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KILLING THE MIDDLE HOMOTOPY GROUPS OF ODD DIMENSIONAL MANIFOLDS

BY
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The main object of this paper is to prove the theorem: If W is an m -parallelisable $(2m+1)$ -manifold, whose boundary has no homology in dimensions $m, m+1$; then W is χ -equivalent to an m -connected manifold.

This is written as a sequel to Milnor's paper *A procedure for killing homotopy groups of differentiable manifolds*. We attempt to preserve the notations of this paper, and refer to it as [M].

Milnor proves in [M] that W is χ -equivalent to an $(m-1)$ -connected manifold, and we show in §1 that we can reduce $H_m(W)$ to a finite group. §2 is devoted to the definition and study of a nonsingular bilinear form on this group, symmetric if m is odd, and skew if m is even. §3 applies these results to prove the theorem above. It follows, in the notation of [8], that $\Theta_{2m}(\partial\pi) = 0$ (this has also been proved by Milnor and Kervaire). In §4 we prove a more precise version of Milnor's reduction of $(m-1)$ -parallelisable to $(m-1)$ -connected manifolds; this is applied in §5 to obtain results about the topology of certain $(m-1)$ -parallelisable $(2m+1)$ -manifolds. Our results are complete for a class of 5-manifolds, and yield an interesting test for cobordism.

Throughout this paper, "manifold" shall mean "compact connected differential manifold." Here, "differential" means "endowed with differential structure"; it seems a more suitable word for this concept than "differentiable," which ought to mean "admitting at least one differential structure."

1. Preliminaries. We consider manifolds W of dimension $2m+1$ (where $1 < m$). We suppose that W is m -parallelisable, and that we have already killed the homotopy groups $\pi_i(W)$ for $i < m$; we will study the possibility of killing $\pi_m(W)$. Since $m < (1/2) \dim W$, every element of $\pi_m(W)$ is representable by an imbedding $f_0: S^m \rightarrow W$. The induced bundle $f_0^*(\tau^{2m+1})$ is trivial, so by Lemma 3 of [M] there exists an imbedding $f_1: S^m \times D^{m+1} \rightarrow W$ extending f_0 . In this case we can carry out surgery without trouble; the only snag is that we are not sure of simplifying $\pi_m(W)$ when we do it.

We reconsider the proof of Lemma 2 of [M]. It is convenient to give it a somewhat different form. We first pass from W to the manifold W'' obtained by removing the interior of $f_1(S^m \times D^{m+1})$ from it, and then to the manifold W' obtained by glueing $D^{m+1} \times S^m$ in its place. It is easy to see that $\pi_m(W') \rightarrow \pi_m(W)$ is onto, and its kernel is generated by the class of $f_2(e \times S^m)$, where $f_2: S^m \times S^m \rightarrow W''$ is induced by f_1 (and we use e indiscriminately to denote an

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unspecified base point). For the same reason, $\pi_m(W'') \rightarrow \pi_m(W')$ is onto, and its kernel is generated by the class of $f_2(S^m \times e)$. We will usually denote these two classes in $\pi_m(W'')$, or rather, the corresponding ones in $H_m(W'')$, by z, x .

We suppose, as in §6 of [M], that $H_{m+1}(\text{Bd } W)$ and $H_m(\text{Bd } W)$ vanish, hence so also does $H_{m-1}(\text{Bd } W)$. The homology sequence for the pair $(W, \text{Bd } W)$ and the Universal Coefficient Theorem, imply that $H_m(W) = H_m(W, \text{Bd } W)$ and $H_{m+1}(W) = H_{m+1}(W, \text{Bd } W)$ with any coefficient group, and similarly for cohomology, so in these dimensions Poincaré duality for W has the same form as for a closed manifold.

We first consider the case when $H_m(W)$ has elements of infinite order.

LEMMA 1. *Let x be of infinite order in $H_m(W)$ and be indivisible. Then if we perform surgery on W starting from x , the new class z in $H_m(W'')$ vanishes, so $H_m(W')$ is obtained from $H_m(W)$ by killing x .*

Proof. Denote the chain $f_1(e \times D^{m+1})$ by \bar{q} . Since x is indivisible, by Poincaré duality there is a class $p \in H_{m+1}(W)$ with unit intersection number with x . Since $f_1((S^m - e) \times D^{m+1})$ is a cell, we may choose a representative cycle \bar{p} for p which avoids it: and clearly we may suppose that the only simplexes of \bar{p} contained in $f_1(S^m \times D^{m+1})$ form \bar{q} , since the intersection number of p with x is unity. But now $\bar{q} - \bar{p}$ defines a chain in W'' whose boundary $\bar{z} = f_2(e \times S^m)$ determines z . Hence $z = 0$ in $H_m(W'')$ and a fortiori also in $H_m(W')$.

