EXTENSIONS OF THE GUDKOV-ROHLIN CONGRUENCE

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TABLE OF CONTENTS

§ 1. INTRODUCTION

1.1. The subject of the paper.

What pictures on the real projective plane $\mathbb{R}P^2$, up to homeomorphism, can be realized by a real algebraic curve ? The answer is not dufficult, unless we put a restriction on the degree of the curve (or a restriction of some other kind on the complexity of its equation).

However, for a fixed degree the question is very difficult and far from being solved in the complete generality, see e.g. G.Wilson [24] and O.Viro [21] (as for the other restrictions, see A.G.Khovansky [10]). The most complicated situation appears if the number of branches is great enough. Curves which have the maximal number of branches for a given degree (so called M-curves) are most remarkable from the topological point of view. It is the Gudkov-Rohlin congruence that makes one of the main features of the topology of M-curves of even degree.

The notion of M-curve, the Gudkov-Rohlin congruence, as well as many other results on nonsingular plane curves, permit appropriate extensions to the case of real algebraic manifolds of higher dimensions and to the case of real algebraic varieties (i.e. manifolds with singular points). Generalization of the notion of M-curve and the Gudkov-Rohlin congruence to the case of nonsingular real algebraic manifolds of arbitrary dimension were given by V.A.Rohlin [17] , [18] . Some extensions of the Gudkov-Rohlin congruence to the singular case were outlined in our note [9] . The present paper is devoted to extension of the Gudkov-Rohlin congruence and some related theorems to the singular case. Our results are fairly complete for plane curves, but higher dimensions appear only incidentally.

1.2. The Gudkov-Rohlin congruence and related ones.

Let A be a nonsingular plane projective real algebraic curve of degree m . It is said to be of *type I* or *dividing* if its real point set $\mathbb{R}A$ bounds in its complex point set $\mathbb{C}A$ (in this case $\mathbb{R}A$ divides $\mathbb{C}A$ into two parts, which are interchanged by the complex conjugation $\mathrm{conj}: \mathbb{C}P^2 \to \mathbb{C}P^2 : (z_0 : z_1 : z_2) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$ Otherwise it is said to be of *type II* or *non-dividing*. Below in this section the degree m of A is even, $m = 2k$. Then $\mathbb{R}A$ divides $\mathbb{R}P^2$ into two parts having $\mathbb{R}A$ as their common boundary. Only one of the parts is orientable; we denote it by $\mathbb{R}P^2_+$. The non-orientable part is denoted by $\mathbb{R}P^2_-$

By the well-known Harnack inequality [24] the number of components of $\mathbb{R}A$ is not more than $\frac{(m-1)(m-2)}{2}+1$. If it equals $\frac{(m-1)(m-2)}{2}+1$ then A is called an *M-curve*; if it equals $\frac{(m-1)(m-2)}{2}+1-i$ then A is called an *$(M-i)$-curve*.

(1.A) *If* A *is an* M *-curve, then*

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \bmod 8 \tag{1}$$

That is the Gudkov-Rohlin congruence. It was conjectured by D.A. Gudkov. He proved it for $m=6$ in [5]. The weakened congruence $\chi(\mathbb{R}P^2_+) \equiv k^2 \bmod 4$ under a weaker hypethesis (see 1.D below) was proved by V.I.Arnold [1]. To the full extent it was proved by V.A.Rohlin [17].

There are several related congruences (also for a nonsingular A). We formulate three of them as (1.B) - (1.D). For the others, see Viro's survey [21] and the original papers by V.V.Nikulin [13] and T.Fiedler [4].

(1.B) *If* A *is an* *$(M-1)$-curve, then*

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \pm 1 \bmod 8 \tag{2}$$

(1.C) *If* A *is an* *$(M-2)$-curve of type* II, *then*

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \quad \textit{or} \quad k^2 \pm 2 \bmod 8 \tag{3}$$

(1.D) *If* A *is a curve of type* I, *then*

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \bmod 4 \tag{4}$$

Proofs of (1.A)-(1.D) are reproduced below in 6.1. First, (1.B) was proved by D.A.Gudkov and A.D.Krahnov [6] and V.M.Kharlamov [8] independently, (1.C) by V.M.Kharlamov, see [19, 3.4], and A.Marin [12] independently; (1.D) is due to V.I.Arnold [1].

1.3. Two approaches.

Three proofs of the Gudkov-Rohlin congruence have been published. They are due to V.A.Rohlin [16], [17] and A.Marin [12]. The first [16] contains a mistake. The third [12] appears to be an improvement of the first. The example considered by Marin [12] seems to show that there is no correct proof of (1.A) which is closer to Rohlin's arguments [16] than Marin's proof.

Marin's [12] and Rohlin's second [17] approaches based on quite different techniques. Rohlin's proof works in any dimension while no generalization of Marin's proof to higher dimensions is known. Nevertheless the approaches seem to be closely related. Rohlin asked his students to find a relation and said that an understanding of it might lead to essential progress.

Both approaches admit extension to the case of singular curves. We did not seek identification of the results in their complete generality obtained for singular curves by those two approaches, although for all concrete situations considered the results coincide. Marin's approach seems to be simpler for our purposes, so we adopt it as the basic one. Rohlin's approach also has some important advantages. First, it is applicable to real algebraic varieties of arbitrary dimension; second, for some classes of singularities it gives results, which are more easy to formulate and use. In the last part of the paper we discuss these topics.

1.4. Two levels of results.

Our extensions of the Gudkov-Rohlin congruence, as many other statements on the topology of singular curves, involve some characteistics of the curve singularities. For efficient formulation of these results some additional investigation of the singularities is to be done. Due to a great diversity of singularities it is impossible to do this work once for all cases. Thus we distinguish two levels of results: first, general theorems (see § 3), which involve curves of vast classes and rather complicated characteristics of singularities (introduced in 2.3), and second, efficient theorems on curves of more special classes with singularities of some special types, formulations in this case involve only simplest characteristics of singularities (see § 4). The results of the first level are useful not only as initial steps to the results of the second level. In applications it is sometimes sufficient to know that some congruence is to be satisfied, for its efficient statement is obvious from known examples. See A.B.Korchagin [11] and sections 4.1 - 4.4 below.

1.5. Acknowledgements.

G.M.Polotovsky's work [14] on splitting curves of degree 6 suggested that there must be some congruences for singular curves, which are close to the Gudkov-Rohlin congruence but can not be straightforwardly reduced to it. Our first results in this direction were met by D.A. Gudkov, G.M.Polotovsky, E.I.Shustin and A.B.Korchagin with a stimulating interest. We are indebted to them for their encouragement.

§ 2. PREREQUISITE FOR STATING OF RESULTS

2.1. Preliminary arithmetics: $\mathbb{Z}/_4$ -quadratic spaces.

By $\mathbb{Z}/_4$-quadratic space we mean a triple $(V, \circ, q)$ consisting of a finite-dimensional vector space V over $\mathbb{Z}/_2$, a symmetric bilinear form $V \times V \to \mathbb{Z}/_2 : (x, y) \mapsto x \circ y$ and a function $q: V \to \mathbb{Z}/_4$, which is quadratic with respect to that bilinear form, i.e.

$$q(x+y) = q(x) + q(y) + 2 \cdot x \circ y \tag{5}$$

for $x, y \in V$, where $2\cdot : \mathbb{Z}/_2 \longrightarrow \mathbb{Z}/_4$ is the unique non-zero homomorphism. The bilinear form $\circ$ is certainly determined by q via (5).

A $\mathbb{Z}/_4$ -quadratic space $Q = (V, \circ, q)$ is said to be nonsingular if its bilinear form $\circ$ is nonsingular, i.e. its radical $R(Q) = \{ x \in V \mid \forall y \in V \;\; x \circ y = 0 \}$ is the zero-subspace. We say that a $\mathbb{Z}/_4$ -quadratic space $Q(V, \circ, q)$ is informative, if q vanishes on $R(Q)$. In this case $\circ$ and q induce well-defined bilinear and quadratic forms on $V/R(Q)$. The $\mathbb{Z}/_4$ -quadratic space appeared is nonsingular and it is called a nonsingular $\mathbb{Z}/_4$ -quadratic space associated with Q .

The isomorphism classes of nonsingular $\mathbb{Z}/_4$ -quadratic spaces form a commutative semigroup under the orthogonal sum operation. To obtain a group, one introduces the relation $(V, \circ, q)$ for any $\mathbb{Z}/_4$ -quadratic space $(V, \circ, q)$ with V containing a vector subspace H such that $\dim H = \frac{1}{2} \dim V$ and $q|_H = 0$ (and consequently $H \circ H = 0$). The resulting factor-group is called the Witt group $WQ(\mathbb{Z}/_2, \mathbb{Z}/_4)$. It is isomorphic to $\mathbb{Z}/_8$ (see e.g. [2]). The isomorphism is set up by the van der Blij-Brown invariant $(V, \circ, q) \longmapsto B(q)$ defined by the formula

$$\exp\left(\frac{i\pi B(q)}{4}\right) = 2^{-\frac{\dim V}{2}} \sum_{x \in V} \exp\left(\frac{i\pi q(x)}{2}\right) \tag{6}$$

see e.g. L.Guillou and A-Marin [7] .

Nonsingular $\mathbb{Z}/_4$ -quadratic spaces which determine the same element of $WQ(\mathbb{Z}/_2, \mathbb{Z}/_4$ are said to be cobordant. Informative $\mathbb{Z}/_4$ -quadratic spaces with cobordant associated nonsingular $\mathbb{Z}/_4$ -

quadratic spaces are also said to be cobordant. If $Q=(V,\circ,q)$ is an informative $\mathbb{Z}/_4$ -quadratic space, then the van der Blij-Brown invariant of its associated nonsingular $\mathbb{Z}/_4$ -quadratic space is denoted by $B(q)$. It can be calculated by the formula

$$\exp\left(\frac{i\pi B(q)}{4}\right) = 2^{-\frac{\dim V + \dim R(Q)}{2}} \sum \exp\left(\frac{i\pi\, q(x)}{2}\right) \tag{7}$$

2.2. Preliminary topology: the Rohlin-Guillou-Marin form.

Let X be an oriented smooth compact four-dimensional manifold, let F be its smooth compact two-dimensional submanifold (not necessarily orientable) with $\partial F = F \cap \partial X$ such that $in_* H_1(F;\mathbb{Z}/_2) = \{0\} \subset H_1(X;\mathbb{Z}/_2)$ (as usual in = inclusion), and let F realize in $H_2(X,\partial X;\mathbb{Z}/_2)$ the class which is the Poincaré dual to the Stiefel-Whitney class $w_2(X)\in H^2(X;\mathbb{Z}/_2)$.

Then there is a natural function $q: H_1(F;\mathbb{Z}/_2) \longrightarrow \mathbb{Z}/_4$,which is quadratic in the sense of 2.1 with respect to the intersection form $H_1(F;\mathbb{Z}/_2)\times H_1(F;\mathbb{Z}/_2) \longrightarrow \mathbb{Z}/_2$, see e.g. [7] . We call it the Rohlin-Guillou-Marin form of the pair (X,F) . This q may be defined as follows. To define $q(\alpha)$ for $\alpha\in H_1(F;\mathbb{Z}/_2)$, realize α by an embedded closed smooth curve $\ell \subset F$, span ℓ by a surface $P\subset X$, which is normal to F at $\ell = \partial P$ and transversal at inner points. Consider on ℓ a field of lines tangent to F and normal to ℓ and denote by $\varkappa$ the obstruction to extending this field to a field of lines normal to P . Then

$$q(\alpha) = \varkappa + 2\,(\mathrm{Int}\; P\circ F) \bmod 4 \tag{8}$$

where by $\mathrm{Int}\; P\circ F$ we mean the $\bmod 2$ -intersection number.

We like to consider here a slinghtly more general situation allowing F to have a corner, which is a smooth curve transversal to ∂X. The definition of q is naturally generalized to this situation. One may obtain q by smoothing F and checking that the result is independent on the choice of the smoothing. However there is a clear direct generalization of the definition of q given above. For α represented by ℓ, which does not meet the corner, $q(\alpha)$ is defined exactly as above.

