Seifert's Algorithm, Châtelet Bases and the Alexander Ideals of Classical Knots

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A Thesis presented for the degree of
Doctor of Philosophy

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Dedicated to
My Parents

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Abstract

I begin by developing a procedure for the construction of a Seifert surface, using Seifert's algorithm, and the calculation of a Seifert matrix for a knot from a suitable encoding of a knot diagram. This procedure deals with the inherent indeterminacy of the diagram encoding and is fully implementable.

From a Seifert matrix one can form a presentation matrix for the Alexander module of a knot and calculate generators for the Alexander ideals. But to use the Alexander ideals to their full potential to distinguish pairs of knots one needs a Gröbner basis type theory for $\Lambda = \mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials with integer coefficients.

I prove the existence of what I call *Châtelet bases* for ideals in Λ. These are types of Gröbner bases. I then develop an algorithm for the calculation of a Châtelet basis of an ideal from any set of generators for that ideal. This is closely related to Buchberger's algorithm for Gröbner bases in other polynomial rings.

Using these algorithms and the knot diagram tables in the program *Knotscape* I calculate Châtelet bases for the Alexander ideals of all prime knots of up to 14 crossings. We determine the number of distinct ideals that occur and find examples of pairs of mutant knots distinguished by the higher Alexander ideals but not by any of the polynomials of Alexander, Jones, Kauffman or HOMFLY.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences of the University of Durham, England from 1998 to 2002. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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My studies were supported by awards from the University of Durham and the Engineering and Physical Sciences Research Council of the United Kingdom. And my parents of course, have always helped me as I spent my time doing mathematics.

Last and by all means most I thank Geraldine for her love and help.

The computations described in Chapter 5 of this thesis were performed with the aid of *Maple*(TM) and *Knotscape*.

"Knotting ought to be reckoned, in the scale of insignificance, next to mere idleness."

-Dr. Samuel Johnson [1]

[1] *Dictionary*, 1755 (seen in [13]).

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Process.mws		

Directions for the use of the CD-ROM are in appendix B.

Chapter 1

Introduction

1.1 Knot Theory

In this section we give the definitions and relevant facts on the knot theory used in the rest of the thesis. It has mainly been gathered from the books of Burde and Zieschang [5] and Kawauchi [21].

1.1.1 Knots, Equivalences and Diagrams.

Definition 1.1.1 A *knot* k, is an embedding of the circle S^1 in the 3-sphere S^3, or Euclidean space $\mathbb{R}^3$. The *knot exterior* $K = \overline{S^3 - n(k)}$, is the closure of the complement in S^3 of a regular open neighbourhood $n(k)$, of k.

There is a natural notion of equivalence among knots.

Definition 1.1.2 Two knots k_1 and k_2 are *topologically equivalent* if there is an ambient isotopy carrying k_1 to k_2.

This is an equivalence relation on the set of knots. If we take our knots to be oriented, i.e. an embedding of an oriented S^1, then we would ask that the ambient isotopy carry the orientation of k_1 to that of k_2.

To avoid the peculiarities of so called *wild* knots, attention is normally restricted to the piecewise linear setting.

Definition 1.1.3 A *piecewise linear knot* is a piecewise linear embedding of S^1 into S^3. Two piecewise linear knots k_1 and k_2 are *piecewise linear equivalent* if there is a piecewise linear ambient isotopy carrying k_1 to k_2.

So a *tame*(=*non-wild*) knot is defined as follows.

Definition 1.1.4 A *tame* knot is one that is topologically equivalent to a piecewise linear knot.

This might leave us in two possible situations; on the one hand with equivalence classes of tame knots with respect to topological equivalence and on the other hand with equivalence classes of piecewise linear knots with respect to the piecewise linear equivalence. However, these two situations coincide.

Proposition 1.1.1 (Corollary 3.16 in [5]) Two tame knots are equivalent if and only if the piecewise linear knots in their topological equivalence classes are piecewise linear equivalent.

So in the rest of this thesis we will understand knot to mean a tame knot.

Knots are often studied through their diagrams.

Definition 1.1.5 A *regular projection* of a knot k to the plane is a projection of k whose self-intersections are all transversal.

Definition 1.1.6 A *diagram* of a knot k is a regular projection of k to the plane $\mathbb{R}^2$ in which the over-crossing and under-crossing parts of k are indicated, usually by omitting a small portion of the under-crossing part.

In figure 1.1 we give a diagram of the figure eight knot. We shall regard two diagrams as essentially the same if there is an isotopy between the projections corresponding to the diagrams that preserves the over-crossing and under-crossing information of the diagrams.

1.1.2 Seifert Surfaces, the Seifert Form and Seifert Matrices

Definition 1.1.7 A Seifert surface of a knot k is an orientable surface F embedded in S^3 such that the boundary of F is equal to k.

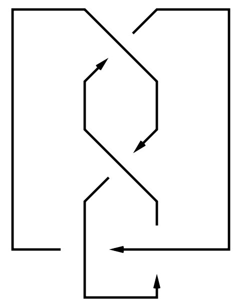


Figure 1.1: A diagram of the figure eight knot.

That a Seifert surface exists for any knot is a consequence of Seifert's algorithm, which is described in chapter 2. This is a simple strategy for constructing a Seifert surface from a diagram of the knot. Given an embedding of a Seifert surface $\psi : F \to S^3$ we can always thicken the embedding to produce a *bi-collar* of F.

Definition 1.1.8 A *bi-collar* of F is an embedding $\hat{\psi} : F \times [-1, 1] \to S^3$ such that $\hat{\psi}(F \times \{0\}) = \phi(F)$. We will use the notation F^+ for $\hat{\psi}((F \times \{1\})$ and F^- for $\hat{\psi}((F \times \{-1\})$. We will assume further that the bi-collar is such that F^+ is in the positive normal direction from the surface F.

There is essentially only one way of doing this as given a second bi-collar ξ, there will always be an isotopy of S^3 between $\hat{\psi}$ and ξ.

Given a 1-cycle α on F we will use the notation α^+ for the copy of α on F^+, and α^- for the copy of α on F^{-1}. We note that if α_1 and α_2 are two 1-cycles on F then the cycles α_1^+ and α_2 will be a pair of disjoint 1-cycles in S^3.

Now given any pair of disjoint 1-cycles β and γ in S^3 we can define their linking number Link(β, γ)(=Link(γ, β)) as the algebraic intersection number of γ with b, where b is any 2-cycle in S^3 with $\partial b = \beta$ such that γ and b intersect transversally.

So to any pair of cycles α_1 and α_2 on F we can associate the two linking numbers Link(α_1^+, α_2) and Link(α_2^+, α_1). These linking numbers only depend on the homology classes of the 1-cycles α_1 and α_2. One can also show that

$$\mathrm{Link}(\alpha_1^+, \alpha_2) = \mathrm{Link}(\alpha_2^+, \alpha_1) + \mathrm{Int}(\alpha_1, \alpha_2) \tag{1.1}$$

where Int(α_1, α_2) is the algebraic intersection number of the 1-cycles on F.

This allows the definition of the bilinear *Seifert form.*

Definition 1.1.9 The *Seifert form* associated to a Seifert surface F is the form

$$\phi : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z} \tag{1.2}$$

defined by

$$\phi(a_1, a_2) = \mathrm{Link}(\alpha_1^+, \alpha_2) \tag{1.3}$$

where α_1 and α_2 are cycles on F representing the elements a_1 and a_2 of $H_1(F)$.

Now $H_1(F)$ is a free abelian group of finite rank. Let $\{a_1, \ldots, a_n\}$ be a basis for $H_1(F)$. Then we can represent the Seifert form with the $n \times n$ matrix S with entries

$$S_{i,j} = \phi(a_i, a_j) \tag{1.4}$$

Definition 1.1.10 Such a matrix S is called a *Seifert matrix* of k.

Of course a Seifert matrix S is not an invariant k as it depends on the Seifert surface F of k and the chosen basis of $H_1(F)$. There is an equivalence relation called *S-equivalence* among Seifert matrices whose equivalence classes are knot invariants. However we will be interested in some knot invariants that can be obtained from S in another way.

1.1.3 The Alexander module and the Alexander Ideals.

Here we will define the Alexander module, Alexander ideals and Alexander polynomials which are all knot invariants. Let $\pi_1(K)$ be the fundamental group of the exterior of a knot. Let $h : \pi_1(K) \to H_1(K)$ be the Hurewicz homomorphism. The covering space of K corresponding the subgroup $\mathrm{Ker}(h) = [\pi_1(K), \pi_1(K)]$ of $\pi_1(K)$ is known as the *universal abelian covering space* of K. It is also known as the *infinite cyclic covering space* of K since the group of covering transformations is $H_1(K) = \langle t \rangle$, the infinite cyclic group. We shall refer to it thus and denote it by K_∞. There is a well known construction of K_∞ using a Seifert surface for k which we now give.

Let F be a Seifert surface for k. Let K' be K cut open along $K \cap F$, with F^+ and F^- denoting the two copies of F in K'. We take an infinite family of copies of K', $\{K'_i\}_{i \in \mathbb{Z}}$, and let $F_i^\pm$ denote the copy of $F^\pm$ contained in K'_i. Form the quotient

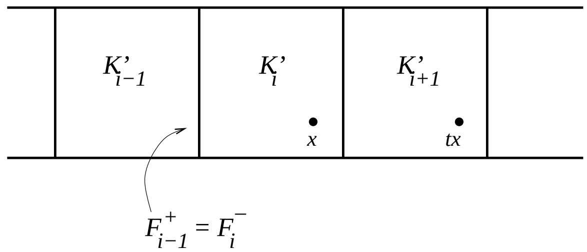


Figure 1.2: Construction of K_∞.

space of $\bigcup_{i\in\mathbb{Z}} K'_i$ by identifying, for each i, F_i^+ with F_{i+1}^-. The resulting space is K_∞, see figure 1.2, and the action of $H_1(K)$ on K_∞ can be visualised as follows: if $t \in H_1(K)$ is the class of a meridian of k, oriented so that it has linking number $+1$ with the knot k, then t acts on K_∞ by sending $x \in K'_i$ to the corresponding point in K'_{i+1}. This generates an action of $H_1(K)$ on $H_1(K_\infty)$. We shall denote the integer group ring of $H_1(K)$ as $\Lambda = \mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials in the variable t with integer coefficients. So the action of $H_1(K)$ on $H_1(K_\infty)$ gives $H_1(K_\infty)$ the structure of a Λ-module.

Definition 1.1.11 The *Alexander module* of k is the Λ-module $H_1(K_\infty)$.

We can obtain a presentation matrix for the Alexander module from a Seifert matrix as shown by theorem 1.1.12 below. Consider the two maps induced by inclusion

$$\zeta_* : H_1(F^+) \to H_1(K') \tag{1.5}$$

$$\xi_* : H_1(F^-) \to H_1(K') \tag{1.6}$$

Let $\{a_1, \ldots, a_n\}$ be a basis for $H_1(F) \cong H_1(F^+) \cong H_1(F^-)$. Now $H_1(K')$ has a basis $\{b_1, \ldots, b_n\}$ that is dual to the basis $\{a_1, \ldots, a_n\}$, i.e. $\mathrm{Link}(a_i, b_j) = \delta_{ij}$. Note that

$$\zeta_*(a_i) = \sum_{j=1}^{n} \mathrm{Link}(\alpha_i^+, \alpha_j) b_j \tag{1.7}$$

$$\xi_*(a_i) = \sum_{j=1}^{n} \mathrm{Link}(\alpha_i^-, \alpha_j) b_j \tag{1.8}$$

$$= \sum_{j=1}^{n} \mathrm{Link}(\alpha_j^+, \alpha_i) b_j \tag{1.9}$$

So if S is the Seifert matrix of k with respect to the surface F and basis $\{a_1, \ldots, a_n\}$ of $H_1(F)$ then ζ_* and ξ_* are represented by the matrices S and S^T respectively, with respect to the bases $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$.

Theorem 1.1.12 $A = tS - S^T$ is a presentation matrix for the Alexander module $H_1(K_\infty)$.

Proof: This is from chapter 6 of [23]. Define $L, M \subset K_\infty$ as the following disjoint unions

$$L = \bigcup_{i\in\mathbb{Z}} K'_{2i} \tag{1.10}$$

$$M = \bigcup_{i\in\mathbb{Z}} K'_{2i+1} \tag{1.11}$$

Note that $L \cap M$ is a countable disjoint union of copies of F and that $K_\infty = L \cup M$. The Meyer-Vietoris sequence gives us the long exact sequence of homology groups with integer coefficients

$$\ldots \longrightarrow H_1(L\cap M) \xrightarrow{\theta_*} H_1(L) \oplus H_1(M) \xrightarrow{\psi_*} H_1(K_\infty) \tag{1.12}$$

$$\longrightarrow H_0(L\cap M) \xrightarrow{\theta_*} H_0(L) \oplus H_0(M) \tag{1.13}$$

Now F and K' are connected, so $H_0(F) = H_0(K') = \mathbb{Z}$. Therefore

$$H_0(L\cap M) = H_0(L) \oplus H_0(M) = \Lambda \tag{1.14}$$

as Λ-modules since both are a direct sum of countable copies of $\mathbb{Z}$.

Now suppose $1 \in H_0(L \cap M)$ is represented by F_0^+ say and K_0 represents $1 \in H_1(L) \oplus H_1(M)$ then

$$\theta_*(1) = t - 1 \tag{1.15}$$

Hence $\theta_* : H_0(L\cap M) \to H_0(L) \oplus H_0(M)$ is injective. So $\psi_* : H_1(L) \oplus H_1(M) \to H_1(K_\infty)$ is surjective.

Which gives us the short exact sequence of Λ-modules

$$H_1(L\cap M) \xrightarrow{\theta_*} H_1(L) \oplus H_1(M) \xrightarrow{\psi_*} H_1(K_\infty) \tag{1.16}$$

where $\theta_* : H_1(L\cap M) \to H_1(L) \oplus H_1(M)$ is represented by the matrix $tS - S^T$. But $H_1(L \cap M)$ and $H_1(L) \oplus H_1(M)$ are both finitely generated free Λ-modules. Hence the Alexander module $H_1(K_\infty)$ is presented by $tS - S^T$. ■

But of course the Alexander module of k can have many different presentation matrices. It is a fact that any two presentation matrices of an R-module, where R is any ring, are related by a sequence of the following matrix transformations and their inverses. Let A be a matrix with entries from R.

1. Permuting the rows or permuting the columns of A.

2. Multiplying a row or column of A by a unit of R.

3. Adding to a row an R-linear combination of the other rows of A or adding to a column an R-linear combination of the other columns of A.

4. Replacing A with the matrix
$$\begin{pmatrix} A \\ r \end{pmatrix}$$
where r is any R-linear combination of the rows of A.

5. Replacing A with the matrix
$$\begin{pmatrix} A & 0 \\ r & 1 \end{pmatrix}$$
where r is an arbitrary row

In view of this fact we can see that the *elementary ideals* of a presentation matrix A are invariants of the module presented by A.

Definition 1.1.13 Given an $n \times m$ matrix A with entries from R, the r^{th} *elementary ideal* $\mathcal{I}_r$ of A is the ideal of R generated by the $(n-r+1) \times (n-r+1)$ minors of A. By convention we take $\mathcal{I}_r = 0$ for $r \leq 0$ and $\mathcal{I}_r = R$ for $r > n$. The elementary ideals form an ascending chain, i.e. for all r, $\mathcal{I}_{r-1} \subseteq \mathcal{I}_r$.

These ideals are also known as Fitting ideals and determinental ideals. In knot theory we refer to the elementary ideals as the *Alexander ideals.*

Definition 1.1.14 The r^{th} *Alexander ideal* of a knot k is the r^{th} elementary ideal of a presentation matrix for the Alexander module, which we can take as $tS - S^T$ for any Seifert matrix S of k. By the *length* of the chain of Alexander ideals we mean the greatest r for which the r^{th} Alexander ideal is not the whole ring Λ.