Now as in §6 of [M], W' is m -parallelisable if W is, so we can repeat the process to kill all elements of infinite order in $H_m(W)$. Hence we may assume $G = H_m(W)$ finite. Let its exponent (the l.c.m. of the orders of its elements) be θ . We shall take homology and cohomology with coefficient group Z_θ , but still represent the classes by integral chains. Now $H_m(W, Z_\theta) = G$ by the Universal Coefficient Theorem; we shall identify these groups by this isomorphism. Consider the map $\bar{y} \rightarrow \partial \bar{y} / \theta$ of chain groups: this induces a homomorphism $\gamma: H_{m+1}(W, Z_\theta) \rightarrow H_m(W, Z_\theta)$ dual to the Bockstein in cohomology. This is onto since each element of the latter group has a representative θ times which is a boundary, and (1-1) since if \bar{y} represents a class y with $\gamma y = 0$, there exists a chain \bar{w} with $\partial \bar{w} = \partial \bar{y} / \theta$, and since $H_{m+1}(W) = 0$ (by duality), $\bar{y} - \theta \bar{w}$, being a cycle, is a boundary, so \bar{y} determines the zero element of $H_{m+1}(W, Z_\theta)$.

2. The nonsingular bilinear form. Combining γ with isomorphisms deduced from Poincaré duality and the Universal Coefficient Theorem we now have

$$G = H_m(W) \cong H_m(W, Z_\theta) \cong H_{m+1}(W, Z_\theta) \cong H^m(W, Z_\theta) = \text{Hom}(G, Z_\theta).$$

Hence we have a pairing of G with itself to Z_θ . Write $b: G \otimes G \rightarrow Z_\theta$.

LEMMA 2. *b is a nonsingular bilinear form on G , symmetric if m is odd and skew if m is even.*

Proof. We have already proved the first part. For the second it is more convenient to work in cohomology (isomorphic to homology by the above). Here, b is given by $b(x, y) = \beta x \cdot y$, evaluated on the fundamental class of $(W, \text{Bd } W)$, where β denotes the Bockstein. Now

$$\begin{aligned} b(x, y) + (-1)^m b(y, x) &= \beta x \cdot y + (-1)^m \beta y \cdot x \\ &= \beta x \cdot y + (-1)^m x \cdot \beta y \\ &= \beta(xy). \end{aligned}$$

But $xy \in H^{2m}(W, Z_\theta)$, so reverting to homology we get $H_1(W, \text{Bd } W; Z_\theta)$. But every element of this is the restriction of an integer class, so applying ∂/θ gives zero, as required.

Note. This result also follows by interpreting $b(x, y)$ as a linking number (mod θ).

We shall now show how the form b determines the effect of surgery on $H_m(W)$. Let x be the element chosen to operate on, and let y be of order r in $H_m(W)$. Since $ry = 0$, $\theta \mid rb(y, x)$, so $(r/\theta)b(y, x)$ is an integer defined modulo r . (\mid denotes divisibility.) Represent y by an m -cycle \bar{y} not meeting $f_1(S^m \times D^{m+1})$. In W'' , \bar{y} represents a homology class y' , and ry' is a multiple of z .

LEMMA 3. *If we write $ry' = \lambda z$, we have $\lambda \equiv (r/\theta)b(y, x) \pmod{r}$.*

Proof. Let \bar{p} be an $(m+1)$ -chain with $\partial \bar{p} = r\bar{y}$. As in the proof of Lemma 1, if the intersection number of \bar{p} and x is λ , we may suppose that the only simplexes of \bar{p} contained in $f_1(S^m \times D^{m+1})$ form $\lambda \bar{q}$. Now $\bar{p} - \lambda \bar{q}$ defines a chain in W'' , of boundary $r\bar{y} - \lambda \bar{z}$, hence $ry' = \lambda z$. But as $(\partial/\theta)(\theta/r)\bar{p} = \bar{y}$, the class mod θ of $\theta \bar{p}/r$ corresponds under γ to y , so by definition of b ,

$$b(y, x) \equiv (\theta \bar{p}/r) \cap x \equiv \theta \lambda / r \pmod{\theta}$$

i.e.

$$\lambda \equiv \frac{r}{\theta} b(y, x) \pmod{r}.$$

COROLLARY. *Let $b(y, x) = 0$. Then there exists a class y'' in $H_m(W'')$ inducing y in $H_m(W)$ and also of order r .*

Proof. $ry' = krz$ for some integer k . We may choose $y'' = y' - kz$.

Before we can prove our main theorem we need a number-theoretic lemma about bilinear forms b .

