# A SURVEY OF MULTIDIMENSIONAL KNOTS

by

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## CHAPTER I : INTRODUCTION

### § 1. Some historical landmarks.

Knotted n-spheres $K = f(S^n) \subset S^{n+2}$ with $n \geqslant 2$ make what seems to be their first appearance in a famous paper by E. Artin published in 1925, where he describes a construction which produces examples of non-trivial n-knots for arbitrary $n \geqslant 2$. (Detailed reference data are provided at the end of the survey). In today's terminology, introduced by E.C. Zeeman (1959), the construction is called spinning and it goes as follows.

Let $K \subset S^{n+2}$ be an n-knot, i.e. a smoothly embedded n-sphere K in $S^{n+2}$. Take the associated knotted disk pair $(B, bB) \subset (D^{n+2}, S^{n+1})$ obtained by removing from $S^{n+2}$ a small open disk U centered at a point of K. Here, $D^{n+2} = S^{n+2} - U$ and $B = K - K \cap U$ . The subset $D = \{ \Sigma_{i=1}^{n+4} x_i^{\ 2} = 1, \ \ x_{n+3} \geqslant 0, \ x_{n+4} = 0 \}$ in $S^{n+3} \subset R^{n+4}$ is an (n+2)-dimensional disk which we identify with $D^{n+2}$. Thus, $B \subset D$ . Now, the sphere $S^{n+3}$ can be obtained by rotating this disk D in $R^{n+4}$ around the (n+2)-plane $P = \{x_{n+3} = 0, \ x_{n+4} = 0\}$ . Note that P contains the unknotted boundary sphere $bD = S^{n+1} \subset S^{n+3}$ which thus remains point-wise fixed during the rotation. In the process, the set $B \subset D$ will sweep out a smooth (n+1)-dimensional sphere embedded in $S^{n+3}$. This is the spun knot $\Sigma_K \subset S^{n+3}$ of the knot $K \subset S^{n+2}$ .

E.Artin observed in his paper that

$$\pi_1(S^{n+3} - \Sigma_K) \cong \pi_1(S^{n+2} - K) .$$

Thus, $\Sigma_K \subset S^{n+3}$ is certainly knotted if $\pi_1(S^{n+2} - K) \not\cong \mathbb{Z}$ .

Starting with a non-trivial "classical" knot (i.e. $n = 1$) and iterating the construction, one gets non-trivial n-knots for all n .

A similar construction can be performed on linked spheres and it also leaves unchanged the fundamental group of the complement. See van Kampen (1928) and Zeemann (1959) for details.

The objective of multidimensional knot theory is, as for classical knots, to perform classification, ultimately (and ideally) with respect to isotopy, and meanwhile with respect to weaker equivalence relations. There is however with higher dimensional knots the additional difficulty that the construction of a knot cannot merely be described by the simple-minded drawing up of a knot projection. Thus, efforts at classification (i.e. finding invariants) now have to be complemented by construction methods (i.e. showing that the invariants are realizable). This is why Artin's paper is so significant. It gives the first construction showing that the groups of classical knots are all realizable as fundamental groups of the complement of n-knots for arbitrary n.

After Artin's paper, multidimendional knot theory went into a long sleep. Strangely enough, the theory awoke subsequently to Papa's proof of the sphere theorem. One of the consequences of this famous result is that classical knots have aspherical complements, i.e. : $\pi_i(S^3 - f(S^1)) = 0$ for $i > 1$. Hence a natural question : What about multidimensional knots ? The answer came quickly : In 1959, J.J. Andrews and M. L.Curtis showed that the complement of the spun trefoil has a non-vanishing second homotopy group. In fact their result is more general and also better : there is an embedded 2-sphere which represents a non-zero element.

This paper was followed less than a month later by D.Epstein (1959) who gave a formula expressing $\pi_2$, the second homotopy group of the complement of any spun 2-knot. A corollary of Epstein's result is that the complement of a non-trivial spun 2-knot has a $\pi_2$ which is not finitely generated as an abelian group.

The question was then raised by R.H.Fox (1961) to describe $\pi_2$ as a $\pi_1$-module. This gave the impulse for the subsequent research in that direction. (See for example S.J. Lomonaco Jr (1968)).

One thus began to suspect that multidimensional knots would behave quite differently from the classical ones. The major breakthrough came from the development of surgery techniques which made it possible to get a general method of constructing knots with prescribed properties of their complements. In a perhaps subtler way, surgery techniques were also decisive in classification problems. See our chapters III and IV.

Here is an illustration of the power of surgery techniques. A common feature to the examples (all based on spinning) known in 1960 was that $\pi_2$ was $\neq 0$ because $\pi_1 \neq \mathbb{Z}$. It was thus natural to ask : Can one produce an n-knot with $\pi_2 \neq 0$ but $\pi_1 = \mathbb{Z}$ ? Clearly such an example cannot be obtained by spinning a classical knot. However, J. Stallings (see M. Kervaire's paper (1963), p. 115) and C.T.C. Wall (see his book : Surgery on compact manifolds, p.18 ) proved in 1963 that for all $n \geqslant 3$, there exist many knots $K \subset S^{n+2}$ with $\pi_1(S^{n+2}-K) = \mathbb{Z}$ but $\pi_2(S^{n+2} - K) \neq 0$ . The construction is an easy exercise in surgery.

At the same time, another construction method was invented by E.C. Zeeman (1963). It is a deep generalization of Artin's spinning called twist-spinning. We shall talk about it in chap. V § 4.

To close this short historical survey, we ought to mention Kinoshita's paper (1960). It gives a construction of 2-knots by pasting together discs in 4-space which is probably the unique knot construction prior to 1963 not based on spinning. There is also the somewhat related method used by R. Fox (1961a), where a 2-knot is described slice by slice, by the moving picture of its intersection with a 3-dimensional hyperplace sliding across $R^4$.

Hovewer, one cannot expect Kinoshita's nor Fox's level curve methods to be applicable in higher dimensions because they still rely on drawings and intuitive descriptions in the next lower-dimensional 3-space.

As a conclusion, let us make a few remarks :

1) The use of surgery techniques showed that multidimensional knot theory could do well without direct appeal to 3-dimensional goemetric intuition nor immediate computability. There resulted a useful kickback for classical knot theory which benefited much, since 1965, from the use of geometrical tools borrowed from higher dimensional topology and from a partial relinquisment of computational methods.

2) Around 1964, it became generally accepted that the theory of imbeddings in codimensions $\geqslant 3$ was well understood. Piecewise linear imbeddings $S^n \longrightarrow S^{n+q}$ with $q \geqslant 3$ are all unknotted by a theorem of E.C. Zeeman of 1962, published in "Unknotting combinatorial balls", Ann. of Math. 78 (1963)p.501-526. The differentiable theory was in good shape with the works, both in 1964, of J. Levine, "A classification of Differentiable Knots", Ann. of Math. 82 (1965), 15-50 on the one hand, and A. Haefliger, "Differentiable Embeddings of $S^n$ in $S^{n+q}$ for $q > 2$", Ann. of Math. 83(1966)p.402-436 on the other.

These impressive pieces of work provided a decisive encouragement to take up the certainly less tractable codimension 2 case. A lot of effort went into it and since then the growth of the subject has been so important that we cannot follow a chronological presentation. We have chosen instead to talk about articles published after 1964 in the chapters corresponding to their subject as listed in the table of contents below. Of course, at some points, whenever convenient, we did go back again to papers which appeared before this date.

For the same reason we had to delete from this survey the mention of many beautiful papers. In particular, we have mostly disregarded the papers centering around a discussion of the equivalence (or non-equivalence) of various possible definitions. We have rather tried to emphasize the moving aspect of the subject.

## § 2 . Some definitions and notations.

Do we now have to tell the reader what a knot is ?

Usually an n-knot is a codimension 2 submanifold K in $S^{n+2}$. Most of the time $S^{n+2}$ will be the standard (n+2)-dimensional smooth sphere. However, in some cases, one is forced to relax this condition. (For instance, when n+2 = 4, in order to get the realization theorems for $\pi_1$. (See Chap.II, § 3).

What K should be is a little harder to make definite. For us, it will be a locally flat, oriented, PL-submanifold of $S^{n+2}$, PL-homeomorphic to the standard n-sphere or a differential submanifold homeomorphic (or diffeomorphic) to the standard n-sphere.

The reason for such hesitations can easily be explained. The proof of the algebraic properties of the various knot invariants usually does not require a very restrictive definition of a knot. In some cases, $S^{n+2}$ could as well be replaced by a homotopy sphere and K by a homology sphere, or even less (see chap. V, § 5), sometimes not even locally flat.

On the other hand, to be able to perform geometrical constructions we usually need more restrictions. For instance the proof of the existence of a Seifert surface requires local flatness in order to get a normal bundle (which will be trivial).

Moreover, when one wants to prove realization theorems for the algebraic invariants, the stronger the restrictions on the knot definition, the better the theorems.

So we decided to let a little haze about the definition of a knot, leaving to the reader the task to get to the original papers whenever needed and see what is really required (or used).

The dimension of a knot is n if it is an n-dimensional sphere K in $S^{n+2}$. We also say an n-knot.

We refer to 1-knots as being "classical" ; n-knots with $n \geqslant 2$ are "multidimensional".

NOTATIONS :

$X_o = S^{n+2} - K$ is the complement of the knot.

X is the exterior of the knot. (See beginning of Chap. II for the definition).

bX is the boundary of X.

C denotes an infinite cyclic group, written multiplicatively.

t is a generator of C. When $C = H_1(X_o)$, t is usually chosen according to orientation conventions.

$\Lambda = \mathbb{Z}C$ is the integral group ring of C. If t has been chosen, $\Lambda$ is canonically isomorphic to the ring $\mathbb{Z}[t,t^{-1}]$.

This paper is mainly intended to topologists not working in multidimensional knot theory. As the standard joke goes : the specialist will find here nothing new, except mistakes.

Therefore, in this spirit,

1) We have often written up in some detail elementary arguments which are well known to people working in the field, but perhaps not so easy to find in the literature.

2) We did not attempt to talk about everything in the subject, but rather tried to emphasize what seems to be its most exciting aspects.

3) The latest news is often not here. Other parts of this book should fill this gap and provide references.

On the other hand, we have assumed that the reader knows some algebraic and geometric topology, and even sometimes that he is moderately familiar with classical knot theory.

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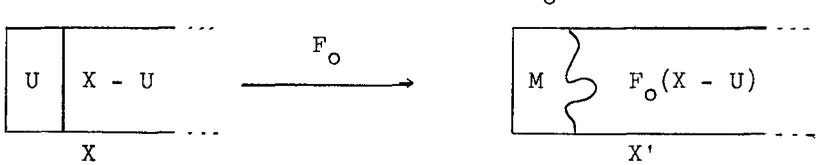
# CHAPTER II : THE COMPLEMENT OF A KNOT AS AN INVARIANT

## § 1 : Completeness theorems

The idea of distinguishing knots by the topology of their complements goes back at least to M. Dehn (Ueber die Topologie des dreidimensionalen Raumes, Math. Annalen 69 (1910), 137-168).

However, the question to decide just how close the complement comes to be a complete invariant of the knot does not seem to have occured (for higher knots) before the paper of H. Gluck in 1962.

Actually it is technically advantageous to replace the complement $X_o = S^{n+2} - K$ by the so-called exterior , that is the complement $X = S^{n+2} - N$ of an open tubular neighborhood N of K. Observe that K has trivial normal bundle $\nu$ so that N is diffeomorphic to $S^n \times D^2$ and a trivialization of $\nu$ will give an identification $N \cong S^n \times D^2$. Observe also that $X_o$ is diffeomorphic to the interior of the compact manifold X and that $bX = b\overline{N} \cong S^n \times S^1$. Thus X determines $X_o$. The converse is true at least for $n \geqslant 3$ . Since this point seems left in the dark in the printed literature, the following explanations may perhaps be helpfu Suppose X and X' are knot exteriors and let $F_o : X_o \longrightarrow X_o{}'$ be a diffeomorphism. Take a neighborhood U of bX of the form $U \cong bX \times [0,1]$, i.e. a collar. Look at the submanifold $M = X' - F_o(X-U)$.