The Alexander ideals satisfy two conditions. Let $\bar{\ } : \Lambda \to \Lambda$ be the ring homomorphism that sends t to t^{-1}. Let $\epsilon : \Lambda \to \mathbb{Z}$ be the ring homomorphism that sends a polynomial f to $f(1)$. The following lemma is from [9].

Lemma 1.1.15 If $\mathcal{I}$ is an Alexander ideal then $\mathcal{I}$ satisfies

1. $\overline{\mathcal{I}} = \mathcal{I}$
2. $\epsilon(\mathcal{I}) = \mathbb{Z}$

And this actually characterises the Alexander Ideals. Any ideal satisfying 1 and 2 is an Alexander ideal of some knot as shown by Kearton [22].

As shown by theorem 1.1.12 an Alexander module has a square presentation matrix so that the 1^{st} Alexander ideal $\mathcal{I}_1$ of a knot is always principal. A generator, $\Delta_1 \in \Lambda$ of this is known as the *Alexander polynomial* of k. Δ_1 is obviously only defined up to multiplication by units of Λ, i.e. elements of the form $\pm t^i$, where $i \in \mathbb{Z}$. A higher Alexander ideal $\mathcal{I}_r$, where $r > 1$, of a knot k need not be principal, but as Λ is a unique factorisation domain, every ideal of Λ is contained in a unique principal ideal.

Definition 1.1.16 The r^{th} *Alexander polynomial* Δ_r, is a generator of the unique principal ideal containing the r^{th} Alexander ideal $\mathcal{I}_r$.

We would usually normalise the Alexander polynomials so that they lie in $\mathbb{Z}[t]$ and have non-zero positive constant term.

1.2 Gröbner Bases and Using the Alexander Invariants

Suppose we have two knots k and l with diagrams d_k and d_l and we want to use the Alexander invariants to try and decide if k is not equivalent to l. We could proceed as follows. We would construct Seifert surfaces F_k and F_l for k and l from their diagrams. Then we would choose bases for $H_1(F_k)$ and $H_1(F_l)$ and calculate linking numbers to obtain Seifert matrices S_k and S_l. Next we would form the presentation matrices $A_k = tS_k - S_k^T$ and $A_l = tS_l - S_l^T$ for the Alexander modules of k and l. Taking determinants of A_k and A_l we would obtain the Alexander polynomials $\Delta_1(k)$ and $\Delta_1(l)$. By inspection we could decide whether these are the same or not (remembering to normalise them as above). If they were unequal we could conclude that k and l were inequivalent knots. If they were equal then we would need more information. We could then try calculating the second Alexander ideals $\mathcal{I}_2(k)$ and $\mathcal{I}_2(l)$. Calculating all the various minors of A_k and A_l would give us sets of generators for the second Alexander ideals,

$$\mathcal{I}_2(k) = \langle G_k \rangle_\Lambda \quad , \quad \mathcal{I}_2(l) = \langle G_l \rangle_\Lambda \tag{1.17}$$

where G_k and G_l are the sets of minors. But we cannot tell by simple inspection of G_k and G_l whether the ideals $\mathcal{I}_2(k)$ and $\mathcal{I}_2(l)$ are the same or not; for quite different sets of polynomials can generate the same ideal. We could calculate the second Alexander polynomials as

$$\Delta_2(k) = \gcd(G_k) \quad , \quad \Delta_2(l) = \gcd(G_l) \tag{1.18}$$

And inspection of these might allow us to decide the equivalence or not of k and l. The problem here is that by passing to the Alexander polynomials we are losing information. Different Alexander ideals can posses the same Alexander polynomials, i.e. be contained in the same principal ideal.

So it would be nice if there were a practical way to decide whether G_k and G_l generate the same ideal or not.

1.2.1 Gröbner Bases

This problem of deciding whether two sets of polynomials G and H, of a polynomial ring R say, generate the same ideal is the subject of Gröbner bases. To give a *very* brief sketch, a *Gröbner basis* $\mathcal{B}$ for an ideal $\mathcal{G}$, is a set of polynomials that generate the ideal and such that there is a procedure for deciding whether or not any polynomial $f \in R$ is in the ideal $\mathcal{G}$. This procedure is a type of *division process* of f with respect to $\mathcal{B}$ that relies on fixing order on the monomials of R. We prefer not to give a definition here as it is quite detailed and can vary depending on the context; as for polynomial rings of more than one variable there are many different ways of ordering the monomials and different choices of these orderings lead to different *division processes.*

But the important point is that a Gröbner basis for $\mathcal{G}$ allows the *decision* of whether f is a member of $\mathcal{G}$ or not. So having Gröbner bases for two ideals would allow one to decide whether one ideal is contained in the other or vice-versa by testing each element of a basis for membership of the other ideal.

In his 1965 Ph.D. thesis [3] Buchberger introduces the subject and gives an algorithm for generating a Gröbner basis for an ideal $\mathcal{G} = \langle G \rangle_R$ from any set of generators G for the ideal in the case of $R = k[x_1, \ldots, x_n]$, the ring of polynomials in n variables with coefficients from a field k. The algorithm generates and adds certain polynomials to G until G becomes a Gröbner basis.

Buchberger and others have extended this result to other polynomial rings and in [1] Gröbner bases and algorithms for generating them are given for the case of $R = S[t]$, where S is a principal ideal domain, which includes the case $R = \mathbb{Z}[t]$ of course. Rings of Laurent polynomials do not seem to have attracted much attention and in the two sources we've used, [1] and [2], they receive no mention.

In Chapter 3 we will define what we have called Châtelet bases. These are a type of Gröbner basis for ideals of the ring Λ. The definition is based on work of Albert Châtelet on the ring $Z[t]$ which was published posthumously in [6] by François Châtelet in 1967. He called his basis a *reduced basis* and it is actually equivalent to what is called a *minimal strong Gröbner basis* in [1]. Unfortunately Châtelet's work does not seem to have been noticed by others and we have not been able to find it in any of the bibliographies on Gröbner bases. We continue in Chapter 3 to give a Buchberger type algorithm; one that generates a Châtelet basis for an ideal $\mathcal{G} = \langle G \rangle_\Lambda$ of Λ from any set G of generators for the ideal. The form of our algorithm is very similar to, and was developed from, that for $S[t]$, where S is a principal ideal domain, found in [1]. But there are slight differences in the concept of *division process* used, and we replace the concept of monomial orders with the related concept of the *length* of Laurent polynomials.

A recent paper by Pauer and Unterkircher [24] did give definitions and algorithms for Gröbner bases in Laurent polynomial rings by developing appropriate extensions of the definition of term orders and monomial orders.

Chapter 2

Implementing Seifert's Algorithm and the Calculation of the Seifert Matrix

2.1 Seifert's Algorithm

We begin this chapter by describing Seifert's algorithm, the well known procedure for constructing a Seifert surface of a knot from a diagram of the knot. So let D be a diagram in the plane, of a knot K. At each of the crossing points of D we modify D as indicated in figure 2.1. The result will be to change D to a collection $\mathcal{C}$ of disjoint oriented simple closed curves.

Definition 2.1.1 We shall refer to these closed curves as the *Seifert circles* of the diagram D.

We span each of the Seifert circles with a disk which we orient in such a way to agree with the orientation on the Seifert circle.

Definition 2.1.2 We shall refer to these disks as the *Seifert disks* of the diagram D.

Now the Seifert disks of D are not necessarily disjoint as some of them may contain others. We remedy this by lifting, as necessary, some of the disks directly up-wards

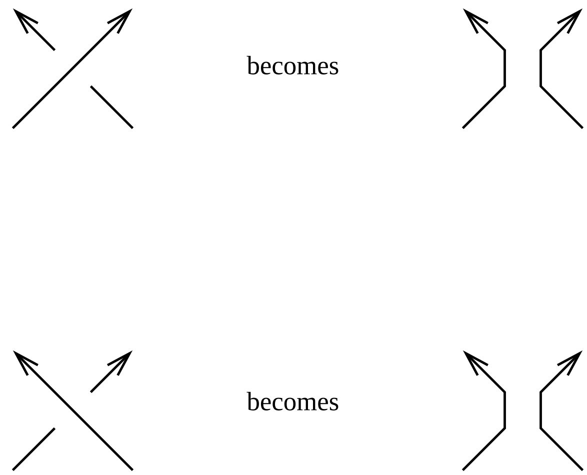


Figure 2.1: Modifying the crossing points of a knot diagram.

off of the plane, into $\mathbb{R}^3$ so that the Seifert disks of D are disjoint and located in various 'levels', $\mathbb{R}^2 \times \{x\}$ of $\mathbb{R}^3$. For each of the crossing points of D we paste a twisted band to the Seifert disks lying either side of the crossing as in figure 2.2. So now we have a surface F that is the union of the Seifert disks and the twisted bands. That the surface F is orientable can be seen from figure 2.2 where we see how the pasted bands preserve the orientation from disk to disk. Finally we notice that the boundary of F is a knot of the same type of K and that the boundary of F projects down onto the original diagram D. We can be imprecise and refer to F as a Seifert surface of K. In figure 2.3 we see these stages applied to the figure eight knot where we have raised the disk spanning the circle C_2 above that spanning the circle C_1.

Having constructed F we can calculate a Seifert matrix of K by choosing a set of cycles on F that form a basis for $H_1(F)$ and then working out the associated linking numbers. At first sight this strategy sounds simple and straightforward. Indeed it is readily carried out by hand for diagrams with not too many crossings. However there are lots of arbitrary choices made, in lifting the Seifert disks and choosing a basis for $H_1(F)$. So for a complicated diagram with many crossings and levels of nesting it might not be so straightforward. Indeed for a particularly nasty diagram it might be a task even to recognise when and how Seifert circles are nested one inside the other. So we would like to give a more precise description of how to carry out Seifert's algorithm and the calculation of the Seifert matrix, ideally producing something that could be programmed on to a computer and produce a Seifert matrix from some type of encoding of a knot diagram. That is what we do in this chapter and such a program can be found on the accompanying CD-ROM.

Others have recognised this difficulty with Seifert's algorithm and have devised ways around the problem. In chapter 7 of [19] Kauffman gives a procedure that involves changing the knot diagram to a link diagram by adding simple closed curves (that are unknotted and unlinked from the original knot diagram) such that the link diagram will not have any nested Seifert circles. The Seifert surface of this link is closely related to the Seifert surface, F, associated to the original diagram and it allows one to find a basis for $H_1(F)$ and to read off the linking numbers from the new link diagram. In chapter 13 of [5] Burde and Zieschang present another method. There it is proved that any knot diagram can be changed to a type of diagram (of the same knot), what they call a *special diagram*, that again, has no nesting of Seifert circles. Then they describe, in a similar way to Kauffman, a way to read off a basis for H_1 of the surface associated to a special diagram and the associated linking numbers. But given an arbitrary diagram one is still left with the task of recognising nesting and constructing a special diagram. Also, both of these solutions seem to still rely on a person looking at actual diagrams and making certain observations about them.

In contrast the procedure we will describe is algorithmic in the true sense of the word. It operates on an encoding of a knot diagram which we call the *combinatorial data* and we have implemented it with the mathematical computer package Maple.

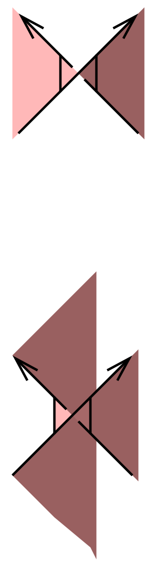


Figure 2.2: Pasting the twisted band to the Seifert disks.

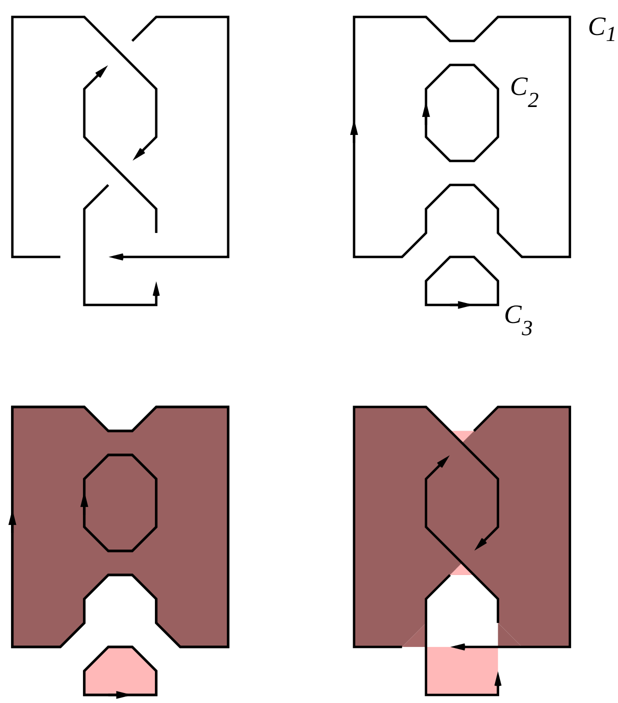


Figure 2.3: Seifert's algorithm on a diagram of the figure eight knot.

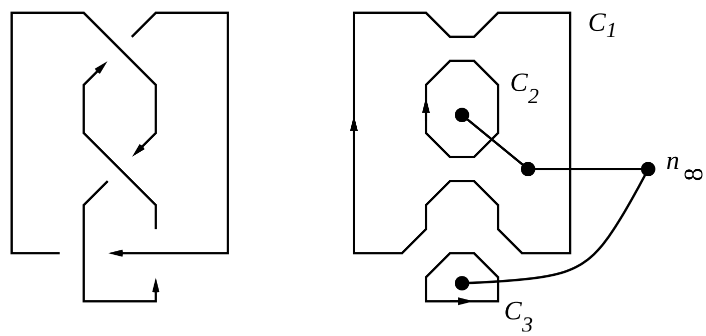


Figure 2.4: The nesting graph of a diagram of the figure eight knot.

2.2 The Nesting Graph and Standard Lift of a Knot Diagram

Let D be a knot diagram in the plane $\mathbb{R}^2$. We are going to describe here a particular knot K in $\mathbb{R}^2 \times [0, \infty)$ and Seifert surface F spanning K, such that K projects to D in $\mathbb{R}^2 \times \{0\}$. The main point that we are making rigorous from the basic description of Seifert's algorithm is deciding how to arrange the disks that span nested Seifert circles from D.

The *standard lift* of D is the knot K and Seifert surface F of K defined as follows. Let $\mathcal{C}$ be the collection of Seifert circles of D. For each $C \in \mathcal{C}$ let F_C be a disk spanning C. We shall locate each F_C in a level plane $\mathbb{R}^2 \times \{n\}$ for some $n \in \mathbb{N}$ according to the location of C in $\mathbb{R}^2$ with respect to the other Seifert circles. This is done with the use of the following graph which we associate to the diagram D.

Definition 2.2.1 Let D be a knot diagram in the plane $\mathbb{R}^2$. The *nesting graph* $\mathcal{N}(D)$ of D is the graph consisting of one node $n(U)$ for each component U of $\mathbb{R}^2 - \mathcal{C}$ and one edge $e(C)$ for each Seifert circle C. The edge $e(C)$ joins the nodes $n(U_1)$ and $n(U_2)$ where $C \in \partial U_1 \cap \partial U_2$.

In figure 2.4 we show the nesting graph of the diagram of the figure eight knot.

We note that $\mathcal{N}(D)$ is an acyclic graph. Let n_∞ be the node of $\mathcal{N}(D)$ corresponding to the unbounded component of $\mathbb{R}^2 - \mathcal{C}$. The nesting graph $\mathcal{N}(D)$ can be thought of as a tree, the *nesting tree*, with n_∞ as the root. The nodes of a tree have depth. The root has depth 0, the children of the root have depth 1, their children

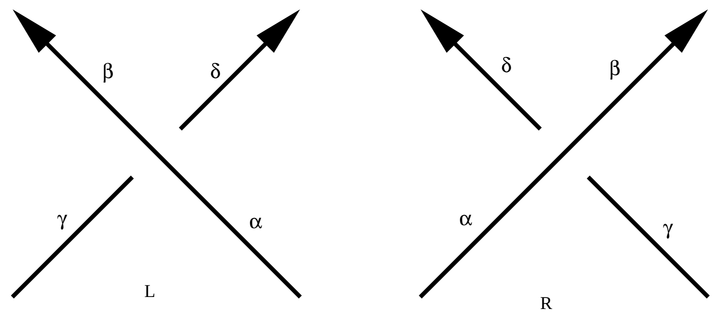


Figure 2.5: L and R crossings

have depth 2 etc. Now we can say at what level $\mathbb{R}^2 \times \{n\}$ we shall locate each disk F_C. The disk F_C is located at level n, where the edge $e(C)$ of $\mathcal{N}(D)$ joins nodes at depth n and $n+1$. So in our figure eight example in figure 2.4 the disks spanning the Seifert circles C_1 and C_3 will be in $\mathbb{R}^2 \times \{0\}$, while C_2 will be in $\mathbb{R}^2 \times \{1\}$.