LEMMA 4. *Let $b: G \otimes G \rightarrow Z_\theta$ be a nonsingular bilinear form on the finite Abelian group G . Write $c(x)$ for $b(x, x)$.*

(i) *If b is symmetric and $c(x) = 0$ for all x , then $\theta = 2$ and we can find a basis $\{x_i, y_i: 1 \leq i \leq r\}$ for G such that*

$$b(x_i, y_j) = \delta_{ij} b(x_i, x_j) = b(y_i, y_j) = 0.$$

(ii) If b is skew-symmetric, we can find elements x_i, y_i of order θ_i in G ($1 \leq i \leq r$) such that

$$\begin{aligned} b(x_i, x_j) &= b(x_i, y_j) = b(y_i, y_j) = 0 \text{ for } i \neq j; \\ c(x_i) &= 0, \quad b(x_i, y_i) \text{ has order } \theta_i, \end{aligned}$$

and G contains the direct sum of the cyclic subgroups generated by the x_i, y_i as a direct summand of index at most 2.

COROLLARY. Under the conditions of (ii), if B is the subgroup generated by the x_i , then either

$$G \cong B \oplus B \quad \text{or} \quad G \cong B \oplus B \oplus Z_2.$$

Proof. (i) Under these hypotheses, for all x, y in G ,

$$2b(x, y) = b(x, y) + b(y, x) = c(x + y) - c(x) - c(y) = 0.$$

Hence the exponent of G is 2. We now pick x_i, y_i by induction. Choose any nonzero x_1 , then since b is nonsingular there exists y_1 with $b(x_1, y_1) = 1$. Since $c(x_1) = 0, y_1 \neq x_1$. Now G is the direct sum of the subgroup $Gp\{x_1, y_1\}$ and H , the annihilator of $Gp\{x_1, y_1\}$, and b induces a nonsingular form on H , so we may continue the induction. (All this is of course well known.)

Note. If x_1, x_2, \dots belong to a group, $Gp\{x_1, x_2, \dots\}$ denotes the subgroup which they generate.

(ii) Since b is skew, $c(x) = b(x, x) = -c(x)$, so has order 2. Moreover, $c(x + y) - c(x) - c(y) = b(x, y) + b(y, x) = 0$, so c is a homomorphism $G \rightarrow Z_2$. Now since G is a finite Abelian group it is the direct sum of its Sylow subgroups S_p , and these are clearly orthogonal under b , so we can take them separately.

First, suppose p odd. Let x_1 be an element of maximal order p^r in S_p . Then since b is nonsingular there exists y_1 such that $b(x_1, y_1)$ has order p^r . Then y_1 has order p^r (not greater, since this was maximal) and G contains the direct sum of the cyclic groups generated by x_1, y_1 ; for if $0 = \lambda x_1 + \mu y_1$, then

$$0 = b(\lambda x_1 + \mu y_1, y_1) = \lambda b(x_1, y_1) + \mu c(y_1) = \lambda(bx_1, y_1)$$

so λ is divisible by p^r ; similarly, so is μ . Again we have $G = Gp\{x_1, y_1\} \oplus H$, where H is the annihilator of x_1, y_1 , since any $z \in G$ can be written as

$$z = b(z, y_1)x_1 - b(z, x_1)y_1 + h$$

with $h \in H$. b induces a nonsingular form on H , so we may apply induction to obtain our theorem.

For $p = 2$ we apply the same argument, if $1 < r$. The proof of independence of x_1, y_1 must be modified as follows. By the equation above, $b(x_1, y_1)$ has order at most 2, so λ is divisible by 2^{r-1} , so by 2. Similarly, so is μ . Hence $\mu c(y_1) = 0$, and we may proceed as before. (The modification of the direct sum argument

is left to the reader.) We may suppose that $c(x_1) = 0$, for if not, and $c(y_1) = 0$, we interchange x_1, y_1 ; whereas if $c(x_1) = c(y_1) \neq 0$, we may replace x_1 by $x_1 + y_1$.

Finally, suppose G has exponent 2. If the order of G is two, G has the required form. If it is greater, let x_1 be any nonzero element of $\text{Ker } c$, and y_1 such that $b(x_1, y_1) \neq 0$; then we can split off the direct summand $Gp\{x_1, y_1\}$ as before. This concludes the proof.

Note. (i) We can be somewhat more precise in our reduction of (G, b) , but this is of no advantage for the applications we shall make of the lemma.

(ii) The above proof is complicated by the possibility $c \neq 0$ in (ii). We shall show in §5 that for m -parallelisable W , c must in fact vanish.

