q}$ in $J^{0,q}/J^{1,q-1}$. By [3], § 4 (c), we have $k^{-1}(\overline{\sigma(f)})=1\otimes i^*\sigma(f)=1\otimes f$ since σ is a right inverse to i^*. Thus, again $\bar{\omega}(1\otimes f)=k(1\otimes f)$.

If σ preserves the product, then

$$\begin{aligned}\omega[(x\otimes f)\cdot(x'\otimes f')]&=(-1)^{qp'}\omega(xx'\otimes ff')=(-1)^{qp'}\pi^*(x\cdot x')\cdot\sigma(f\cdot f')\\&=(-1)^{qp'}\pi^*(x)\cdot\pi^*(x')\cdot\sigma(f)\cdot\sigma(f')=\pi^*(x)\cdot\sigma(f)\cdot\pi^*(x')\cdot\sigma(f')\\&=\omega(x\otimes f)\cdot\omega(x'\otimes f').\end{aligned}$$

In the situation of Theorem (11.6), the existence of the homomorphism σ is obvious (one has $H^*(\boldsymbol{F}(n);\mathbf{Z})=\mathbf{Z}[u_1,\cdots,u_n]/(S^+(u_1,\cdots,n_n))$, where $u_k=i^*x_k$; let $h_1=1,h_2,\cdots,h_t$ be a basis for the vector space

$H^*(\boldsymbol{F}(n);\boldsymbol{Z})$, the h_i being represented by polynomials $P_i(u_1,\cdots,u_n)$ in $u_1,\cdots,u_n$, and define $\sigma(h_i)$ as the polynomial in $x_1,\cdots,x_n$ obtained from P_i by the substitution $u_k\to x_k$; extend then σ by linearity). This completes the proof of Theorem (11.6) part (a).

The proof of part (b) is similar to the proof of (b) in Theorem (5.1). Use has to be made of the results of [6] (in particular Prop. 4.1) rather than from [4]. A slight change occurs at the end of the proof: we have to show that $\mu_R{}^*S_R{}^{q+1}(x_1,\cdots,x_n)=0$, where $\mu_R{}^*\colon H^*(B_{Q(n)},\bar{A})\to H^*(B_{Q(r)},\bar{A}^1)$. The argument used for Stiefel-Whitney classes does not work now, because $\deg x_i=2$ and $a^*\colon H^2(B_{Q(r)},\bar{A}^1)\to H^2(B_{Q(r)})$ need not be a monomorphism. However, $\mu_R{}^*x_{r+j}=0$ for $j=1,\cdots,n-r$ may be seen as follows. Consider for some j $(1\leqq j\leqq n-r)$ the $\boldsymbol{U}(1)$-bundle $\mathfrak{E}^{r+j}$ over $B_{Q(n)}$; it admits over $\bar{A}$ the section θ_j. The map $\mu\colon B_{Q(r)}\to B_{Q(n)}$ induces over $B_{Q(r)}$ a $\boldsymbol{U}(1)$-bundle ("counter-image of $\mathfrak{E}^{r+j}$") and explicit construction shows that the induced cross-section ψ over $\bar{A}^1$ (defined by $\psi(u)=(u,\theta_j\mu(u))$) can be extended all over $B_{Q(r)}$. Thus, by naturality, $\mu_R{}^*x_{r+j}=0$ (from this $\mu_R{}^*S_R{}^{q+1}(x_1,\cdots,x_n)=0$ follows because of $q\geqq r$).

11.b. *Relative Pontryagin classes.*

From now on, the coefficients are integers mod p, where p is prime and >2. The following definition and naturality property would be valid without alteration with integer coefficients, but Whitney duality is not.

Let $\mathfrak{B}=(E,p,X,\boldsymbol{SO}(n))$ be a principal $\boldsymbol{SO}(n)$-bundle induced by a map $\tau\colon X\to B_{SO(n)}$. Let $\sigma\colon B_{SO(n)}\to B_{U(n)}$ be the mapping corresponding to the inclusion $\boldsymbol{SO}(n)\to\boldsymbol{U}(n)$. Then $\sigma\circ\tau$ induces over X a principal $\boldsymbol{U}(n)$-bundle $\mathfrak{B}_C=(E_C,\pi,X,\boldsymbol{U}(n))$.