If U has been taken narrow enough, M is contained in a collar around bX' and it is easy to construct continuous retractions of M onto each of its two boundary components $M_o = bX'$ and $M_1 = bF_o(X-U)$. Thus $M$ is an h-cobordism between $M_o$ and $M_1$. Now, $M_o$ and $M_1$ are both diffeomorphic to $S^n \times S^1$, and $\pi_1 M_o = \mathbb{Z}$. A basic theorem of differential topology, the s-cobordism theorem, now states that under these conditions tions and if $\dim M \geq 6$, then the diffeomorphism $F_o : b(X-U) \longrightarrow M_1$ can be extended to a diffeomorphism $F : X \longrightarrow X'$. (For a proof of the s-cobordism theorem see M. Kervaire, Comm. Math. Helv. 40 (1965),31-42. Here one also needs the fact that the Whitehead group of the infinite cyclic group is trivial. For this fact, see H. Bass, A. Heller and R.G. Swan, Publications mathématiques, I.H.E.S. No 22. In the case $n = 3$, the s-cobordims theorem does not apply and one needs Theorem 16.1 in C.T.C. Wall's book, Surgery on compact manifolds, p. 232).

Now, if $K$ and $K'$ are two knots and a diffeomorphism $F : X \longrightarrow X'$ is given between their exteriors, then after choosing identifications $N \cong S^n \times D^2 \cong N'$, $F$ will restrict on boundaries to a diffeomorphism $f : S^n \times S^1 \longrightarrow S^n \times S^1$. The equivalence of $K$ and $K'$ thus reduces to a question of extendability of $f$ to a (core preserving) diffeomorphism $S^n \times D^2 \longrightarrow S^n \times D^2$.

One is then led to study the group $\mathcal{D}(S^n \times S^1)$ of concordance classes of diffeomorphisms of $S^n \times S^1$ onto itself. Two diffeomorphisms $h_o$, $h_1 : M \longrightarrow M$ are concordant if there exists a diffeomorphism $h : M \times [0,1] \longrightarrow M \times [0,1]$ such that $h(x, 0) = (h_o(x), 0)$ and $h(x, 1) = (h_1(x), 1)$.

It is clear that indeed, only the concordance class of $f : S^n \times S^1 \longrightarrow S^n \times S^1$ in $\mathcal{D}(S^n \times S^1)$ matters for the extension problem at hand.

The final result is then

<u>THEOREM</u> : <u>For</u> $n > 1$, <u>there exist at most two</u> n-<u>knots with a given exterior</u>.

Sketch of proof. The group $\mathcal{D}(S^n \times S^1)$ projects onto the group of concordance classes of homeomorphisms $\mathcal{H}(S^n \times S^1)$ and it turns out that the extendability question for the above $f : S^n \times S^1 \longrightarrow S^n \times S^1$ depends only on its image in $\mathcal{H}(S^n \times S^1)$.

H. Gluck (1961) calculated $\mathcal{H}(S^2 \times S^1)$ and proved that $\mathcal{H}(S^2 \times S^1) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which means that there are at most eight 2-knots with a given exterior.

This number can however be cut down to two, as H. Gluck observed, since $\mathcal{H}(S^2 \times S^1)$ has a subgroup of order 4 generated by

a reflection on $S^2$ $\times$ the identity on $S^1$, and

the identity on $S^2$ $\times$ a reflection on $S^1$,

which both obviously extend to core preserving diffeomorphisms $S^2 \times D^2 \longrightarrow S^2 \times D^2$.

The calculation of $\mathcal{H}(S^n \times S^1)$ for $n \geqslant 5$ was achieved by W.Browder (1966) and finally completed to include the cases $n = 3$ and $n = 4$ by R.K. Lashof and J. Shaneson (1969). In all cases $\mathcal{H}(S^n \times S^1) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with generators which are the obvious generalizations of those for $n = 2$ .

It still remained the question whether inequivalent knots with diffeomorphic complements do actually exist.

Examples of such knots were more recently produced by S. Cappell and J. Shaneson (1975) in dimensions $n = 3$, $4$, (and possibly $5$) and by C. Gordon (1975) for $n = 2$.

The method of S. Cappell and J. Shaneson is general and should yield examples of non-equivalent knots with diffeomorphic complements for all $n \geqslant 3$. It stumbles for $n \geqslant 6$ on the following purely algebraic open problem : does there exist for all $n$ an automorphism $A$ of $\mathbb{Z}^{n+1}$ without any real-negative eigenvalue and with determinant $+1$ such that for all exterior powers $\lambda^i A$, $i = 1,\ldots,n$ , the endomorphism $\lambda^i A - 1 : \lambda^i Z^{n+1} \longrightarrow \lambda^i Z^{n+1}$ is again an automorphism ?

Such an A can be concocted fairly easily for $n = 3,4$ and if one finds other values of n for which A exists with the required properties, it can be fed into the machinery of S. Cappell and J. Shaneson to produce new examples of inequivalent n-knots with diffeomorphic complements.

## § 2 . Unknotting theorems

There is one case where one would certainly like the complement $X_o = S^{n+2} - K$ to determine the knot. That is the case where $X_o$ has the homotopy type of $S^1$, i.e. the homotopy type of the complement of the trivial, unknotted imbedding $K_o = S^n \subset S^{n+2}$ . Is it then true that K is isotopic to $K_o$ ?

In 1957 this was known to hold in the classical case (n = 1) as a consequence of the so-called Dehn's lemma proved by C.Papakyriakopoulos. (See Ann. of Math., 66 (1957)p.1-26).

For n = 2, this problem is still unsolved today, as far as we know.

For $n \geq 3$, it was solved by J. Stallings in 1962 for topological knots. If $K \subset S^{n+2}$ is a locally flat, topologically imbedded n-sphere with $n \geq 3$ and if $S^{n+2} - K$ has the homotopy type of $S^1$, then there exists a homeomorphism $h : S^{n+2} \longrightarrow S^{n+2}$ such that $hK = K_o$.

From the point of view of differential topology however, the major problem is whether a smooth knot $K \subset S^{n+2}$ with $S^{n+2} - K \simeq S^1$ is smoothly unknotted, i.e. whether there exists a <u>diffeomorphism</u> $h : S^{n+2} \longrightarrow S^{n+2}$ such that $hK = K_o$ .

J. Levine's paper proving this and a little more in 1964 certainly played a decisive role in getting multidimensional knot theory off the ground.

His precise result is as follows.

<u>**LEVINE'S UNKNOTTING THEOREM**</u> : - <u>Let</u> $K \subset S^{n+2}$ <u>be a smooth n-knot with</u> $n \geq 4$ <u>and let</u> $X_o = S^{n+2} - K$. <u>Suppose that</u> $\pi_i(X_o) \cong \pi_i(S^1)$ <u>for</u> $i \leq \frac{1}{2}(n+1)$ <u>Then there is a diffeomorphism</u> h <u>of</u> $S^{n+2}$ <u>onto itself such that</u> hK <u>is</u>

<u>the standard</u> n-<u>sphere</u> $S^n$ <u>in</u> $S^{n+2}$ .

The proof shows in fact that under the stated hypotheses, K is the boundary of a contractible (n+1)-manifold V smoothly imbedded in $S^{n+2}$. We come back on this in the section on Seifert surfaces. See Chap. III, § 2.

By a theorem of S. Smale, the manifold V is then diffeomorphic to a disc. (See Ann. of Math. 74 (1961) p. 391-406). Thus, under the stated hypotheses, K bounds an (n+1)-disc smoothly imbedded in $S^{n+2}$.

The remainder of the proof is then relatively easy and has to do with the equivalence of various definitions of isotopy.

There remained the case of a smooth 3-knot $K^3 \subset S^5$. It was solved by C.T.C. Wall (1965) and independently by J. Shaneson (1968). (Note that these two references are only announcements of results. For a complete proof see C.T.C. Wall's book : Surgery on compact manifolds, § 16, p. 232).

<u>Remark</u>. The reader has perhaps noticed that we have slided from the homeomorphism type of the complement to its homotopy type, in the beginning of this paragraph. The invariants we are going to talk about in the next paragraph are invariants of the homotopy type of the complement. So, the question arises whether the homotopy type determines the topology of the complement. There are several results in this direction. See S. Cappell (1969) for a discussion. Here are some striking results :

Let us treat the exterior as a pair (X, bX). Then, the homotopy type of (X, bX) determines the homeomorphism type :

1) For classical knots. This is a beautiful result due to F. Waldhausen : "On irreductible 3-manifolds which are sufficiently large"

Annals of Math. 87 (1968) p. 56-88.

2) When $n \geqslant 4$ and $\pi_1(X) = \mathbb{Z}$. See R.K. Lashof and J.L. Shaneson (1968).

## § 3 . Invariants derived from the complement.

In view of the importance of the complement $X_o = S^{n+2} - K$, or the exterior X, as an invariant of the knot, it is desirable to extract from X weaker but calculable invariants such as for example the Alexander polynomial in the case of classical knots.

The homology of X is uninteresting. By Alexander duality, $H_*(X) \cong H_*(S^1)$, and thus $H_*(X)$ is in fact independent of the knot.

It was then natural to turn attention to the homotopy groups $\pi_i(X)$ of X .

$\pi_1(X)$ was easy to understand once surgery techniques were available to perform the necessary knot constructions. (See M. Kervaire (1963)). The fundamental group $\pi$ of the complement of an n-knot, $n \geqslant 3$, is characterized by the following properties :

(1) $\pi$ is finitely presented.

(2) $H_1(\pi) = \mathbf{Z}$ , $H_2(\pi) = 0$ ,

(3) There is an element in $\pi$ whose set of conjugates generates $\pi$.

Surgery techniques (for instance ) enable one to construct an (n+2) dimensional oriented manifold M with $\pi_1(M) \cong \pi$ , and $H_i(M) = 0$ for $i \neq 0, 1, n+1, n+2$ . (For this the properties (1) and (2) of $\pi$ are used. Surgery is not essential here) .

Then one takes an imbedding $\varphi : S^1 \times D^{n+1} \longrightarrow M$ representing an element $\alpha \in \pi$ whose conjugates generate $\pi$. One constructs a new manifold $\Sigma$ by removing from M the interior of the image $\varphi(S^1 \times D^{n+1})$, say $X = M - \text{int}\, \varphi(S^1 \times D^{n+1})$, and replacing it by $D^2 \times S^n$. Since $D^2 \times S^n$

and $S^1 \times D^{n+1}$ have the same boundary $S^1 \times S^n$, it follows that $D^2 \times S^n$ can be glued to X along $S^1 \times S^n$ by the map $\varphi$ . The resulting manifold $\Sigma = X \cup \varphi(D^2 \times S^n)$ has the homotopy type of $S^{n+2}$, and for $n \geqslant 3$ is therefore homeomorphic to $S^{n+2}$ by the theorems of S. Smale (Annals of Math. 74(1961)p.391-406). Actually, with some patching up one can even assume that $\Sigma$ is diffeomorphic to $S^{n+2}$. By construction $\Sigma$ contains a beautifully imbedded n-sphere, namely the core $K = \{0\} \times S^n$ in the subspace $N = D^2 \times S^n \subset \Sigma$ . The subspace $X = \Sigma - N$ is just the exterior of the obtained n-knot $K \subset \Sigma$ and $\pi_1(X) \cong \pi_1(M) \cong \pi$ .

The construction of $\Sigma$ from M is one of the simplest examples of surgery.

For a discussion of the case n = 2, see M. Kervaire (1963) as well as J. Levine's article : "Some results on higher- dimensional knot groups" in this volume.

These references also contain some analysis of the above algebraic conditions (1), (2), (3) on a group. For further work in this direction see J.-Cl. Hausmann et M. Kervaire : "Sous-groupes dérivés des groupes de noeuds", l'Enseignement Mathématique XXIV (1978), pp. 111-123.

As to the higher $\pi_i$, $i > 1$, we have already mentionned in the introduction the papers of J.J. Andrews and M.L. Curtis (1959) and D.B.A. Epstein (1959).

More recently the subject has been taken up again. See E. Dyer and A. Vasquez (1972) and B. Eckmann (1975). Their result is that for $n > 1$, the space $X_o = S^{n+2} - K$ is never aspherical unless the knot is trivial.