2.3. Singular point data.

Let $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a holomorphic function, which is real (in the sence that $f(\bar{x},\bar{y}) = \overline{f(x,y)}$ for $(x,y) \in \mathbb{C}^2$). Let p be its real isolated singular point with $f(p) = 0$

In this section to any such situation we assign $\mathbb{Z}/_2$ -vector spaces L_p and $L^{\sim}_p$, a homomorphism $\nu_p: L^{\sim}_p \longrightarrow L_p$, a $\mathbb{Z}/_4$ - quadratic space $(V_p, \circ, q_p)$, a bilinear pairing $L_p \times V_p \longrightarrow \mathbb{Z}/_2 : (x,y) \longmapsto x \sqcap y$, a subspace W_p of V_p, a subspace X_p of W_p and homomorphism $\omega_p: L_p \longrightarrow V_p / W_p$ and $\chi_p: L^{\sim}_p \longrightarrow V_p$. These objects are involved in formulation of our main theorems. We shall call them singular point data of p. We can reduce the number of them, but for this we'll be made to pay with more heavy calculation in applications. In the corresponding simplified versions of formulations (see (3.A) and (3.C)) only L_p, $(V_p, \circ, q_p)$, $\sqcap$ of the singular point data are involved.

Denote by Φ the curve defined by the equation $f(x,y) = 0$ and let $\nu: \Phi^{\sim} \longrightarrow \Phi$ be a normalization. Set

$$L_p = H_1(\mathbb{R}\Phi, \mathbb{R}\Phi \setminus p\ ; \mathbb{Z}/_2),$$

$$L^{\sim}_p = H_1(\mathbb{R}\Phi^{\sim}, \mathbb{R}\Phi^{\sim} \setminus \nu^{-1}(p); \mathbb{Z}/_2),$$

$$\nu_p = \nu_* .$$

Let $\mathcal{D}$ be a ball in $\mathbb{C}^2$ centered at p and so small that the pair $(\mathcal{D}, \mathbb{C}\Phi \cap \mathcal{D})$ is homemorphic to the cone over $(\partial\mathcal{D}, \mathbb{C}\Phi \cap \partial\mathcal{D})$. Let $\varepsilon > 0$ be such that for any $t \in (0, \varepsilon]$ the curve defined by the equation $f(x,y) = -t$ is nonsingular and transversal to $\partial\mathcal{D}$. Denote this curve by Φ_t. Set

$$R = \{ (x,y) \in \mathcal{D} \cap \mathbb{R}^2 \mid f(x,y) \geqslant -\varepsilon \}$$

see fig. 1

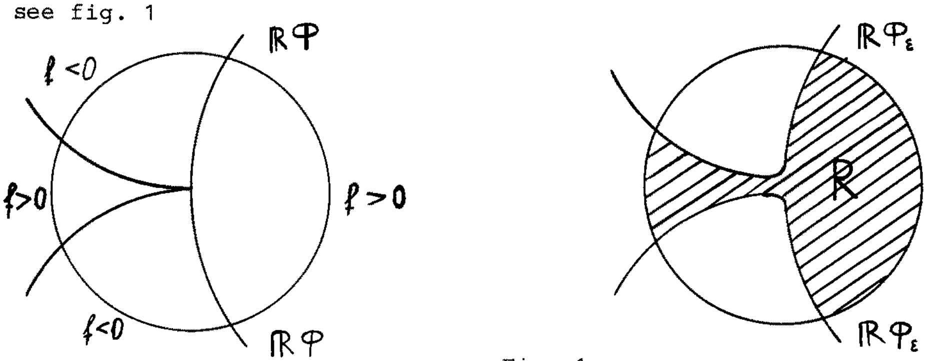


Fig. 1.

Now let us factorize by the complex conjugation $conj : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$. The ball $\mathcal{D}$ gives a ball $\mathcal{D}^* = \mathcal{D}/conj$. The surface R is not changed: it is contained in $fix(conj) = \mathbb{R}^2$ and so the natural projection $R \longrightarrow R/conj$ is a homeomorphism. We shall use the notation R for both R and $R/conj$. The surface $\mathbb{C}\Phi_\varepsilon \cap \mathcal{D}$ gives a compact surface $\mathcal{Y} = (\mathbb{C}\Phi_\varepsilon \cap \mathcal{D})/conj$ with a boundary $[(\mathbb{R}\Phi_\varepsilon \cap \mathcal{D}) \cup (\mathbb{C}\Phi_\varepsilon \cap \partial\Phi)]/conj$. The surfaces R and $\mathcal{Y}$ intersect in a curve $C = (\mathbb{R}\Phi_\varepsilon \cap \mathcal{D})/conj$, which is the common part of their boundaries. The union $\Sigma = R \cup \mathcal{Y}$ is a compact surface with a corner C.

The promised $\mathbb{Z}/_4$-quadratic space $(V_p, \circ, q_p)$ is formed of $V_p = H_1(\Sigma; \mathbb{Z}/_2)$, the intersection form $\circ$ of Σ and the Rohlin-Guillou-Marin form q_p of $(\mathcal{D}^*, \Sigma)$. As to the subspaces W_p and X_p they are nothing but in $in_* H_1(\mathcal{Y};\mathbb{Z}/_2)$ and in $in_* H_1(\partial\mathcal{Y}\setminus R;\mathbb{Z}/_2)$.

The promised pairing $\sqcap : L_p \times V_p \longrightarrow \mathbb{Z}/_2$ is defined by the intersection pairing

$$H_1(\Sigma,\partial\Sigma;\mathbb{Z}/_2)\times H_1(\Sigma;\mathbb{Z}/_2)$$

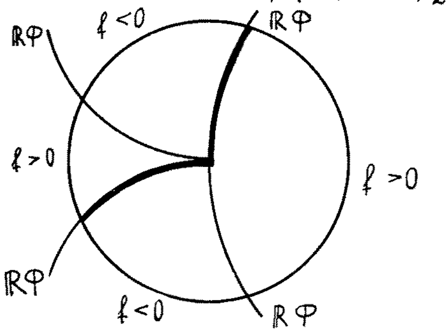
combined with a natural homomorphism

$$L_p = H_1(\mathbb{R}\Phi,\mathbb{R}\Phi\setminus p;\mathbb{Z}/_2) \xrightarrow[\text{(excision)}]{in_*^{-1}} H_1(\mathbb{R}\Phi\cap\mathcal{D},\mathbb{R}\Phi\cap\mathcal{D}\setminus p;\mathbb{Z}/_2) \xrightarrow{in_*^{-1}} H_1(\mathbb{R}\Phi\cap\mathcal{D},\mathbb{R}\Phi\cap\partial\mathcal{D};\mathbb{Z}/_2) \xrightarrow{in_*} H_1(\Sigma,\partial\Sigma;\mathbb{Z}/_2).$$

Since $\mathcal{Y}$ and Σ are connected, the factor-space $V_p/W_p = H_1(\Sigma;\mathbb{Z}/_2)/in_* H_1(\mathcal{Y};\mathbb{Z}/_2)$ is isomorphic to $H_1(\Sigma,\mathcal{Y};\mathbb{Z}/_2)$ and by excision, to $H_1(R,C;\mathbb{Z}/_2)$. To define ω_p, we combine these isomorphisms with the composition of the following isomorphisms

$$L_p = H_1(\mathbb{R}\Phi,\mathbb{R}\Phi\setminus p;\mathbb{Z}/_2) \xrightarrow[\text{(excision)}]{in_*^{-1}} H_1(\mathbb{R}\Phi\cap\mathcal{D},\mathbb{R}\Phi\cap\mathcal{D}\setminus p;\mathbb{Z}/_2) \xrightarrow{\partial} \widetilde{H}_0(\mathbb{R}\Phi\cap\mathcal{D}\setminus p;\mathbb{Z}/_2) \xrightarrow{in_*^{-1}} \widetilde{H}_0(\mathbb{R}\Phi\cap\partial\mathcal{D};\mathbb{Z}/_2) \xrightarrow{in_*} \widetilde{H}_0(\{x\in\partial\mathcal{D}\cap\mathbb{R}^2 \mid 0\geq f(x)\geq-\varepsilon\};\mathbb{Z}/_2) \xrightarrow{in_*^{-1}} \widetilde{H}_0(\partial C;\mathbb{Z}/_2),$$

homomorphism $in_*:\widetilde{H}_0(\partial C;\mathbb{Z}/_2)\longrightarrow\widetilde{H}_0(C,\mathbb{Z}/_2)$ and isomorphism $\partial^{-1}:\widetilde{H}_0(C;\mathbb{Z}/_2)\longrightarrow H_1(R,C,\mathbb{Z}/_2)$. This definition is presented visually at fig. 2: given two components of $\mathbb{R}\Phi\cap\mathcal{D}\setminus p$ they determine an element of $H_1(\mathbb{R}\Phi\cap\mathcal{D},\mathbb{R}\Phi\cap\mathcal{D}\setminus p;\mathbb{Z}/_2)$, ω_p add to every component it adjycent arc of $\{x \in \partial\mathcal{D} \cap \mathbb{R}^2 \mid 0 \geq f(x) \geq -\varepsilon\}$ to give the element of $H_1(R, C; \mathbb{Z}/_2)$



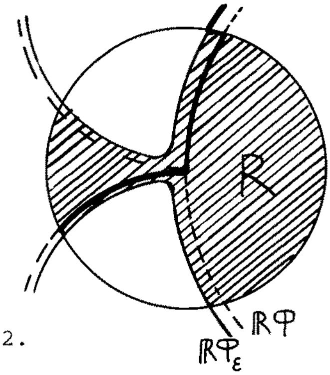


Fig. 2.

The group $H_1(\mathbb{R}\varphi^{\sim}, \mathbb{R}\varphi^{\sim} \setminus \nu^{-1}(p); \mathbb{Z}/_2)$ is generated by fundamental classes $[\ell]$ of components ℓ of $\mathbb{R}\varphi^{\sim} \cap \nu^{-1}(\mathbb{R}\varphi \cap \mathcal{D})$. For a component ℓ of $\mathbb{R}\varphi^{\sim} \cap \nu^{-1}(\mathbb{R}\varphi \cap \mathcal{D})$ both end points lie on one boundary circle of $\mathbb{C}\varphi^{\sim} \cap \nu^{-1}(\mathbb{R}\varphi \cap \mathcal{D})$. The image of the circle under ν is a boundary circle of $\mathbb{C}\varphi \cap \mathcal{D}$ and under the deformation $\mathbb{C}\varphi_t \cap \partial\mathcal{D}$, $0 \leq t \leq \varepsilon$ it remains to be the corresponding boundary circle C_t of the moving surface $\mathbb{C}\varphi_t \cap \mathcal{D}$. The image of C_t in $\partial\mathcal{D}^*$ is an arc C_t^* with end points lying in $\{x \in \partial\mathcal{D} \cap \mathbb{R}^2 \mid 0 \geq f(x) \geq -\varepsilon\}$. Hence, C_ε^* represents the element of $H_1(J, C; \mathbb{Z}/_2)$, which has the same boundary as $\omega_p(\nu_*[\ell]) \in H_1(R, C; \mathbb{Z}/_2)$. By exactness of the sequence

$$0 \to H_1(\Sigma; \mathbb{Z}/_2) \to H_1(R, C; \mathbb{Z}/_2) \oplus H_1(J, C; \mathbb{Z}/_2) \to H_0(C; \mathbb{Z}/_2)$$

these elements determine an element of $H_1(\Sigma; \mathbb{Z}/_2)$. We set it to be $\chi_p([\ell])$.

2.4. Singular point diagram and its $\mathbb{Z}/_4$-quadratic spaces.

Let A be a reduced (i.e. without multiple components) plane projective real algebraic curve of degree $m = 2k$. Then its real point set $\mathbb{R}A$ divided $\mathbb{R}P^2$ into two parts having $\mathbb{R}A$ as their common boundary. Let us fix one of the parts and denote it by

$\mathbb{R}P^2_+$. The choice of the part is equivalent to choice, up to positive constant factor, of an equation $a = 0$ of the curve (here a is a real homogeneous polynomial of degree m). Since the sign is fixed, singular point data is well defined for each real singular point of the curve A.