Let θ^r be a cross-section over the closed subset A of X in the bundle $\mathfrak{B}^r$ associated to $\mathfrak{B}$ with fibre $V_{n,n-2r+1}$. For $k\geqq r$, we have a cross-section θ^k over A in $\mathfrak{B}^k$ (with fibre $V_{n,n-2k+1}$) induced by θ^r. These cross-sections provide cross-sections $\bar{\theta}_C{}^k$ over A in the corresponding "complex" bundles $\mathfrak{B}_C{}^k$,associated to $\mathfrak{B}_C$ with fibre $W_{n,n-2k+1}$) as follows. Let $\bar{\tau}$, $\bar{\sigma}$ be the maps of the total space covering τ, σ respectively ($\bar{\sigma}$ is not a bundle map), and let h be the bundle map covering $\sigma\circ\tau$. Then h is injective (actually a homeomorphism) on each fibre $W_{n,n-2k+1}$, and $\theta_C{}^k$ may be defined by

$$(11.8)\qquad \pi\theta_C{}^k(a)=a,\qquad \theta_C{}^k(a)=h^{-1}\bar{\sigma}\bar{\tau}\theta^k(a)$$

Definition. The $4k$-dimensional relative Pontryagin class

$$P_R{}^k\in H^{4k}(X,X;\boldsymbol{Z}_p)\bmod A$$

corresponding to the cross-section θ^r, defined for $k \geqq r$, is given by

$$P_R^k = (-1)^k C_R^{2k}, \tag{11.9}$$

where C_R^{2k} is the relative Chern class mod A of $\mathfrak{B}_C$ corresponding to θ_C^k.

Naturality. Let $\mathfrak{B}$, $\mathfrak{B}'$ be two $\boldsymbol{SO}(n)$-bundles over X and X' respectively, such that $\mathfrak{B}$ is induced from $\mathfrak{B}'$ by a map $f\colon X \to X'$. Let $\mathfrak{B}^k = (E^k, X)$, $\mathfrak{B}'^k = (E'^k, X')$ be the associated bundles with fibre $V_{n,n-2k+1}$ and assume that cross-sections θ^r, θ'^r in $\mathfrak{B}^r$, $\mathfrak{B}'^r$ are given over closed subsets $A \subset X$, $A' \subset X'$ respectively, such that $f(A) \subset A'$ and $\theta' f(a) = \bar{f}\theta(a)$, ($\bar{f}\colon E^k \to E'^k$ cover f) for every $a \in A$.

Let P^k and P'^k be the relative Pontryagin classes ($k \geqq r$) of $\mathfrak{B}$ and $\mathfrak{B}'$ corresponding to θ and θ' respectively. Then

$$P^k = f^*(P'^k). \tag{11.10}$$

Proof. The above formula follows from the naturality property of relative Chern classes, provided we prove that $f_C\theta_C(a) = \theta'_C f(a)$, where $f_C\colon E_C{}^k \to E_C{}'^k$ covers f (Remark that $(E_C{}^k, X)$ is induced from $(E'_C{}^k, X')$ by $f\colon X \to X'$). We have

$$h\theta'_C f(a) = \bar{\sigma}\bar{\tau}\theta' f(a) = \bar{\sigma}\bar{\tau}\bar{f}\theta(a) = h f_C \theta_C(a)$$

and, since h is homeomorphic on the fibres,

$$\theta'_C f(a) = f\ \theta_C(a) \text{ for every } a \in A.$$

We are thus in position to apply the naturality property of relative Chern classes and (11.10) follows.

Whitney *duality* for Pontryagin classes will follow from Whitney duality for relative Chern classes reduced mod p. Because of the naturality property, we may restrict attention to the case of a classifying $\boldsymbol{SO}(n)$ bundle $\mathfrak{B} = (E_{SO(n)}, p, B_{SO(n)}, \boldsymbol{SO}(n))$ assuming a cross-section to be given over A (closed subset of $B_{SO(n)}$) in the associated bundle $\mathfrak{B}^r$ with fibre $V_{n,n-2r+1}$. Let $\mathfrak{B}_C$ be the $\boldsymbol{U}(n)$-bundle over $B_{SO(n)}$ induced by $\sigma\colon B_{SO(n)} \to B_{U(n)}$, and let $\theta_C{}^r$ be the cross-section over A induced by θ^r in the bundle $\mathfrak{B}_C{}^r$ associated to $\mathfrak{B}_C$ with fibre $W_{n,n-2r+1}$.