Nevertheless, a complete understanding of the higher homotopy groups of knot complements seems out of reach today.

The most gratifying invariants at present are the homology modules of coverings of X and in particular those of the maximal abelian cover $\tilde{X}$ corresponding to the kernel of the surjection $\pi_1(X) \longrightarrow H_1(X)$.

These are simple enough to be tractable and yet non-trivial enough to provide a beautiful theory.

The homology modules $A_q(K) = H_q(\tilde{X})$ are modules over the integral group ring $\Lambda$ of $H_1(X)$ which operates on $\tilde{X}$ as the group of covering transformations. The group $H_1(X)$ is infinite cyclic and if we denote by t a generator of $H_1(X)$, then $\Lambda$ is the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials in t. Observe that $H_1(X)$ is generated by a fibre of the normal circle-bundle over $K \subset S^{n+2}$ and thus a choice of generator t is provided by the orientations of K and $S^{n+2}$.

We shall follow M. Hirsch and L. Neuwirth (1964) in calling $A_q(K)$ $q \geqslant 1$ , the Alexander modules of the knot or simply, following J. Levine (1974), the <u>knot modules</u>.

The general problem is : What sequences of $\Lambda$-modules $A_1, \ldots, A_n$ are modules of n-knots ? (It turns out that $A_q = 0$ for $q > n$).

Observe that $\pi_1(\tilde{X})$ is just the commutator subgroup $G = [\pi, \pi]$ of the knot group $\pi = \pi_1(X)$. Therefore $H_1(\tilde{X})$ is G/G' viewed as a group with operators from $H_1(X)$ via the extension $1 \longrightarrow G/G' \longrightarrow \pi/G' \longrightarrow H_1(X) \longrightarrow 1$. Thus $A_1(K)$ is determined by the knot group $\pi$ .

In the classical case, $A_1(K)$ is the only (non-zero) Alexander module. It possesses a square presentation matrix (over $\Lambda$ ) whose determinant is the familiar Alexander polynomial.

The fundamental group $\pi = \pi_1(X)$ influences $A_2(K)$ also . Since $G = \pi_1(\tilde{X})$, there is an exact sequence

$$\pi_2(\tilde{X}) \longrightarrow H_2(\tilde{X}) \longrightarrow H_2(G) \longrightarrow 0$$

by a celebrated theorem of H. Hopf (Fundamentalgruppe und zweite Betti'sche Gruppe, Comm. Math. Helv. 14 (1941), 257-309) and thus $A_2(K)$ must surject onto $H_2(G)$.

It may then perhaps be more appropriate to ask : what set $\{\pi, A_1, A_2, \ldots, A_n\}$ with $A_1 = H_1(G)$ and surjection $A_2 \longrightarrow H_2(G)$, $G = [\pi,\pi]$, is realizable with $\pi$ the knot group and $A_q$ the knot modules for $q = 1, \ldots, n$ ?

A start on this question with $\pi$ infinite cyclic was made by M. Kervaire (1964). The formulation (in terms of the homotopy modules of the knot complement) was however very ackward. The decisive break-through was accomplished by J. Levine (1974) which we now follow.

Let $X$ again be the exterior of a knot $K \subset S^{n+2}$. Assume $X$ is triangulated as a finite complex and let $\tilde{X}$ be the infinite cyclic covering of $X$ with the natural triangulation (such that $\tilde{X} \longrightarrow X$ is a simplicial map). We denote by $C$ the multiplicative infinite cyclic group with generator $t$ . $C$ operates on $\tilde{X}$ without fixed point and the chain groups $C_q(\tilde{X})$ are finitely generated free $\mathbb{Z}C$-modules.

Since $\Lambda$ has no divisors of zero, the multiplication by 1-t induces an injection $1-t : C_*(\tilde{X}) \longrightarrow C_*(\tilde{X})$. The quotient module is (canonically) isomorphic to the chain group of $X$ (regarded as $\Lambda$-module with trivial action) and we get an exact sequence of complexes :

$$0 \longrightarrow C_*(\tilde{X}) \xrightarrow{1-t} C_*(\tilde{X}) \longrightarrow C_*(X) \longrightarrow 0 .$$

Passing to the associated long homology sequence

$$\cdots \longrightarrow H_{q+1}(X) \longrightarrow H_q(\tilde{X}) \xrightarrow{1-t} H_q(\tilde{X}) \longrightarrow H_q(X) \longrightarrow \cdots$$

in which $H_q(X) = 0$ for $q > 1$ by Alexander duality, one obtains that

$$1 - t : H_q(\tilde{X}) \longrightarrow H_q(\tilde{X})$$

is an isomorphism for $q \geqslant 2$ . Inspection of the sequence near $q = 1$, i.e.

$$0 \to H_1(\tilde{X}) \xrightarrow{1-t} H_1(\tilde{X}) \longrightarrow H_1(X) \longrightarrow H_o(\tilde{X}) \xrightarrow{1-t} H_o(\tilde{X})$$

reveals that $1\text{-}t : H_1(\tilde{X}) \longrightarrow H_1(X)$ is also an isomorphism.

Following J. Levine (1974), we shall say that a $\Lambda$-module A is <u>of type</u> K if

(1) A is finitely generated (over $\Lambda$), and

(2) $1\text{-}t : A \longrightarrow A$ is an isomorphism.

We have just seen that all knot modules are of type K.

Of course, one cannot expect this property to characterize the Alexander modules of knots.

It is a remarkable theorem of J. Levine that there is however just one property missing : Blanchfield duality. (Except perhaps for a condition on the $\mathbb{Z}$-torsion submoduble of $A_1$).

In order to understand Blanchfield duality, recall that an oriented, triangulated, m-dimensional manifold M possesses an intersection pairing

$$I : C_q(M, bM) \otimes C_{m-q}(M^*) \longrightarrow \mathbb{Z} ,$$

where $C.(M^*)$ is the chain complex of the dual cellular subdivision $M^*$ of M. If M is compact, this gives rise to Poincaré duality. Here, we shall take $M = \tilde{X}$, the infinite cyclic cover of the exterior of a knot $K \subset S^{n+2}$. Of course, $\tilde{X}$ is non-compact but $C \cong H_1(X)$ operates on

$\tilde{X}$ simplicially with compact quotient $X$.

One first uses the action of $C$ on $\tilde{X}$ to construct a $\Lambda$-valued intersection pairing on $\tilde{X}$

$$C_q(\tilde{X},\ b\tilde{X}) \otimes C_{n+2-q}(\tilde{X}^*) \longrightarrow \Lambda = \mathbb{Z}C$$

defined by

$$(x,y^*) = \Sigma_{s\epsilon C}\ I(x,\ sy^*)s\ .$$

This construction actually goes back to K. Reidemeister (Durchschnitt und Schnitt von Homotopieketten, Monathefte Math. 48(1939), 226-239).

The above pairing has nice algebraic properties and because $X=\tilde{X}/C$ is a finite complex, it is a completely orthogonal pairing and one gets an isomorphism

$$H_q(\tilde{X},\ b\tilde{X}) \cong H^{n+2-q}(\tilde{X},\Lambda).$$

The left hand side is the ordinary homology of the pair $(\tilde{X},\ b\tilde{X})$ with integral coefficients. The right hand side is the cohomology of the complex $\mathrm{Hom}_\Lambda(C.(\tilde{X}^*),\Lambda)$. The isomorphism is an isomorphism of $\Lambda$-modules provided that $H^{n+2-q}(\tilde{X},\Lambda)$ is given its natural right-module structure and $H_q(\tilde{X},\ b\tilde{X})$ is turned into a right module by the usual formula $x\ .\ \lambda = \overline{\lambda}\ .\ x$, $\lambda \epsilon \Lambda$, where $\lambda \longmapsto \overline{\lambda}$ is the obvious involution on $\Lambda$ sending the elements of $C$ to their inverses.

An elegant reformulation due to J. Levine (1974) using $H_q(\tilde{X},\ b\tilde{X}) = H_q(\tilde{X})$ for $0 < q \leqslant n$ and some non-trivial homological algebra yields the following statements.

Recall $A_q = H_q(\tilde{X})$. Let $T_q$ be the $\mathbb{Z}$-torsion submodule of $A_q$ and $F_q =$ $= A_q/T_q$ . Then,

(1) There is a $(-1)^{q(n-q)}$-hermitian completely orthogonal pairing

$$F_q \otimes F_{n-q+1} \longrightarrow Q(\Lambda)/\Lambda$$

over $\Lambda$, where $Q(\Lambda)$ is the field of fractions of $\Lambda$ and $Q(\Lambda)/\Lambda$ is the quotient $\Lambda$-module. (Note that $Q(\Lambda)$ is merely the field of rational fractions $\mathbb{Q}(t)$. The hermitian property of the pairing is of course with respect to the involution of $\Lambda$ defined above).

(2) There is a $(-1)^{q(n-q)}$-symmetric completely orthogonal pairing

$$[\ ,\ ] : T_q \otimes T_{n-q} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

with respect to which C operates by isometries, i.e.

$$[t\alpha,\ t\beta] = [\alpha,\beta] \ .$$

This second pairing has also been discovered by M.Š. Farber(1974).

Now, J. Levine's realization theorem reads as follows.

THEOREM. - <u>Given a sequence</u> $A_1, \ldots, A_n$ <u>of</u> $\Lambda$-<u>modules of type</u> K. <u>Let</u> $T_q$ <u>be the torsion submodule of</u> $A_q$ <u>and</u> $F_q = A_q/T_q$ . <u>Suppose that</u> $T_1 = 0$ <u>and that the families</u> $F_q$ <u>and</u> $T_q$ <u>are provided with pairing as in</u> (1) <u>and</u> (2) <u>above. Then, there exists an</u> n-<u>knot</u> K <u>such that</u> $A_1, \ldots, A_n$ <u>is the sequence of Alexander modules of</u> K.

Hopefully the unfortunate assumption $T_1 = 0$ will turn out to be removable. It is known that this assumption is not a necessary condition on $T_1$.

# CHAP. III : TOWARDS A CLASSIFICATION UP TO ISOTOPY .

## § 1 . Seifert surfaces.

A basic concept for any attempt at classification is that of a Seifert surface.

A Seifert surface for an n-knot K is a compact, orientable submanifold $V \subset S^{n+2}$, such that $bV = K$.

The fact that V should be orientable is important and was first emphasized by H. Seifert (1934) who introduced the concept and proved existence in the classical case.

For multidimensional knots, the existence of a Seifert surface seems to have become public knowledge during the Morse Symposium at Princeton in 1963. (However, H. Gluck had proved it earlier for 2-knots. See H. Gluck (1961)). It appears in print in M. Kervaire (1963) and E.C. Zeeman (1963).

Here is a sketch of proof. Recall that a trivialization of the normal bundle of the knot K provides an identification $bX \cong S^n \times S^1$, and thus a projection $bX \longrightarrow S^1$.

The first step consists in showing that with a proper choice of trivialization above, the projection $bX \longrightarrow S^1$ extends to a map $X \longrightarrow S^1$. This is not difficult. The homotopy classes of maps into $S^1$ are classified by the first cohomology group $H^1$ with integral coefficients and one has enough control on both $H^1(X) \cong \mathbb{Z}$ and the restriction homomorphism $i^* : H^1(X) \longrightarrow H^1(bX)$ .

The existence of a Seifert surface now follows by transversality. One chooses an extension $X \longrightarrow S^1$ which is transverse regular to the point $1 \in S^1$. The inverse image of 1 is then a codimension one submanifold W in $S^{n+2}$ equipped with a non-vanishing normal vector field (pulled back from a tangent vector to $S^1$ at 1). Hence, W is orientable. The boundary of W is precisely $S^n \times \{1\} \subset bX$. We can then add a collar to W, joining bW to K along the radii of the normal bundle to K and get the submanifold V we are looking for.

Many constructions in knot theory depend on a Seifert surface. We collect in this section some of the notions derived from a Seifert surface which we shall need in the subsequent chapters of this survey (even though they may not pertain directly to the subject of the present chapter).

First, a Seifert surface enables one to perform a paste and scissors construction of the infinite cyclic cover of a knot.

Let V be a Seifert surface for the knot K. Let N be an open tubular neighborhood of K and set $X = S^{n+2} - N$. We assume V to be radial inside N and set $W = V \cap X$.