The scheme of joining of real singular points by real branches is nothing but a one-dimensional graph. It will be denoted by Γ_A It can be thought of as $\mathbb{R}A$ with all the non-singular components deleted. We supply it by additional structures. The first one is the homomorphism $i : H_1(\Gamma_A ; \mathbb{Z}/_2) \longrightarrow H_1(\mathbb{R}P^2_+ ; \mathbb{Z}/_2)$ induced by the natural inclusions $\Gamma_A \hookrightarrow \mathbb{R}A \hookrightarrow \mathbb{R}P^2$, the second one-singular point data for each vertex of Γ_A and the third one-homomorphisms λ_p : $H_1(\Gamma_A ; \mathbb{Z}/_2) \longrightarrow L_p$ induced by the composition of the inclusion $\Gamma_A \hookrightarrow \mathbb{R}A$ and localization. The graph Γ_A supplied with these structures will be called the singular point diagram of the curve and will be denoted by Δ.

At the rest part of the section we assign to Δ two $\mathbb{Z}/_4$-qudratic spaces $\widetilde{Q}_\Delta = (\widetilde{V}_\Delta , \circ , \tilde{q}_\Delta)$ and $Q_\Delta = (V_\Delta , \circ , q_\Delta)$ and a subspace B_Δ of V_Δ . $\widetilde{Q}_\Delta$ is involved in the simplified versions of the main formulations and does not involve $\tilde{L}_p$, X_p, ν_p, χ_p and ω_p. It is well defined by the following

(i) $\widetilde{V}_\Delta = H_1(\Gamma_A ; \mathbb{Z}/_2) \oplus \bigoplus\limits_p V_p$

(ii) the restriction of $\circ$ to the summand $\bigoplus\limits_p V_p$ is equal to the orthogonal sum of bilinear forms from singular point data, the restriction of $\circ$ to $H_1(\Gamma_A ; \mathbb{Z}/_2)$ is induced from the intersection form of $\mathbb{R}P^2$ via i :

$$x \circ y = i(x) \circ i(y)$$

and for $x \in H_1(\Gamma_A ; \mathbb{Z}/_2)$, $y \in V_p$

$$x \circ y = \lambda_p(x) \sqcap y \tag{9}$$

(iii) the restriction of $\tilde{q}_\Delta$ to $\bigoplus_p V_p$ is equal to the orthogonal sum $\bigoplus_p q_p$ of quadratic forms from the singular point data and the restriction of $\tilde{q}_\Delta$ to $H_1(\Gamma_A ; \mathbb{Z}/_2)$ is expressed via i :

$$\tilde{q}_\Delta(x) = \begin{cases} (-1)^k & \text{, if } i(x) \neq 0 \\ 0 & \text{, if } i(x) = 0 \end{cases} \tag{10}$$

The $\mathbb{Z}/_4$ -quadratic space Q_Δ is a shortened substitute for $\tilde{Q}_\Delta$. It and B_Δ are not involved in the simplified versions of the main formulations and involve $\tilde{L}_p , \nu_p , \chi_p$ and ω_p . When simplified reading, one may omit them.

To define Q_Δ let us take the subspace of $\tilde{Q}_\Delta$ with the underlying space $V_\Delta' \subset V_\Delta$,

$$V_\Delta' = \{ x + \sum_p v_p \in H_1(\Gamma_A ; \mathbb{Z}/_2) \oplus \bigoplus_p V_p \mid \omega_p \lambda_p(x) = \\ = v_p \bmod W_p \quad \text{for each } p \},$$

and factor it by the following part of its radical :

$$R_\Delta = \{ x + \sum_p v_p \in V_\Delta' \mid v_p = 0 \quad \text{for each } p \}.$$

Thus $V_\Delta = V_\Delta' / R_\Delta$

To define B_Δ let us take

$$B_\Delta' = \left\{ \begin{array}{c|l} x + \sum_p v_p \in H_1(\Gamma_A ; \mathbb{Z}/_2) \oplus & \text{for each } p \text{ there exists} \\ & x_p \in \tilde{L}_p \text{ such that } \nu_p(x_p) = \\ \oplus \bigoplus_p V_p & = \lambda_p(x) , v_p - \chi_p(x_p) \in X_p \end{array} \right\}$$

and set $B_\Delta = B'_\Delta / B'_\Delta \cap R_\Delta$.

2.5. Extension of notions: M-curve, $(M-i)$-curve, types I and II.

Here we extend these notions (see 1.2) from nonsingular Plane curves to general (not necessarily nonsingular and plane) curves.

A nonsingular real algebraic curve A is called an M-curve if the number of components of $\mathbb{R}A$ is equal to the genus of A enlarged by 1. For the given genus the number of components can not be more than in that case. The curve A is called an $(M-i)$-curve if the deficiency is equal to i. An irreducible singular curve is called an M-curve (respectively $(M-i)$-curve) if its nonsingular model (the result of normalization) is an M-curve (respectively $(M-i)$-curve). A reduced curve is called an M-curve if nonsingular models of all irreducible components are M-curves and is called an $(M-i)$-curve if the sum (over all irreducible components) of the deficiencies is equal to i.

A reduced real algebraic curve is said to be of type I if nonsingular models of all irreducible components are of type I. Otherwise it is said to be of type II.

§ 3. STATEMENT OF GENERAL RESULTS

3.1. Projective curves.

Let A be a reduced real plane projective curve of degree $m = 2k$ without non-real singular points and let $\mathbb{R}P^2_+$ be one of two parts of $\mathbb{R}P^2$ bounded by $\mathbb{R}A$. Let Δ be a singular point diagram of A related with $\mathbb{R}P^2_+$.

(3.A). Suppose the $\mathbb{Z}/_4$-quadratic space $\tilde{\mathfrak{a}}_\Delta$ is informative. Let $\tilde{k}$ be zero, if $\operatorname{Int} \mathbb{R}P^2_+$ is orientable, and $\tilde{k} = (-1)^k$

otherwise. If A is an M-curve, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + B(\tilde{q}_\Delta) + \tilde{b} \mod 8 \tag{11}$$

If A is an $(M-1)$-curve, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \pm 1 + B(\tilde{q}_\Delta) + \tilde{b} \mod 8 \tag{12}$$

If A is an $(M-2)$-curve of type II, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + d + B(\tilde{q}_\Delta) + \tilde{b} \mod 8 ,$$

$$\text{where} \quad d \in \{0, 2, -2\} \tag{13}$$

If A is of type I, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + B(\tilde{q}_\Delta) + \tilde{b} \mod 4 \tag{14}$$

We present another variant of this theorem. In all application it leads to the same results but usually through easier calculations.

(3.B). Suppose q_Δ vanishes on B_Δ. Let b be zero if $\mathbb{R}P^2_+$ is contractible in $\mathbb{R}P^2$ and $b = (-1)^k$ otherwise. If A is an M-curve then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + B(q_\Delta) + b \mod 8 \tag{15}$$

If A is an $(M-1)$-curve, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 \pm 1 + B(q_\Delta) + b \mod 8 \tag{16}$$

If A is an $(M-2)$-curve of type II, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + d + B(q_\Delta) + b \mod 8, \tag{17}$$

where $d \in \{0, 2, -2\}$

If A is of type I, then

$$\chi(\mathbb{R}P^2_+) \equiv k^2 + B(q_\Delta) + b \mod 4 \tag{18}$$

3.2. Smoothings of a plane curve singularity.

As above in 2.3, let $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a real holomorphic function and p its real isolated singular point with $f(p)=0$. Denote by Φ the curve defined by the equation $f(x,y)=0$. Let $\mathcal{D}$ be a ball in $\mathbb{C}^2$ centered at p and so small that the pair $(\mathcal{D}, \mathbb{C}\Phi \cap \mathcal{D})$ is homeomorphic to the cone over $(\partial\mathcal{D}, \mathbb{C}\Phi \cap \partial\mathcal{D})$. Let $\phi_t: \mathbb{C}^2 \longrightarrow \mathbb{C}$, $t \in \mathbb{R}$ be a continuous family of real holomorphic functions with $\phi_0 = f$. Denote by Ψ_t the curve defined by the equation $\phi_t(x,y)=0$. Suppose that $\mathbb{C}\Psi_t$ has no singular points in $\mathcal{D}$ and is transversal to $\partial\mathcal{D}$ for $t \in (0, \varepsilon]$. Set

$$\mathcal{D}_+ = \{(x,y) \in \mathcal{D} \cap \mathbb{R}^2 \mid \phi_\varepsilon(x,y) \geq 0\}$$

In this section we state results on topology of $\mathcal{D}_+$ similar to (3.A- and (3.B). The main idea of the transfering is to glue pairs $(\mathcal{D}, \mathbb{C}\Psi_\varepsilon \cap \mathcal{D})$ and $(\mathcal{D}, \mathbb{C}\Phi \cap \mathcal{D})$ by an equivariant diffeomorphism of their boundaries arisen from the deformation $\mathbb{C}\Psi_t \cap \partial\mathcal{D}$, $t \in [0, \varepsilon]$. The gluing gives a 4-dimensional sphere with an involution and a subset which is a smooth submanifold at each point except one and is invariant under the involution. This situation is similar to that of the projective plane and a singular real curve in it. Moreover, we observe two simplifications: first, S^4 is simpler than $\mathbb{C}P^2$, second, here we have only one singular point.

Before stating the results we ought to describe modification of auxiliary notions (such as $\mathbb{Z}/_4$ -quadratic space of the singular point diagram) involved in (3.A) and (3.B).

Let Γ be a bouquet of circles which are in 1 - 1 correspondence with components of $\mathbb{R}\Psi_\varepsilon \cap \mathcal{D}$ homeomorphic to I. It can be throught of as the union of these components of $\mathbb{R}\Psi_\varepsilon \cap \mathcal{D}$ glued to $\mathbb{R}\Phi \cap \mathcal{D}$ by the natural bijection of the boundaries. The number of the circles is denoted by τ, it is equal to the number of real branches of Φ passing through p.

Let $\lambda : H_1(\Gamma ; \mathbb{Z}/_2) \longrightarrow L_p$ be the composition

$$H_1(\Gamma ; \mathbb{Z}/_2) \xrightarrow{in_*} H_1(\Gamma, \Gamma \setminus p ; \mathbb{Z}/_2) \xrightarrow{in_*^{-1}} H_1(\mathbb{R}\Phi \cap \mathcal{D}, \mathbb{R}\Phi \cap \mathcal{D} \setminus p ;$$

$$\mathbb{Z}/_2) \xrightarrow{in_*} H_1(\mathbb{R}\Phi, \mathbb{R}\Phi \setminus p ; \mathbb{Z}/_2) = L_p$$

The graph Γ supplied with the singular point data of p and the homomorphism λ will be denoted by Δ. Now we assign to it $\mathbb{Z}/_4$ -quadratic spaces $\widetilde{Q}_\Delta = (\widetilde{V}_\Delta, \circ, \tilde{q}_\Delta)$ and $Q_\Delta = (V_\Delta, \circ, q_\Delta)$ and a subspace B_Δ of V_Δ, cf. 2.4. The space $\widetilde{Q}_\Delta$ is involved in the simplified version of formulation and does not require $\tilde{L}_p$, $\chi_p, \nu_p, \varkappa_p$ and ω_p for its definition. It is well defined by the following

(i) $\widetilde{V}_\Delta = H_1(\Gamma ; \mathbb{Z}/_2) \oplus V_p$

(ii) the restriction of $\circ$ to the summand V_p is the bilinear form from the singular point data.

$$x \circ y \qquad \text{for} \qquad x, y \in H_1(\Gamma ; \mathbb{Z}/_2),$$

$$x \circ y = \lambda_p(x) \cap y \qquad \text{for} \qquad x \in H_1(\Gamma ; \mathbb{Z}/_2),\ y \in V_p \tag{19}$$

(iii) the restriction of $\tilde{q}_\Delta$ to V_p is the quadratic form q_p

from the singular point data, the restriction of $\tilde{q}_\Delta$ to $H_1(\Gamma;\mathbb{Z}/_2)$ is equal to zero.