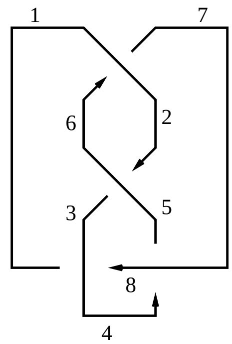
To complete the standard lift we add in the twisted bands that connect up the F_C, as described in the previous section.

2.3 The Combinatorial Data of a Knot Diagram

Let D be an oriented knot diagram with m crossings. In this section we will define the *combinatorial data* $\mathcal{P}(D)$ of the diagram D. This data will form the input to the algorithm that calculates the Seifert matrix.

We treat D as a union of arcs, where an arc is a segment of the diagram joining two crossing points without going through any other crossing points. As D has m crossing points it will consist of $2m$ arcs. We label the arcs with the integers $1, \ldots, 2m$ so that they run upwards as one traverses the diagram in the direction of orientation.

Now to each crossing c of D we associate a sequence of five labels $P_c = (\alpha, \beta, \gamma, \delta, S)$. Where $\alpha, \beta, \gamma, \delta$ are respectively, the labels of the incoming over-cross, outgoing over-cross, incoming under-cross and outgoing under-cross, and S is the letter L or R accordingly as the under-cross crosses the over-cross from left to right (L), or from the right to the left (R), see figure 2.5.



$$\{\ (1,2,6,7,R),\ \ (5,6,2,3,R),\ \ (3,4,8,1,L),\ \ (7,8,4,5,L)\ \}$$

Figure 2.6: The combinatorial data of a diagram of the figure eight knot.

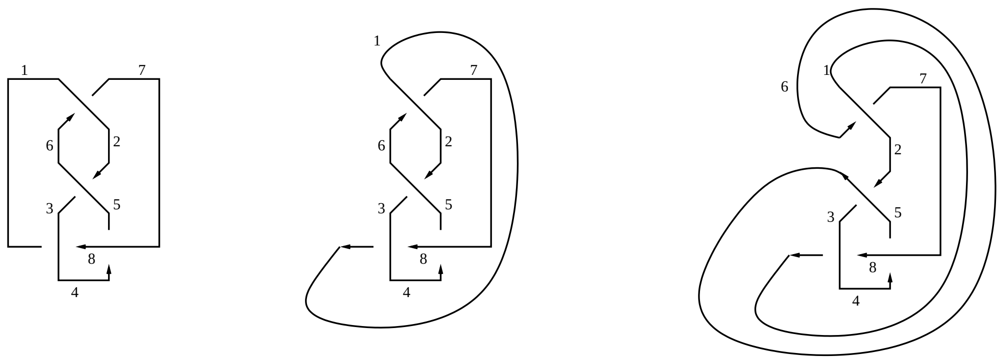


Figure 2.7: Different diagrams with the same combinatorial data.

Definition 2.3.1 Now we can define the *combinatorial data* $\mathcal{P}(D)$ of D as the set

$$\mathcal{P}(D) = \{P_c | c \text{ a crossing of } D\} \tag{2.1}$$

In figure 2.6 we show a labelling of a diagram of the figure eight knot and the associated combinatorial data. The combinatorial data does not uniquely determine the diagram as the diagrams in figure 2.7 for the figure eight knot show. These diagrams are not equivalent (related by an isotopy of the plane), as can be seen if you draw their Seifert circles. However they are all related by isotopies of the extended plane $\mathbb{R}^2 \cup \infty$. We might conjecture that this is true in general and in this chapter we will see how quite a lot of information about a diagram is calculated from its combinatorial data. For instance in section 2.5 we will see how diagrams with the same combinatorial data must have the same nesting graph.

2.4 Determining the Seifert Circles

In this section we describe how to determine the Seifert circles from the combinatorial data $\mathcal{P}(D)$. Each Seifert circle will be represented as a sequence $S(C) = (\alpha_1, \ldots, \alpha_n)$ of arc labels. Notice of course that the circle is represented also by any cyclic permutation of the labels, i.e. $(\alpha_1, \ldots, \alpha_n)$ and $(\alpha_2, \ldots, \alpha_n, \alpha_1)$ represent the same circle. This method was pointed out to me by Cherry Kearton.

Start with any arc α_1 that has not yet been identified as belonging to a Seifert circle. The sequence (α_1) is the initial segment of a representation $S(C)$, of some Seifert circle C of D. We shall denote this initial segment as $\hat{S}(C)$.

Now we show how to extend the initial segment $\hat{S}(C) = (\alpha_1, \ldots, \alpha_i)$ that describes a portion of the Seifert circle to $(\alpha_1, \ldots, \alpha_i, \alpha_{i+1})$. Now α_i is the incoming arc to one of the crossings c_i of D. We identify c_i by searching the P_c of $\mathcal{P}(D)$ to find α_i in either the incoming over-cross or incoming under-cross positions. If α_i is the incoming over-cross to c_i then let α_{i+1} be the outgoing under-cross from c_i. On the other hand, if α_i is the incoming under-cross to c_i then let α_{i+1} be the outgoing over-cross from c_i. Now if $\alpha_{i+1} = \alpha_1$ then the Seifert circle is complete and we record it's representation as the sequence $S(C) = (\alpha_1, \ldots, \alpha_i)$. On the other hand if $\alpha_{i+1} \neq \alpha_1$ then we set $\hat{S}(C) = (\alpha_1, \ldots, \alpha_i, \alpha_{i+1})$ and repeat the steps above until we have the complete representation for C.

We continue these steps until all of the arcs have been used. So we will have recorded representations for the Seifert circles as sequences of the arc labels.

2.5 Determining the Nesting Graph

In this section we show how to calculate from $\mathcal{P}(D)$, the nesting graph $\mathcal{N}(D)$ defined in section 2.2. For a crossing c of D we will say that the two Seifert circles involving the arcs at c are *adjacent* to c and that the two Seifert circles are *adjacent* to one another. Considering the two types, L and R, of crossing shown in figure 2.5 we can also say to which side of c the adjacent Seifert circles lie.

Definition 2.5.1 For a crossing c of type L we say that the Seifert circle containing the incoming over-cross and outgoing under-cross lies to the *right* of c and the Seifert circle containing the incoming under-cross and outgoing over-cross lies to the *left* of c. For a crossing of type R the situation is reversed. We say that the Seifert circle containing the incoming over-cross and outgoing under-cross lies to the *left* of c while the Seifert circle containing the incoming under-cross and outgoing over-cross lies to the *right* of c.

Note that these definitions are in agreement with what we actually see when we look at crossings the way they are presented in figure 2.5.

Given that we have calculated representations for the Seifert circles as in section 2.4, it is a straightforward manner to deduce, from $\mathcal{P}(D)$, for each crossing c of D and each Seifert circle C whether C lies to the left of c, to the right of c or is not adjacent to c.

Recall the Jordan curve theorem that states that any simple closed curve in the plane divides the plane into two regions, one bounded and the other unbounded. Applying this to our Seifert circles gives us the following lemma.

Lemma 2.5.2 Each Seifert circle C splits the plane into two regions. One of these contains all the crossings that lie to the left of C and the other contains all the crossings that lie to the right of C.

We can equally well speak of the two regions as lying to the left and right of C. This gives us a way to refer to the components of $\mathbb{R}^2 - \mathcal{C}$. Each Seifert circle C forms part of the boundary of two of the components, U and V say, of $\mathbb{R}^2 - \mathcal{C}$. One of these components will lie in the region of $\mathbb{R}^2 - C$ that lies to the right of C and the other will lie in the region of $\mathbb{R}^2 - C$ that lies to the left of C. In what follows we shall refer to these components U and V of $\mathbb{R}^2 - \mathcal{C}$ as lying to the left or right of C and use the notation $U_L(C)$ and $U_R(C)$ to denote the regions of $\mathbb{R}^2 - \mathcal{C}$ lying to the left and right respectively of C.

Now we show how to determine $\mathcal{N}(D)$ from $\mathcal{P}(D)$. We start by forming the graph $\hat{\mathcal{N}_0}(D)$, shown in figure 2.8. $\hat{\mathcal{N}_0}(D)$ is the graph consisting of just two nodes and an edge connecting the two nodes. We select any Seifert circle C and label the edge of $\hat{\mathcal{N}_0}(D)$ as $e(C)$, i.e. that edge corresponding to C. We label one of the nodes

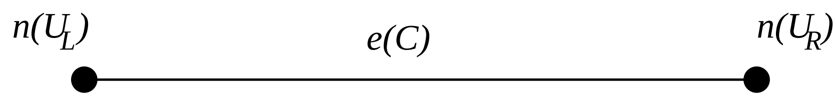


Figure 2.8: The graph $\hat{\mathcal{N}_0}(D)$.

as $n(U_L(C))$, i.e. that corresponding to the component of $\mathbb{R}^2 - \mathcal{C}$ lying to the left of C and the other node as $n(U_R(C))$, i.e. that corresponding to the component of $\mathbb{R}^2 - \mathcal{C}$ lying to the right of C. Clearly $\hat{\mathcal{N}_0}(D)$ is a subgraph of $\mathcal{N}(D)$.

We now show how to take a strict subgraph $\hat{\mathcal{N}}(D)$ of $\mathcal{N}(D)$ and add a new node and edge to it to leave a strictly larger subgraph $\hat{\mathcal{N}}(D)$ of $\mathcal{N}(D)$. Choose a Seifert circle C' such that $e(C') \notin \hat{\mathcal{N}}(D)$ but that C' is adjacent to a circle C for which $e(C) \in \hat{\mathcal{N}}(D)$, (since D is the diagram of a knot, and hence connected, such a C' exists). Let $U_L(C')$ and $U_R(C')$ denote the components of $\mathbb{R}^2 - \mathcal{C}$ lying to the left and right of C' respectively. So now we want to add the edge $e(C')$ and one of the nodes $n(U_L(C'))$ and $n(U_R(C'))$ to $\hat{\mathcal{N}}(D)$. Let $U_L(C)$ and $U_R(C)$ denote the components of $\mathbb{R}^2 - \mathcal{C}$ lying to the left and right of C respectively. Now since C' and C are adjacent, C' lies either to the left or right of C and this can be worked out from the combinatorial data of D as pointed out above. Suppose C' lies to the left of C, so $U_R(C') = U_L(C)$. In this situation we will add the edge $e(C')$ and the node $n(U_L(C'))$ to $\hat{\mathcal{N}}(D)$ with $e(C')$ connecting $n(U_L(C))(= n(U_R(C')))$ and $n(U_L(C'))$, as shown in figure 2.9. Otherwise C' will lie to the right of C, so $U_L(C') = U_R(C)$ and we will add the edge $e(C')$ and the node $n(U_R(C'))$ to $\hat{\mathcal{N}}(D)$ with $e(C')$ connecting $n(U_R(C))(= n(U_L(C')))$ and $n(U_R(C'))$, as shown in figure 2.10. It should be said, where we have indicated the Seifert circles C and C' as dotted circles in these two figures that this is only to illustrate how $U_L(C)$ and $U_R(C)$ say, lie to the left and right of C. How C and C' *actually* appear in D, i.e. which of the regions is the bounded one and which is the unbounded one, can not be deduced from $\mathcal{P}(D)$.

So starting with $\hat{\mathcal{N}_0}(D)$ we apply the process described in the previous paragraph until all the edges corresponding to the Seifert circles have been added and we are left with $\hat{\mathcal{N}}(D) = \mathcal{N}(D)$.

That the nesting graph of D is determined by it's combinatorial data proves the result mentioned at the end of section 2.3; that diagrams with the same combinato-

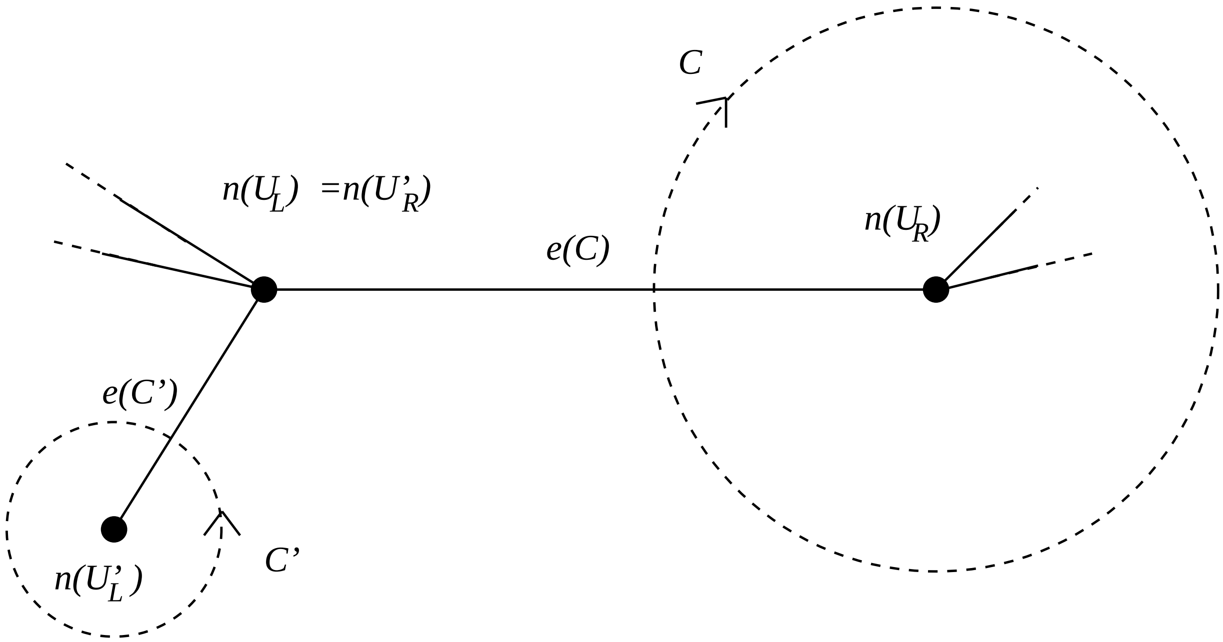


Figure 2.9: Extending the graph $\hat{\mathcal{N}}(D)$.