Let Y be the manifold with boundary obtained by cutting X along W. Equivalently, Y is obtained from X by removing a small tubular neighborhood of W, homeomorphic to $W \times [-1, +1]$. Notice that it is here that the orientability of V comes in.

The boundary of Y is the union of two copies of W, i.e. $W \times \{-1\} = W_-$ and $W \times \{+1\} = W_+$ together with $bW \times I$, where $I = [-1, +1]$. These pieces are glued together to form bY in the obvious way.

Notice also that there is a natural projection map $\pi$ from Y onto X which sends the two copies of W onto W and is otherwise injective.

(Glue again what you had cut ! ).

Now, let $\{Y_i\}_{i \in \mathbb{Z}}$ be a collection of copies of Y, indexed by the integers $\mathbb{Z}$ . Let $\tilde{X}$ be the quotient of the disjoint union $\coprod_{i \in \mathbb{Z}} Y_i$ by the obvious identification of $(W_-)_i$ with $(W_+)_{i+1}$ for all $i \in \mathbb{Z}$ . The maps $\pi_i : Y_i \longrightarrow X$ are compatible with these identifications and provide a map $p : \tilde{X} \longrightarrow X$.

It is not hard to verify that p is a covering map. The covering is regular and its Galois group is C. (We denote by C the group of integers written multiplicatively).

Hence, $p : \tilde{X} \longrightarrow X$ is "the" infinite cyclic covering of X.

This construction has been used by L.P. Neuwirth (1959) to give a description of the knot group. It is also the first step in proving the Neuwirth-Stallings fibration theorem. (We come back on this in the chapter on fibered knots, Chapter V, § 3).

The above description of the infinite cyclic cover leads of course to a computation of the homology of this covering by a Mayer-Vietoris sequence. (See M. Hirsch and L. Neuwirth (1964)).

Indeed, let $\tilde{X}_{odd}$ be the subspace of $\tilde{X}$ which is equal to the canonical image of $\coprod_{i \text{ odd}} Y_i$ in $\tilde{X}$, and let $\tilde{X}_{even}$ be the analogous subspace for i even. Obviously $\tilde{X}_{odd} \cup \tilde{X}_{even} = \tilde{X}$ and $\tilde{X}_{odd} \cap \tilde{X}_{even} = \coprod_{i \in \mathbb{Z}} W_i$, $W_i$ being identified with $(W_-)_i$, say.

Let now $H_j$ denote homology with some fixed coefficient group and let $\Lambda = \mathbb{Z}\, C$ be the integral group ring of C. One has

$$H_j(\tilde{X}_{odd}) \oplus H_j(\tilde{X}_{even}) = H_j(Y) \otimes \Lambda ,$$

$$H_j(\textstyle\coprod_{i \in \mathbb{Z}} W_i) = H_j(W) \otimes \Lambda ,$$

the isomorphisms being $\Lambda$-isomorphisms, C acting on the left hand side via the Galois operations.

The Mayer-Vietoris sequence for the decomposition $\tilde{X} = \tilde{X}_{odd} \cup \tilde{X}_{even}$ produces the following exact sequence :

$$(*) \quad \ldots \longrightarrow H_j(W) \otimes \Lambda \xrightarrow{\alpha} H_j(Y) \otimes \Lambda \xrightarrow{\beta} H_j(\tilde{X}) \longrightarrow \quad \ldots \quad .$$

The homomorphisms are all $\Lambda$-modules homomorphisms.

Moreover, if we denote by $i_+$ the homomorphisms $H_j(W) \longrightarrow H_j(Y)$ induced by the inclusion $W_+ \subset Y$, and similarly with $i_-$, then

$$\alpha(x \otimes \lambda) = i_+(x) \otimes t\lambda \ - i_-(x) \otimes \lambda \ ,$$

the minus sign coming from the Mayer-Vietoris sequence. Here, $t$ is a correctly chosen generator for $C$ .

Caution. Different identifications in the construction may lead to slightly different formulas.

A useful fact, due to J. Levine, is that this sequence always breaks up into short exact sequences

$$0 \longrightarrow H_j(W) \otimes \Lambda \xrightarrow{\alpha} H_j(Y) \otimes \Lambda \xrightarrow{\beta} H_j(\tilde{X}) \longrightarrow \ 0 \ .$$

In some circumstances, we may thus be on the way to get a free resolution of the module $H_j(\tilde{X})$. See J. Levine (1976).

Remarks.

1. For a very nice application of this sequence to the symmetry properties of the Alexander polynomials, see also J. Levine (1966).

2. A variant of this process gives a description of the g-th cyclic covering $X_g$ of $X$, $g$ an integer $> 1$. Alternatively, $X_g$ can be obtained as a quotient of $X$ via the automorphism $t^g$, where $t$ is a generator of

the Galois group. One then gets for the homology of $X_g$ a sequence analogous to the one described above in chap. II, § 3, p. 21.

A notion of paramount importance for all classification problems of knots is that of the Seifert pairing associated with a Seifert surface for an odd dimensional knot.

This notion was introduced in the classical case by H. Seifert (1934). We proceed to describe it in general.

Let $K \subset S^{2m+1}$ be a $(2m-1)$-knot. Choose a trivialization of the normal bundle of a (truncated) Seifert surface W for the knot K. The trivialization determines a map

$$i_+ : W \longrightarrow Y ,$$

where Y, as above, is the complement of a neighborhood of V.

There is a pairing

$$L : H_m(W) \times H_m(Y) \longrightarrow \mathbb{Z}$$

defined by the linking number in $S^{2m+1}$. Now, define

$$A : H_m(W) \times H_m(W) \longrightarrow \mathbb{Z}$$

by the formula $A(x, y) = L(x, i_+(y))$.

Observe that A is bilinear and thus vanishes on the torsion subgroup of $H_m(W)$.

We note $F_m$ the free part of the integral homology $H_m$, i.e. $F_m = H_m/\text{Torsion}$.

Since $H_m(V) = H_m(W)$, we have obtained a bilinear pairing

$$A : F_m(V) \times F_m(V) \longrightarrow \mathbf{Z} \quad .$$

By definition, A is called the Seifert pairing associated with the Seifert surface V .

In general, there is no symmetry nor non-degeneracy properties satisfied by A itself. However, let $A^T$ denote the transpose of A. One has

$$\begin{aligned} (A + (-1)^m A^T)(x, y) &= L(x, i_+ y) + (-1)^m L(y, i_+ x) \\ &= L(x, i_+ y) - L(x, i_- y) \\ &= L(x, i_+ y - i_- y) \ , \end{aligned}$$

and this is equal to the intersection number of x and y in V .

So, $A + (-1)^m A^T = I$ is the intersection pairing on $F_m(V) = H_m(V)/\text{Torsion}$. Since bV is a sphere, Poincaré duality on V implies that $A + (-1)^m \mathbf{A}^T$ is unimodular.

We shall come back to the study of the Seifert pairing in § 3 below in the case of simple knots, and in Chap. IV again, where we talk about knot cobordism.

## § 2 . Improving a Seifert surface.

For a given knot, there are many possible Seifert surfaces. The surfaces may be abstractly different (non homeomorphic), or abstractly the same but imbedded differently. (However, the existence proof shows that they are all cobordant).

It is hence natural to look for Seifert surfaces which are "minimal in some sense. For classical knots, it is clear what "minimal" should mean : V should be connected and its genus as small as possible. But, for multidimensional knots, the notion is not so clear, except under special circumstances (such as for the odd dimensional simple knots which we discuss in § 3 below).

We shall now review some cases in which one can "improve" or "simplify" a Seifert surface. The main point is that there is a strong connection between the connectivity of $\widetilde{X}$ and the best possible connectivity of a potential Seifert surface.

a) For all $n \geqslant 1$, if a knot has a 1-connected Seifert surface, then $\widetilde{X}$ is 1-connected. The first proof of this fact is due to M. Hirsch and L. Neuwirth (1964) and it goes as follows : if V is 1-connected, then by van Kampen, $\pi_1(X) \cong \pi_1(Y) * \mathbb{Z}$ and a generator z of $\mathbb{Z}$ represents a meridian of the knot. It follows that the normal closure of z in $\pi_1(X)$ should be the entire group. (Compare the characterization of knot groups in Chap. I, § 3). We see immediately that this is possible only if $\pi_1(Y) = \{1\}$, and thus $\pi_1(X) = \mathbb{Z}$ .

Caution . It is essential in this proof to be able to identify a generator of the factor $\mathbb{Z}$ as a meridian of the knot. The question whether in general a free product $G * \mathbb{Z}$ with $G \neq \{1\}$ , may or may not contain an element whose normal closure is the whole group is still an unsolved problem.

b) The converse of a) is almost true. In fact, M. Hirsch and L. Neuwirth (1964) proved by an argument of exchange of handles that if $\pi_1(\tilde{X}) = \{1\}$ and if $n \geqslant 3$, then there exists a 1-connected Seifert surface for the knot.

The case $n = 1$ is also true. (Dehn's lemma). So there remains only the case $n = 2$ which is still open.

c) By the above case a), Alexander duality and the homology exact sequence (*) of the preceeding paragraph one sees immediately that if there exists a k-connected Seifert surface for a knot, then $\tilde{X}$ is also k-connected.

d) Now, again the converse is almost true. But this is the content of a deep theorem of J. Levine (1964). For clarity we separate the statements in two parts :

Part. 1 : Let $n \geqslant 2k+1$ and suppose that $\tilde{X}$ is k-connected. Then, there exists a k-connected Seifert surface for the knot.

Part 2 : Let $n = 2m$ or $n = 2m-1$ and suppose that $\tilde{X}$ is m-connected. Then, if $n \geqslant 4$, there exists a m-connected Seifert surface V for the knot.

Observe that by Blanchfield duality the condition on $\tilde{X}$ in Part 2 is equivalent to $\tilde{X}$ being contractible. Similarly, Poincaré duality and the Hurewicz theorem imply that the Seifert surface V in Part 2 must be contractible.

These statements constitute the essential part of J. Levine's unknotting theorem. Suppose that X has the homotopy type of $S^1$ and that $n \geq 4$. Then $\widetilde{X}$ is contractible and so K bounds a contractible Seifert surface V. By S. Smale, V is a P.L. disk and so K is P.L. unknotted. If $n \geq 5$ and K is differentiable, then K is differentiably unknotted and so has the standard differential structure.

## § 3. Simple knots.

In view of Levine's unknotting theorem, it is natural to study the n-knots which are "almost" trivial ; that is those for which $\pi_i(\widetilde{X}) = 0$ for $i < m$ with $n = 2m$ or $n = 2m-1$. These knots have been called simple by J. Levine. Their study breaks up into two cases, depending upon the parity of n .

### First case : n odd.

This case has been much studied by J. Levine (1969). We describe now the content of his paper.

By the statement under d), Part. 1, in the preceeding paragraph, one can find for any simple knot $K \subset S^{2m+1}$ a (m-1)-connected Seifert surface V. As dim V = 2m the only non-trivial homology group of V is $H_m(V)$, where we use integer coefficients.

It is not difficult, using Poincaré duality and the parallelizability of V in the case m even, to see that $H_m(V)$ is a free abelian group of even rank. Moreover, for $m \neq 2$, the conditions we have on V imply that V is obtained from a 2-dimensional disk by attachning handles of type m. (See C.T.C. Wall : "Classification of (n-1)-connected 2n-manifolds" in Annals of Math. 75 (1962), p. 163-198).

So, odd dimensional simple knots have a tendency to look like classical knots. For instance, it is obvious how to define a minimal Seifert surface V for them : V should be (m-1)-connected and the rank of $H_m(V)$ as small as possible.

In order to classify odd dimensional simple knots, J. Levine undertakes to classify all (m-1)-connected Seifert surfaces whether minimal or not, which are associated to such a knot.

It turns out that the Seifert pairing does the job. Let K be a simple (2m-1)-knot and let $V = V^{2m}$ be a (m-1)-connected Seifert surface for K. Since $H_m(V)$ is torsion free, the Seifert pairing is a bilinear map

$$A : H_m(V) \times H_m(V) \longrightarrow \mathbb{Z}$$

such that $A + (-1)^m A^T$ is $(-1)^m$-symmetric and unimodular.