The $\mathbb{Z}/_4$ -quadratic space Q_Δ is a shortened substitute for $\tilde{Q}_\Delta$. Together with B_Δ it is not involved in the simplified statement. When simplified reading one may omit them.

To define Q_Δ let us take the subspace of $\tilde{Q}_\Delta$ with the underlying space $V'_\Delta \subset V_\Delta$

$$V'_\Delta = \{ x + v \in H_1(\Gamma;\mathbb{Z}/_2) \oplus V_p \mid \omega_p \lambda(x) = v \mod W_p \}$$

and factor it by the following part of its radical :

$$R_\Delta = \{ x + v \in V'_\Delta \mid \omega_p \lambda(x) = 0 \}$$

Thus $V_\Delta = \tilde{V}_\Delta / V'_\Delta$. To define B_Δ let us take

$$B'_\Delta = \{ x + v \in H_1(\Gamma;\mathbb{Z}/_2) \oplus V_p \mid \exists\, y \in L^{\sim}_p : \nu_p(y) = \lambda(x) \ ,$$
$$v - \chi_p(y) \in X_p \}$$

and set $B_\Delta = B'_\Delta / B'_\Delta \cap R_\Delta$

Now transfer the notions of M-, $(M-i)$ -curve and type to the case of smoothings. A smoothing Ψ_ε of a singular point of Φ is called an M -smoothing, if the number of components of $(\mathbb{R}\Psi_\varepsilon \cap \mathcal{D}) \cup (\mathbb{C}\Psi_\varepsilon \cap \partial\mathcal{D})$ is equal to the genus (number of handles) of $\mathbb{C}\Psi_\varepsilon$ enlarged by 1. This number can not be more than in that case. The smoothing is called an $(M-i)$ -smoothing if the deficiency is equal to i . The smoothing is said to be of type I if $\mathbb{C}\Psi_\varepsilon \cap \mathcal{D}$ is divided by $\mathbb{R}\Psi_\varepsilon \cap \mathcal{D}$ into two path components. Otherwise it is said to be of type II .

(3.C). Suppose the $\mathbb{Z}/_4$-quadratic space $\tilde{Q}_\Delta$ is informative. If Ψ is an M -smoothing, then

$$\chi(\mathcal{D}_+) \equiv B(\tilde{q}_\Delta) \bmod 8 \tag{20}$$

If Ψ_ε is an $(M-1)$-smoothing, then

$$\chi(\mathcal{D}_+) \equiv B(\tilde{q}_\Delta) \pm 1 \bmod 8 \tag{21}$$

If Ψ_ε is an $(M-2)$-smoothing of type II, then

$$\chi(\mathcal{D}_+) \equiv B(\tilde{q}_\Delta) + d \bmod 8 \qquad \text{where } d \in \{-1, 1, -3\} \tag{22}$$

If Ψ_ε is of type I, then

$$\chi(\mathcal{D}_+) \equiv B(\tilde{q}_\Delta) \bmod 4 \tag{23}$$

We present another variant of this theorem. For all applications it leads to the same result but usually through easier calculations. Remind that τ involved below is the number of real branches of φ passing through p.

(3.D) Suppose q_Δ vanishes of B_Δ . If Ψ_ε is an M -smoothing, then

$$\chi(\mathcal{D}_+) \equiv B(q_\Delta) + \tau - 1 \bmod 8 \tag{24}$$

If Ψ_ε is an $(M-1)$-smoothing, then

$$\chi(\mathcal{D}_+) \equiv B(q_\Delta) + \tau - 1 \pm 1 \bmod 8 \tag{25}$$

If Ψ_ε is an $(M-2)$-smoothing of type II, then

$$\chi(\mathcal{D}_+) \equiv B(q_\Delta) + \tau + d \bmod 8, \text{ where } d \in \{-1, 1, -3\} \tag{26}$$

If Ψ_ε is of type I, then

$$\chi(\mathcal{D}_+) \equiv B(q_\Delta) + \tau - 1 \mod 4 \tag{27}$$

§ 4. APPLICATIONS

4.1. Sufficient conditions for applicability.

As in Theorem (3.B), let A be a reduced real plane projective curve of even degree without non-real singular points and let Δ be a singular point diagram of A. Let us consider the following condition

(4.A) *For each irreducible component of the non-singular model of A the images of all real path components except at most one do not contain a singular point of A.*

Sometimes (4.A) makes Theorem (3.B) to work:

(4.B) *Under* (4.A), *if each singular point of A has no non-real branch then q_Δ vanishes on B_Δ.*

(4.B) is generalized below. The generalization is not applied in this paper. We present it for the sake of completeness only.

(4.C) *Under* (4.A), *if at each singular point of A each non-real branch β has an even intersection number with the union of all branches different from β and* $\operatorname{conj}\beta$, *then q_Δ vanishes on B_Δ.*

Now as in Theorem (3.D), let Ψ_ε be a smoothing of Φ. Let us consider the condition

(4.A') *The* $(\mathbb{C}\Psi_\varepsilon \cap \partial\mathcal{D})$ *united with all components of $\mathbb{R}\Psi_\varepsilon \cap \mathcal{D}$ homeomorphic to* I *is connected.* It is a substitute for (4.A):

(4.C') *Under* (4.A'), *if each non-real branch β of $\mathbb{C}\Phi \cap \mathcal{D}$ has even intersection number with the union of all branches different* from β and $conj\ \beta$, then q_Δ vanishes on B_Δ.

The following criteria (4.E) and (4.F) require the condition (4.D) instead of (4.A') [(4.E) is slightly less general than (4.F)]. Let $\mathcal{D}^* = \mathcal{D}/conj$, $\mathcal{Y}_\varepsilon = \mathbb{C}\Psi_\varepsilon/conj$, $\Sigma_\varepsilon = \mathcal{Y}_\varepsilon \cup \mathcal{D}_+$ (cf.3.2).

(4.D) Each boundary component of Σ_ε has even linking number with the union of the others boundary components of Σ_ε.

(4.E) Under (4.D), if each component of $\mathcal{D}_+$ contains only one non-closed component of $\mathbb{R}\Psi_\varepsilon$, then q_Δ vanishes on B_Δ.

(4.F) Under (4.D), if the subgroup $in_* H_1(\partial \mathcal{Y}_\varepsilon; \mathbb{Z}/2)$ of $H_1(\Sigma; \mathbb{Z}/2)$ is contained in $in_* H_1(\partial\Sigma; \mathbb{Z}/2) + in_* H_1(\mathcal{D}_+; \mathbb{Z}/2)$, then q_Δ vanishes on B_Δ.

Proofs are given in 6.6.

4.2. Korchagin's curves.

Let A be a real plane projective curve of degree 7 with only one singular point and let there be 4 branches at the singular point, all branches be real, and one of them be transversal to the others, which are ordinarily tangent to each other (such a singular point is denoted by Z_{15} in Arnold's notations). In some (perhaps non-linear) coordinates x, y in some neighbourhood of the point the curve is defined by an equation

$$x(y-ax^2)(y-bx^2)(y-cx^2)=0 \quad (a\neq b,\ b\neq c,\ c\neq a).$$

Real schemes of such curves were treated by Korchagin [11]. He conjectured some congruences. They constitute a part of the following propositions which does not involve type of the curve (purely real point of view).

(4.G) Let A has a real scheme outlined in the fig. 3 ($\langle\alpha\rangle$ designates a set of α ovals each lying outside others)

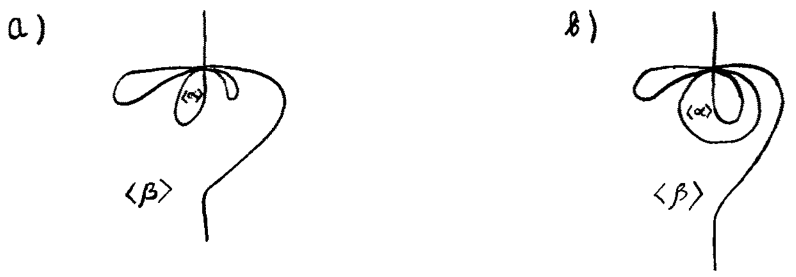


Fig. 3.

If $\alpha+\beta=6$ *then* $\alpha-\beta\equiv 4 \bmod 8$, *if* $\alpha+\beta=5$ *then* $\alpha-\beta\equiv 4\pm 1 \bmod 8$, *if* $\alpha+\beta=4$ *and* A *is of type* II *then* $\alpha-\beta\not\equiv 0 \bmod 8$, *if* A *is of type* I *then* $\alpha-\beta\equiv 0 \bmod 4$.

(4.H) *Let* A *has a real scheme represented on the fig.* 4

Fig. 4.

If $\alpha+\beta=6$ *then* $\alpha-\beta\equiv -2 \bmod 8$, *if* $\alpha+\beta=5$ *then* $\alpha-\beta\equiv -2\pm 1 \bmod 8$, *if* $\alpha+\beta=4$ *and* A *is of type* II *then* $\alpha-\beta\equiv 2 \bmod 8$, *if* A *is of type* I *then* $\alpha-\beta\equiv 2 \bmod 4$

The condition $\alpha+\beta=6-i$ means that A is an $(M-i)$-curve.

To prove (4.G) and (4.H) it is enough to add the straight line tangent at the singular point to three (pairwise tangent) branches of A and to apply the theorem (3.B) to the reducible curve $A\cup\ell$

of degree 8. The theorem is applicable due to (4.A). We have a possibility not to do a straightforward calculation of the invariant $B(q_\Delta)$. Really, the theorem implies that for each scheme from fig. 3 and 4 there are a congruence of the form $\alpha - \beta \equiv u \mod 8$ in M-case, a congruence of the form $\alpha - \beta \equiv u \pm 1 \mod 8$ in $(M-1)$-case etc; the true values of u can be taken from examples constructed by Korchagin [11].

4.3. The case of non-degenerate double points.

Let A be a real plane projective curve of degree $m = 2k$ and let real non-degenerate double points with real tangents be the only singularities of A. Let $\mathbb{R}P^2_+$ be a half of $\mathbb{R}P^2$ bounded by $\mathbb{R}A$. As in 2.4, denote by Γ_A the union of components $\Gamma^1_1, \dots, \Gamma^r$ of $\mathbb{R}A$ containing singular points. Denote by $C^i_1, \dots, C^i_{r(i)}$ $(1 \leqslant i \leqslant r)$ components of $\mathbb{R}P^2 \setminus (\Gamma_A \cup \Theta)$ which lie on the other side of Γ^i than $\mathbb{R}P^2_+$ and by L some simple loop in Γ_A non-contractible in $\mathbb{R}P^2$ (if such a loop exists).

We present a straightforward independent description of $(V_\Delta, \circ, q_\Delta)$ and B_Δ constructed in 2.4.

(4.I) *Let ∂^i_j be an element of $H_1(\Gamma_A; \mathbb{Z}/_2)$ realized by the boundary of C^i_j and let ℓ be an element of $H_1(\Gamma_A; \mathbb{Z}/_2)$ realized by L. Then V_Δ is a subspace of $H_1(\Gamma_A; \mathbb{Z}/_2)$ generated by ℓ, ∂^i_j $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant r(i))$. The bilinear form $\circ$ and the quadratic function q_Δ are determined by the following:*

(i) *$q_\Delta(\partial^i_j)$ is equal modulo 4 to a number of singular points through which the boundary of C^i_j passes only once; $q_\Delta(\ell)$ is equal modulo 4 to $(-1)^k$ plus the number of singular points through which L passes as in fig. 5 (not as in fig. 6);*

Fig. 5.

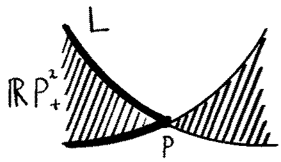


Fig. 6.

(ii) $\partial^i_j \circ \partial^k_n$ $((i,j)\neq(k,n))$ is equal modulo 2 to the number of singular points common to the boundaries of C^i_j and C^k_n

(iii) $\partial^i_j \circ l$ is equal modulo 2 to the number of singular points through which the boundary of C^i passes only once and L passes as in fig. 5.