3. Proof of theorem.

THEOREM. *Let W be m -parallelisable, of dimension $2m+1$. If the boundary of W has no homology in dimensions $m, m+1$, W is χ -equivalent to an m -connected manifold.*

Proof, m even. By Theorem 3 of [M], we may suppose W $(m-1)$ -connected, and by Lemma 1, $H_m(W)$ finite. By Lemma 2 it admits a nonsingular skew form b , so by Lemma 4, we may express G in the special form there given. First suppose B is not zero. Take the class x_1 , represent by a sphere, and perform surgery. Then $H_m(W'')$ is generated by elements x'_i, y'_i, z ; where x'_i, y'_i are classes mapping to x_i, y_i in $H_m(W)$, for uniformity of notation we have denoted the generator of the "extra" Z_2 in G (if there is one) by x_0 , and x'_i, z are the classes of $f_2(S^m \times e), f_2(e \times S^m)$. By the corollary to Lemma 3, we may suppose that for $i \neq 1$, x'_i, y'_i have the same orders as x_i, y_i . Also by Lemma 3, we may choose y'_1 such that $\theta_1 y'_1 = -z$, and since $c(x_1) = 0$, $\theta_1 x'_1 = \lambda \theta_1 z$, for some integer λ .

Suppose if possible $\lambda \neq 0$. Then in W' , x'_1 becomes zero, so we have (using primes to denote corresponding elements)

$$\theta_1 y'_1 = -z', \quad \lambda \theta_1 z' = 0$$

so y'_1 has order $\lambda \theta_1^2$. The orders of other basic elements are unchanged from G , and there are no new ones. We see that the resulting group fails to have the form required by the corollary to Lemma 4. Hence $\lambda = 0$. Then in W' we have $\theta_1 y'_1 = -z'$, and y'_1 has infinite order. By Lemma 1, we may now kill y'_1 , and we have then simplified the finite group G . Hence by induction we may simplify till G is 0 or Z_2 . In the latter case perform surgery starting with the nonzero element x of G . Then $2x' = \lambda z$ for some odd λ . Hence $H_m(W')$ is cyclic of some odd order, which by Lemma 4 must be unity, so in this case also we can make W m -connected.

We must now consider the case when m is odd. The main difference from the earlier case is that there (using Lemma 4) the effect of surgery was already determined by the choice of the class x . But for m odd there is the additional question of product structure for $S^m \times S^m$. Now $H^m(S^m \times S^m)$ is the

free Abelian group on two generators induced from the projections on the factors. Any autohomeomorphism of $S^m \times S^m$ induces an automorphism of this group and so a linear transformation of determinant ± 1 . We represent this by the appropriate matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

over \mathbb{Z} . If $m \neq 1, 3, 7$, there is no element of Hopf invariant odd in $\pi_{2m+1}(S^{m+1})$ and so no map $S^m \times S^m \rightarrow S^m$ with both degrees odd (by [2; 5]). Hence ab, cd are even, i.e., a, d have the opposite parity to b, c . However,

LEMMA 5. $S^m \times S^m$ admits diffeomorphisms corresponding to any matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which is unimodular, and with a, d of opposite parity to b, c .

Proof. This falls naturally into two parts. First we produce a diffeomorphism for the matrix

$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

and then prove that this, together with the trivially representable matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

generates the group of all matrices satisfying the conditions above. We define the diffeomorphism using a map of Hopf [5]. Let $(p, q) \in S^m \times S^m$. Then draw the great circle through the points p, q of S^m , and let q' be the other point of it at the same distance from p as q is. Thus if q is p or its antipode, $q' = q$ is unique. Then consider the map $S^m \times S^m \rightarrow S^m \times S^m$ defined by $(p, q) \rightarrow (p, q')$. It is clearly (1-1) and infinitely differentiable (and its own inverse), and since m is odd it corresponds to the matrix

$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix},$$

as promised.

Alternatively we may use a map $f: S^m \rightarrow SO_{m+1}$ of index 2 (it is well known that such exist), and define a diffeomorphism by $F(x, y) = (x, f(x) \cdot y)$: this corresponds to the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The proof about generators for the group parallels Kuroš [6, Appendix B]. The only change is where he sets $a = qc + a'$, $0 \leq a' < c$, we must put $a = 2q'c + a''$, $-c < a'' \leq c$. But $a'' = c$ is impossible, as this would imply that a had the same parity as c . The remainder of the proof is unaltered (working with

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

which is easily expressed by the matrices above). In fact the corresponding projective group, a subgroup of index 3 in the modular group, is $Z_2 * Z$.