THEOREM . For $m \geq 3$, the isotopy class of an (m-1)-connected Seifert surface V for a simple (2m-1)-knot is determined by its associated Seifert pairing.

For a proof, see J. Levine (1969), p. 191, sections 14 to 16.

Furthermore, using the fact that two Seifert surfaces for the same knot are cobordant, J. Levine shows :

Fact 1 : For $m \geq 1$, any two Seifert pairings for a given knot are S-equivalent.

S-equivalence is the equivalence relation generated by isomorphisms and by the following elementary operations : replace the underlying $\mathbb{Z}$-module H by $H \times \mathbb{Z} \times \mathbb{Z}$ and A by A' or A", where A', A" are expressed matricially by

$$A' = \left(\begin{array}{c|cc} & 0 & 0 \\ & \vdots & \vdots \\ A & \vdots & \vdots \\ & \vdots & \vdots \\ & 0 & 0 \\ \hline * \; \dots \; * & 0 & 0 \\ 0 \; \dots \; 0 & 1 & 0 \end{array}\right) \qquad A'' = \left(\begin{array}{c|cc} & * & 0 \\ & \vdots & \vdots \\ A & \vdots & \vdots \\ & \vdots & \vdots \\ & * & 0 \\ \hline 0 \dots 0 & 0 & 1 \\ 0 \dots 0 & 0 & 0 \end{array}\right)$$

Fact. 2 : Suppose $m \geqslant 2$ . Let K and K' be two simple (2m-1)-knots, each equipped with a (m-1)-connected Seifert surface. Suppose that the two corresponding Seifert pairings are S-equivalent. Then the two knots are isotopic.

This is of course the most difficult part of the theory. It relies heavily on the classification of Seifert surfaces described in the above theorem.

Definition : Given an integer $m \neq 2$, define a Seifert form (for m) to be a bilinear form

$$A \quad : \quad E \times E \longrightarrow \mathbb{Z}$$

on a finitely generated free $\mathbb{Z}$-module E such that $A + (-1)^m A^T$ is unimodular.

For $m = 2$, observe that the Seifert surface is a smooth, parallelizable 4-manifold, with boundary a sphere, and therefore, by V.Rochlin's theorem its intersection pairing has a signature divisible by 16. (For V. Rochlin's theorem, see J. Milnor and M. Kervaire, Bernoulli numbers, Homotopy groups and a theorem of Rochlin, Proc. of the Int. Congress of Math., 1958, p. 454-458). Thus, for $m = 2$, a Seifert form will be defined as a bilinear map A as above subject to the additional condition that signature $(A + A^T) \equiv 0 \bmod 16$.

We can now state the last needed fact.

Fact 3 : Given a Seifert form A for m . Then, if $m \neq 2$, there exists a (m-1)-connected (orientable) submanifold $V^{2m} \subset S^{2m+1}$ such that bV is homeomorphic to the (2m-1)-sphere and A is the associated Seifert pairing. For $m = 2$, the same statement holds, except that now A is only S-equivalent to the Seifert pairing of the constructed Seifert surface V.

In the classical case ($m = 1$), this fact is due to H. Seifert himself. For multidimensional knots, see M. Kervaire (1964) in the case $m \neq 2$, and J. Levine (1969) in general.

Putting all these facts together, J. Levine obtains the theorem :

For $m \geqslant 2$, the isotopy classes of simple $(2m-1)$-knots are in one-to-one correspondence with the S-equivalence classes of Seifert forms.

In the classical case ($m = 1$), the isotopy classes of 1-knots are mapped onto the set of S-equivalence classes of Seifert forms. This fact was known already to H. Trotter (1960) and to K. Murasugi (1963). But the mapping is not injective. For instance, knots with trivial Alexander polynomial are mapped into the trivial S-equivalence class.

Remarks : From J. Levine's theorem, the set of simple 5-knots is isomorphic to the set of 9-knots, to the set of 13-knots, etc... The bijection is well defined. So, it is natural to ask whether one can define this bijection directly. In the case of fibered knots, such a construction is provided by L. Kauffman and W. Neumann (1976).

Let K be a simple $(2m-1)$-knot. Look at the set of all its minimal Seifert surfaces. Question : Are all these surfaces isotopic ? If they are, we would say that the minimal Seifert surface for K is (essentially unique.

By J. Levine's theorem this question can now be attacked algebraically. Look at the (minimal) Seifert pairing associated with the minimal surfaces. We know that they are all S-equivalent. But if the answer to the question is "yes", they should all be isomorphic ($m \geqslant 1$). Conversely, for $m \geqslant 2$, if they are isomorphic, the Seifert surfaces are isotopic. Thus the problem is to determine the isomorphism classes of of Seifert forms within a given S-equivalence class. This algebraic problem has been attacked by H. Trotter in several papers (1960), (1970) and (1972). Sometimes the S-equivalence class determines the isomorphism class, sometimes it does not. Sometimes the answer is unknown. The problem involves the determinant of a minimal Seifert pairing (which is an invariant of the knot and therefore of the S-equivalence class of the Seifert form). As an example, there is only one isomorphism class in the given S-equivalence class if this determinant is $\pm 1$, a result which can be interpreted (and proved) geometrically, using fibered knots.

H. Trotter's papers give also nice answers to other old questions. For instance, it is easy to see that if we change the orientation of the knot, we must also change the orientation of the Seifert surface because K and V are given orientations which correspond each other via the homology exact sequence, and then, the normal vectors to V have to change direction. It is then easy to see that the initial Seifert form is changed into its transposed (up to a sign which seems to be $(-1)^{m+1}$). H. Trotter then gives examples of Seifert forms which are not S-equivalent to their transpose, showing thus that non-invertible knots exist for $m \geq 1$. For $m = 1$ this is the famous result first proved by H.Trotter (1963).

For $m \geq 2$, it is also rather nice, because it is not based on the non-symmetry of some Alexander invariant. Related reference : C. Kearton (1974).

Second case : n even . The case of even dimensional simple knots is much harder than the first case because there is no such simple algebraic invariant as the Seifert pairing. There is only a complicated invariant consisting of a composite algebraic object. However, the classification has almost been completed by C. Kearton (1975). The problem has also been taken up by S. Kojima (1977) and A. Ranicki(1977).

## § 4 . Seifert pairings and the infinite cyclic covering .

As may perhaps be expected, the Seifert pairing is related to the calculation of $H_*(\tilde{X})$ using a Seifert surface as explained in § 1.

Let $K \subset S^{2m+1}$ be an odd dimensional simple knot and V a (m-1)-connected Seifert surface for K .

Recall that the Alexander duality gives an isomorphism

$$d \; : \; H_m(Y) \longrightarrow H^m(W) \; ,$$

where we keep the notations of § 1. (W is the truncated Seifert surface and Y is the exterior X of the knot cut along W).

Because W is (m-1)-connected, the evaluation map

$$e \; : \; H^m(W) \longrightarrow \mathrm{Hom}(H_m(W), \mathbb{Z})$$

is also an isomorphism.

Now, let $a \in H_m(V)$ and $b \in H_m(Y)$ be given. Then $L(a,b)$ is by definition the integer obtained by evaluating on $a \in H_m(V)$ the homomorphism $ed(b)$. (Recall $H_m(V) = H_m(W)$) .

In other words, the (right) adjoint to $L : H_mV \times H_mY \longrightarrow \mathbb{Z}$ is precisely ed.

So, the adjoint to $A : H_mV \times H_mV \longrightarrow \mathbb{Z}$ is $edi_+ : H_m(V) \longrightarrow \mathrm{Hom}(H_m(V),\mathbb{Z})$.

As e and d are (canonical) isomorphisms, we see that the algebraic properties of $i_+$ will be reflected by those of A.

Now, if we start from a $\mathbb{Z}$- basis of $H_m(V)$, we can take the dual basis for $\mathrm{Hom}(H_m(V), \mathbb{Z})$ and get via d and e a basis for $H_m(Y)$.

With respect to these bases, the matrix expressing the bilinear form A will be precisely the matrix for the map $i_+ : H_m(V) \longrightarrow H_m(Y)$.

Returning to the short exact sequence

$$(*) \quad 0 \longrightarrow H_m(V) \otimes \Lambda \xrightarrow{\alpha} H_m(Y) \otimes \Lambda \xrightarrow{\beta} H_m(\tilde{X}) \longrightarrow 0$$

of § 1, with integer coefficients, the $\mathbb{Z}$-bases for $H_m(V)$ and $H_m(Y)$ give $\Lambda$-bases for the tensor products.

It now follows from the formula

$$\alpha\ (\chi \otimes \lambda) = i_+(x) \otimes t\lambda - i_-(x) \otimes \lambda$$

of § 1, that the matrix expressing $\alpha$ is $At + (-1)^m A^T$ .

So, from a Seifert matrix for K (i.e. the matrix of a Seifert pairing for K), one can get a presentation matrix for $H_m(\tilde{X})$. For classical knots this result is due to H. Seifert.

We now consider the Blanchfield pairing on $H_m(\tilde{X})$, still assuming that K is a simple (2m-1)-knot. A study of the above exact sequence (*) with various coefficients reveals that for simple knots, $H_m(\tilde{X})$ is $\mathbb{Z}$-torsion free. See, for instance, the thorough study made by J. Levine (1976), § 14.

So, the Blanchfield pairing reduces to a pairing

$$H_m(\tilde{X}) \times H_m(\tilde{X}) \longrightarrow Q(\Lambda)/\Lambda$$

(Compare § 3 in Chap. I).

Now, H. Trotter (1972) and C. Kearton (1973) have shown that this Blanchfield pairing is determined by the Seifert form in the following way. Let us take as generators for $H_m(\tilde{X})$ the images by $\beta : H_m(Y) \otimes \Lambda \longrightarrow H_m(\tilde{X})$ of the basis elements chosen for $H_m(Y) \otimes \Lambda$ . Of course, they do not form a basis for $H_m(\tilde{X})$, but there still is a matrix representative of the Blanchfield pairing with respect to this set of generators, and it is

$$(1-t)(At + (-1)^m A^T)^{-1} \ .$$

(See the exposition in J. Levine (1976), prop. 14.3).

Again, different conventions will lead to slightly different formulas.

This result is the starting point of H. Trotter's paper (1972). More precisely, to every free abelian group equipped with a Seifert form A, H. Trotter associates a $\mathbb{Z}C$-module with presentation matrix $At + (-1)^m A^T$ and equipped with a Blanchfield pairing represented by the matrix $(1 - t)(At + (-1)^m A^T)^{-1}$ .

He then goes on to prove that

(1) S-equivalent Seifert forms give rise to isomorphic Blanchfield pairings, and the deep result :

(2) If two Seifert forms give rise to isomorphic Blanchfield pairings, then they are S-equivalent.

A nice geometric consequence of this result is that simple $(2m-1)$-knots (for $m \geqslant 2$) are classified by their Blanchfield duality. This furnishes an intrinsic classification for these knots. The same result has also been proved by C. Kearton (1973).

An interesting question, asked by C. Kearton, and which provides our conclusion to this chapter, is whether the same is true for simple even dimensional knots. Possibly, A. Ranicki will tell you the answer.

# CHAP. IV : KNOT COBORDISM.

## § 1 : Prehistory .

The notion of knot cobordism was invented in the context of classical knots around 1954 by R. Fox and J. Milnor.

An announcement appeared in 1957 but the paper itself (with simplified proofs) was only published in 1966.

Knot cobordism is a weaker equivalence relation between knots than isotopy and part of the motivation for introducing it certainly is the discouraging difficulties involved in the classification up to isotopy. But there is another motivation. The idea of knot cobordism is also related to the topological study of isolated codimension two singularities.

Suppose that $M^{n+1} \subset N^{n+3}$ is an embedded submanifold which is locally flat except at one point $x_o \in M$. Intersecting M with the boundary of a small disk neighborhood U of $x_o$ in N will yield a (knotted)sphere K of dimension n in $bU = S^{n+2}$. Thus an n -knot.

Definition : A knotted n - sphere $K \subset S^{n+2}$ is null-cobordant if K is the boundary of a locally flat embedded disk $B \subset D^{n+3}$.