The space B_Δ is generated by elements realized by smoothly immersed circles.

We have described all ingradients of Theorem (3.B). As a result, we obtain that in the situation considered in this section the theorem works iff the following condition is satisfied.

(4.J) Each real branch of A (i.e. smoothly immersed circle) contractible in $\mathbb{R}P^2$ passes through $n \equiv 0 \bmod 4$ singular points and each real branch of A non-contractible in $\mathbb{R}P^2$ passes through $n \equiv (-1)^{k+1} \bmod 4$ singular points.

To prove the equivalence of (4.I) and the definitions from 2.3, 2.4 it is sufficient to observe the following: for each non-degenerate double point p with real tangents the surfaces R_p, $\mathcal{Y}_p$ are homeomorphic to a disk, Σ_p to a Möbius band and further, $V_p = \mathbb{Z}/_2$, $W_p = 0$ and $q_p(v) = -1$ for $v \in V_p$, $v \neq 0$.

Let us apply the criterion (4.J) to curves of degree 6 represented by figures 7-11 (the first four curves are supposed to be reducible)

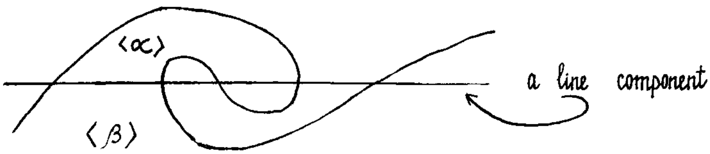


Fig. 7.

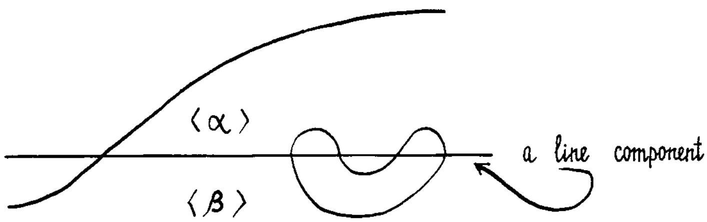


Fig. 8.

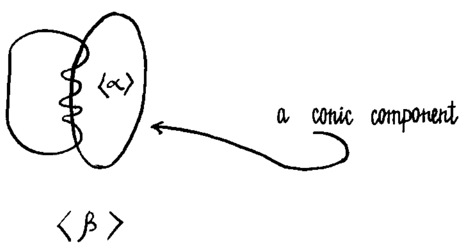


Fig. 9.

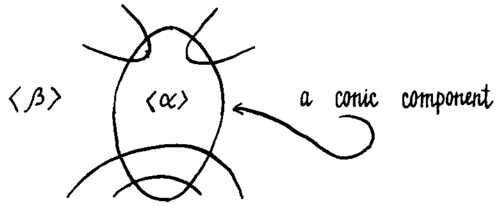


Fig. 10.

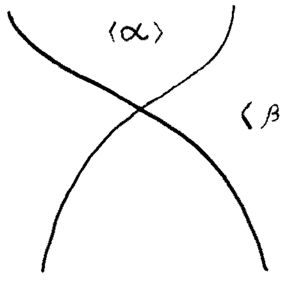


Fig. 11.

According to (4.J) Theorem (3.B) is aplicable to all these situations. As in 4.2, the Theorem immediately implies that the appropriate congruences for $\alpha - \beta$ are to be satisfied. The concrete form of congruences may be obtained by computation of $B(q_\Delta|)$ or by known examples (cf. 4.2)

	Assumption	Assertion
fig.7	$\alpha+\beta=6$	$\alpha-\beta\equiv 6 \mod 8$
	$\alpha+\beta=5$	$\alpha-\beta\equiv 5$ or $7 \mod 8$
	$\alpha+\beta=4$ and A is of type II	$\alpha-\beta\not\equiv 2 \mod 8$
	A is of type I	$\alpha-\beta\equiv 2 \mod 4$
fig.8	$\alpha+\beta=5$	$\alpha-\beta\equiv 5 \mod 8$
	$\alpha+\beta=4$	$\alpha-\beta\equiv 4$ or $6 \mod 8$
	$\alpha+\beta=3$ and A is of type II	$\alpha-\beta\not\equiv 1 \mod 8$
	A is of type I	$\alpha-\beta\equiv 1 \mod 4$
fig.9	$\alpha+\beta=3$	$\alpha-\beta\equiv 5 \mod 8$ (i.e. $\alpha=2, \beta=1$)
	$\alpha+\beta=2$	$\alpha-\beta\equiv 0$ or $2 \mod 8$
	A is of type I	$\alpha-\beta\equiv 1 \mod 4$

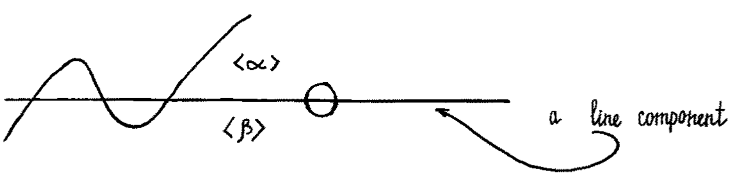
	Assumption	Assertion
fig.10	$\alpha+\beta=3$ $\alpha+\beta=2$ $\alpha+\beta=1$ and A is of type II A is of type I	$\alpha-\beta\equiv 3 \mod 8$ (i.e. $\alpha=3, \beta=0$) $\alpha-\beta\equiv 2$ or $4 \mod 8$ (i.e. $\alpha=2, \beta=0$) $\alpha-\beta\equiv -1 \mod 8$ (i.e. $\alpha=1, \beta=0$) $\alpha-\beta\equiv -1 \mod 4$
fig.11	$\alpha+\beta=8$ $\alpha+\beta=7$ $\alpha+\beta=6$ and A is of type II A is of type I	$\alpha-\beta\equiv 4 \mod 8$ $\alpha-\beta\equiv 3$ or $5 \mod 8$ $\alpha-\beta\not\equiv 0 \mod 8$ $\alpha-\beta\equiv 0 \mod 4$

Table 1.

Purely real part of first four blocks (results not refering to the type of the curve) was originally obtained by G.M.Polotovskij [14] (via a different approach). In [14] he considered curves of degree 6 decomposed into curves being non-singular and transversal each other. One can find there a big stock of situations in which (3.B) works.

The first row of the last block was conjectures by I.V.Itenberg when classifying curves of degree 6 with one non-degenerated double point.

Numerous examples prove necessity of (4.J), see e.g. [14]. In particular, there exist reducible curves of degree 6 shown in fig. 12



any α, β with $\alpha+\beta\leq 5$

Fig. 12.

4.4. Smoothings of some plane curve singularities.

Here we consider cases when in some (perhaps nonlinear) coordinates x, y in some neighbourhood of the singular point the curve is defined by a real equation.

$$(y - ax^2)(y - bx^2)(y - cx^2) = 0$$

(a, b, c are distinct numbers not necessarily real)

(Y_{10} in Arnold's notations; three nonsingular branches ordinarily tangent to each other) or by a real equation

$$(y - ax^2)(y - bx^2)(y - cx^2)(y - dx^2) = 0$$

(a, b, c, d are distinct numbers not necessarily real)

(X_{21} ; four nonsingular branches ordinarily tangent to each other). We have chose namely these singularities because of their applications in constructing curves with presecribed topological properties, see [21] . For N_{16} (five nonsingular branches transversal to each other), which is the other singularity involved there, our congruences could be applied too. We have omitted these applications since, as it was shown by E.J. Shustin [20] , smoothings of N_{16} are essentially affine nonsingular plane curves of degree 5 with 5 different asymptotes . Such curves are considered above in section 4.3. For Y_{10} as for N_{16} , the classification is completed (see [22], [20]), for X_{21} it is closed to completion (see [23] , [20]). In these classification achievements, congruences of the sort considered in our paper play important role.

Let us apply criteria (4.C') and (4.E) to smoothings outlined in the figures 13 - 17

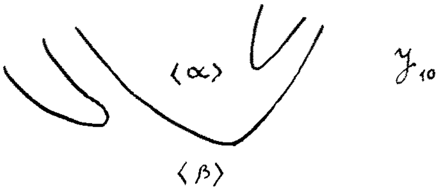


Fig. 13.

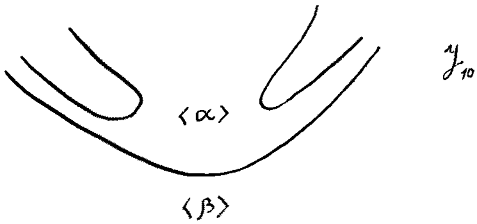


Fig. 14.

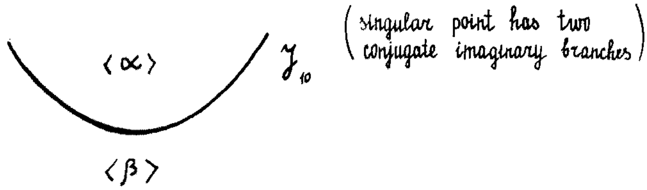


Fig. 15.

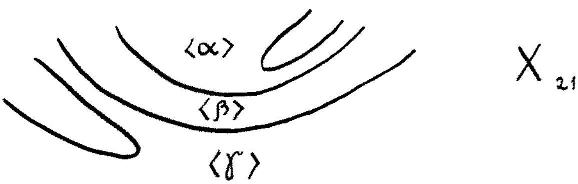


Fig. 16.

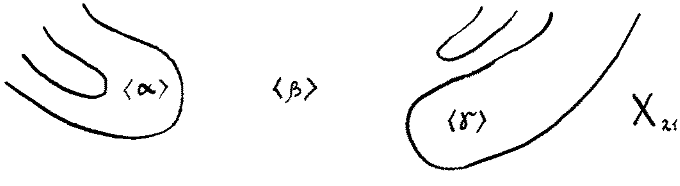


Fig. 17.

According to (4.C') theorem (3.D) is applicable to situations of figures 13, 15-17 and according to (4.E) it is applicable to one of fig. 14. As usually, the concrete form of congruences may be obtained by a computation of $B(q_\Delta)$ or by known examples (see the latters in [22] and [23])

	Assumption	Assertion
fig.13 and fig.15	$\alpha+\beta=4$	$\alpha\equiv 0 \mod 4$
	$\alpha+\beta=3$	$\alpha=0$ or 3
	$\alpha+\beta=2$ and the smoothing is of type II	$\alpha=0$ or 2
	the smoothing is of type I	$\alpha-\beta\equiv 0 \mod 4$
fig.14	$\alpha+\beta=3$	$\alpha=3$
	$\alpha+\beta=2$	$\alpha=2$
	$\alpha+\beta=1$ and the smoothing is of type II	$\alpha=1$
	the smoothing is of type I	$\alpha-\beta\equiv -1 \mod 4$
fig.16	$\alpha+\beta+\gamma=9$	$\beta\equiv 1 \mod 4$
	$\alpha+\beta+\gamma=8$	$\beta\equiv 0$ or $2 \mod 4$
	$\alpha+\beta+\gamma=7$ and the smoothing is of type II	$\beta\not\equiv 2 \mod 4$
	the smoothing is of type I	$\alpha-\beta+\gamma\equiv -1 \mod 4$
fig.17	$\alpha+\beta+\gamma=9$	$\beta\equiv 0 \mod 4$
	$\alpha+\beta+\gamma=8$	$\beta\equiv -1$ or $1 \mod 4$
	$\alpha+\beta+\gamma=7$ and the smoothing is of type II	$\beta\not\equiv 1 \mod 4$
	the smoothing is of type I	$\alpha-\beta+\gamma\equiv 1 \mod 8$

In [20] one can find some more isotopy types of smoothing of X_{21} and appropriate congruences which can be derived from our theorems. They can be supplemented by congruences refering to the type of smoothing.