Proof of theorem, m odd. First suppose $c(x)$ not identically zero. Choose x such that $c(x) \neq 0$. We represent x by an S^m and perform surgery. We shall adhere to our earlier notation, denoting corresponding classes with primes. We consider the elements x', z of $H_m(W'')$. By Lemma 3, $rx' = sz$, say, where r is the order of x , and $r \nmid s$ since $c(x) \neq 0$. Hence the h.c.f. $(r, s) = h < r$. Set $r = r'h$, $s = s'h$. Choose λ, μ such that $\lambda r' + \mu s' = 1$: we may suppose λ, μ of opposite parity since if they are both odd, r', s' must be of opposite parity and we may take $\lambda + s', \mu - r'$.

Write $y = \lambda z + \mu x'$. Since λ, μ have opposite parity, by Lemma 5 we may choose the product structure in $S^m \times S^m$ so that y corresponds to one of the factors. Then glue in $D^{m+1} \times S^m$ to kill y and give $W'(1)$. Now in W' ,

$$rx'' = sz' \quad \text{i.e., } h(r'x'' - s'z') = 0$$

and $0 = y' = \lambda z' + \mu x''$, so

$$\begin{aligned} x'' &= (\lambda r' + \mu s')x'' = \lambda(r'x'' - s'z'), \\ z' &= (\lambda r' + \mu s')z' = -\mu(r'x'' - s'z'), \end{aligned}$$

hence the group generated by x'', z' has order a factor of h (in fact equal to it) which is less than r . Since the index of this group in $H_m(W')$ equals that of $Gp\{x\}$ in $H_m(W)$, (for $Gp\{x', z\}$ contains the kernels of both $H_m(W'') \rightarrow H_m(W)$ and $H_m(W'') \rightarrow H_m(W')$) we have succeeded in decreasing the order of $H_m(W)$, or more precisely, in replacing it by a divisor of itself.

We may repeat the above process as long as c is not zero. Hence by induction (G being finite) we may suppose $c = 0$, and (G, b) as in (i) of Lemma 4. Perform surgery on the class x_1 . If A denotes the subgroup of G generated by x_i, y_i for $1 < i \leq r$, then by Lemma 3, and corollary, $H_m(W'') = A \oplus Gp\{x'_1, y'_1, z\}$, $2x'_1$ is an even multiple of z , and we may suppose $2y'_1 = z$.

Write $2x'_1 = (4k + d)z$, where $d = 2$ or 4 , and kill $x'_1 - 2kz$. Then $dz' = 0$, and so $H_m(W') = A \oplus Gp\{y'_1\}$, and y'_1 has order $2d$.

(¹) This form of surgery is rather more general than that used in [M], but it follows from our proof of Lemma 5 that it is equivalent to a series of the spherical modifications of [M].

Write U for W' , u for y_1'' . Then since b is nonsingular, and u is the only basis element of $H_m(U)$ of order greater than 2, the order of $c(u)$ equals the order of u . Now perform surgery starting with the class u . Then in $H_m(U'')$ we have $2du$ equal to an odd multiple of the new class w . Then we kill u (we have no need to worry about the product structure this time), and

$$H_m(U') = A \oplus Z_k \quad \text{if } k \text{ is the odd order of } w.$$

(The sum is direct as A is a 2-group.) But now, by the first part of the proof, we can replace the order of the group by a divisor of itself such that the new group has the form of Lemma 4 (i), and so has order not exceeding that of A . Hence in the second case also we have succeeded in decreasing the order of $H_m(W)$, so our induction is complete, and we may reduce the group to zero.

COROLLARY. *Let T^{2m} be a homotopy sphere which bounds a π -manifold. Then it bounds a contractible manifold.*

For the result is trivial if $m=1$, and otherwise we may apply the theorem to find an m -connected manifold with boundary T . But by relative Poincaré duality, such a manifold must be contractible.

COMPLEMENT. Let T^{4m} be a homotopy sphere, and W a π -manifold with boundary T . Then there is a contractible manifold C with boundary T , such that if W' is formed by glueing W to C along T , there is a parallelisable manifold M , with boundary W' .

Proof. Our construction of C from W by surgery was by choosing at each stage a class on which to perform the construction. By Lemma 5 of [M], if we choose the correct trivialisation of the normal bundle at each stage, the manifolds $\omega(W, f)$ are parallelisable: this goes also for the proof of Theorem 2 of [M]. Since the trivialisations given for the tangent bundles of these manifolds fit together on the boundary, we may form M by glueing these manifolds together, and it will then be parallelisable.