LEMMA (11.11). *The* $(4i+2)$*-dimensional relative Chern classes of* $\mathfrak{B}_C$ mod A *corresponding to* $\theta_C{}^r$ *are zero* mod p: $C_R^{2i+1} = 0$, $i \geqq r$.

Proof. Consider the diagram

$$\begin{tikzcd}
H^{4i+1}(E) \arrow[r, "\bar{i}^*"] & H^{4i+1}(\theta^i A) \arrow[r, "\delta"] \arrow[d, "\theta^*"] & H^{4i+2}(E^i, \theta^i A) \arrow[r] & H^{4i+2}(E^i) \\
H^{4i+1}(B_{SO(n)}) \arrow[u, "\omega^*"] \arrow[r, "i^*"] & H^{4i+1}(A) \arrow[r, "\delta"] & H^{4i+2}(B_{SO(n)}, A) \arrow[u, "\omega_R^*"] \arrow[r, "a^*"] & H^{4i+2}(B_{SO(n)}). \arrow[u, "\omega^*"]
\end{tikzcd} \tag{11.12}$$

Since $a^*(C_R^{2i+1}) = C^{2i+1}$, we have (by [5], Proposition 25.4) $a^* C_R^{2i+1} = 0$ (coefficients mod p). Therefore $C_R^{2i+1} = \delta x$, $x \in H^{4i+1}(A; \boldsymbol{Z}_p)$, and since θ^* is an isomorphism, there exists a class $u \in H^{4i+1}(\theta^i A)$, with $\theta^* u = x$. We have

$$\delta u = \delta \theta^{*-1} x = \omega_R^* \delta x = \omega_R^* C_R^{2i+1} = 0.$$

The last equality follows from consideration of the diagram

$$\begin{tikzcd}
H^{4i+2}(E^i, \theta^i A) & H^{4i+2}(E_C^i, \theta_C^i A) \arrow[l, "\bar{\sigma}^*"'] \\
H^{4i+2}(B_{SO(n)}, A) \arrow[u, "\omega_R^*"] & H^{4i+2}(B_{U(n)}, A). \arrow[l, "\sigma^*"'] \arrow[u, "\pi^*"']
\end{tikzcd}$$

Indeed, $\omega_R^* C_R^{2i+1} = \omega_R^* \sigma^* c_R^{2i+1} = \bar{\sigma}^* \pi^* c_R^{2i+1} = 0$ (by (11.4)).

By exactness of the rows in diagram (11.12), there exists an element $z \in H^{4i+1}(E^i)$ such that $\bar{i}^* z = u$. The assertion $C_R^{2i+1} = 0$ follows from the fact that ω^* is an *epimorphism* in every dimension (By [5], Theorem 23.2, $H^*(B_{SO(2m-1)}; \boldsymbol{Z}_p) = \boldsymbol{Z}_p[P_1, \cdots, P_{m-1}]$ and

$$H^*(B_{SO(2m)}; \boldsymbol{Z}_p) = \boldsymbol{Z}_p[P_1, \cdots, P_{m-1}, W_{2m}].$$

Notice that E^i is classifying space for $\boldsymbol{SO}(2i-1)$.).

Remark. Using the fact that ω^* is still an epimorphism if *integer* coefficients are used (see a forthcoming paper by A. Borel and F. Hirzebruch), the same method would give $2C_R^{2i+1} = 0$, where C_R^{2i+1} is the integer relative Chern class of a $\boldsymbol{U}(n)$-bundle obtained from an $\boldsymbol{SO}(n)$-bundle.

Whitney duality for relative Pontryagin classes is an immediate consequence of the same property for relative Chern classes with coefficients mod p, making use of Lemma (11.11).

Let $\mathfrak{B}_i$ be two $\boldsymbol{SO}(n_i)$-bundles $(i = 1, 2)$ over X and let θ_i be cross-sections over (closed subsets) $A_i \subset X$ in the associated bundles $\mathfrak{B}_i^{r_i} = (E_i^{r_i}, p_i, X, V_{n_i, n_i - 2r_i + 1})$.