§ 5. PREREQUISITE FOR PROOFS

5.1. Rohlin-Guillou-Marin congruence . (see [7]).

Let X be an oriented smooth closed four-dimensional manifold and let F be its smooth closed two-dimensional submanifold realizing in $H_2(X; \mathbb{Z}/_2)$ the class which is the Poincaré dual to the Stiefel-Whitney class $w_2(X)$. Then

$$2B(q) \equiv \sigma(X) - F \circ F \mod 16 \tag{28}$$

where q is the Rohlin-Guillou-Marin form of the pair (X,F), $\sigma(X)$ is the signature of X and $F \circ F$ is the natural Euler number of F in X (self-intersection number).

5.2. Informative subspaces.

Let $(V, \circ, q)$ be a nonsingular $\mathbb{Z}/_4$ -quadratic space. Its subspaces are defined to be quadratic spaces $(\mathcal{U}, \circ|, q|)$, where $\mathcal{U}$ is a vector subspace of V, $q| = q|_{\mathcal{U}}$ and $\circ| = \circ|_{\mathcal{U}\times\mathcal{U}}$ We say that a subspace is *informative* if $\mathcal{U}$ contains its own orthogonal (with respect to $\circ$) complement $\mathcal{U}^{\perp}$ and q vanishes on $\mathcal{U}^{\perp}$. This is conformed to the definition in 2.1: every informative subspace is an informative space (but not vice versa!)

5.A. *A nonsingular* $\mathbb{Z}/_4$*-quadratic space is cobordant to any* its informative subspace.

In various equivalent forms this fact is well known. For the proof it is enough to check that the graph $H \subset \mathcal{U} \oplus V$ os the inclusion $\mathcal{U} \hookrightarrow V$ provides (according to the definition of the Witt group, see 2.1) vanishing of $(\mathcal{U}, \circ|, q|) \oplus (V, \circ, -q)$ in the Witt group.

[The definition of an informative subspace may be transformed with evident alterations from nonsingular to informative ambient spaces. An informative space accurs to be cobordant to any its informative subspace .]

5.3. Additivity for B .

Let F be a closed subspace and let $(V, \circ, q)$ be a $\mathbb{Z}/_4$-quadratic space with $V = H_1(F; \mathbb{Z}/_2)$ and $\circ$ = the intersection form. Given a decomposition of F into two compact subsurfaces F_1 and F_2 (perhaps non-connected) with common boundary $\partial = \partial F_1 = \partial F_2$ there naturally arises subspaces $\mathcal{U}_1 = in_* H_1(F_1; \mathbb{Z}/_2)$, $\mathcal{U}_2 = in_* H_1(F_2; \mathbb{Z}/_2)$, $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2$ of V . It is evident that $\mathcal{U}^\perp = in_* H_1(\partial; \mathbb{Z}/_2)$. So, according to 5.2, if q vanishes on $in_* H_1(\partial; \mathbb{Z}/_2)$ then $(\mathcal{U}, \circ|, q|)$ is an informative subspace of $(V, \circ, q)$, $(\mathcal{U}_1, \circ|, q|)$ and $(\mathcal{U}_2, \circ|, q|)$ are informative spaces and $B(q) = B(q|_{\mathcal{U}_1}) + B(q|_{\mathcal{U}_2})$. In fact, q vanishes on $in_* H_1(\partial; \mathbb{Z}/_2)$ iff $(\mathcal{U}, \circ|, q|)$ is an informative subspace iff $(\mathcal{U}_1, \circ|, q|)$ and $(\mathcal{U}_2, \circ|, q|)$ are informative spaces.

The $\mathbb{Z}/_4$-quadratic space $(H_1(F; \mathbb{Z}/_2), \circ, q)$ induces (by in_*) $\mathbb{Z}/_4$-quadratic spaces $(H_1(F_j; \mathbb{Z}/_2), \circ, q_j)$, where $\circ$ = the intersection form and $q_j = q \circ in_*$. These spaces are informative if (and only if) $(\mathcal{U}_1, \circ|, q|)$ and $(\mathcal{U}_2, \circ|, q|)$ are informative spaces and then they have the same associated nonsingular spaces. Thus if q vanishes on $in_* H_1(\partial;$

$\mathbb{Z}/_2$) then

$$B(q) = B(q_1) + B(q_2) \tag{29}$$

5.4. Low-dimensional $\mathbb{Z}/_4$-quadratic spaces. (see, e.g. [2]).

Any $\mathbb{Z}/_4$-quadratic space can be decomposed in an orthogonal sum of one and two-dimensional quadratic spaces. Besides if $\mathbb{Z}/_4$-quadratic space $(V, \circ, q)$ is odd (i.e. there exists $v \in V$ with $v \circ v \neq 0$) then it can be decomposed in an orthogonal sum of one-dimensional spaces only. There are only two different (up to isomorphism) one dimensional spaces:

$$(\mathbb{Z}/_2, \circ, q_+) \quad \text{with} \quad a \circ a = 1,\ q_+(a) = 1 \quad \text{for } a \neq 0;$$
$$(\mathbb{Z}/_2, \circ, q_-) \quad \text{with} \quad a \circ a = 1,\ q_-(a) = -1 \quad \text{for } a \neq 0.$$

Both spaces are odd. It is clear that $B(q_+) = 1$ and $B(q_-) = -1$. Two-dimensional spaces are even iff they are indecomposable, we have no need of their precise form and remark only that in this case values of B are 0 and 4. Since a two-dimensional odd space $(V, \circ, q)$ is a sum of two one-dimensional spaces, it has $B(q) = 0$ or ± 2.

§ 6. PROOF OF THE MAIN THEOREMS

6.1. Prototype. Marin's proof for nonsingular curves.

The factor-space $\mathbb{C}P^2/conj$ carries a natural smooth structure (as always when the fixed point set of a smooth involution has (real) codimension 2). It is well known that this manifold is diffeomorphic to S^4. The complex point set $\mathbb{C}A$ of the (real) curve

A is invariant under the complex conjugation. Its image $\mathbb{C}A/\mathrm{conj}$ is a compact two-dimensional submanifold of $\mathbb{C}P^2/\mathrm{conj} = S^4$ with boundary $\mathbb{R}A$. It is clear that $\mathbb{C}A/\mathrm{conj}$ is orientable iff the curve is of type I.

Apply the congruence (28) to $F = \mathbb{C}A/\mathrm{conj} \cup \mathbb{R}P^2_+$. Straightforward calculations show that

$$F \circ F = 2k^2 - 2\chi(\mathbb{R}P^2_+)$$

(see, e.g. [16]) and so (28) turns into

$$\chi(\mathbb{R}P^2_+) - k^2 \equiv B(q) \mod 8 .$$

It remains to calculate $B(q)$.

Now apply 5.3 to $F_1 = \mathbb{R}P^2_+$, $F_2 = \mathbb{C}A/\mathrm{conj}$ and $\partial = \mathbb{R}A$. Some straightforward calculations more show that q vanishes on $in_* H_1(F_1; \mathbb{Z}/_2)$ and thus on $in_* H_1(\partial; \mathbb{Z}/_2)$ (see [12]). So we get $B(q) = B(q_2)$, see (29).

If A is an M-curve then the genus of F_2 is zero. Consequently, $q_2 = 0$ and $B(q_2) = 0$. So we get (1.A). If A is an $(M-1)$-curve then F_2 is homeomorphic to the projective plane with several holes. So in this case the associated nonsingular space has dimension one and we get $B(q_2) = \pm 1$ (see 5.4) and (1.B). If A is an $(M-2)$-curve of type II then F_2 is homeomorphic to the Klein bottle with several holes. In this case the associated nonsingular space has dimension 2 and it is odd, so that we get $B(q_2) = 0$ or 2 (see 5.4 and (1.C)). If A is of type I, then q_2 is even and $B(q_2) \equiv 0 \mod 4$. This gives (1.D).

6.2. An auxiliary surface and its decomposition.

For every singular point p of the curve A let us fix a sufficiently small ball $\mathcal{D}_p \subset \mathbb{C}P^2$ with center in p . Introduce an auxiliary curve A_ε defined by an equation $a = -\varepsilon(x_0^{2k} + x_1^{2k} + x_2^{2k})$ where a is the polynomial chosen as in 2.4 up to a positive constant factor and ε is a positive number such that for any $t \in (0, \varepsilon]$ the curve defined by the equation $a = -t(x_0^{2k} + x_1^{2k} + x_2^{2k})$ has no singular point and is transversal to $\partial \mathcal{D}_p$ for each p , cf. 2.3.

Let us factorize $\mathbb{C}P^2$ by $conj$ and take in $S^4 = \mathbb{C}P^2/conj$ surface $F = \mathbb{C}A_\varepsilon/conj \cup \mathbb{R}P^2_\varepsilon$ where $\mathbb{R}P^2_\varepsilon \subset \mathbb{R}P^2$ is defined by the inequality $a \geqslant -\varepsilon(x_0^{2k} + x_1^{2k} + x_2^{2k})$. This auxiliary surface has a corner $\mathbb{R}A_\varepsilon$. The intersection of F with $\mathcal{D}_p^* = \mathcal{D}_p/conj$ is nothing but the surface Σ_p constructed in 2.4. Let us set $\Sigma = \cup \Sigma_p$ and $\mathcal{Y} = \cup \mathcal{Y}_p$

We decompose F into three pieces Π, Π' and Π'' where Π' coincides with $\mathbb{R}P^2_\varepsilon$, Π is a union of $\mathcal{Y}$ with a small color C of $\mathbb{R}A_\varepsilon \setminus \cup \operatorname{Int} \mathcal{D}_p^*$ in $\mathbb{C}A_\varepsilon/conj \setminus \cup \operatorname{Int} \mathcal{D}_p^*$ and $\Pi'' = Cl(F \setminus (\Pi' \cup \Pi))$. Each piece is a compact surface. Boundaries ∂' and ∂'' of Π' and Π'' form together the boundary ∂ of Π . An interior of Π'' is homeomorphic to $(\mathbb{C}A \setminus \mathbb{R}A)/conj$.

The surface Π plays first fiddle and we need to know some its details. Each oval of $\mathbb{R}A_\varepsilon$ begotten by an oval of $\mathbb{R}A$ gives rise to a component of Π homeomorphic to an annulus. Let us remove these components and denote the remainder by Π_Δ . A boundary of Π_Δ consists of real (contained in ∂') and imaginary (contained in ∂'') circles. There exist natural isomorphisms

$$\varphi : H_1(\Pi_\Delta \cup \Sigma; \mathbb{Z}/2) \longrightarrow \tilde{V}_\Delta \tag{30}$$

$$H_1(\Pi_\Delta; \mathbb{Z}/2) \longrightarrow V'_\Delta$$

$$H_1(\partial' \cap \Pi_\Delta; \mathbb{Z}/2) \longrightarrow R_\Delta$$

$$
\begin{aligned}
&H_1(\partial''\cap\Pi_\Delta;\mathbb{Z}/_2) \longrightarrow B'_\Delta \\
&H_1(\Pi_\Delta;\mathbb{Z}/_2)/in_*H_1(\partial'\cap\Pi_\Delta;\mathbb{Z}/_2) \longrightarrow V_\Delta \\
&H_1(\Pi_\Delta;\mathbb{Z}/_2)/in_*H_1(\partial\Pi_\Delta;\mathbb{Z}/_2) \longrightarrow V_\Delta/B_\Delta
\end{aligned}
\tag{31}
$$

($\widetilde{V}_\Delta, V'_\Delta, V_\Delta, R_\Delta, B'_\Delta, B_\Delta$ are as in 2.4) such that the diagram

$$
\begin{tikzcd}
H_1(\partial''\cap\Pi_\Delta;\mathbb{Z}/_2) \arrow[rr, "in_*"] \arrow[dddd] & & H_1(\Pi_\Delta;\mathbb{Z}/_2) \arrow[r, "in_*"] \arrow[ddd] & H_1(\Pi_\Delta\cup\Sigma;\mathbb{Z}/_2) \arrow[ddd] \\
& H_1(\partial'\cap\Pi_\Delta;\mathbb{Z}/_2) \arrow[ur, "in_*"] & & \\
& & & \\
& R_\Delta \arrow[r, "in"] & V'_\Delta \arrow[r, "in"] & \widetilde{V}_\Delta \\
B'_\Delta \arrow[urr, "in"] & & &
\end{tikzcd}
$$

is commutative.