These results are of use for computing the groups Θ_m of J -equivalence classes of homotopy spheres. Our reference is [7]. In the notation of those notes, the above corollary states $\Theta_{2m}(\partial\pi)=0$. Since Milnor proves that $\Theta_{2m}/\Theta_{2m}(\partial\pi)$ is finite, it follows that for each m , Θ_{2m} is a finite group. Also, using other results of Milnor, Θ_4 and Θ_{12} vanish. We may also show $\Theta_6=0$, and will sketch the proof (we omit details since a simpler proof is known). By Thom [8], the spinor cobordism group in dimension 6 is isomorphic to the stable homotopy group $\pi_{n+6}(M(\text{Spin } n))$. Results of Adams [1] relate these to a spectral sequence which starts with

$$\text{Ext}_{A_2}^{**}(H^*(M(\text{Spin } n), Z_2), Z_2),$$

where A_2 denotes the Steenrod algebra mod 2. A straightforward computation of this in low dimensions now shows that the group in question vanishes.

Hence a homotopy 6-sphere, being a spin manifold, bounds another, W say. But W is a spin manifold, and so 3-parallelisable, and the result now follows by the theorem above.

All these results have been obtained independently by M. Kervaire (including a stronger form of the above complement), and will appear in a joint paper by M. Kervaire and J. Milnor entitled *Groups of homotopy spheres*, which will also contain the substance of [7]. Recent results of Smale and Munkres have emphasised the importance of the groups Θ_m .

4. Simplifying certain $(m-1)$ -parallelisable $(2m+1)$ -manifolds. Suppose that U is an $(m-1)$ -parallelisable $(2m+1)$ -manifold, and in addition that $H_{m-1}(U)$ is torsion free, hence free Abelian. By Theorem 3 of [M], U is χ -equivalent to an $(m-1)$ -connected manifold. We wish to obtain a slight refinement of this result. Now since $H_{m-1}(U)$ is free, $H^m(U, A) = \text{Hom}(H_m(U), A)$. The obstruction p to m -parallelisability of U lies in $H^m(U, \pi_{m-1}(O))$, where O denotes the stable orthogonal group. We make the convention of regarding p as a function on $H_m(U)$.

We may now state the reduction lemma.

LEMMA 6. *If U is a compact $(m-1)$ -parallelisable $(2m+1)$ -manifold, with $H_{m-1}(U)$ torsion free, then there is a sequence of surgeries taking U to an $(m-1)$ -connected manifold U^* , and such that*

(i) *If $m > 2$, there are induced isomorphisms of $H_m(W)$, $H_{m+1}(W, Z_\theta)$ at each stage, which commute with the Bockstein operator, with intersection numbers mod θ , and with p .*

(ii) *If $m = 2$, there are forwards maps of $H_2(W)$ at each stage, inducing isomorphisms of its torsion subgroup, and backwards maps of $H_3(W, Z_\theta)$, commuting with the same three invariants, and inducing isomorphisms*

$$H_2(U) = H_2(U^*), \quad H_3(U^*, Z_\theta) = H_3(U, Z_\theta).$$

Proof. If $m < 2$, we can take $U^* = U$ (supposed connected).

(i) If $m > 2$, we may first use the procedure of [M] to kill successively the $\pi_i(U)$: $0 < i < m-1$. We note that this induces natural isomorphisms of $H_m(W)$, $H_{m+1}(W, Z_\theta)$ at each stage, and if the resulting manifold is U_1 , $H_{m-1}(U_1)$ is naturally imbedded in $H_{m-1}(U)$, hence it also is torsion free. Since $m-1 > 1$, by the Hurewicz isomorphism, $\pi_{m-1}(U) = H_{m-1}(U)$, so is free Abelian. Now since U_1 is $(m-1)$ -parallelisable, by construction, we may kill the generators in turn: it is easy to see that $H_m(W)$ and $H^{m+1}(W, Z_\theta)$ remain unaltered. The required commutativities now follow from the naturality of the several invariants for the successive inclusion maps $W'' \rightarrow W$ and $W'' \rightarrow W'$.

(ii) If $m = 2$, we may first choose elements of $\pi_1(U)$ inducing generators of $H_1(U)$, and kill these as before. Hence we may assume $H_1(U) = 0$. We now select a set of generators of $\pi_1(U)$ and kill them in order. At each stage, we have exact sequences

$$\begin{array}{c}
 0 \rightarrow H_2(W'') \rightarrow H_2(W') \rightarrow Z \rightarrow 0 \\
 \parallel \\
 H_2(W) \\
 0 \rightarrow Z_\theta \rightarrow H_3(W'', Z_\theta) \rightarrow H_3(W, Z_\theta) \rightarrow 0. \\
 \parallel \\
 H_3(W', Z_\theta)
 \end{array}$$

Let the resulting manifold be U_1 . $H_2(U)$ is contained in $H_2(U_1)$ with free Abelian quotient group. We lift a set of generators of this quotient group to $H_2(U_1)$: we may suppose that p vanishes on each. For if U is 2-parallelisable, by Theorem 3 of [M], we may suppose that U_1 is also, so p vanishes identically; yet if not, p is a nonzero homomorphism $H_2(U) \rightarrow Z_2$, and to each lifted generator on which p does not vanish we may add an element of $H_2(U)$ with the same property.