Let $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$ be the Whitney sum and θ the cross-section over $A = A_1 \cap A_2$ in $\mathfrak{B}^r$ (with fibre $V_{n,n-2r+1}$, where $n = n_1 + n_2$, $r = r_1 + r_2$) obtained using $\theta_1^{r_1}$ and $\theta_2^{r_2}$.

P_R^k, the relative Pontryagin class of dimension $4k$ of corresponding to θ is defined for $k \geqq r$. Similarly, $P^1{}_R^k$, $P^2{}_R^k$ are defined for $k \geqq r_1$ and $k \geqq r_2$ respectively.

One has in $H^{4k}(X, A; \mathbf{Z}_p)$:

$$P_R^k = P^1{}_R^k + \cdots + P^1{}_R^{r_1} \cdot P^2{}_R^{k-r_1} + \cdots + P^2{}_R^k \qquad (k \geqq r). \tag{11.13}$$

Although some absolute classes might appear in the above formula, each cup-product contains at least one relative class. The sum on the right consists only of *relative* classes.

Pontryagin classes as symmetric functions. Consider again a principal $\boldsymbol{SO}(n)$-bundle $\mathfrak{B} = (E, p, X, \boldsymbol{SO}(n))$ and define the subgroup $\boldsymbol{Q}(n)$ of $\boldsymbol{SO}(n)$ by $\boldsymbol{Q}(n) = \boldsymbol{SO}(2) \times \boldsymbol{SO}(2) \times \cdots \times \boldsymbol{SO}(2)$ or $\boldsymbol{Q}(n) = \boldsymbol{SO}(2) \times \boldsymbol{SO}(2) \times \cdots \times \boldsymbol{SO}(2) \times \boldsymbol{SO}(1)$ according as $n = 2m$ or $n = 2m + 1$ (m factors $\boldsymbol{SO}(2)$ in both cases). Consider the quotient space $E/\boldsymbol{Q}(n)$. The principal fibre bundle $(E, \pi, E/\boldsymbol{Q}(n), \boldsymbol{Q}(n))$ is the Whitney sum of m principal $\boldsymbol{SO}(2)$-bundles $\mathfrak{E}^1, \mathfrak{E}^2, \cdots, \mathfrak{E}^m$. Let $x_1, x_2, \cdots, x_m$ be their Chern classes ($\boldsymbol{SO}(2)$ being identified with $\boldsymbol{U}(1)$).

Assuming a cross-section θ over $A \subset X$ (closed subset) to be given in $\mathfrak{B}^r = (E^r, p, X, V_{n,n-2r+1})$ we obtain cross-sections in $\mathfrak{E}^1$ for $i = r, r+1, \cdots, m$ over $\bar{A} \subset B_{\boldsymbol{Q}(n)}$ as follows: E^r is the base space of a principal $\boldsymbol{SO}(2r-1)$-bundle $(E, E^r, \boldsymbol{SO}(2r-1))$. Let $\bar{E}^r$ be the space of flags over $\bar{E}^r$, i.e. $\bar{E}^r = E/\boldsymbol{Q}(2r-1)$, where $\boldsymbol{Q}(2r-1)$ is the subgroup of $\boldsymbol{SO}(n)$ consisting of the matrices of the type

$$\begin{bmatrix} D_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & 0 & & \\ & & & \cdot & & & & \\ & & & & D_{r-1} & & & \\ & 0 & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \\ & & & & & & & & & \cdot \\ & & & & & & & & & & 1 \end{bmatrix}, \text{ where } D_i = \begin{pmatrix} \cos(2\pi x_i) & \sin(2\pi x_i) \\ -\sin(2\pi x_i) & \cos(2\pi x_i) \end{pmatrix}.$$

We have the diagram

$$(11.14)\qquad \begin{array}{ccccc} & & \bar{E}^r & \xrightarrow{\omega} & \bar{X} \\ & & \rho_E \downarrow & & \downarrow \rho_X \\ A & \xrightarrow{\theta} & E^r & \xrightarrow{p} & X \end{array}$$