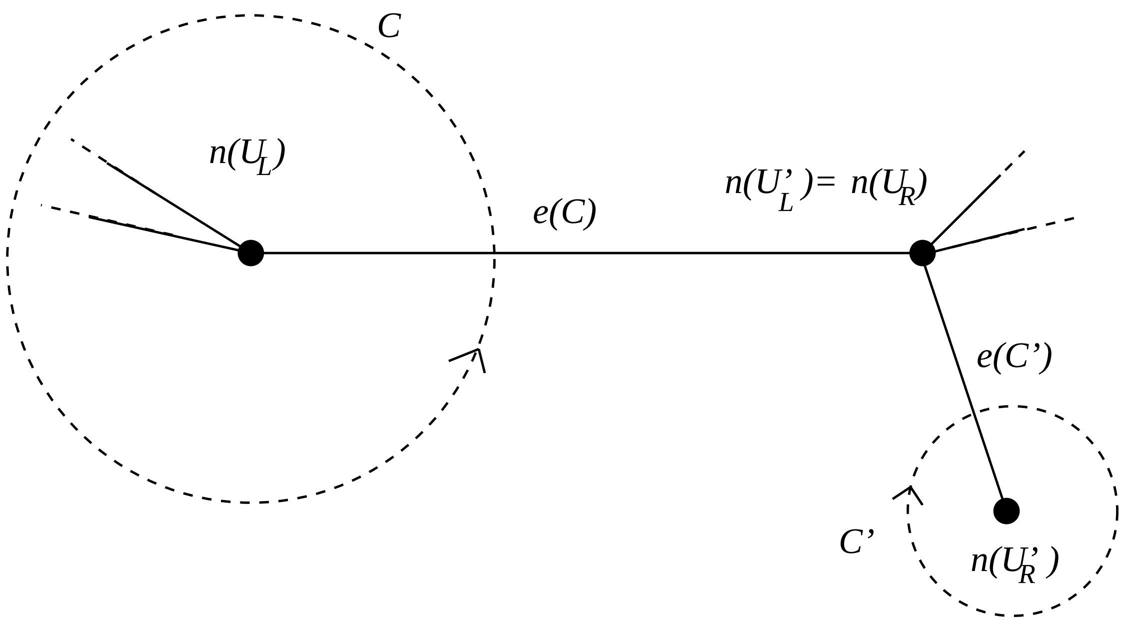


Figure 2.10: Extending the graph $\hat{\mathcal{N}}(D)$.

rial data possess the same nesting graph.

2.6 Fixing a Diagram with Combinatorial Data $\mathcal{P}(D)$

In this section we reduce the ambiguity about which actual diagram our implementation is calculating a Seifert matrix for and we identify the important features of the standard lift of that diagram that our implementation will record. These features are the *configurations* of the twisted bands associated to the crossing points.

As described in section 2.2 the nesting graph of a diagram can be thought of as a tree when we specify the node corresponding to the unbounded component of $\mathbb{R}^2 - \mathcal{C}$ as the root. Now our algorithm has calculated $\mathcal{N}(D)$ but does not know which is the unbounded component of $\mathbb{R}^2 - \mathcal{C}$. To any component of U of $\mathbb{R}^2 - \mathcal{C}$ there corresponds an isotopy ϕ, of the extended plane that carries D to a new diagram D' with $\phi(U)$ now being the unbounded component of $\mathbb{R}^2 - \phi(\mathcal{C})$, see the three diagrams of the figure eight knot in figure 2.7 for some examples.

So we just let the algorithm arbitrarily choose one of the nodes, $n_0 = n(U_0)$ say, of $\mathcal{N}(D)$ to be the root and we let D' denote a diagram $\phi(D)$ where ϕ is any isotopy of the extended plane such that $\phi(U_0)$ is the unbounded component of $\mathbb{R}^2 - \phi(\mathcal{C})$.

We shall denote the resulting tree by $\mathcal{N}^T(D')$. We can tell from $\mathcal{N}^T(D')$ in what level of $\mathbb{R}^2 \times [0, \infty)$ each Seifert disk F_C will be located in the standard lift of D'. The disk F_C will be in the level $\mathbb{R}^2 \times \{n\}$, where the edge $e(C)$ joins nodes of depth n and $n+1$ in the tree $\mathcal{N}^T(D')$.

We remark here that we can actually now work out the orientations on the Seifert circles of D', i.e. which go clockwise and which go counterclockwise, from the information at hand. We select one of the Seifert disks, F_{C_0} say, where $e(C_0)$ joins the root of $\mathcal{N}^T(D')$ with a node of depth 1. This means that F_{C_0} must lie in the level $\mathbb{R}^2 \times \{0\}$. In the construction of $\mathcal{N}(D)(= \mathcal{N}(D'))$ the root n_0 will have been recorded as either $n(U_L(C_0))$ or $n(U_R(C_0))$. If $n_0 = n(U_L(C_0))$ then C_0 has a clockwise orientation. On the other hand if $n_0 = n(U_R(C_0))$ then C_0 has a counterclockwise orientation. The orientation on C_0 is extended out to the other Seifert circles according to the following rule. Two adjacent circles in the same level will have opposite orientations while adjacent circles in consecutive levels will have the same orientation.

This information about D'; the levels of the Seifert disks and their orientation, can be combined and recorded in the form of the *configuration* of each of the twisted bands associated to the crossing points of D'.

Definition 2.6.1 Let c be a crossing of $\mathcal{P}(D)$ and let B_c denote the twisted band associated to c. Recall that c can be a type L or type R crossing, and we shall also refer to B_c as type L or type R accordingly. In addition, B_c will connect either two Seifert disks that lie in the same level or connect two that lie in consecutive levels. So in total there are six possibilities and we define the *configuration* of B_c as L_1, L_2, L_3, R_1, R_2 or R_3 according to figure 2.11 in which the twisted bands together with a small portion of the Seifert disks on either side are shown.

The diagrams in this figure should be self explanatory. Any twisted band of a Seifert surface of a standard lift will look like one and only one of those in figure 2.11. For example, a band of type L_2 is one associated to a crossing c of type L, where the Seifert disk to the left of c lies one level below the Seifert disk lying to the right of c.

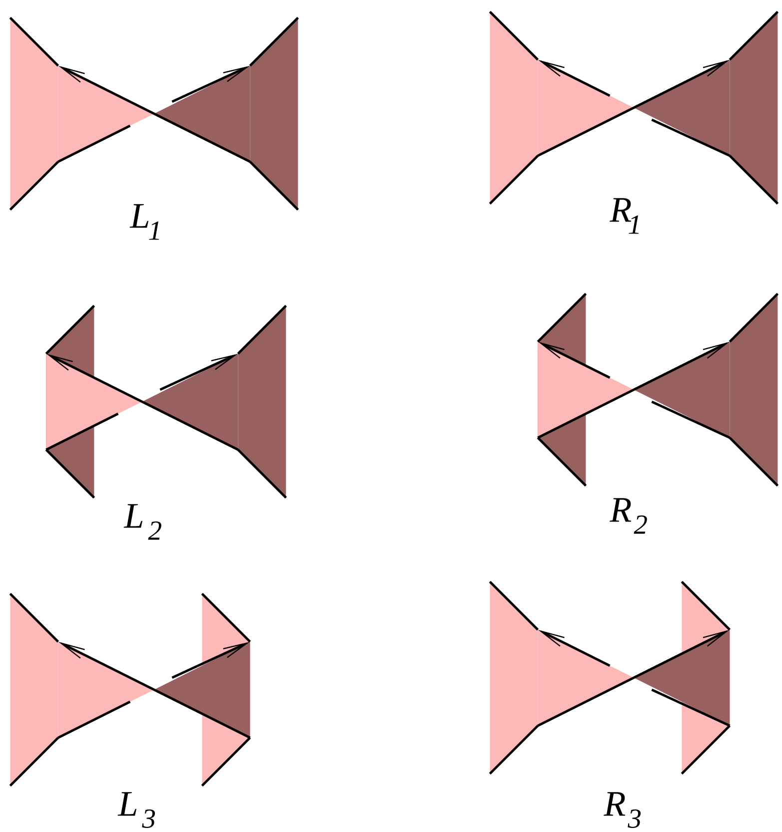


Figure 2.11: The six configurations of a twisted band.

2.7 Seifert's Graph and Generating Cycles for $H_1(F)$

We describe a second graph associated to a diagram D.

Definition 2.7.1 The *Seifert Graph* $\mathcal{S}(D)$ of a diagram D is the graph:

- ⋆ with one node $n(C)$ for each Seifert circle of D
- ⋆ and one edge $e(c)$ for each crossing c of D, connecting the nodes corresponding to the two Seifert circles adjacent to c.

It is clear once again that two diagrams related by an isotopy of the extended plane have the same Seifert graph. Also it should be clear that to calculate $\mathcal{S}(D)$ from $\mathcal{P}(D)$ is a straight forward matter. We have the following lemma about the Seifert graph.

Lemma 2.7.2 If F is the Seifert surface of the standard lift of a diagram D then

$$H_1(\mathcal{S}(D)) = H_1(F) \tag{2.2}$$

Proof: This is true since $\mathcal{S}(D)$ embeds in F and is a deformation retract of F. ■

A cycle in $\mathcal{S}(D)$ is a sequence of edges of $\mathcal{S}(D)$ and an equivalent cycle on F will be any curve that goes through, in the same order the twisted bands, and only those twisted bands, associated to the edges in the sequence.

Given a graph G there is a well known algorithm for generating a set of cycles on G that generate $H_1(G)$, which runs as follows. We assume that G is connected. First we find a spanning tree $T(G)$ of G. A spanning tree $T(G)$ of G is an acyclic subgraph of G that includes all the nodes of G, or equivalently, a maximal acyclic subgraph of G.

A spanning tree is found by beginning with any node n_0 of G and form the tree $\hat{T}(G)$ consisting of just n_0 as the root. Then repeat the following procedure: for each node n of $\hat{T}(G)$, let $n_1, \ldots, n_r$ be the neighbours of n in G that are not already present in $\hat{T}(G)$. For each $1 \leq i \leq r$ we add the node n_i to $\hat{T}(G)$ together with just one of the edges connecting n and n_i, and let $\hat{T}(G)$ denote the new strictly larger tree, which is still a subgraph of $\mathcal{S}(G)$. When this procedure can no longer be applied then $\hat{T}(G) = T(G)$, a spanning tree of G.

Then to each of the edges $\{e \in G | e \notin T(G)\}$ there will correspond a generating cycle of $H_1(G)$. For each node n of $T(G)$ we let s_n be the sequence of edges from the root n_0 to n. Let e be one of the edges in $\{e \in G | e \notin T(G)\}$. Suppose that e connects the nodes n_1 and n_2. Then the sequence of edges $(s_{n_1}, e, s_{n_2}^{-1})$ will be a non-trivial cycle in G, where s^{-1} is the sequence s reversed. This cycle is non trivial because $T(G)$ is a maximal acyclic subgraph of G.

Using this algorithm, our implementation can record a set Z of cycles that generate $H_1(\mathcal{S}(D))$. The cycles $z \in Z$ being recorded as sequences of the edges of $\mathcal{S}(D)$ which we recall, correspond to sequences of the crossings of the diagram D. Our implementation will remove any redundancy in these representations by simplifying the sequences as much as possible using the following rule.

Let e be an edge and s and t any sequences of edges. A sequence of the form (s, e, e, t) or (e, s, e) can be simplified to (s, t) or (s) respectively. Of course this does not change the homological equivalence class of the cycle.

2.8 Tracks on the Seifert Surface

In this section we will describe a system of oriented curves on a standard lift Seifert surface F. We refer to the system of curves as the *tracks* on F. There shall be an inner and outer track. We define the tracks in two stages. First defining the tracks on a Seifert disk and then what the tracks look like on the twisted bands of F.

In order to define the tracks of a Seifert disk F_C we first define the *cycle band* of a Seifert disk.

Definition 2.8.1 The *cycle band* on a Seifert disk F_C is a band on F_C extending a small way from the boundary toward the centre with the following property. When projected down to $\mathbb{R}^2 \times \{0\}$ the cycle bands are all disjoint so in particular any other Seifert circles of D that lie within C do not intersect the projection of the cycle band of F_C, see figure 2.12.

Figure 2.13 shows the inner and outer tracks on a portion of a Seifert disk with representation $(\alpha_1, \ldots, \alpha_n)$ say. They lie in the cycle band. We mark four points on

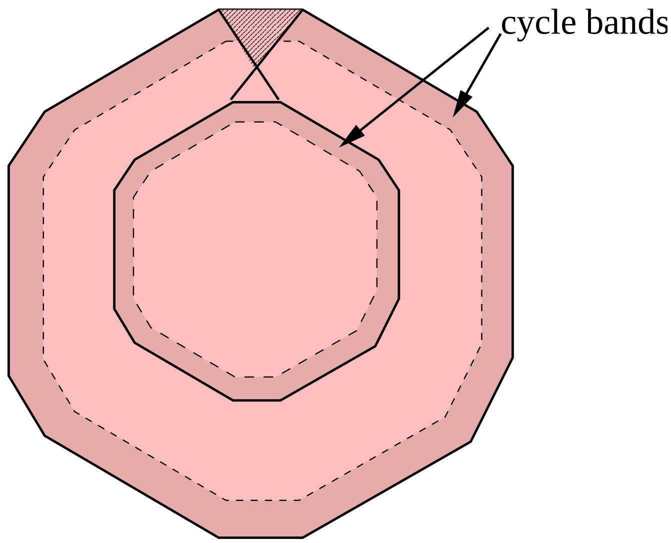


Figure 2.12: Cycle Bands

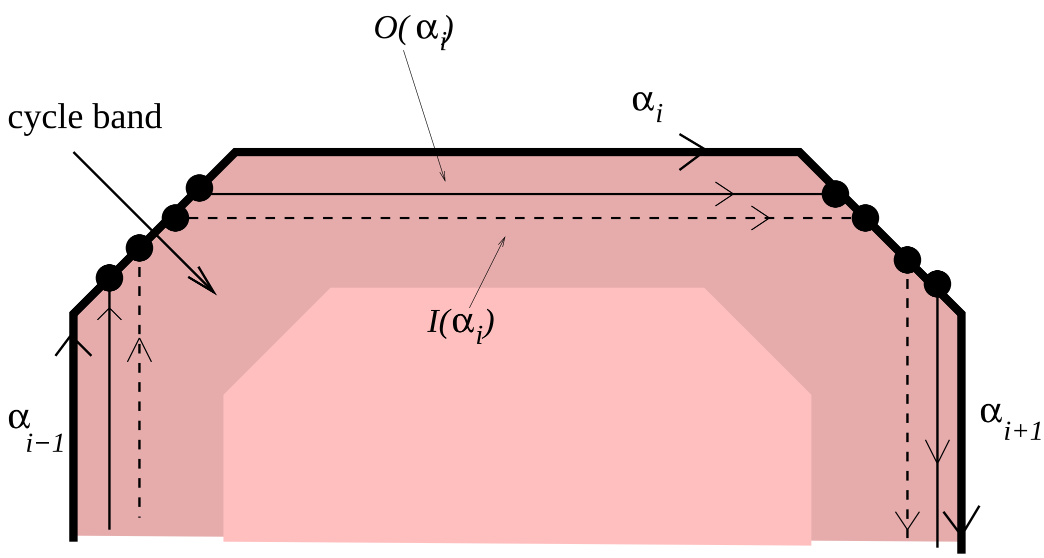


Figure 2.13: Tracks on a disk.

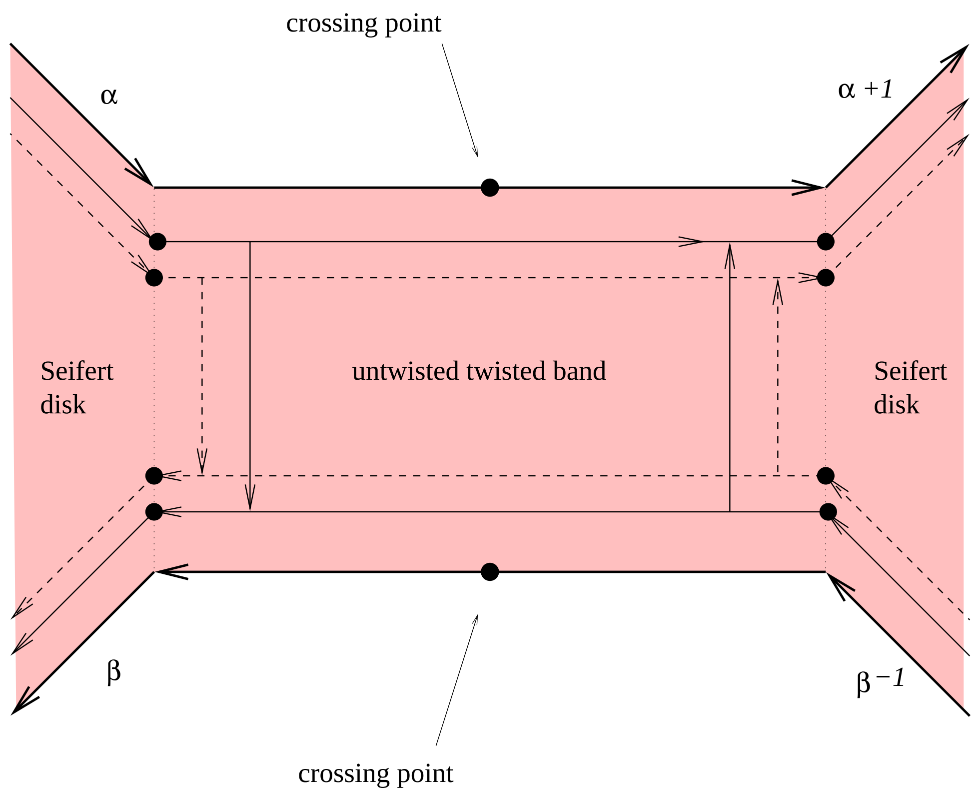


Figure 2.14: Tracks on a band.

each of the 'jumps' made at the crossing points in the formation of the Seifert circles, see figure 2.1 page 13. The outer track (solid lines) is the union of some non-self intersecting lines $O(\alpha_j)$ that run roughly parallel to the arcs α_j, $j = 1, \ldots, n$ and then terminate at the marked points on the 'jumps' of the crossing points. Similarly, the inner track (dashed lines) is the union of some non-self intersecting lines $I(\alpha_j)$ that run roughly parallel to the $O(\alpha_j)$, $j = 1, \ldots, n$ but slightly nearer the centre of the Seifert disk and terminating at the marked points as shown. The important point is that they lie in the cycle band and the outer and inner track never intersect.