Since p vanishes on these generators, they are representable by imbeddings of $S^2 \times D^3$, and we may perform χ -constructions to kill them. At each stage of this process we have exact sequences (by Lemma 1)

$$\begin{array}{c}
 0 \rightarrow Z \rightarrow H_2(W'') \rightarrow H_2(W') \rightarrow 0, \\
 \parallel \\
 H_2(W) \\
 0 \rightarrow H_3(W'', Z_\theta) \rightarrow H_3(W, Z_\theta) \rightarrow Z_\theta \rightarrow 0. \\
 \parallel \\
 H_3(W', Z_\theta)
 \end{array}$$

The resulting manifold is the required U^* . We have exhibited maps of the homology groups as stated, which induce isomorphisms as stated (this is clear for H_2 and will follow by duality for H_3). From the diagrams above, and from the naturality of the invariants for the inclusion maps, follow again the various commutation relations.

COROLLARY. *Suppose in addition that the boundary of U has no homology in dimensions $m, m+1$, so that a bilinear form can be set up as in §2. Then the transition from U to U^* preserves the bilinear form.*

This is clear, since the form is defined by Bocksteins and intersection numbers.

5. Topology of certain $(m-1)$ -parallelisable $(2m+1)$ -manifolds, $(m \text{ even})$.

We may now apply the above lemma to make our manifolds $(m-1)$ -connected, and the methods of the rest of this paper will then apply. We shall study the homomorphism c of Lemma 4 (ii), and show in particular that if W is m -parallelisable, then $c = 0$.

We shall suppose in the following that W satisfies the condition:

(A) W is a compact $(m-1)$ -parallelisable $(2m+1)$ -manifold, such that $H_{m-1}(W)$ is torsion free and $H_m(W)$ finite, and the boundary of W has no homology in dimensions $m, m+1$; where m is even.

The obstruction p to m -parallelisability has coefficient group $\pi_{m-1}(O)$, which was evaluated by Bott [3] as Z if $m \equiv 0 \pmod{4}$; as 0 if $m \equiv 6 \pmod{8}$; and as Z_2 if $m \equiv 2 \pmod{8}$. But under (A), $H^m(W)$ vanishes, so $p=0$ unless $m \equiv 2 \pmod{8}$, when the coefficient group is Z_2 .

LEMMA 7. *If W satisfies (A), $x \in H_m(W)$ and $p(x)=0$, then $c(x)=0$.*

Proof. By Lemma 6, we may suppose W $(m-1)$ -connected. Note that $p(x)=0$ is the condition that x be representable (by an imbedding of $S^m \times D^{m+1}$). We take a base of $H_m(W)$ as in Lemma 4. Let y_1 be an element of this base of order greater than 2 with $p(y_1)=0$, $c(y_1) \neq 0$. We shall deduce a contradiction.

Let $2n$ be the order of y_1 (it is even since $c(y_1) \neq 0$). Let A be the subgroup of $H_m(W)$ generated by x_j, y_j for $j \neq 1$. Since y_1 is representable, we can perform surgery. As in the proof of the theorem, using Lemma 3, we have $H_m(W'') = A \oplus Gp\{x'_1, y'_1, z\}$ where $2nx'_1 = z$, $2ny'_1 = (2\lambda n + n)z$. Hence $H_m(W') = A \oplus Gp\{x''_1, z'\}$ and this last group is cyclic of order $\geq 2n > 2$, which contradicts the corollary to Lemma 4 (A being of the type admitted by that corollary).

Now suppose that $H_m(W)$ contains an element x for which $p(x)=0$, $c(x) \neq 0$. Let $M_{2\theta}$ be obtained from $S^m \times S^{m+1}$ by performing surgery on 2θ times a generator of $H_m(S^m \times S^{m+1})$. Clearly, M satisfies (A). It is easy to see that $H_m(M) = 2Z_{2\theta}$, and since $p=0$ for $S^m \times S^{m+1}$, by [M] we may suppose that it is 0 for M , hence $c=0$, since by what we have already proved c vanishes on each generator. Let x_0 be a generator of $H_m(M)$ (of order 2θ).