6.3. Computation of the $\mathbb{Z}/_4$-quadratic form.

In this section we compute a form $q^*: H_1(\Pi_\Delta\cup\Sigma;\mathbb{Z}/_2)\longrightarrow\mathbb{Z}/_4$ induced by inclusion $\Pi_\Delta\cup\Sigma\hookrightarrow F$ from the Rohlin-Guillou-Marin form q of (S^4, F). More precisely we prove that the isomorphism φ (see (30)) identifies q^* with $\tilde{q}_\Delta$.

Recall that $\widetilde{V}_\Delta = H_1(\Gamma_A;\mathbb{Z}/_2)\oplus H_1(\Sigma;\mathbb{Z}/_2)$ and remark that φ^{-1} coincides with a sum of composite homomorphism

$$
H_1(\Gamma_A;\mathbb{Z}/_2)\xrightarrow{in_*} H_1(W;\mathbb{Z}/_2)\xrightarrow{in_*^{-1}} H_1(\Pi_\Delta\cup\Sigma;\mathbb{Z}/_2)
$$

(here Γ_A is as in 2.4 and W is a regular neighbourhood of $\Pi_\Delta\cup\Sigma$ containing Γ_A) and $in_*: H_1(\Sigma;\mathbb{Z}/_2)\longrightarrow H_1(\Pi_\Delta\cup\Sigma;\mathbb{Z}/_2)$.

Now it is clear that to prove the coincidence of the quadratic forms it is sufficient to check that $q^{*}\circ\varphi^{-1}$ satisfies condition (iii) from 2.4, which determines $\tilde{q}_{\Delta}$. Moreover only formula (10) requires to be verified since the others are immediate consequences of the definition of the Rohlin-Guillou-Marin form.

In fact (10) is nothing but a special case of the general rule: if $\xi\in H_1(\mathbb{R}P^2_{\varepsilon};\mathbb{Z}/_2)$ then

$$q(in^F_*\xi) = \begin{cases} (-1)^k, & \text{if } in^P_*(\xi)\neq 0 \\ 0, & \text{if } in^P_*(\xi)=0 \end{cases} \tag{32}$$

(here in^F is the inclusion $\mathbb{R}P^2_{\varepsilon}\hookrightarrow F$ and in^P is the inclusion $\mathbb{R}P^2_{\varepsilon}\hookrightarrow\mathbb{R}P^2$).

In the case where $in^P_*(\xi)=0$ the equality turns into $q(in^F_*(\xi))=0$ and was proved by Marin [12] (cf. 6.1). It remains to show that $q(in^F_*(\xi))=(-1)^k$ if $in^P_*(\xi)\neq 0$. It may be done as in the previous case using a special membrane. We like to do it in another way.

Let us suppose that $in^P_*H_1(\mathbb{R}P^2_{\varepsilon};\mathbb{Z}/_2)\neq 0$ and denote by $q_{\mathcal{Y}}$ and q_R the forms induced by $in_*:H_1(\mathbb{C}A_{\varepsilon}/conj;\mathbb{Z}/_2)\longrightarrow H_1(F;\mathbb{Z}/_2)$ and $in_*:H_1(\mathbb{R}P^2_{\varepsilon};\mathbb{Z}/_2)\longrightarrow H_1(F;\mathbb{Z}/_2)$ from q. As it follows from Marin's result there exists $\tau\in\{-1,1\}$ such that

$$q(in^F_*(\xi)) = \begin{cases} \tau & \text{for any } \xi \text{ with } in^P_*(\xi)\neq 0 \\ 0 & \text{for any } \xi \text{ with } in^P_*(\xi)=0 \end{cases}$$

Hence, firstly, by 5.3 invariants $B(q_{\mathcal{Y}}), B(q_R)$ are well defined and

$$B(q)=B(q_{\mathcal{Y}})+B(q_R) \tag{33}$$

and, secondly,

$$B(q_R) = \tau \mod 8 \tag{34}$$

Furthermore, as in 6.1 applying of (28) we get

$$\chi(\mathbb{R}P^2_\varepsilon) = k^2 + B(q) \mod 8 \tag{35}$$

Now ler us introduce $F' = \mathbb{C}A_\varepsilon/\mathrm{conj} \cup \mathbb{R}P^2_{\geqslant\varepsilon}$ where $\mathbb{R}P^2_{\geqslant\varepsilon} = Cl(\mathbb{R}P^2\setminus \setminus \mathbb{R}P^2_\varepsilon)$. Repeating previous notations and arguments obviously modified we obtain

$$q'_R = 0$$

$$B(q') = B(q'_y) \tag{36}$$

$$\chi(\mathbb{R}P^2_{\geqslant\varepsilon}) = k^2 + B(q') \mod 8 \tag{37}$$

Both forms q_y, q'_y are defined on $H_1(\mathbb{C}A_\varepsilon/\mathrm{conj}; \mathbb{Z}/_2)$. For $\xi \in H_1(\mathbb{C}A_\varepsilon/\mathrm{conj}; \mathbb{Z}/_2)$ the difference $q_y(\xi) - q'_y(\xi)$ coincides with the linking number of ξ with $\mathbb{R}P^2$ multiplied by $2(:\mathbb{Z}/_2 \to \mathbb{Z}/_4)$. This linking number is 0 iff $\xi \circ \xi = 0$. So $q_y = -q'_y$ and thus

$$B(q_y) + B(q'_y) = 0 \mod 8 \tag{38}$$

Let us sum (33), (34), (35), (36), (37) and (38). Then taking into account that $\chi(\mathbb{R}P^2_\varepsilon) + \chi(\mathbb{R}P^2_{\geqslant\varepsilon}) = 1$ we get $\tau = 1 - 2k^2 \mod 8$. Since $\tau \in \{1,-1\}$ this implies $\tau = (-1)^k$.

6.4. Proof of the theorem (3.A).

Let $F, \Pi, \Pi'', \Sigma, \mathbb{R}P^2_\varepsilon, \Pi_\Delta, q$ and q^* be as in 6.2 and 6.3. According to 6.3 the $\mathbb{Z}/_4$-quadratic space $(H_1(\Pi_\Delta \cup \Sigma; \mathbb{Z}/_2), \circ, q^*)$ is isomorphic to $(\tilde{V}_\Delta, \circ, \tilde{q}_\Delta)$ and consequently by the hypothesis of the theorem it is informative. So we can apply 5.3 to the decomposition $F = F_1 \cup F_2$ with $F_1 = \Pi \cup \Sigma$ and $F_2 = \Pi'' \cup (\mathbb{R}P^2_\varepsilon \setminus \operatorname{Int} \Sigma)$. We get

$$B(q) = B(q_1) + B(q_2)$$

where $q_i = q \circ \{ in_* : H_1(F_i ; \mathbb{Z}/_2) \to H_1(F ; \mathbb{Z}/_2)\}$. The q_1 is zero on $in_* H_1(\Pi \setminus \Pi_\Delta ; \mathbb{Z}/_2) \subset H_1(F_1 ; \mathbb{Z}/_2)$ since $\Pi \setminus \Pi_\Delta$ consists of components of Π which are homeomorphic to an annulus and each contains an oval in its boundary. It follows

$$B(q_1) = B(q^*) = B(\tilde{q}_\Delta)$$

The restriction of q_2 to $in_* H_1(\mathbb{R}P^2_\varepsilon \setminus \operatorname{Int} \Sigma)$ are defined by (32). Thus

$$B(q_2) = \tilde{\ell} + B(q'')$$

where $\tilde{\ell}$ is as in (3.A) and $q'' = q \circ \{in_* : H_1(\Pi'' ; \mathbb{Z}/_2) \to H_1(F ; \mathbb{Z}/_2)\}$
To finish the proof we repeat Marin's arguments reproduced at the end of 6.1. Here we use that the interior of Π'' is homeomorphic to $(\mathbb{C}A \setminus \mathbb{R}A)/\mathrm{conj}$ and that the number $F \circ F$ is determined by the formulae

$$F \circ F = 2k^2 - 2\chi(\mathbb{R}P^2_\varepsilon)$$

$$\chi(\mathbb{R}P^2_\varepsilon) = \chi(\mathbb{R}P^2_+)$$

6.5. Proof of the theorem (3.B).

It is similar to that of (3.A). They differ in the choice of decomposition $F = F_1 \cup F_2$ only. Here we take $F_1 = \Pi$ and $F_2 = \Pi' \cup \Pi''$
By (32) the q is zero on $in_* H_1(\partial' ; \mathbb{Z}/_2)$ and by the hypothesis of the theorem it is zero on $in_* H_1(\partial'' ; \mathbb{Z}/_2)$. So applying 5.3 we get

$$B(q) = B(q_1) + B(q_2)$$

where $q_i = q \circ \{in_* : H_1(F_i; \mathbb{Z}_{/2}) \to H_1(F; \mathbb{Z}_{/2})$. Isomorphisms given in 6.3 show that the q_1 factorized from $H_1(F_1; \mathbb{Z}_{/2})$ to $H_1(F_1; \mathbb{Z}_{/2})/in_* H_1(\partial F_1; \mathbb{Z}_{/2})$ is isomorphic to q_Δ . At last $B(q_2) = b + B(q'')$ where b is as in (3.B) and q'' is as an 6.4. So we get

$$B(q) = B(q_\Delta) + b + B(q'')$$

and the end of the proof is fairly the same as in 6.4.

6.6. Proof of (4.B), (4.C) and (4.F).

First we are going to check that under assumptions of (4.B) or (4.C) the q_Δ vanishes on B_Δ . Since there is an isomorphism (31) between B'_Δ and $H_1(\partial'' \cap \Pi_\Delta; \mathbb{Z}_2)$ transfering $\tilde{q}_\Delta$ to $q \circ in_*$, where q is the Rohlin-Guillou-Marin form of (S^4, F) , it is sufficient to check the vanishing of q on $in_* H_1(\partial'' \cap \Pi_\Delta; \mathbb{Z}_{/2})$

Under assumptions of (4.B) each component of Π'' contains no more than one component of $\partial'' \cap \Pi_\Delta$. Consequently the homology class realized in $H_1(F; \mathbb{Z}_{/2})$ by that component of $\partial'' \cap \Pi_\Delta$ is equal to the sum of the others boundary components of Π'' and so this class lies in $in_* H_1(\partial'; \mathbb{Z}_{/2})$, where q vanishes.

Now let assumptions of (4.C) be fulfilled. Then for each component of Π'' any homology class realized in $H_1(F; \mathbb{Z}_{/2})$ by boundary components of that component of Π'' reduces to the sum of an element of $in_* H_1(\partial'; \mathbb{Z}_{/2})$ and elements realized by circle components of $\mathcal{Y} \cap \cap (\cup \partial \mathcal{D}_p^*)$ (ones begotten by imaginary branches of the curve A at its singular points). Consider one such component γ of $\mathcal{Y} \cap \cap \partial \mathcal{D}_p^*$, the class $[\gamma] \in H_1(F; \mathbb{Z}_{/2})$ realized by γ and the boundary δ of $F \cap \mathcal{D}_p^*$. Then

$$q(\gamma) = 2 lk(\gamma, \delta \setminus \gamma) \bmod 4 ,$$

where lk is linking number in $\partial \mathcal{D}_p^* = S^3$ and hence

$$q(\gamma) = 2\, lk\,(\gamma^*, pr^{-1}(\delta\setminus\gamma)) \mod 4$$

where $pr : \mathcal{D}_p \longrightarrow \mathcal{D}_p^*$ is the natural projection and γ^* is any component of $pr^{-1}(\gamma)$. The latter linking number is equal to the intersection number of the non-real branch β of A at p which begots γ^* with the union of all branches different from β and $conj\,\beta$. Thus $q(\gamma)=0$ and we have finished the proof.