With figure 2.14 we show the tracks as they appear on the twisted bands that connect the Seifert disks together. For convenience the twisted band is shown untwisted together with a portion of the Seifert disks on either side of the band, the Seifert circles having representations $(\ldots, \alpha, \beta, \ldots)$ and $(\ldots, \beta - 1, \alpha + 1, \ldots)$ for some $1 \leq \alpha, \beta \leq 2m$. The outer track is the union of the solid lines and the inner track is the union of the dashed lines. Notice that the outer and inner tracks meet up with the corresponding tracks on the Seifert disks as shown in the figure.

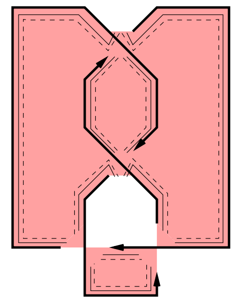


Figure 2.15: Tracks on Seifert disks of the figure eight knot.

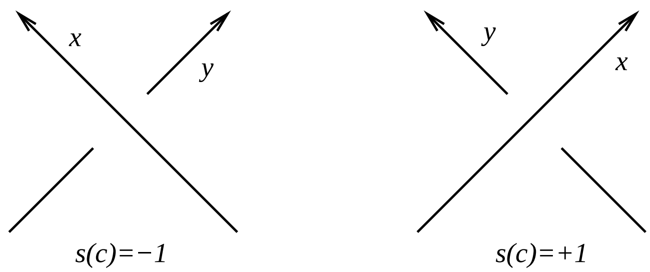


Figure 2.16: Definition of linking number.

As an example we show in figure 2.15 how the tracks would appear on the Seifert disks of the diagram of the figure eight knot. To make it easier to see we have removed the shading indicating the different orientation on the disks.

2.9 Calculation of the Linking Numbers

In this section we show how to calculate the Seifert matrix of the standard lift Seifert surface F of D' with respect to the cycle basis Z. We first remark that the linking number $\mathrm{Link}(x, y)$ of two disjoint 1-cycles can be calculated from a diagram E, of x and y,

$$\mathrm{Link}(x, y) = \sum_{c \in E_x} s(c) \tag{2.3}$$

where the sum is taken over all crossings $c \in E_x = \{c | c \text{ is an overpassing of } x \text{ over } y\}$ and $s(c) = -1$ if c is a crossing of type L, $s(c) = +1$ if c is a crossing of type R, see figure 2.16.

Let $z \in Z$ be one of the generating cycles of $H_1(\mathcal{S}(D')) = H_1(F)$. We will now describe how to construct specific cycles on F which represent the same homological equivalence class as z.

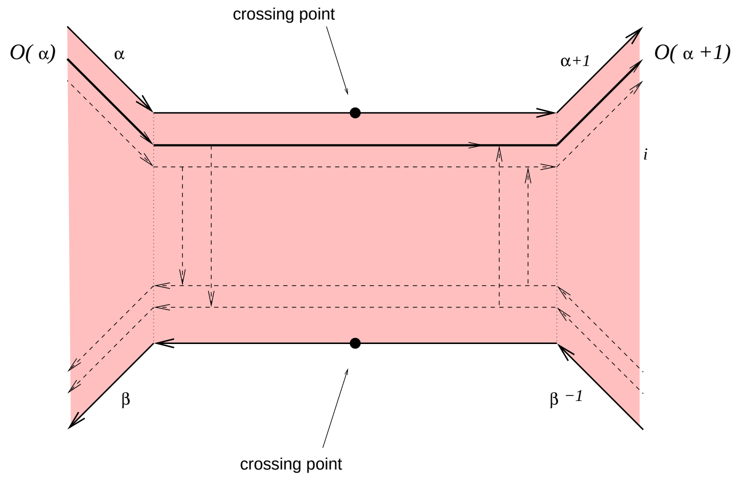


Figure 2.17: Extending $\hat{R}_O(z)$.

Definition 2.9.1 Let $R(z) = (c_1, c_2, \dots, c_r)$ be the recorded representation of z as a sequence of crossings of D'. We can obtain another representation for z, which we denote as $R_O(z)$, by *pushing z onto the outer track* as follows.

We start by setting $\hat{R}_O(z) := (O(\alpha_1))$ where α_1 is the outgoing over-cross from c_1. (We could equally well choose the other outgoing arc from c_1.) $\hat{R}_O(z)$ is an initial segment of what will become $R_O(z)$. We repeat the following procedure that extends $\hat{R}_O(z)$: Let $O(\alpha)$ be the last element of $\hat{R}_O(z)$. Let c be the crossing that has α as one of it's incoming arcs. Let C be the Seifert circle that has α as part of it's boundary, so C has the representation $(\dots, \alpha, \beta, \dots)$ (or $(\beta, \dots, \alpha)$) of arc labels of D' and the other Seifert circle adjacent to c has a representation $(\dots, \beta-1, \alpha+1, \dots)$ (or $(\alpha+1, \dots, \beta+1)$).

Now if $c \in R(z)$ then we extend the cycle by adding $O(\alpha+1)$ together with the line on the outer track on the twisted band joining $O(\alpha)$ and $O(\alpha+1)$, (so in effect the cycle crosses the twisted band corresponding to c). We record this by extending the representation $\hat{R}_O(z)$ to $\hat{R}_O(z) = (\dots, O(\alpha), O(\alpha+1))$, see figure 2.17.

On the other hand if $c \notin R(z)$ then we extend $\hat{R}_O(z)$ by adding $O(\beta)$ together with the line of the outer track on the twisted band joining $O(\alpha)$ with $O(\beta)$, (so in effect the cycle does not cross the twisted band corresponding to c, instead it stays on the Seifert disk spanning C). We record this by extending the representation $\hat{R}_O(z)$ to $\hat{R}_O(z) = (\dots, O(\alpha), O(\beta))$, see figure 2.18.

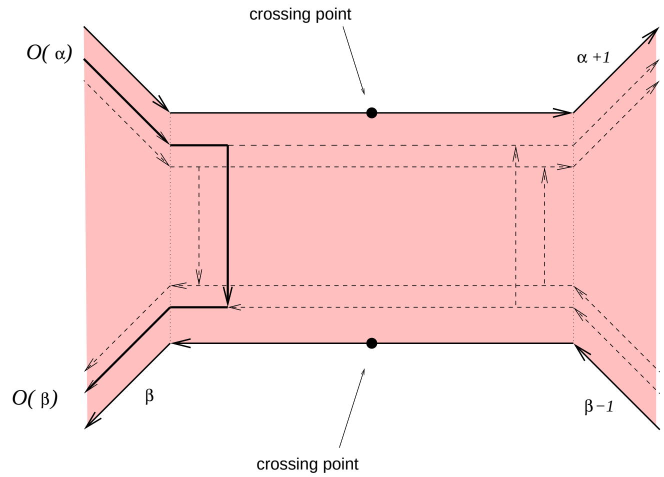


Figure 2.18: Extending $\hat{R}_O(z)$.

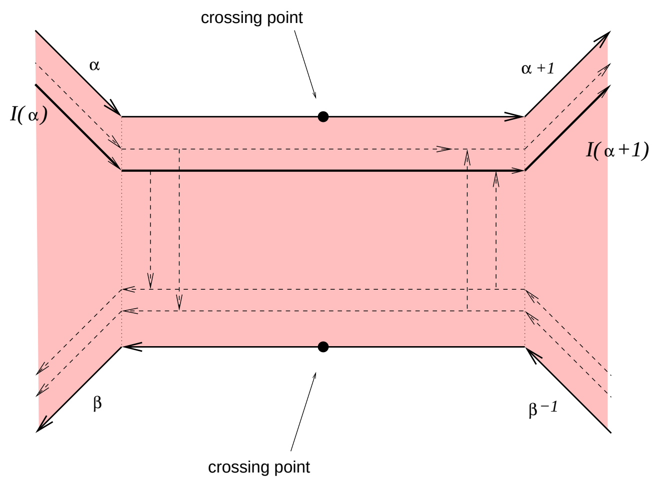


Figure 2.19: Extending $\hat{R}_I(z)$.

We repeat this procedure until the representation is complete, i.e. just before we add the label $O(\alpha_1)$ to the end of $\hat{R}_O(z)$. This leaves us with the representation

$$R_O(z) = (O(\alpha_1), O(\alpha_2), \dots) \tag{2.4}$$

for the cycle z.

In exactly the same way we can *push z onto the inner track* to give a representation

$$R_I(z) = (I(\alpha_1), I(\alpha_2), \dots) \tag{2.5}$$

as in figures 2.19 and 2.20.

We will use the notation z^O and z^I to denote the closed curve that is a subset of

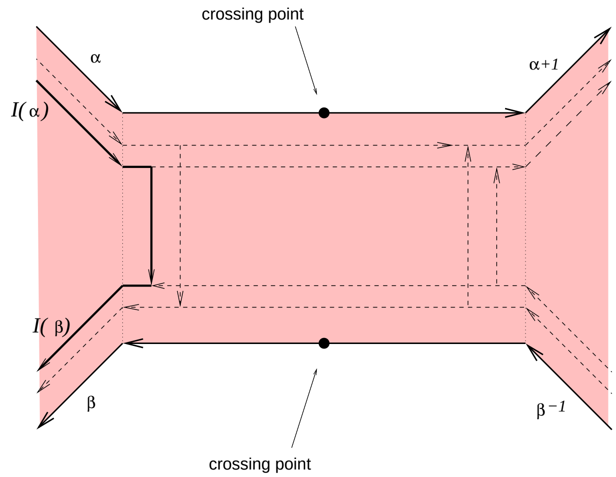


Figure 2.20: Extending $\hat{R}_I(z)$.

the outer / inner track produced by this process, as opposed to the representations $R_O(z)$ and $R_I(z)$ which are just sequences of labels. Of course the representations $R_O(z)$ and $R_I(z)$ are essentially the same, being just sequences of the arc labels so in what follows we will refer to a representation $R(z)$. But note that the cycles z^O and z^I are not the same (but homologically equivalent) for one runs on the outer track, the other on the inner track.

The outer and inner tracks were constructed in such a way so that when we take two of the generating cycles z_i and z_j (allowing $j = i$) then the projections of z_i^O and z_j^I to $\mathbb{R}^2$ will intersect only in the vicinity of each twisted band. And we will now show how our implementation can tell which of these intersections are over and under-crossings and from which direction they pass, i.e. all the information needed to calculate $\mathrm{Link}(z_i^{O+}, z_j^I)$ using the formula (2.3).

First we simplify our notation. We let x denote one of the generating cycles pushed on to the outer track and y denote one of the generating cycles pushed onto the inner track. For each twisted band B_c of F, where c is a crossing of D', we say that the cycle x (or y) takes one of five forms, denoted by the numbers $0, 1, 2, 3, 4$. These are shown in figure 2.21 The cycle x is of form 0 near B if x does not pass near B, 1 if it follows the incoming and outgoing arc belonging to the the Seifert circle to the left of c, 2 if it passes across B_c from the Seifert disk on the left to the Seifert disk on the right of B_c, 3 if it follows the incoming and outgoing arcs

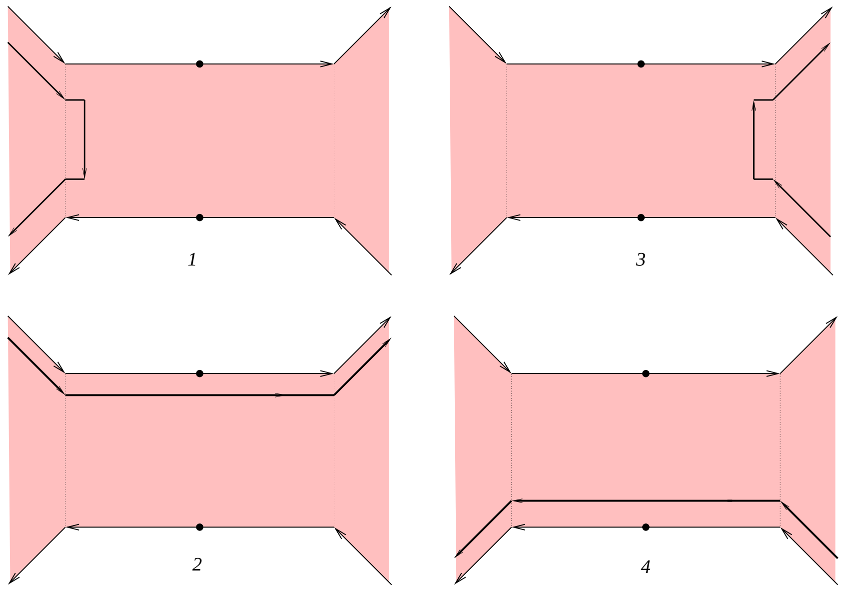


Figure 2.21: Forms near a twisted band.

belonging to the Seifert circle to the right of c and of form 4 if it passes across B_c from the Seifert disk on the right to the Seifert disk on the left of B_c. We say that the form of x and y near B_c is (i, j), where i is the form of x and j is the form of y near B_c.

So there are 25 possibilities, (i, j) where $0 \leq i, j \leq 4$, to specify the form of the two cycles x and y at a band. In the following figures 2.22 through 2.33 we show how each of these possibilities will appear at bands of each of the six possible configurations L_1, L_2, L_3, R_1, R_2 and R_3. The cycle x is on the outer track and will be shown as a solid line. The cycle y is on the inner track and will be shown as a dashed line. The first column shows the portions of the cycles as they actually appear on the Seifert surface F, the second column shows the corresponding portion of the diagram (i.e. a projection to $\mathbb{R}^2$ together with over/under-cross information) of the curves x^+ and y and the third column shows the portion of the diagram of x and y^+. In total there are $25 \times 6 = 150$ possible diagrams but we omit the ones where the form of x and y near the band is one of $(0, i), (i, 0), (1, 3)$ or $(3, 1)$, where $0 \leq i \leq 4$, since for these forms there can be no crossings in the diagram of x^+ and y or in the diagram of y^+ and x, as either some of the curves are not present or they are wholly on opposite sides of the band. In the figures we have adopted the usual convention of having the positive normal direction point out of the page toward the reader.

Then in the three tables, 2.1, 2.2 and 2.3 following those figures we summarise how each of these possibilities will contribute to the linking numbers $\mathrm{Link}(x^+, y)$ and $\mathrm{Link}(y^+, x)$ as defined in equation (2.3) (remember x is the outer cycle and y the inner). That is, the entry in the table under $L(x^+, y)$ shows the sum $\sum s(c)$ where the sum is taken over all of the over-crossings, c, of x^+ over y that occur at that band and similarly the entry in the table under $L(y^+, x)$ is $\sum s(c)$ where the sum is taken over all of the over-crossings, c, of y^+ over x that occur at the band.