Form $W \# M$. Now $H_m(W \# M) = H_m(W) \oplus H_m(M)$, and it is clear that b admits the direct sum decomposition and c and p are additive. Consider the element $x+x_0$ of order 2θ . We have

$$p(x+x_0) = p(x) + p(x_0) = 0, \quad c(x+x_0) = c(x) + c(x_0) = c(x) \neq 0.$$

By the proof of Lemma 4, an odd multiple y of $x+x_0$ can be chosen as a basis element of $H_m(W \# M)$; this will have order greater than 2, and $p(y)=0$, $c(y) \neq 0$, which contradicts what we proved above. This proves the lemma.

The lemma may be rephrased: $c=p$ or $c=0$. For if the kernel of c properly contains that of p , which has index at most 2, the kernel of c is the whole group, so $c=0$. If $m \not\equiv 2 \pmod{8}$, this simply states $c=0$. If $m \equiv 2 \pmod{8}$, we shall now show that whether c is p or 0 depends only on m . In fact we shall produce a closed manifold V satisfying (A), and with $p(V) \neq 0$. Form $W \# V$. p and c are additive. There are now two cases.

If $c(V)=p(V)$, $c(W \# V) = c(W) + c(V) \neq 0$ since $c(V) \neq 0$. Hence it equals $p(W \# V) = p(W) + p(V)$, and we deduce $c(W) = p(W)$.

If $c(V)=0$, $c(W) + c(V) \neq p(W) + p(V)$ since $c(V) \neq p(V)$. Hence $c(W) = c(W \# V) = 0$.

The manifold V may be constructed as follows. Take the nontrivial S^{m+1}

bundle U over S^m (defined since $m \equiv 2 \pmod{8}$). Let x generate $H_m(U)$. $p(2x) = 2p(x) = 0$, so we may perform surgery and kill $2x$. This yields a manifold V which satisfies (A), and x determines a class x' in V with $p(x') \neq 0$.

In the case $m = 2$, we can show that $c = p$. (We have not yet succeeded in deciding the question in any other cases.) For the Wu manifold $P(1, 2)$ (see [4]) satisfies (A) and has $H_m(P) = Z_2$. Since b is nonsingular, $c \neq 0$. We may sum up these results as

PROPOSITION 1. *Let W satisfy (A). If $m \not\equiv 2 \pmod{8}$, $c(W) = p(W) = 0$. If $m \equiv 2 \pmod{8}$, there is an integer $r_m \pmod{2}$ such that $c(W) = r_m p(W)$ for all W . Moreover, $r_2 = 1$.*

Now for $m = 2$, p is the second Stiefel class w^2 . For any closed 5-manifold W satisfying (A), we know c by elementary homology theory, and may now use Wu's formulae to deduce from w^2 the operation of the Steenrod squares in W .

We finally turn to the problem of deciding when in Lemma 4 (ii) there is an extra Z_2 . Since c is a homomorphism and b nonsingular, G has an element y_0 with $c(x) = b(x, y_0)$ for all x . It is easy to show that the extra Z_2 appears if and only if $c(y_0) \neq 0$. For 5-manifolds, this fact admits an interesting interpretation. We know that $c = p = w^2$. Now we have the commutative diagram

$$\begin{array}{ccccc}
 & H^2(W, Z_2) & \xrightarrow{\quad} & H^2(W, Z_\theta) & \\
 \beta_2 \swarrow & & \searrow D & & \downarrow D \\
 H^3(W, Z_2) & & H_3(W, Z_2) & \xrightarrow{\quad} & H_3(W, Z_\theta) \\
 & \searrow D & \swarrow \beta_2 & & \downarrow \beta_\theta \\
 & H_2(W, Z_2) & \xleftarrow{\quad} & H_2(W, Z_\theta) &
 \end{array}$$

where D denotes duality isomorphisms, β_2 is the Bockstein, and the horizontal maps are induced by the obvious homomorphisms of coefficient groups. But $w^2 \in H^2(W, Z_2)$ maps under β_2 to $w^3 \in H^3(W, Z_2)$, and $c \in H^2(W, Z_2)$ maps to $y_0 \in H_2(W, Z_\theta)$, so each of $w^2 w^3$, $c(y_0)$ is equal to the Kronecker product of c with its image in $H_2(W, Z_2)$. Now since a closed oriented 5-manifold W is cobordant to zero if and only if the Stiefel number $w^2 w^3 [W]$ vanishes by [8], we have proved

PROPOSITION 2. *Let W be a closed oriented 5-manifold such that $H_1(W)$ is torsion free, $H_2(W)$ finite. Then there exists a finite Abelian group B such that either*

$$(i) \ H_2(W) = B \oplus B \quad \text{or} \quad (ii) \ H_2(W) = B \oplus B \oplus Z_2.$$

W is cobordant to zero if and only if (i) holds.

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