The map $\omega\colon \bar{E}^r \to \bar{X}$ is induced by the identity $E \to E(\boldsymbol{Q}(2r-1) \subset \boldsymbol{Q}(n))$. Let $\bar{A}$ be the space of the bundle $(\bar{A}, \rho_A, A, \boldsymbol{F}(2r-1), \boldsymbol{SO}(2r-1))$ induced by θ. Notations: $\boldsymbol{F}(2r-1) = \boldsymbol{SO}(2r-1)/\boldsymbol{Q}(2r-1)$, $\bar{\theta}\colon \bar{A} \to \bar{E}^r$ is the bundle map covering θ. The maps $p\theta$ and $\omega\bar{\theta}$ are injective and we may consider $\bar{A}$ as a subset of $\bar{X}$. Geometrically, a point of $\bar{A} \subset \bar{X}$ consists of a point a of A together with a sequence of m oriented 2-planes $\pi_1, \pi_2, \cdots, \pi_m$, such that π_r contains the first, π_{r+1} the second and the third, $\cdots$, π_m the $(n-2r)$-th and the $(n-2r+1)$-st vectors of $\theta(a)$, assuming $n=2m$. If $n = 2m+1$, π_r contains the first and second, etc., π_m the $(n-2r)$-th and the $(n-2r+1)$-st vectors of $\theta(a)$. In both cases ($n=2m$ or $2m+1$), since π_r is oriented, $\mathfrak{E}^r$ admits a cross-section over $\bar{A}$ and so do $\mathfrak{E}^{r+1}, \cdots, \mathfrak{E}^m$. We define $x_r, x_{r+1}, \cdots, x_m$ (the characteristic classes of $\mathfrak{E}^r, \cdots, \mathfrak{E}^m$) as *relative* classes corresponding to the cross-sections given by θ over $\bar{A}$. The elementary symmetric functions $S^k(x_1{}^2, \cdots, x_m{}^2)$ are then relative classes $\operatorname{mod} \bar{A}$ (of dimension $4k$) for $k \geqq r$ and will, consequently, be denoted by $S_R{}^k(x_1{}^2, \cdots, x_m{}^2)$. One has the

THEOREM (11.15). *Let $\rho_R{}^*\colon H^*(X, A; \boldsymbol{Z}_p) \to H^*(\bar{X}, \bar{A}; \boldsymbol{Z}_p)$ be the homomorphism induced by $\rho_R\colon (\bar{X}, \bar{A}) \to (X, A)$. Then* (a) *$\rho_R{}^*$ is a monomorphism, and* (b) *$\rho_R{}^* P_R{}^k = S_R{}^k(x_1{}^2, \cdots, x_m{}^2)$ for $k \geqq r$.*

Proof of (a). Consider the diagram

$$\begin{array}{ccccccc} H^{4k-1}(\bar{X}) & \xrightarrow{\bar{i}^*} & H^{4k-1}(\bar{A}) & \xrightarrow{\delta} & H^{4k}(\bar{X}, \bar{A}) & \xrightarrow{\bar{a}^*} & H^{4k}(\bar{X}) \\ \uparrow \rho_X{}^* & & \uparrow \rho_A{}^* & & \uparrow \rho_R{}^* & & \uparrow \rho_X{}^* \\ H^{4k-1}(X) & \xrightarrow{i^*} & H^{4k-1}(A) & \xrightarrow{\delta} & H^{4k}(X, A) & \xrightarrow{a^*} & H^{4k}(X) \end{array}$$

(coefficients = remainders $\operatorname{mod} p$, $p = \text{prime} > 2$), where $\rho_X{}^*$ and $\rho_A{}^*$ are monomorphisms in every dimension (see [5], Theorem 23.2) and i, $\bar{i}$ are the inclusions $i = p\theta$, $\bar{i} = \omega\bar{\theta}$.

The situation is entirely similar to the one in the proof of Theorem (5.1). A straightforward exactness argument shows that (a) follows from the Lemma: *If $b \in H^*(A)$ and $w \in H^*(\bar{X})$ are such that $\rho_A{}^* b = \bar{i}^* w$, then there exists a class $v \in H^*(X)$, such that $i^* v = b$.*

The proof of the lemma is entirely similar to the one given in the proof of Theorem (5.1).