The requirement of local flatness for B is of course essential, or else the cone over K from the center of $D^{n+3}$ would trivially do the job.

It has been believed that at least for n = 1 , the singularity at the vertex of the cone may be removable, yielding a null-cobordism.

This is definitely not the case. In fact we shall see that it is even worse than its higher dimensional analogues.

Going back to the embedded $M^{n+1} \subset N^{n+3}$ with $K = M \cap bU$ as defined above, it is clear that if the knot $K \subset S^{n+2}$ is null-cobordant, then the embedding $M \subset N$ can be replaced near $x_o$ by a locally flat embedding of M.

Conversely, if the embedding $M \subset N$ can be changed near $x_o$ within some neighborhood of $x_o$ in N, to produce a locally flat embedding, the above knot was null-cobordant.

Thus in some sense, the local singularity of M at $x_o$ is described by the knot cobordism class of K.

An additional pleasant feature is that the set of cobordism classes has nicer algebraic properties than the set of isotopy classes. The set $K_n$ of isotopy classes of n-knots forms a commutative monoid under ambiant connected sum (joining the knotted spheres by a tube). It turns out that modulo null-cobordism, the quotient monoid actually is an abelian group $C_n$. (Incidentally, it does not seem to have attracted attention to investigate whether or not $C_n$ is in any sense the largest quotient group of $K_n$).

R. Fox and J. Milnor looked at $C_1$ and after proving that the Alexander polynomial of a null-cobordant knot must be of the form $t^g f(t)f(1/t)$ for some polynomial $f \in \mathbb{Z}[t]$, they recognized that $C_1$ could not be finitely generated.

A good surprise came with the simple result

$$C_{2m} = 0 \qquad \text{for all} \quad m \geqslant 1 ,$$

proved in M. Kervaire (1964). But it soon appeared that (contrary to tempting dreams) the groups $C_{2m-1}$ are indeed non-finitely generated for all $m \geq 1$.

Much effort was then devoted to the rather formidable task of computing $C_{2m-1}$.

§ 2. The algebraization of the problem.

By analysing the obstructions which arise if one tries to apply to the odd dimensional case the surgery methods used to calculate $C_{2m}$, J. Levine (1969) extracted a purely algebraic description of $C_{2m-1}$ which again hinges on the concept of a Seifert form.

Recall from Chap. III, § 1, that a Seifert form for m is a bilinear pairing $A : E \times E \longrightarrow \mathbb{Z}$ on some finitely generated free $\mathbb{Z}$-module E such that $A + (-1)^m A^T$ is unimodular. (If m = 2, there is also a condition on the signature).

If $K \subset S^{2m+1}$ is a (2m-1)-knot and V is a Seifert surface for K, then the Seifert pairing

$$A \;:\; F_m(V) \times F_m(V) \longrightarrow \mathbb{Z}$$

on the torsion free part $F_m(V)$ of $H_m(V)$ is a Seifert form.

Moreover, by our discussion of simple knots in Chap. III, § 3, Fact 3, every Seifert form is (essentially) the Seifert pairing of a (simple) knot.

The first step is to carry over to Seifert forms the notion of cobordism.

Definition : A Seifert form $A : E \times E \longrightarrow \mathbb{Z}$ is said to be null-cobordant (or split) if there exists a totally isotopic subspace $E_o \subset E$ such that $E_o = E_o^{\perp}$, where

$$E_o^{\perp} = \{ x \in E \mid A(y,x) = A(x,y) = 0 \text{ for all } y \in E_o \} .$$

It turns out that the monoid of Seifert forms (for a given m) modulo null-cobordant forms becomes a group under the operation of direct sum.

Of course the definition of this group resembles much the definition of the Witt group of $\mathbb{Z}$ . But here the forms A are not assumed to be symmetric, and the resulting group is tremendously more complicated than $W(\mathbb{Z})$.

J. Levine's theorem says that for a given $m \geq 2$, the group of Seifert forms modulo split forms is isomorphic to the knot cobordism group $C_{2m-1}$ of (2m-1)-knots.

For $m = 1$, there is a surjection of $C_1$ onto the cobordism group of Seifert forms. But it is known that the kernel is non-zero. (Compare C. Gordon's survey of classical knot theory in this volume).

A corollary of J. Levine's theorem is that $C_n$ is periodic of period 4 for $n \geq 4$. Again, it was natural to try and explain the periodicity by direct geometric arguments. This was done by S. Cappell and J. Shaneson (1972) and by G. Bredon (1972).

In order to prove that $C_{2m} = 0$, one takes a Seifert surface V for the given knot $K \subset S^{2m+2}$. Thus, $\dim V = 2m+1$. The method consists in performing surgery on V, increasing its connectivity by attaching handles which are imbedded in $D^{2m+3}$. Thus, the effect of surgery is to increase the connectivity of V at the cost of pushing it into $D^{2m+3}$. As the dimension of the core of the handles does not exceed $m + 1$ (because it suffices to make V m-connected), there is no obstruction to imbedding problems in $D^{2m+3}$. It follows that the given knot $K \subset S^{2m+2}$ bounds a contractible submanifold of $D^{2m+3}$, and thus is null-cobordant.

In contrast, for a (2m-1)-knot $K \subset S^{2m+1}$ , the Seifert surface V has dimension 2m . The above method will still enable one to replace $V$ by $M \subset D^{2m+2}$ with $M$ $(m-1)$-connected . In fact, one proves in this way that $K$ is cobordant to a simple knot $K' \subset S^{2m+1}$, where $K'$ bounds an $(m-1)$-connected Seifert surface $V' \subset S^{2m+1}$.

But at the last step, i.e. in the attempt to make $V'$ m-connected, one hits obstructions. They arise from the problem of extending to the interior of a bunch of $(m+1)$-dimensional discs a given embedding of their boundaries into $S^{2m+1}$.

The cobordism group of Seifert forms measures precisely these obstructions to constructing a null knot cobordism.

## § 3. Unraveling the integral knot-cobordism group.

We borrow the title of this section from the paper of N.Stoltzfus (1976) containing a major part of the calculation of $C_{2m-1}$. Here is a summary of some of this work:

The reader will have guessed that together with the purely algebraic definition of $C_{2m-1}$ comes the possibility of defining cobordism groups of bilinear forms over other coefficient domains than $\mathbb{Z}$ . It was J. Levine (1969) who recognized this possibility as an essential tool in calculating $C_{2m-1}$ by algebraic methods.

We begin by recasting accordingly the definition of a Seifert form.

First note that our previous definition of a Seifert form for m depended on m via the sign $(-1)^m$ only. The condition on the signature for m = 2 need not really be dragged along as it is easily recaptured at the end of the calculation of $C_3$.

Definition 1 : Let R be a commutative ring and M an R-module. Let $\varepsilon = \pm 1$ . An $\varepsilon$-form is an R-bilinear form

$$A \;:\; E \times E \longrightarrow M$$

on a finitely generated R-module E, satisfying the condition that $S = A + \varepsilon A^T$ is unimodular, i.e.

$$\mathrm{ad}\,(S) \;:\; E \longrightarrow \mathrm{Hom}_R(E,\, M)$$

is an isomorphism, where as usual

$$\mathrm{ad}\,(S)\,(x)\,(y) \;=\; S(x,\, y).$$

As before, A itself is not assumed to possess any symmetry nor non-degeneracy property.

Basic examples are the cases $R = M = \mathbb{Z}$ , and E is then $\mathbb{Z}$-free of finite rank, or $R = M = k$, a field, or $R = \mathbb{Z}$, $M = \mathbb{Q}/\mathbb{Z}$ and E is a finite abelian group.

Definition 2 : An $\varepsilon$-form $A : E \times E \longrightarrow M$ is split (or metabolic) if there exists an R-direct summand $E_o \subset E$ such that $E_o = E_o^{\perp}$ , where $E_o^{\perp} = \{x \in E \mid A(x,y) = A(y,x) = 0 \text{ for all } y \in E_o\}$ .

Note that two $\varepsilon$-forms $A : E \times E \longrightarrow M$ and $B : F \times F \longrightarrow M$ can be added by direct sum $A \oplus B : (E \oplus F) \times (E \oplus F) \longrightarrow M$, where $(A \oplus B)(x \oplus y, x' \oplus y') = A(x,x') + B(y,y')$, and $A \oplus B$ is again an $\varepsilon$ -form. Obviously, if A and B are split forms, so is $A \oplus B$ .

Given a commutative ring R and an R-module M, one can then define the group $C_R{}^{\varepsilon}(M)$ of cobordism classes of M-valued $\varepsilon$-forms $A : E \times E \longrightarrow M$ modulo split forms. Two $\varepsilon$-forms A and A' represent the same element in $C_R{}^{\varepsilon}(M)$ if there exist split forms H and H' such that $A \oplus H \cong A' \oplus H'$ . The addition in $C_R{}^{\varepsilon}(M)$ is induced by the direct sum of $\varepsilon$-forms and the inverse of a class represented by the form $A : E \times E \longrightarrow M$ is represented by the form $-A$ .

We shall abbreviate $C_R{}^{\varepsilon}(R)$ to $C^{\varepsilon}(R)$.

By J. Levine's theorem in the preceeding paragraph, $C_{2m-1} = C^{(-1)^m}(\mathbb{Z})$ for $m \geq 3$. For $m = 2$, $C_3$ is the subgroup of $C^{+1}(\mathbb{Z})$ generated by the (+1)-forms A such that $S = A + A^T$ has signature divisible by 16. For $m = 1$, the group $C_1$ surjects onto $C^{-1}(\mathbb{Z})$ and the kernel is definitely non-zero. (Compare C. Gordon's survey in this volume).

Caution : Unfortunately, $C_R(M)$ is not a functor in M. On the other hand, in the special cases which one needs to consider in order to calculate $C^{\varepsilon}(\mathbb{Z})$, it turns out that if A is equivalent to a split form, i.e. $A \oplus H \cong H'$ , where H and H' are split, then A itself is a split form. However, we do not know how much of this remains true under reasonably general conditions for R and M.

The starting point for the study of $C^{\varepsilon}(\mathbb{Z})$ is the inclusion

$$C^{\varepsilon}(\mathbb{Z}) \subset C^{\varepsilon}(\mathbb{Q})$$

and the calculation, due to J. Levine (1969), of $C^{\varepsilon}(\mathbb{Q})$ which yields a complete system of algebraic invariants detecting the elements of $C^{\varepsilon}(\mathbb{Z})$.

Here is a summary of the method, extended by N. Stoltzfus (1976) to any perfect field k.

Given an $\varepsilon$-form $A : E \times E \longrightarrow M$, let $s : E \longrightarrow E$ be the endomorphism defined by

$$S(sx,y) = A(x,y)$$

for all $x,y \in E$, where $S = A + \varepsilon A^T$ .

We have $S(sx,y) + S(x,sy) = A(x,y) + A(y,x) = S(x,y)$, or
$S(Sx,y) = S(x, (1-s)y)$ .

We propose to call s an additive isometry.

Suppose now that $R = M = k$, a perfect field.

Let $f = f_s$ be the (monic) minimal polynomial of $s : E \longrightarrow E$ . It is easily verified, using the isometric property of s, that f is self-dual, i.e.

$$f(1-X) = (-1)^{\deg f} . f(X).$$

Suppose that $f \in k[X]$ is irreducible. Then E becomes a vector space over the extension field $K = k[X]/(f)$. Denoting by $\sigma$ the element corresponding to X in K, the action of $\sigma$ on E is defined by $\sigma.x = s(x)$ for all $x \in E$, Observe also that K possesses an involution $a \longmapsto \bar{a}$ determined by $\bar{\sigma} = 1 - \sigma$ .

The form S is then lifted to an $\varepsilon$-hermitian, K-valued form

$$( \ , \ ) \ : \ E \times E \longrightarrow K$$

on E, where $(x,y) \in K$ is defined by the formula

$$\mathrm{trace}_{K/k} \{ a.(x,y) \} = S(a.x, y)$$

for all $a \in K$ .

This construction is due to J. Milnor and it plays a decisive role in the calculation of knot cobordism groups. (See J. Milnor, On isometries of inner product spaces, Inventiones Math. 8 (1969), 83-97).