Of course the omitted possibilities $(0, i), (i, 0), (1, 3)$ and $(3, 1)$, where $0 \leq i \leq 4$, all have zero contribution to the linking since no crossings of any kind will appear at a band when cycles of that form are projected to $\mathbb{R}^2$. The forms $(1, 1)$ and $(3, 3)$ also always have zero contribution to the linking number but this is perhaps non-trivially so as they do give rise to crossing points at the bands of configuration L_2, L_3, R_2 and R_3.

Then all our implementation has to do is for each pair of generating cycles $z_i, z_j \in Z$ assign one to be pushed to the outer track and one to the inner track, then work out the form of z_i^O and z_j^I near each of the twisted bands of F and then sum up the relevant contributions to the linking number by reading the tables. In this way the ij^{th} entry $= \mathrm{Link}(z_i^+, z_j)$ and the ji^{th} entry $= \mathrm{Link}(z_j^+, z_i)$ of the Seifert matrix of the standard lift Seifert surface with respect to the generating cycles Z is calculated.

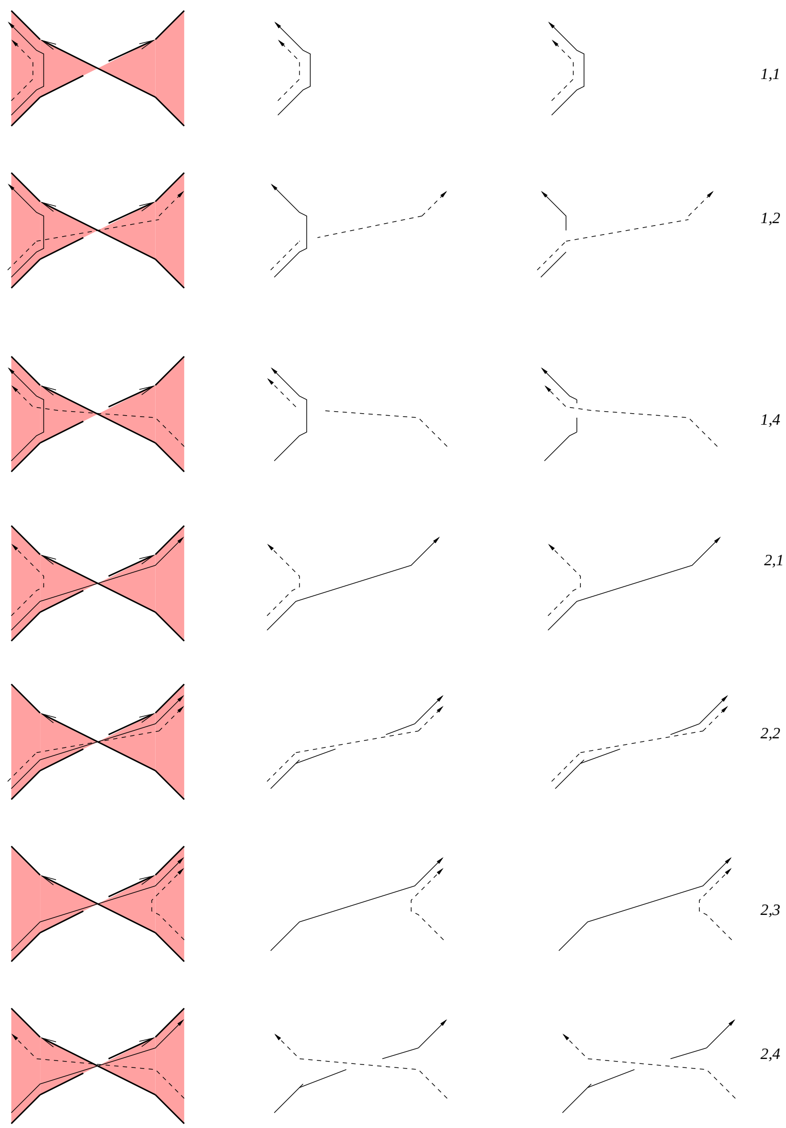


Figure 2.22: The contributions to Linking near a Band with Configuration L_1.

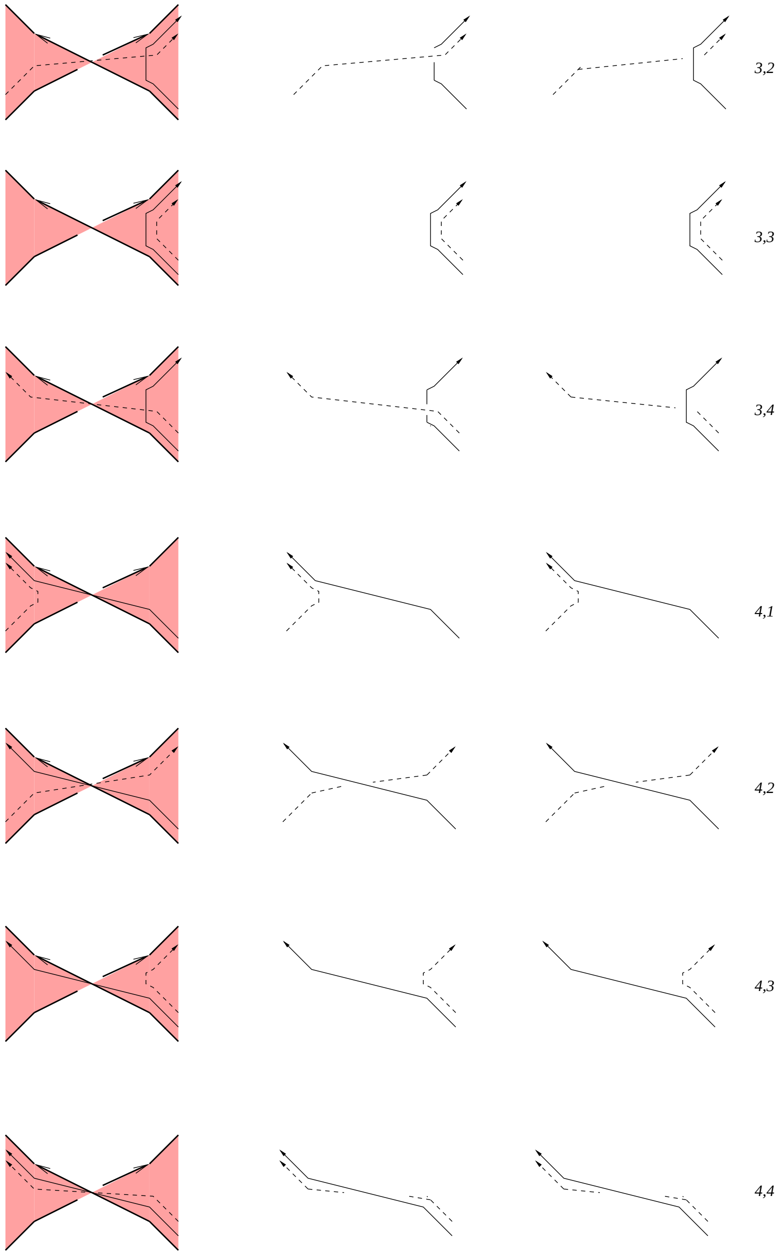


Figure 2.23: The contributions to Linking near a Band with Configuration L_1, continued.

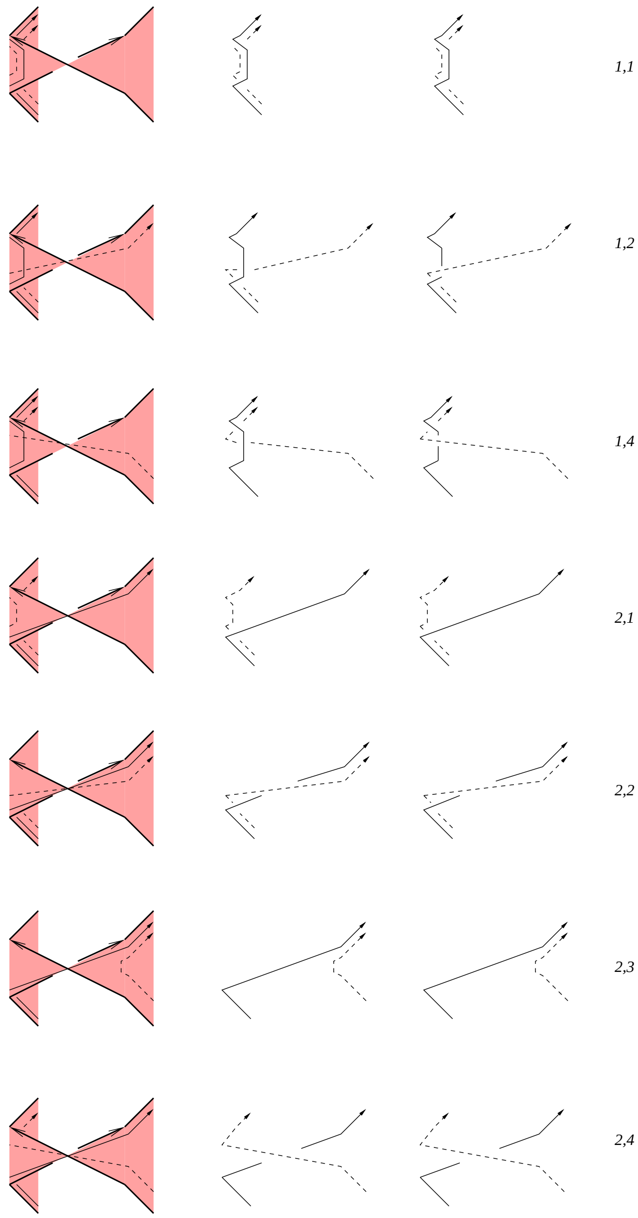


Figure 2.24: The contributions to Linking near a Band with Configuration L_2.

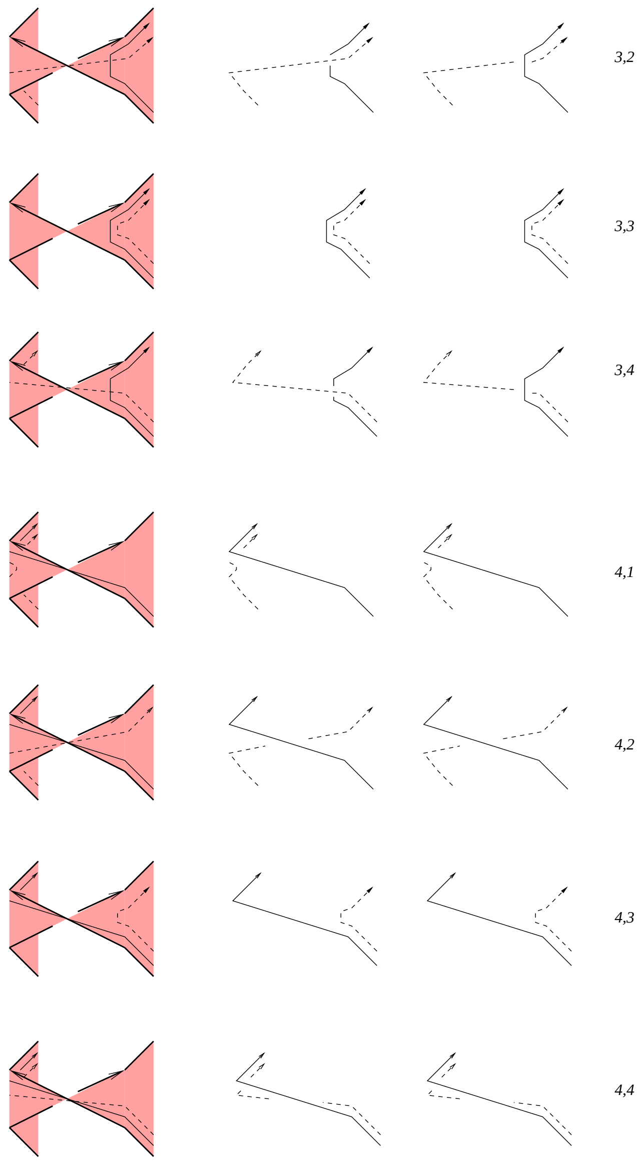


Figure 2.25: The contributions to Linking near a Band with Configuration L_2, continued.

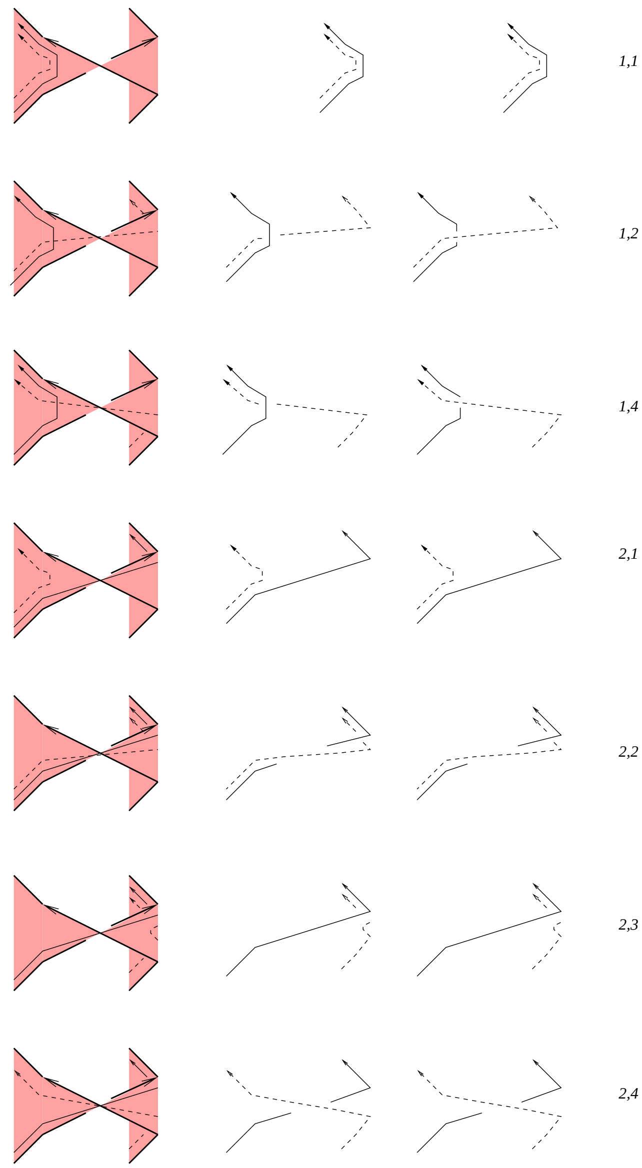


Figure 2.26: The contributions to Linking near a Band with Configuration L_3.

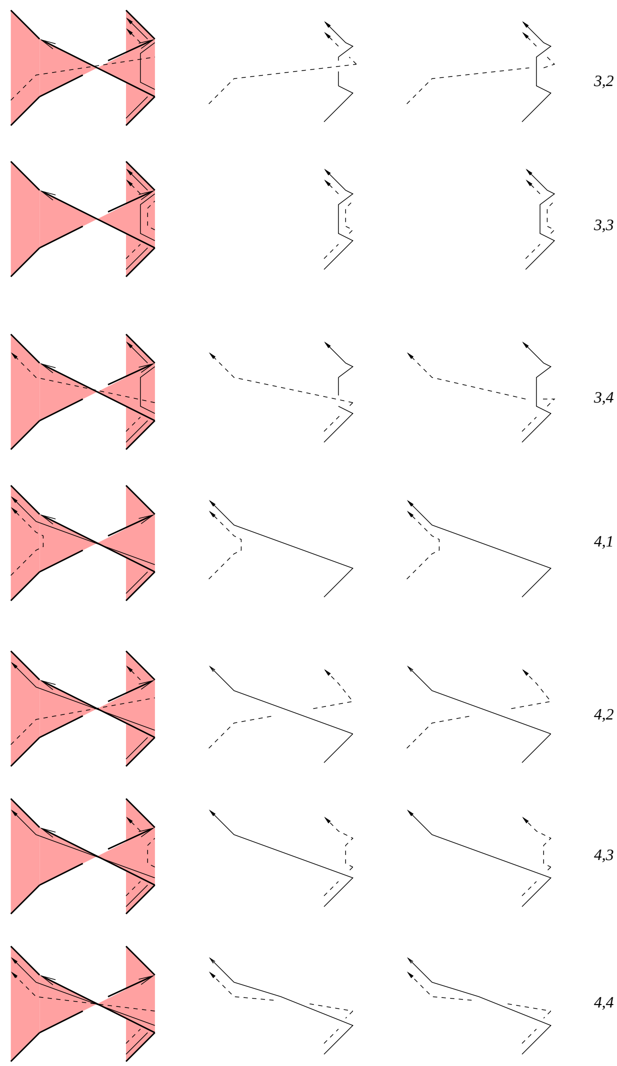


Figure 2.27: The contributions to Linking near a Band with Configuration L_3, continued.