Proof of (b). By naturality, it is sufficient to prove the formula $\rho_R{}^*P_R{}^k = S_R{}^k(x_1{}^2, \cdots, x_m{}^2)$ for the bundle $\mathfrak{B} = (B_{Q(n)}, \rho_n, B_{SO(n)}, \boldsymbol{F}(n), \boldsymbol{SO}(n))$ obtained from the classifying bundle ($B_{Q(n)}$ is the space of flags over $B_{SO(n)}$). The diagram (11.14) reads in this case.

$$\begin{tikzcd} A \arrow[r, "\bar\theta"] \arrow[d, "\rho_A"] & B_{SO(2r-1)} \arrow[r, "\mu"] \arrow[d, "\rho_{2r-1}"] & B_{Q(n)} \arrow[d, "\rho_n"] \\ \bar A \arrow[r, "\theta"] & B_{Q(2r-1)} \arrow[r, "p"] & B_{SO(n)} \end{tikzcd}$$

and there is a similar diagram with k substituted for r for every k such that $r \leqq k \leqq \frac{1}{2}(n+1)$. Considering $\bar A$ as a subset of $B_{Q(2r-1)}$ and $B_{Q(n)}$ by the injections $\bar\theta$ and $\mu\bar\theta$, we have the following diagram

$$\begin{tikzcd} H^{4k-1}(B_{Q(2k-1)}) \arrow[r, "\bar\theta^*"] & H^{4k-1}(\bar A) \arrow[r, "\delta"] \arrow[d, leftrightarrow, "\cong"] & H^{4k}(_{Q(2k-1)}, \bar A) \arrow[r, "a^*"] & H^{4k}(B_{Q(2k-1)}) \\ H^{4k-1}(B_{Q(n)}) \arrow[u, "\mu^*"] \arrow[r, "\theta^*"] & H^{4k-1}(\bar A) \arrow[r, "\delta"] & H^{4k}(B_{Q(n)}, \bar A) \arrow[u, "\mu_R{}^*"] \arrow[r, "a^*"] & H^{4k}(B_{Q(n)}). \arrow[u, "\mu^*"] \end{tikzcd}$$

From the relation $\rho^*(P^k) = S^k(x_1{}^2, \cdots, x_m{}^2)$ for the absolute Pontryagin classes, (see [6], Proposition 5.1), we have $a^*y = 0$, setting $y = \rho_R{}^*P_R{}^k - S_R{}^k(x_1{}^2, \cdots, x_m{}^2)$. Indeed, $a^*P_R{}^k = P^k$ is immediately seen from the definition and the corresponding equality for Chern classes.

By an exactness argument used several times in this paper, (b) follows from $\mu_R{}^*y = 0$. The proof of $\mu_R{}^*S_R{}^k(x_1{}^2, \cdots, x_m{}^2) = 0$ is similar to the one given for Chern classes. The equality $\mu_R{}^*\rho_R{}^*P_R{}^k = 0$ follows from consideration of the diagram

$$\begin{tikzcd} H^{4k}(B_{Q(2k-1)}, \bar A) & H^{4k}(B_{Q(n)}, \bar A) \arrow[l, "\mu_R{}^*"'] \\ H^{4k}(B_{SO(2k-1)}, A) \arrow[u, "\kappa_R{}^*"] & H^{4k}(B_{SO(n)}, A). \arrow[l, "p^*"] \arrow[u, "\rho_R{}^*"'] \end{tikzcd}$$

One has $\mu_R{}^*\rho_R{}^*P_R{}^k = \kappa_R{}^*p^*P_R{}^k$. Now, $p^*(P_R{}^k) = 0$ because

$$P_R{}^k = (-1)^k\sigma^*(C_R{}^{2k})$$

and of the diagram

$$\begin{array}{ccc} H^{4k}(B_{SO(2k-1)}, A) & \xleftarrow{\bar{\sigma}^*} & H^{4k}(B_{U(2k-1)}, \bar{\sigma}A) \\ \uparrow p^* & & \uparrow \pi^* \\ H^{4k}(B_{SO(n)}, A) & \xleftarrow{\sigma^*} & H^{4k}(B_{U(n)}, A) \end{array}$$

$p^*(P_R^k) = (-1)^k p^*\sigma^*(C_R^{2k}) = (-1)^k \bar{\sigma}^*\pi^*(C_R^{2k}) = 0$, by definition of the relative Chern classes.

This completes the proof of Theorem (11.15).

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