Thus, if the minimal polynomial $f_s$ of s is irreducible, there is associated with the $\varepsilon$-form $A : E \times E \longrightarrow k$ an $\varepsilon$-hermitian form over $K = k[X]/(f_s)$. Conversely, the above trace formula redefines S and A if a non-singular $\varepsilon$-hermitian form is given on some K-space E.

It turns out that every $\varepsilon$-form $A : E \times E \longrightarrow k$ over a field k is cobordant to a direct sum of $\varepsilon$-forms $A_i$ whose associated endomorphisms $s_i : E_i \times E_i \longrightarrow k$ have irreducible minimal polynomials.

Denoting by $H^{\varepsilon}(K)$ the Witt group of $\varepsilon$-hermitian forms over the field $K = k[X]/(f)$ with involution induced by $X \longmapsto 1-X$, the result of the calculation of $C^{\varepsilon}(k)$, due to J. Levine (1969) in a somewhat different formulation, is that

$$C^{\varepsilon}(k) = \oplus_{f \in P} H^{\varepsilon}(K_f) ,$$

where P is the set of self-dual irreducible polynomials over k, and $K_f = k[X]/(f)$ .

Note that $f = X - \frac{1}{2}$ if and only if $K_f = k$ with trivial involution. In that case

$$H^{\varepsilon}(K_f) = W^{\varepsilon}(k)$$

is the ordinary Witt group of $\varepsilon$-symmetric forms. If $f \neq X - \frac{1}{2}$, then $K_f/k$ has a non-trivial involution. In this case also the Witt group $H^{\varepsilon}(K_f)$ is well known by the work of W. Landherr (Abh. Math. Sem. Hamburg Univ. 11 (1935) p.245). It is not hard to derive a presentation of $H^{\varepsilon}(K_f)$ by generators and relations similar to the one for Witt groups of symmetric forms. (See the book of J. Milnor and D. Husemoller, Symmetric bilinear forms, Springer Verlag, 1974). More precisely, let $F$ be the fixed field of the involution on $K$. If $a \in F^{\cdot}$, let $< a >$ denote the hermitian form

$$( \; , \; ) : K \times K \longrightarrow K \text{ of rank 1 given by } (x,y) = a x\bar{y}$$

Then, there is a surjection $\mathbb{Z}[F^{\cdot}] \longrightarrow H(K)$ given by $[a] \longrightarrow < a >$. Here $H(K) = H^{+1}(K)$. The kernel is the ideal of $\mathbb{Z}[F^{\cdot}]$ generated by the elements of one of the forms

$$[a] - [a.x.\bar{x}]$$

$$[a] + [-a]$$

$$[a] + [b] - [a+b] - [ab.(a+b)]$$

for $a, b, a+b \in F^{\cdot}$, $x \in K^{\cdot}$ .

For $\varepsilon = -1$, observe that if the involution on $K$ is non-trivial then $H^{-1}(K) \cong H^{+1}(K)$ under the map $( \; , \; ) \longrightarrow \sqrt{d}.( \; , \; )$, where $K = F(\sqrt{d})$ .

The above argument yields in particular

$$C^{\varepsilon}(\mathbb{Q}) = \oplus_{f \in P} H^{\varepsilon}(K_f) ,$$

where $P$ is the set of irreducible polynomials which are self-dual, i.e. $f(1-X) = (-1)^{\deg f} f(X)$ .

Actually, what is needed for the calculation of $C^{\varepsilon}(\mathbb{Z})$ is the group

$$C_o{}^{\varepsilon}(\mathbb{Q}) = \oplus_{f \in I} H^{\varepsilon}(K_f) ,$$

where I is the set of irreducible <u>integral</u> self-dual polynomials .

N. Stoltzfus (1976)observes that there is an exact sequence

$$0 \longrightarrow C^{\varepsilon}(\mathbb{Z}) \longrightarrow C_o{}^{\varepsilon}(\mathbb{Q}) \xrightarrow{\partial} C_{\mathbb{Z}}{}^{\varepsilon}(\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

where $\partial : C_o{}^{\varepsilon}(\mathbb{Q}) \longrightarrow C_{\mathbb{Z}}{}^{\varepsilon}(\mathbb{Q}/\mathbb{Z})$ is defined as follows.

Let $A : E \times E \longrightarrow \mathbb{Q}$ be an $\varepsilon$-form representing some element in $C_o{}^{\varepsilon}(\mathbb{Q})$ . Because $s = S^{-1}A$ has integral characteristic polynomial, there is an integral lattice $L \subset E$ on which S is integral valued and which is invariant by S. Define

$$L' = \{ x \in E \mid S(x,y) \in \mathbb{Z} \text{ for all } y \in L \} .$$

Then $L \subset L'$ and $E^* = L' / L$ is a $\mathbb{Z}$-module with a $\mathbb{Q}/\mathbb{Z}$ -valued form $S^* : E^* \times E^* \to \mathbb{Q}/\mathbb{Z}$ defined by $S^*(x^*,y^*) = S(x,y) \bmod \mathbb{Z}$ for $x,y \in L'$ representing $x^*,y^* \in E^*$ .

The dual L' is also invariant by s and therefore there is an additive isometry $s^* : E^* \longrightarrow E^*$ induced by s and satisfying

$$S^*(s^*x,y) = S^*(x,(1-s^*)y)$$

for all $x,y \in E^*$. It is not hard to verify that

$$\operatorname{ad} S^* : E^* \longrightarrow \operatorname{Hom}(E^*,\mathbb{Q}/\mathbb{Z})$$

is an isomorphism.

By definition

$$\partial [E,S,s] = [E^*, S^*, s^*] .$$

At this point, one has to calculate $C^{\varepsilon}{}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})$. A localization argument gives first

$$C^{\varepsilon}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = \oplus_p \; C^{\varepsilon}(\mathbb{F}_p) ,$$

where p runs over all rational primes. Now, $C^{\varepsilon}(\mathbb{F}_p)$ has been evaluated above since $\mathbb{F}_p$ is a perfect field, and it can be calculated explicitly. We quote Corollary 2.9. of N. Stoltzfus' paper (1976) :

$$C^{\varepsilon}(\mathbb{F}_p) = \oplus \mathbb{Z}/2\mathbb{Z} \; \oplus W^{\varepsilon}(\mathbb{F}_p) ,$$

where $W^{\varepsilon}(\mathbb{F}_p)$ is the Witt group of the prime field and the first direct sum is taken over all irreducible, self-dual, monic polynomials except $X - \frac{1}{2}$ . This is for p an odd prime, if p = 2 only the first summand is present.

In the remainder of his paper N. Stoltzfus uses algebraic number theory to make the above results more explicit. We cannot enter into the details here.

# CHAP. V : FIBERED KNOTS

## § 1 : General properties .

In this chapter we study the very important special case of fibered knots. At least two reasons make this special case worth of study :

1) Knots which appear as local singularities of complex hypersurfaces are fibered knots.

2) The geometry of the complement of fibered knots can be made quite explicit and thus many knot invariants get a very nice geometrical interpretation.

Let us start with the definition. Racall that $H^1(X;\mathbb{Z}) \approx \mathbb{Z}$ and let t be a chosen generator. One says that K is a fibered knot if one is given a representative $p : X \longrightarrow S^1$ for t, which is a locally trivial (differentiable) fibration.

Remark : It is often nice to add the further restriction that $p|bX \longrightarrow S^1$ (which is, by hypothesis, a fibration) should be the projection onto the fiber associated with a trivialization of the sphere normal bundle to K in $S^{n+2}$. A useful remark due to S. Cappell shows that whenever $n \neq 2,3$, one can always change p such that this further requirement is satisfied. See Cappell (1969). In the sequel, we shall usually make this assumption.

The fiber of p is a codimension one submanifold W of $S^{n+2}$. It is connected because p represents a generator of $H^1(X;\mathbb{Z})$. (To see that, consider the end of the homotopy exact sequence of the fibration : $\pi_1(X) \longrightarrow \pi_1(S^1) \longrightarrow \pi_0(F) \longrightarrow 0$ .)

If we add to W a collar inside the normal bundle to K in $S^{n+2}$, we get a Seifert surface V for K.

Choose now a point $1 \in S^1$ and remove a small open interval I centered in 1. Call J the big closed interval that remains. $p^{-1}(I)$ is a trivialised open neighborhood of W in X. So, $p^{-1}(J)$ is what we called Y in Chap. III § 1 . But, as J is contractible, $p^{-1}| \; Y \longrightarrow J$ is a trivial fibration. So, Y is homeomorphic to $W \times J$.

Looking at things a bit differently, we see that we can think of X as being obtained from $W \times [0,1]$ by $W \times \{0\}$ and $W \times \{1\}$ identified together via a homeomorphism $h : W \to W$. More precisely, X is the quotient of $W \times [0,1]$ by the equivalence relation $(x,0) \sim (h(x),1)$.

h is called "the" monodromy of the fibration. p being given, h is well defined up to isotopy. If we insist that p satisfies the restriction condition on bX, we shall get a monodromy map which is the identity on bW.

## § 2. The infinite cyclic covering of a fibered knot.

Let us consider the product $W \times \mathbb{R}$ and the equivalence relation $(x,a) \sim (h^j(x), a+j)$ for any $j \in \mathbb{Z}$. It is immediate to verify that the quotient space is homeomorphic to X. Moreover, the quotient map $W \times \mathbb{R} \longrightarrow X$ is a regular covering map, whose Galois group is C. So this is the infinite cyclic covering of X. We deduce from that :

1) $\widetilde{X}$ has the homotopy type of W, which is a compact C.W. complex.

2) The generator t of the Galois group C acts by the map $(x,a) \longmapsto (h(x), a+1)$. So t acts on $H_*(\widetilde{X})$ as h acts on $H_*(W)$.

As before, let us denote by $F_k(\widetilde{X})$ the torsion-free quotient of $H_k(\widetilde{X};\mathbb{Z})$. By 1), $F_k(\widetilde{X})$ is a finitely generated free abelian group ; and it is also a $\mathbb{Z}C$-module. Under these circumstances,a theorem of algebra says that a generator $\lambda$ of the first elementary ideal of the $\mathbb{Z}C$-module $F_k(\widetilde{X})$ is just the characteristic polynomial of t . Moreover, it is not hard to see that $\lambda$ is just the Alexander polynomial $\Delta_1$ of $H_k(\widetilde{X};\mathbb{Z})$. (The lazy reader can look at Weber's paper in this book). Recalling that t acts like $h_k$ we get the folklore theorem :

When a knot fibers, the Alexander polynomial of $H_k(\widetilde{X};\mathbb{Z})$is just the characteristic polynomial of the monodromy $h_k$ acting on $F_k(W)$,

As it is a characteristic polynomial, its leading coefficient is +1 ; as $h_k$ is an isomorphism on the finitely generated free abelian group $F_k(\widetilde{X})$ its last coefficient is $\pm 1$ .

Remark : A simplified version of the above argument gives the following : Let $\mathbb{F}$ be a field. Then the order of the $\mathbb{F}C$-module $H_k(\widetilde{X};\mathbb{F})$ is just the characteristic polynomial of the automorphism

$h_k : H_k(W;\mathbb{F}) \longrightarrow H_k(W;\mathbb{F})$ . Cf Milnor (1968a)and (1968b).

Using some more algebra, it is not hard to see that the minimal polynomial of the action of $h_k$ on $F_k(W)$ is $\Delta_1/\Delta_2$ , $\Delta_i$ being the g.c.d. of the ith elementary ideal of $H_k(\widetilde{X}; \mathbf{Z})$.

See R. Crowell : "The annihilator of a knot module" Proceedings AMS 15(1964) p. 696-700. This fact is much used by people working on singularities. For instances see N.A'Campo (1972a).

Let us close this paragraph by mentioning that fibered knots give a nice interpretation of the pairing of torsion submodules mentioned in Chap. II §3 : it is the linking pairing induced on the fiber by Poincaré duality (See J. Levine (1974) § 7) .

## § 3 : When does a knot fiber ?

We saw in this chapter § 1 that a knot fibers if and only if one can find a Seifert surface V such that Y is homeomorphic to $W \times [0,1]$. One can choose a homeomorphism which is the "identity" from $W_+$ to $W \times \{0\}$ . The homeomorphism we get from $W_-$ to $W \times \{1\}$ is just h .