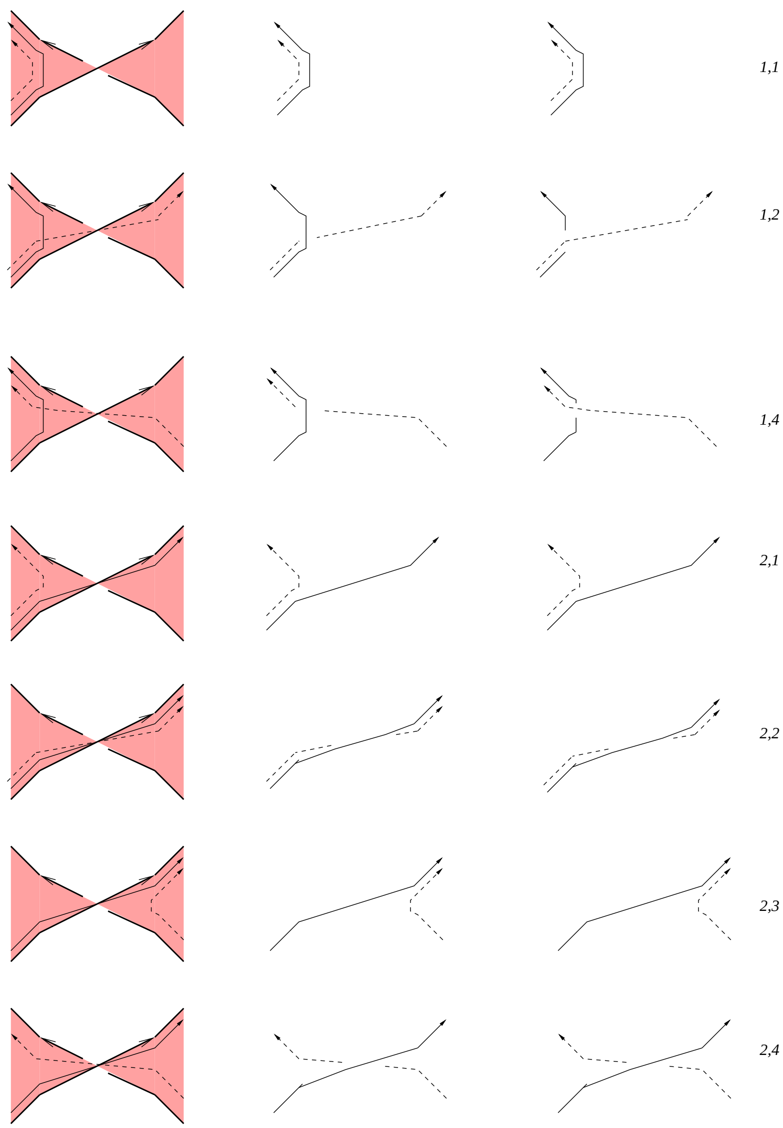


Figure 2.28: The contributions to Linking near a Band with Configuration R_1.

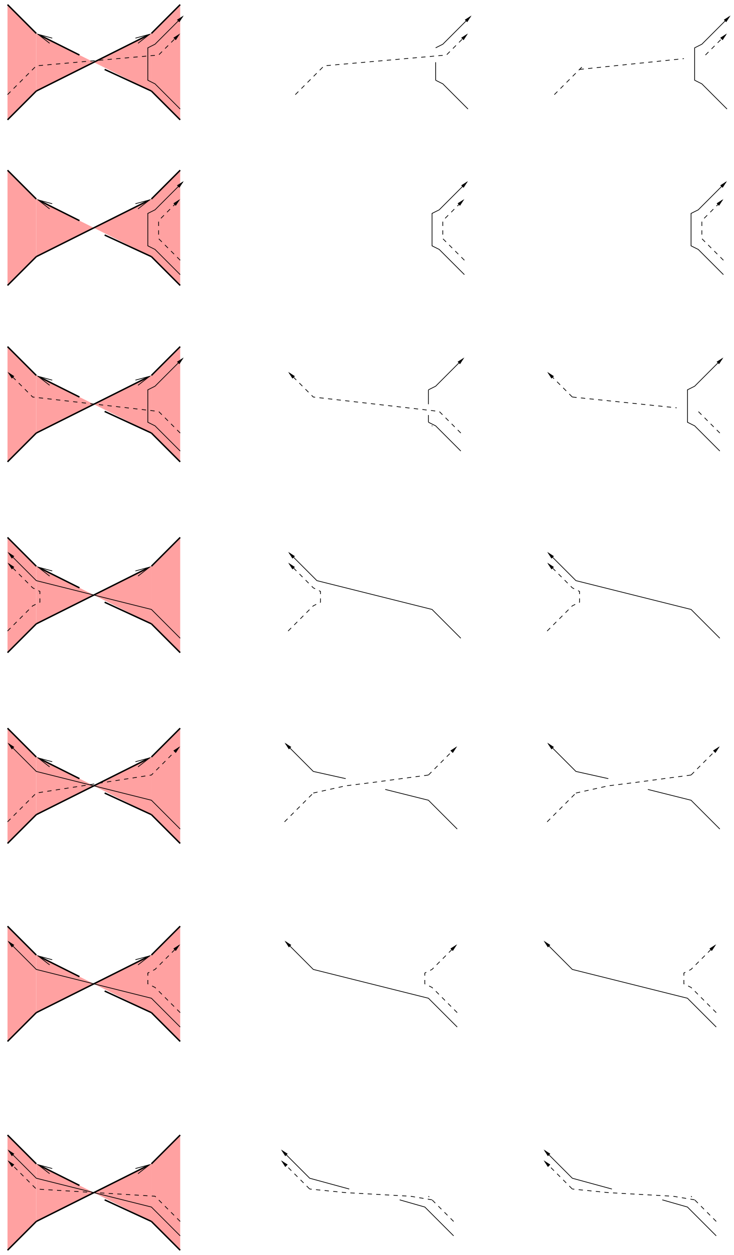


Figure 2.29: The contributions to Linking near a Band with Configuration R_1, continued.

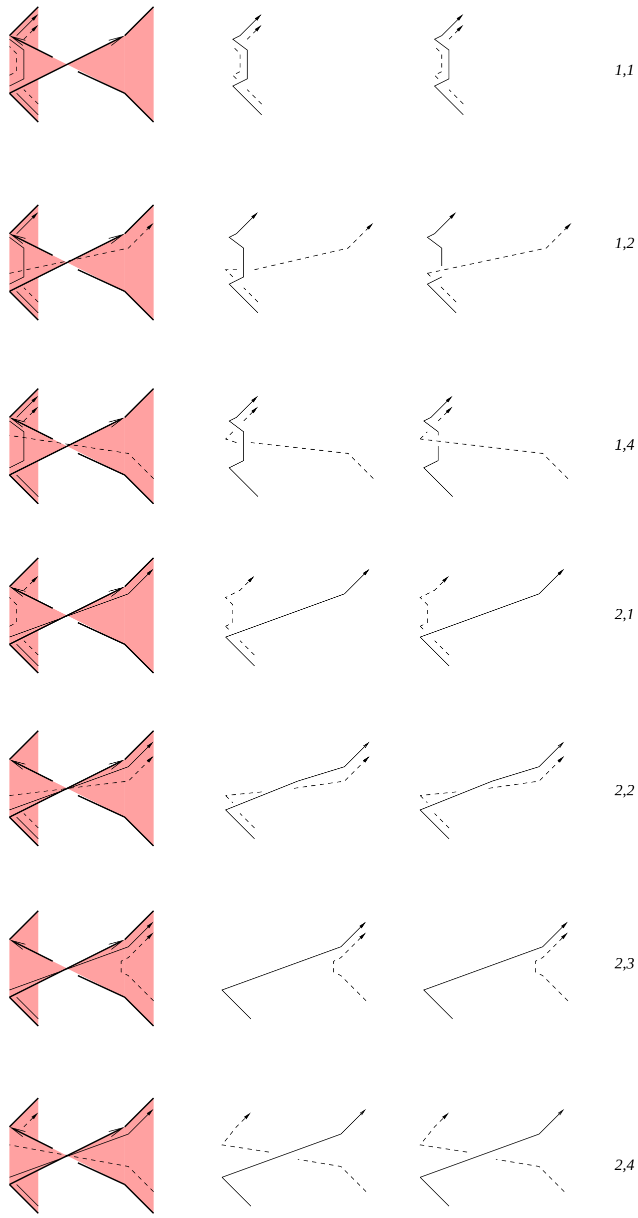


Figure 2.30: The contributions to Linking near a Band with Configuration R_2.

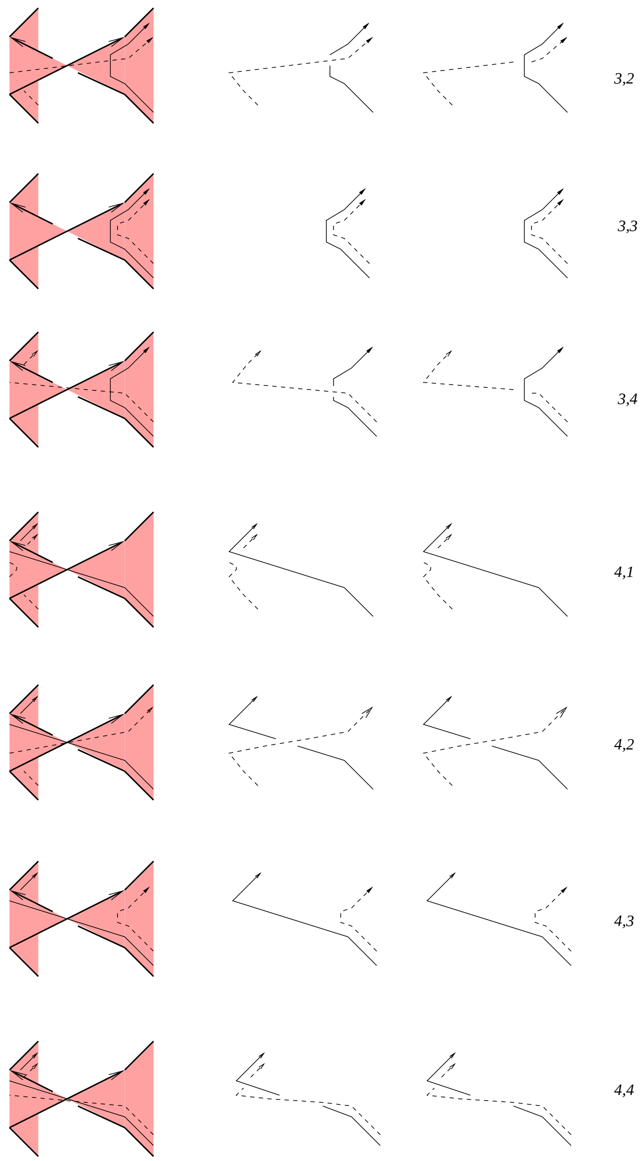


Figure 2.31: The contributions to Linking near a Band with Configuration R_2, continued.

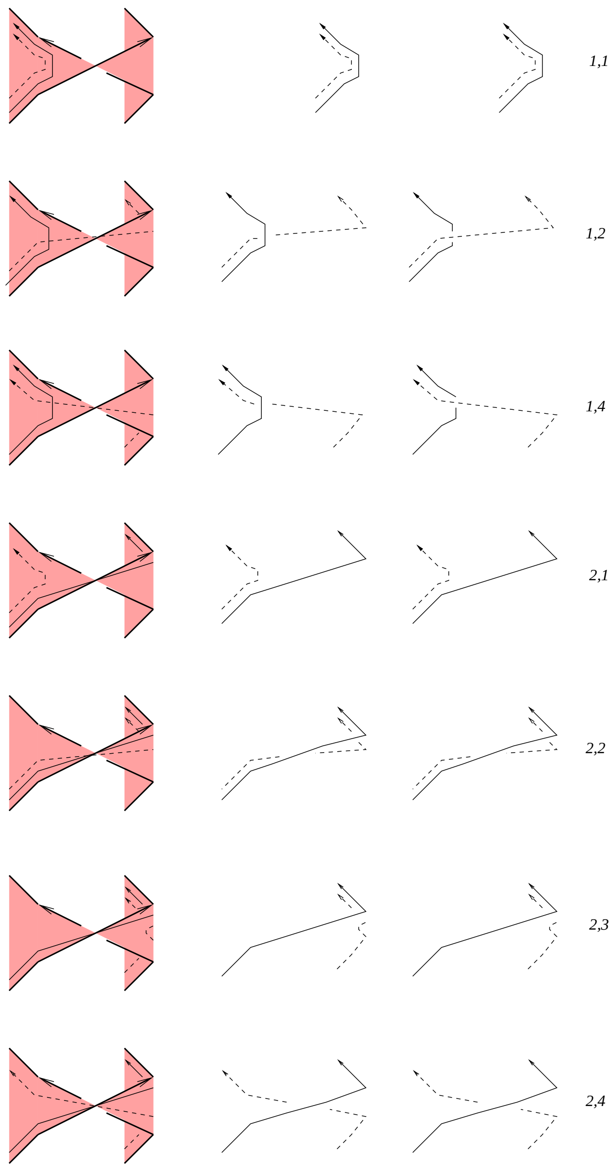


Figure 2.32: The contributions to Linking near a Band with Configuration R_3.

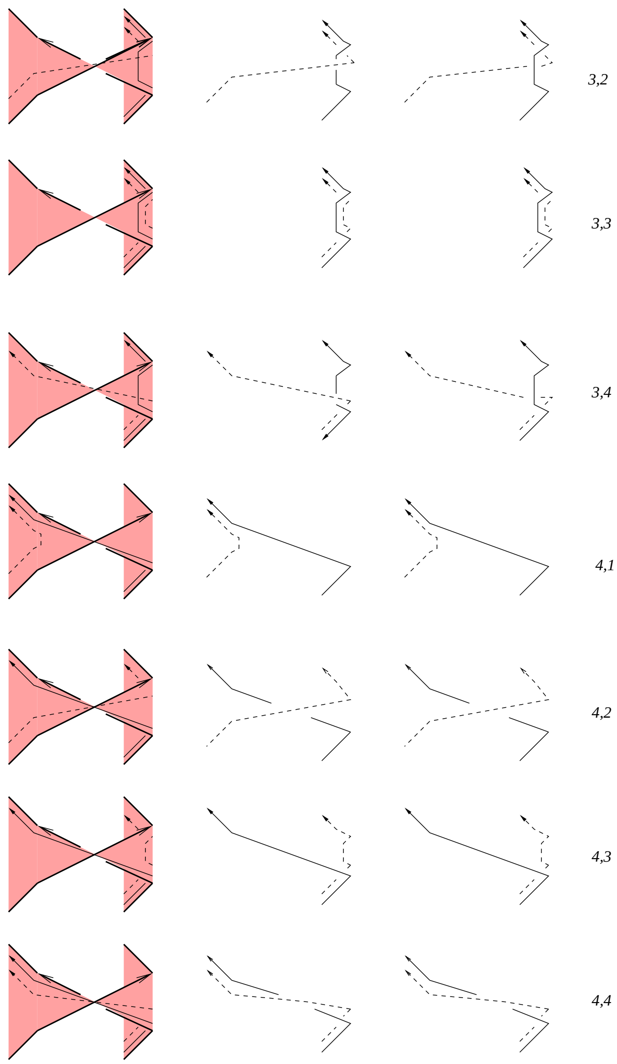


Figure 2.33: The contributions to Linking near a Band with Configuration R_3, continued.