To prove that under assumptions of (4.F) the q_Δ vanishes on B_Δ it is sufficient to note that: boundary components of $\mathcal{Y}_\varepsilon$ give generators of B_Δ; for elements of $in_* H_1(\partial\sum_\varepsilon ; \mathbb{Z}/_2)$ the values of the Rohlin-Guillou-Marin form coincide with the linking numbers involved in (4.D); the Rohlin-Guillou-Marin form vanishes on $in_* H_1(\mathcal{D}_+ ; \mathbb{Z}/_2)$ cf. 6.1.

§ 7. ANOTHER APPROACH

7.1. Prototype: Rohlin's proof.

Let c be an antiholomorphic involution of a closed quasicomplex manifold $\mathcal{Y}$ of complex dimension $2n$. Suppose that

$$\dim H_*(Y; \mathbb{Z}/_2) = \dim H_*(\mathcal{Y}; \mathbb{Z}/_2) \tag{39}$$

where Y is the fixed point set of c.

By the Atiyah-Singer-Hirzebruch formula, it is hold (and this result does not use (39)) the relation

$$\chi(Y) = \sigma(\mathcal{Y}) - 2\,\sigma_{(-1)^{n+1}} \tag{40}$$

where $\sigma_\lambda\ (\lambda = 1 \text{ or } -1)$ is the signature of the bilinear form b_λ obtained by restriction of the intersection form of $\mathcal{Y}$ to $H_\lambda = \mathrm{Ker}\{1 + \lambda c_*: H_{2n}(\mathcal{Y}) \longrightarrow H_{2n}(\mathcal{Y})\}$. By the Smith theory arguments the assumption (39) implies that forms b_λ are unimodular. The form $b_{(-1)^{n+1}}$ is even and, since the signature of any even unimodular form is divisible by 8, from (40) it follows

$$\chi(Y) \equiv \sigma(\mathcal{Y}) \bmod 16 \tag{41}$$

Turning to a real nonsingular plane projective curve A of even degree, one should associate with it the 2-sheeted branched covering space $\mathcal{Y}$ of $\mathbb{C}P^2$ with branch locus $\mathbb{C}A$ and the involution c which covers conj and has Y lying over $\mathbb{R}P^2_-$. Such $\mathcal{Y}, c$ exist and are unique. The condition (39) holds iff A is an M-curve. The congruence (41) applied to these $\mathcal{Y}, c$ reduces to the Gudkov-Rohlin congruence $(\chi(Y) = 2 - 2\chi(\mathbb{R}P^2_+),\ \sigma(\mathcal{Y}) = 2 - 2k^2)$.

7.2. The Atiyah-Singer-Hirzebruch formula for manifolds with boundary.

Let c be an antiholomorphic involution of a compact quasicomplex manifold $\mathcal{Y}$ of complex dimension $2n$ with a boundary $\partial\mathcal{Y}$ Let Y denote, as before, the fixed point set of c. The normal bundle of $\partial\mathcal{Y}$ in $\mathcal{Y}$ is just the oriented one-dimensional bundle. Thus the complex structure in the tangent bundle $T\mathcal{Y}$ induces the complex structure in the direct sum of $T(\partial\mathcal{Y})$ with the trivialized one-dimensional bundle. Let us denote the complex structure introduced by θ .

(7.A) The number δ satisfying the formula

$$\chi(Y) = \sigma(\mathcal{Y}) - 2\sigma_{(-1)^{n+1}} + \delta \tag{42}$$

is an invariant of the triple $(\partial Y, c|_{\partial Y}, \theta)$.

We have to check that the difference $\delta = \chi(Y) - \sigma(Y) + 2\sigma_{(-1)^{k+1}}$ depends only on $(\partial Y, c|_{\partial Y}, \theta)$. Given another pair with the boundary triple isomorphic to $(\partial Y, c|_{\partial Y}, \theta)$, let us glue it to (Y, c) along the boundary. By additivity of σ and χ, (40), applied to the closed manifold obtained by the gluing shows that the numbers δ given by the halves coincide.

7.3. $\mathbb{Q}/2\mathbb{Z}$-quadratic spaces.

By $\mathbb{Q}/2\mathbb{Z}$-quadratic space it is called a triple $(V, \circ, q)$ consisting of a finite group V, a symmetric bilinear form $V \times V \longrightarrow \mathbb{Q}/\mathbb{Z} : (x, y) \longmapsto x \circ y$ and a function $q : V \longrightarrow \mathbb{Q}/2\mathbb{Z}$ which is quadratic with respect to that bilinear form, i.e.

$$q(x + y) = q(x) + q(y) + 2x \circ y$$

for $x, y \in V$, where $2 : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/2\mathbb{Z}$ is the canonical isomorphism. The canonical embedding $1/2 : \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Q}/2\mathbb{Z}$ allows to consider $\mathbb{Z}/_4$-quadratic spaces as $\mathbb{Q}/2\mathbb{Z}$-quadratic spaces.

A $\mathbb{Q}/2\mathbb{Z}$-quadratic space $S = (V, \circ, q)$ is said to be nonsingular if its bilinear form $\circ$ is nonsingular, i.e. its radical $R(S) = \{x \in V \mid \forall y \in V \;\; x \circ y = 0\}$ is the zero-subspace.

Any $\mathbb{Q}/2\mathbb{Z}$-quadratic space can be obtained in the following way. Let L be a finitely generated free abelian group endowed with a non-degenerate even symmetric bilinear form $L \times L \longrightarrow \mathbb{Z} : (x, y) \longmapsto \langle x, y \rangle$. This form has a unique extension to $L \otimes \mathbb{Q}$. By the correlation isomorphism $\phi : L \otimes \mathbb{Q} \longrightarrow L^* = \mathrm{Hom}(L, \mathbb{Q})$ defined by

$$\phi(x) = \langle \;\; , x \rangle$$

the group $L^\vee = \mathrm{Hom}(L, \mathbb{Z})$ can be considered as an intermediate group:

$$L \subset L^\vee \subset L \otimes \mathbb{Q}$$

Then we obtain a $\mathbb{Q}/2\mathbb{Z}$-quadratic space $(V, \circ, q)$ taking

$$V = L^{\vee}/L\ ,$$
$$\bar{x}\circ\bar{y} = \langle x, y\rangle \mod \mathbb{Z} \quad \text{for} \quad x\in\bar{x}\in V,\ y\in\bar{y}\in V$$
$$q(\bar{x}) = \langle x, x\rangle \mod 2\mathbb{Z} \quad \text{for} \quad x\in\bar{x}\in V$$

It is easily checked that $\circ$ and q are well defined.

If V has no element of order 2 then evident relations

$$q(v) \mod \mathbb{Z} = v\circ v,\quad r^2 q(v) = 0 \qquad \text{if } r \text{ is order of } v \tag{43}$$

allow to determine q by $\circ$.

The van der Blij formula [3] states that

$$e^{i\frac{\pi\sigma}{4}} = (\operatorname{card} V)^{-\frac{1}{2}} \sum_{v\in V} e^{2\pi i q} \tag{44}$$

where σ is the signature of the form $\langle\ ,\ \rangle$

7.4. The case where 2-torsion in homology of boundary vanishes.

Let c be an antiholomorphic involution of a compact quasicomplex manifold $\mathcal{Y}$ of complex dimension $2n$. Let Y denote the fixed point set of c. Endow the group $L = H_{2n}(\mathcal{Y})/\operatorname{Tors} + in_* H_{2n}(\partial\mathcal{Y})$ with the form $\langle\ ,\ \rangle$, induced by the intersection form of $\mathcal{Y}$. Consider the $\mathbb{Q}/2\mathbb{Z}$-quadratic space $(V, \circ, q)$ associated with $(L, \langle\ ,\ \rangle)$. Then V is nothing but $\operatorname{Tors} H_{2n-1}(\partial\mathcal{Y}) \cap \partial H_{2n}(\mathcal{Y}, \partial\mathcal{Y})$ and $\circ$ is the linking form. Thus we obtain (see 7.3)

(7.B) *If* $H_{2n-1}(\partial\mathcal{Y})$ *has no element of order* 2 *the space* $(V, \circ, q)$ *is determined by* $\partial\mathcal{Y}$, *namely,*

$$V = \mathrm{Tors}\, H_{2n-1}(\partial Y) \cap \partial H_{2n}(Y, \partial Y)$$

$\circ$ is linking form ,

q is defined via $\circ$ by (43)

Suppose now that $\dim H_*(Y; \mathbb{Z}/2) = \dim H_*(Y; \mathbb{Z}/2)$. Then by the Smith theory $(L, \langle\ ,\ \rangle)$ is an orthogonal sum of subspaces $(L_+, \langle\ ,\ \rangle)$ and $(L_-, \langle\ ,\ \rangle)$, where

$$L_+ = \mathrm{Ker}\,\{1 - c_* : L \to L\},$$

$$L_- = \mathrm{Ker}\,\{1 + c_* : L \to L\}$$

This immediately implies

(7.C) *If* $H_{2n-1}(\partial Y)$ *has no element of order* 2 *and*

$$\dim H_*(Y; \mathbb{Z}/2) = \dim H_*(Y; \mathbb{Z}/2)$$

then the $\mathbb{Q}/2\mathbb{Z}$*-quadratic space* $(V_{(-1)^n}, \circ, q)$ *associated with* $(L_{(-1)^n}, \langle\ ,\ \rangle)$ *is determined by* $(\partial Y, c|_{\partial Y})$, namely,

$$V_{(-1)^n} = \mathrm{Tors}\, H_{2n-1}(\partial Y) \cap \partial H_{2n}(Y, \partial Y) \cap \mathrm{Ker}\,(1 + (-1)^{n+1} c_*),$$

$\circ$ *is the linking form* ,

q *is defined via* $\circ$ *by* (43)

By (7.C) ,the Atiyah-Singer-Hirzebruch and the van der Blij formulae imply

(7.D) *If* $H_{2n-1}(\partial Y)$ *has no element of order* 2 *and*

$$\dim H_*(Y; \mathbb{Z}/2) = \dim H_*(Y; \mathbb{Z}/2)$$

then

$$\chi(Y) \equiv \sigma(\mathcal{Y}) + \delta - 2b \mod 16$$

where $b \bmod 8$ is defined by

$$e^{i\frac{\pi b}{4}} = (\operatorname{card} V_-)^{-\frac{1}{2}} \sum_{v \in V_-} e^{2\pi i q(v)}$$

with V_-, q from (7.C).

7.5. Application.

Let A be a real plane projective curve of degree $m = 2k$ without non-real singular points and let for every singular point in some (perhaps nonlinear) coordinated x, y in some neighbourhood of the point the curve is defined by an equation

$$x^3 + y^5 = 0$$

(E_8 in Arnold's notations) or by an equation

$(y - ax^2)(y - bx^2)(y - cx^2) = 0$ (a, b, c are distinct real numbers)

(J_{10}) . Let $\mathbb{R}P^2_+$ be the half of $\mathbb{R}P^2$ which is not contractible to a point in $\mathbb{R}P^2$ (and, of course, is bounded by $\mathbb{R}A$).

(7.E) *If A is an M-curve then*

$$\chi(\mathbb{R}P^2_+) \equiv 1 - k^2 - 4e + 3j \mod 8$$

where e is the number of points of type E_8 and j is the number of points of type J_{10}.

To prove (7.E) it is sufficient to apply (7.D) to the case where $\mathcal{Y}$ is obtained by removal of neighbourhoods of singular points from the two-sheeted cover of $\mathbb{C}P^2$ with branch locus $\mathbb{C}A$. The c

should be choosen to cover conj and to have $Y = \mathrm{Fix}\, c$ lying over $\mathbb{R}P^2_+$. Then $H_{2n-1}(\partial Y)$ has no torsion and

$$\dim H_*(Y;\mathbb{Z}_{/2}) = \dim H_*(Y;\mathbb{Z}_{/2}) ,$$

$$\chi(Y) = 2\,\chi(\mathbb{R}P^2_+) - e + j ,$$

$$\sigma(Y) = 2 - 2k^2 - 8e - 8j ,$$

$$\delta = -(e + j) .$$

The last equality is a special case of the general rule: any quasihomogeneous singular point makes a contribution -1 to δ. This rule is a straightforward consequence of the definition of δ.

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