Moreover $i_+$ and $i_-$ are homotopy equivalences. So, $(i_+)_k$ and $(i_-)_k : H_k(W) \longrightarrow H_k(Y)$ are isomorphisms and :

$$h_k = (i_+)_k^{-1} \cdot (i_-)_k \quad \text{for all } k .$$

If the fibered knot is (2m-1)-dimensional, the Seifert pairing $A : F_m(W) \times F_m(W) \longrightarrow Z$ associated with the fiber W is unimodular, because $(i_+)_m$ is an isomorphism.

Suppose now that, for a given (2m-1)-knot, we can find an (m-1)-connected Seifert surface W such that its Seifert pairing is unimodular. Then, if $m \geqslant 3$, by the h-cobordism theorem Y is homeomorphic to the product $W \times [0,1]$ and so the knot fibers. Moreover, using notations introduced in chap. III § 1 and § 3, the matrix for $h_m$ is given by $(-1)^{m+1} A^{-1} \cdot A^T$ .

For classical knots (m = 1), the unimodularity of a Seifert matrix is necessary for a knot to fiber, but it is not sufficient. See R. Crowell and D. Trotter (1962). The correct condition, due to L. Neuwirth and J. Stallings is that one should find a Seifert surface such that $i_+$ and $i_-$ induce isomorphisms on the fundamental group.

It is harder to get useful fibration theorems for non-simple knots.

However, we saw in § 2 that a necessary condition for a knot to fiber is that the extremal coefficients of the Alexander polynomial for $H_k(\tilde{X};\mathbb{Z})$ should be $\pm 1$ for all $k \geq 1$. A theorem due to D.W. Sumners says that the converse is true if $\pi_1(X) = \mathbb{Z}$ and $n \geq 4$. See Sumners (1971).

If one spins a fibered knot, one gets again a fibered knot. This fact has been used by J.J. Andrews and D.W. Sumners (1969).

## § 4. Twist-spinning.

An important and striking way to construct a fibered knot is E.C. Zeeman's twist-spinning.

We give a sketched description of the twist-spinning construction and for more details, we refer the reader to Zeeman's original paper (1963), where the geometry of the construction is beautifully described.

Look at the unit closed ball $E^{n+2}$ as being the product $E^n \times E^2$. In $E^2$ use polar coordinates, $(\rho,\Phi)$ being mapped onto $\rho e^{2i\pi\Phi}$, $0 \leqslant \rho \leqslant 1, 0 \leqslant \Phi \leqslant 1$. So, a point in $E^{n+2}$ will be described by a triple $(x,\rho,\Phi)$. Also, $S^1$ is the unit circle in $E^2$, with angular coordinate $\theta$ , $0 \leqslant \theta \leqslant 1$ .

Suppose now that we have a subspace $A \subset E^{n+2}$. Let $r \in \mathbb{Z}$ be given. The full r twist of A is the subspace $A_r \subset E^{n+2} \times S^1$ consisting of the quadruples : $(x,\rho,\Phi + r\theta,\theta)$ for all $(x,\rho,\Phi) \in A$, $\theta \in [0,1]$. It is obvious that $A_r$ is abstractly homeomorphic to $A \times S^1$.

Now, let an n-knot $K \subset S^{n+2}$ be given. Choose a small open (n+2)-disc neighborhood of a point belonging to K such that :

1) The intersection of the disc with K is an open n-disc.
2) The small disc pair thus obtained is standard.

Let us take the complementary pair $(D^{n+2}, B)$. Identify $D^{n+2}$ with $E^{n+2}$. $r \in \mathbb{Z}$ being given, look at the pair $(D^{n+2} \times S^1, B_r)$. On the boundary it is the standard $(S^{n+1} \times S^1, S^{n-1} \times S^1)$, because via the identification bB goes to $bE^n \times \{0\}$ . Glue along the boundary the standard $(S^{n+1} \times D^2, S^{n-1} \times D^2)$ and you get an (n+1)-knot; because, abstractly for any $k \geqslant 0$ $(D^k \times S^1) \amalg (S^{k-1} \times D^2)$ glued along $S^{k-1} \times S^1$ yields $S^{k+1}$.

This is the r-twist spinning of the original knot.

If we are careful that the subball $B \subset D^{n+2}$ is standard near the boundary, the twist-spun knot will be differentiable, if we started with a differentiable knot.

It is not hard to see that, for $r = 0$, the construction is essentially Artin's spinning. Changing $\theta$ into $-\theta$ changes the r-twist spun into the $(-r)$ one, so the construction really depends on $|r|$ . The properties of the twist-spinning operation are given by :

Zeeman's theorem : Suppose $r \neq 0$ . Then :

1) The exterior of the r-twist spun knot fibers on $S^1$, in the sense of § 1.

2) The fiber W is the r-th cyclic branched covering of the original knot, minus an open (n+2)-disc.

3) Let f be a correctly chosen generator of the Galois action on the unbranched r-th cyclic covering of the exterior of the original knot. f extends to an automorphism $\overline{f}$ of the branched cyclic covering (the knot being fixed) and $\overline{f}$ restricts to an automorphism h of the punctered branched cyclic covering W. h is of order r and can be taken as the monodromy of the fibration. Beware : h is not quite the identity on bW.

4) There is an action of $S^1$ on $S^{n+3}$ leaving the twist-spun knot invariant, and acting freely outside the knot. But the action on the knot is not the identity.

Comments :

a) Because of point 4), one is very close to counter-examples to the Smith conjecture, for multidimensional knots. Soon after Zeeman's paper, C.H. Giffen (1964) was able to produce such counter-examples, by using as a start the twist-spinning operation. Several other counter-examples are now known (all for non-classical knots ! ) .

b) By § 2, the infinite cyclic covering of the r-twist spun knot has the homotopy type of the punctered branched r-cyclic covering of the original knot, and the monodromy is "known". So, to compute the invariants of the new knot, one can use classical procedures about branched cyclic coverings.

c) A generalization of Artin's result about $\pi_1$ of a spun knot shows that $\pi_1$ of the r-twist spun knot is obtained from $\pi_1$ of the original knot by adding the relations saying that the r-th power of the meridian commutes with everybody. J. Levine has a very useful way to look at the twist-spinning contruction which yields this result very nicely. (Unpublished).

d) As the 1-branched cyclic covering of an n-knot is the (n+1)-sphere, 1-twist spun knots bound a disc and are thus trivial.

## § 5. Isolated singularities of complex hypersurfaces.

Let $f : \mathbb{C}^{m+1} \longrightarrow \mathbb{C}$ be a $\mathbb{C}$-polynomial map, such that $f(0) = 0$ and that $0 \in \mathbb{C}^{m+1}$ is an isolated singularity of f. (This means that the $\mathbb{C}$-gradient of f does not vanish in a neighborhood of 0 except at 0). J. Milnor (1968b) shows :

1) The intersection K of the hypersurface $f^{-1}(0) = H$ with sufficiently small spheres $S_\varepsilon^{2m+1}$ in $\mathbb{C}^{m+1}$, centered in 0, is transversal. Thus K is a(real) codimension two submanifold of $S_\varepsilon^{2m+1}$, but not necessarily a sphere.

2) The exterior of K in $S^{2m+1}$ fibers in the strong sense, i.e. the restriction of the fibration to bX is the projection onto $S^1$ associated to a trivialization of the normal bundle of K in $S^{2m+1}$.

3) The fiber W has the homotopy type of a wedge of m-dimensional spheres.

If we look at the homology exact sequence of the pair V mod.K, we see that K is not too far from being a homology sphere. Its only (possibly) non-vanishing homology groups are in dimensions (m-1) and m .Their vanishing depends on the intersection pairing on $H_m(V) = H_m(W)$. Moreover one can prove that if $m \geqslant 3$, K is simply-connected .

It is clear that there is a Seifert pairing for W, and that, if we agree to call "knots" submanifolds such as K, we have got an odd dimensional, fibered, simple knot. One can check that Levine's S-equivalence theory works in that case also. So, from a topological point of view, the situation is rather well understood. See A.F. Durfee (1973) for detail.

Remarks :

a) It is known (see Milnor's book) that locally around 0, the pair $(\mathbb{C}^{m+1},H)$ is homeomorphic to the cone on the pair $(S^{2m+1},K)$. Thus, topologically, the singularity is determined by the knot.

b) If $f : U \longrightarrow \mathbb{C}$ is a holomorphic map with $f(0) = 0$, (U open neighborhood of 0 in $\mathbb{C}^{m+1}$) and with 0 as isolated singularity, then the germ of f at 0 is analytically equivalent to a polynomial. For a very detailed discussion of this kind of results see : "Remarks on finitely determined analytic germs" by J. Bochnak and S.Lojasiewicz in Springer Lecture Notes vol. 192 (1970) p. 262-270. So one can apply the theory also to holomorphic germs.

c) Define two holomorphic germs $f_i : U_i \longrightarrow \mathbb{C}$ , $i = 1,2$, $U_i$ open neighborhood of 0 in $\mathbb{C}^{m+1}$, $f_i(0) = 0$, to be topologically equivalent if there exist a germ $\Phi$ of homeomorphism at $0 \in \mathbb{C}^{m+1}$, $\Phi(0)= 0$ and a germ $\varphi$ of homeomorphism at $0 \in \mathbb{C}$, $\varphi(0) = 0$ such that :

$f_2 = \varphi \cdot f_1 \cdot \Phi$ in a suitable neighborhood of $0 \in \mathbb{C}^{m+1}$.

A recent theorem of H.C. King (1977) says that, if $m \neq 2$, two holomorphic germs with isolated singularities at 0 are topologically equivalent if and only if the knots they determine are isotopic. So, roughly speaking, the knot determines the topological type of the germ f, a result much stronger than the classical one stated in a) above.

The main question in the topological study of isolated singularities of complex hypersurfaces is to relate the topological invariants coming from knot theory and the invariants coming from algebraic geometry. (Here, when we say "topological" we mean as well "differential" as opposed to "algebraic" or "analytic").

For instance, the differential structure on K is an interesting invariant. More precisely :

1) One would like to compute the knot invariants from the algebraic data. Historically the whole story began (after O. Zariski's work in the thirties) when F. Pham (1965) and subsequently E. Brieskorn (1966) studied the sigularities :

$$f(z_o, z_1, \dots, z_k) = (z_o)^{i_o} + (z_1)^{i_1} + \dots + (z_k)^{i_k} \quad .$$

In that case, computations can be done. For other results, see P. Orlik and J. Milnor (1969).

2) One would **also like** to know when a given knot is obtained from a singularity. This can be first attacked by trying to determine which restrictions are imposed on the knot invariants when it is "algebraic", besides the fact that it is a fibered knot. Striking examples of such restrictions are :

a) The monodromy theorem, which says that the roots of the Alexander polynomial of $H_m(\tilde{X};\mathbb{Z})$ (which is the characteristic polynomial of $h_m$) are all roots of unity. See E. Brieskorn(1969).

b) There exists a basis for $H_m(W;\mathbb{Z})$such that the Seifert matrix is triangular. See A.F. Durfee (1973).

c) The trace of $h_m$ is equal to $(-1)^{m+1}$. See N.A'Campo (1972b) and, more generally, N. A'Campo(1974).

All these are very deep results about singularities.

Since the beginning of the theory, a lot of work has been spent to get nice geometrical descriptions of some singularities. For recent results, look at L.H.Kauffman (1973) and also at L.H. Kauffman and W.D. Neumann(1976).

For a more detailed exposition and more references about the whole subject, the reader should see J. Milnor's book (1968b), A.H. Durfee (1975), M. Demazure (1974).

Historical remark : The theory of isolated singularities began in the late twenties by the study of singularities of complex plane curves, approximately at the same time as knot theory really started (exception being made for M. Dehn's papers). In fact, progresses were made in knot theory to understand O. Zariski's results about curves and conversely, algebraic geometers found beautiful applications of J.W. Alexander and K. Reidemeister's work. It is amusing to note that a remark (due to W. Wirtinger) about the singularity $z_1^2 + z_2^3 = 0$ being locally the cone on the trefoil knot appears already in E. Artin's paper (1925). This permits us to close this paper at the point where we started it.

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