Configuration of Band	Form of x and y	$L(x^+, y)$	$L(y^+, x)$
L_1	$1, 1$	0	0
L_1	$1, 2$	-1	+1
L_1	$1, 4$	+1	-1
L_1	$2, 1$	0	0
L_1	$2, 2$	0	+1
L_1	$2, 3$	0	0
L_1	$2, 4$	0	-1
L_1	$3, 2$	0	0
L_1	$3, 3$	0	0
L_1	$3, 4$	0	0
L_1	$4, 1$	0	0
L_1	$4, 2$	-1	0
L_1	$4, 3$	0	0
L_1	$4, 4$	+1	0
L_2	$1, 1$	0	0
L_2	$1, 2$	0	+1
L_2	$1, 4$	0	-1
L_2	$2, 1$	+1	0
L_2	$2, 2$	+1	+1
L_2	$2, 3$	0	0
L_2	$2, 4$	0	-1
L_2	$3, 2$	0	0
L_2	$3, 3$	0	0
L_2	$3, 4$	0	0
L_2	$4, 1$	-1	0
L_2	$4, 2$	-1	0
L_2	$4, 3$	0	0
L_2	$4, 4$	0	0

Table 2.1: The linking contributions at bands L_1 and L_2

Configuration of Band	Form of x and y	$L(x^+, y)$	$L(y^+, x)$
L_3	$1, 1$	0	0
L_3	$1, 2$	-1	+1
L_3	$1, 4$	+1	-1
L_3	$2, 1$	0	0
L_3	$2, 2$	+1	+1
L_3	$2, 3$	+1	0
L_3	$2, 4$	0	-1
L_3	$3, 2$	+1	0
L_3	$3, 3$	0	0
L_3	$3, 4$	-1	0
L_3	$4, 1$	0	0
L_3	$4, 2$	-1	0
L_3	$4, 3$	-1	0
L_3	$4, 4$	0	0
R_1	$1, 1$	0	0
R_1	$1, 2$	-1	+1
R_1	$1, 4$	+1	-1
R_1	$2, 1$	0	0
R_1	$2, 2$	-1	0
R_1	$2, 3$	0	0
R_1	$2, 4$	+1	0
R_1	$3, 2$	0	0
R_1	$3, 3$	0	0
R_1	$3, 4$	0	0
R_1	$4, 1$	0	0
R_1	$4, 2$	0	+1
R_1	$4, 3$	0	0
R_1	$4, 4$	0	-1

Table 2.2: The linking contributions at bands L_3 and R_1

Configuration of Band	Form of x and y	$L(x^+, y)$	$L(y^+, x)$
R_2	$1,1$	0	0
R_2	$1,2$	0	+1
R_2	$1,4$	0	-1
R_2	$2,1$	+1	0
R_2	$2,2$	0	0
R_2	$2,3$	0	0
R_2	$2,4$	+1	0
R_2	$3,2$	0	0
R_2	$3,3$	0	0
R_2	$3,4$	0	0
R_2	$4,1$	-1	0
R_2	$4,2$	0	+1
R_2	$4,3$	0	0
R_2	$4,4$	-1	-1
R_3	$1,1$	0	0
R_3	$1,2$	-1	+1
R_3	$1,4$	+1	-1
R_3	$2,1$	0	0
R_3	$2,2$	0	0
R_3	$2,3$	+1	0
R_3	$2,4$	+1	0
R_3	$3,2$	+1	0
R_3	$3,3$	0	0
R_3	$3,4$	-1	0
R_3	$4,1$	0	0
R_3	$4,2$	0	+1
R_3	$4,3$	-1	0
R_3	$4,4$	-1	-1

Table 2.3: The linking contributions at bands R_2 and R_3

2.10 Summary and Example

Here we give a summary of the main steps that the implementation takes to calculate a Seifert matrix associated to a knot diagram with combinatorial data $\mathcal{P}(D)$.

1. Determine the Seifert circles as sequences of the arc labels of D (section 2.4).
2. Determine the nesting graph $\mathcal{N}(D)$ (section 2.5).
3. Choose a node of $\mathcal{N}(D)$ to be the root thus determining the nesting tree $\mathcal{N}^T(D')$ of some diagram D' with combinatorial data $\mathcal{P}(D)$ (section 2.6).
4. Hence determine the configuration of each of the twisted bands of the standard lift of D' (section 2.6).
5. Determine the Seifert graph $\mathcal{S}(D)(=\mathcal{S}(D'))$ and a set Z of generating cycles of $H_1(\mathcal{S}(D))$ (section 2.7).
6. For each pair of cycles $z_i, z_j \in Z$ push one onto the outer track to get a representation $R_O(z_i)$ and the other on to the inner track to get a representation $R_I(z_j)$. From these representations calculate the form of z_i, z_j at each of the twisted bands. Then calculate the ij^{th} and ji^{th} entries of the Seifert matrix by reading off the contributions from the tables 2.1–2.3 (section 2.9).

2.10.1 Example: *The Figure Eight Knot.*

Recall from 2.6 the diagram D of the figure of eight knot which has combinatorial data

$$\mathcal{P}(D) = \{(1,2,6,7,R),(5,6,2,3,R),(3,4,8,1,L),(7,8,4,5,L)\} \tag{2.6}$$

We will number the crossings as

$$c_1 = (1,2,6,7,R) \tag{2.7}$$

$$c_2 = (5,6,2,3,R) \tag{2.8}$$

$$c_3 = (3,4,8,1,L) \tag{2.9}$$

$$c_4 = (7,8,4,5,L) \tag{2.10}$$

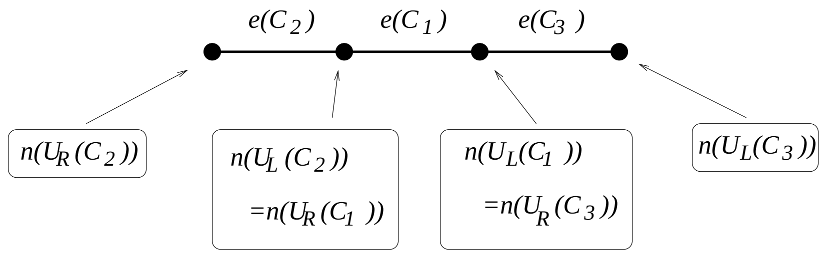


Figure 2.34: The nesting graph $\mathcal{N}(D)$.

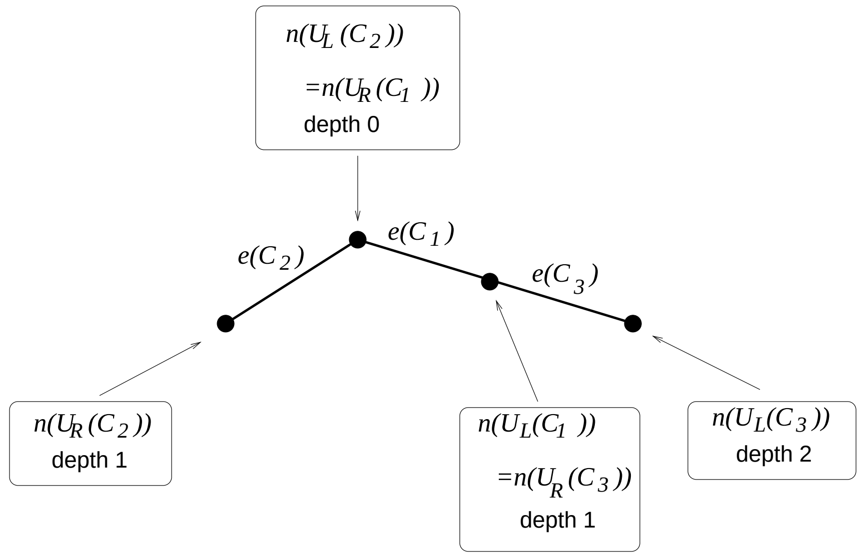


Figure 2.35: The nesting tree $\mathcal{N}^T(D')$.

1. The Seifert circles have representations

$$C_1 = (1, 7, 5, 3) \tag{2.11}$$

$$C_2 = (6, 2) \tag{2.12}$$

$$C_3 = (4, 8) \tag{2.13}$$

2. The nesting graph $\mathcal{N}(D)$ is shown in figure 2.34

3. Say we choose the node $n(U_L(C_2))$ to be the root to give the tree $\mathcal{N}^T(D')$ in figure 2.35.

4. From $\mathcal{N}^T(D')$ we see that the twisted bands B_{c_1} and B_{c_2} both have configuration R_1 as they both join the Seifert disks F_{C_1} and F_{C_2} which are located in level 0. The band B_{c_3} has configuration L_3 since it joins the disks F_{C_1} and F_{C_3}, where F_{C_1} is below F_{C_3} and F_{C_1} is to the right of the crossing c_3. The band B_{c_4} also has configuration L_3 as it joins the disks F_{C_1} and F_{C_3}, where F_{C_1} is below F_{C_3} and F_{C_1}

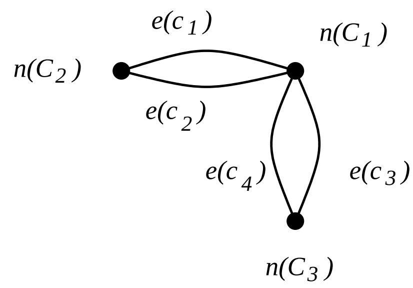


Figure 2.36: The Seifert graph $\mathcal{S}(D)$.

	(z_1, z_1)	(z_1, z_2)	(z_2, z_2)
B_{c_1}	(2,2)	(2,0)	(0,0)
B_{c_2}	(4,4)	(4,1)	(1,1)
B_{c_3}	(3,3)	(3,4)	(4,4)
B_{c_4}	(0,0)	(0,2)	(2,2)

Table 2.4: Forms of the cycles near the bands.

is to the right of the crossing c_4.

5. The Seifert graph $\mathcal{S}(D)$ is shown in figure 2.36. A generating set for $H_1(\mathcal{S}(D))$ is $Z = \{z_1, z_2\}$ where

$$z_1 = (e(c_1), e(c_2)), \qquad z_2 = (e(c_3), e(c_4)) \tag{2.14}$$

6. Pushing z_1 and z_2 out onto the tracks will give representations

$$R(z_1) = (2, 3, 1) \tag{2.15}$$

$$R(z_2) = (4, 5, 3) \tag{2.16}$$

The forms of each pair of cycles at the bands is given in table 2.4. Then using the linking contribution tables we read off the following entries of the Seifert matrix

$$L(z_1^+, z_1) = +1 \tag{2.17}$$

$$L(z_1^+, z_2) = +1 \tag{2.18}$$

$$L(z_2^+, z_1) = 0 \tag{2.19}$$

$$L(z_2^+, z_2) = -1 \tag{2.20}$$

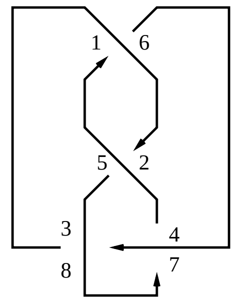


Figure 2.37: Dowker code of the figure eight knot

2.11 Implementation: Dowker Code to Combinatorial Data

The Maple procedure `SeifertMatrixCombData` contained on the CD-ROM (see appendix B for details) will return a Seifert matrix from the combinatorial data of a knot diagram. However for diagrams of more than a few crossings the combinatorial data is cumbersome and the redundancy in it can become infuriating. One is better advised to use the *Dowker code* of the unoriented knot diagram, see [10]. This is a much more compact form of encoding. For a diagram D with n crossings the Dowker code is just a sequence, with signs, of n even integers. It is formed as follows.

Number the crossings of D with the integers $1, \ldots, 2n$ in order as one traverses the diagram, so each crossing will get two numbers, one associated to the overpass and the other to the underpass. This produces a parity reversing permutation on the numbers $1, \ldots, 2n$ in that each odd number is sent to an even number and vice-versa. The *Dowker code* E is the sequence of the n even numbers paired to the sequence $(1, 3, 5, \ldots, 2n-1)$ of the odd numbers. The even numbers in the Dowker code are signed $+$ if they belong to an underpass and $-$ if they belong to an overpass. As an example, the Dowker code of the figure eight knot numbered as in figure 2.37 is $E = (6, 8, 2, 4)$. As with the combinatorial data there are many diagrams with the same Dowker code, see [10] for details.

Now under certain conditions, which correspond to D not being obviously the sum of two non-trivial diagrams, the combinatorial data of a diagram with Dowker code E can be derived from E. The algorithm for this is given in [10] and we have implemented it as the Maple procedure `CombData`. Unfortunately we have not incorporated the checking into `CombData` so that if it is invoked on a Dowker code not satisfying the conditions it is likely to return an error, or even worse, apparent combinatorial data. But the intention is that one use the Seifert matrix procedure on the Dowker codes contained in the program *Knotscape* [12]. These are Dowker codes of nice diagrams of prime knots and as such will produce valid combinatorial data.

Chapter 3

Châtelet Bases and Algorithms

3.1 Introduction

In section 3.2 we lay out some notation for Laurent polynomials that will be used throughout the rest of this chapter. In section 3.3 we prove the existence of certain special bases for ideals of the ring of Laurent polynomials, called *Châtelet bases*. These bases are special kinds of Gröbner bases and allow the decision, algorithmically, of the membership question for ideals, i.e. whether or not a given polynomial f is an element of a given ideal. In addition to this Châtelet bases have a very nice compact form that often allows one to distinguish different ideals by simple inspection of the polynomials appearing in the Châtelet bases.

The rest of the chapter is devoted to developing an algorithm that will calculate a Châtelet basis for the ideal $\mathcal{G} = \langle G \rangle_\Lambda$, given a finite set of generators, G, for the ideal. In section 3.4 we introduce a simple diagrammatic way of representing sums of polynomials that will be used in later sections. The final algorithm is then built up in three stages in sections 3.5, 3.6 and 3.7. The algorithms in these sections work by constructing successive 'approximations' to a Châtelet basis for $\mathcal{G}$. The 'algorithm' given in section 3.5 should not really be called an algorithm at all as it involves the manipulation of infinite sets of polynomials. But it is simple and shows some of the key ideas that are involved in the final implementable algorithm. In section 3.6 we modify the algorithm from the previous section so that it no longer involves infinite sets. This algorithm is now implementable yet perhaps it, also, should not be called an algorithm for, while it does produce a Châtelet basis for $\mathcal{G}$ after a finite number of steps, it has no way of recognising when this has been achieved and so will keep executing forever, vainly trying to improve the 'approximation'. In section 3.7 we modify this second algorithm to one that is both implementable and that terminates after a finite number of steps with a Châtelet basis for $\mathcal{G}$; an algorithm in the true sense of the word and the one that was used to produce the tables of Châtelet bases for the Alexander ideals of knots.

3.2 Some Notation for Polynomials

Let Λ denote the ring of Laurent polynomials, $\mathbb{Z}[t, t^{-1}]$. Throughout the rest of this chapter we shall understand the word polynomial to mean an element of Λ.

Definition 3.2.1 Let $f(t)$ be a non-zero element of Λ. We can write $f(t)$ as

$$f(t) = \phi_0 t^a + \phi_1 t^{a-1} + \cdots + \phi_n t^{a-n} \tag{3.1}$$

where $n \in \mathbb{N}$, $a \in \mathbb{Z}$ and $\phi_0, \ldots, \phi_n \in \mathbb{Z}$ such that $\phi_0, \phi_n \neq 0$. We make the following definitions:

- $\star$ $\text{length}(f(t)) = n$, the *length* of $f(t)$.
- $\star$ $\text{lcoeff}(f(t)) = \phi_0$, the *leading coefficient* of $f(t)$.
- $\star$ $\text{tcoeff}(f(t)) = \phi_n$, the *trailing coefficient* of $f(t)$.
- $\star$ $\text{ldeg}(f(t)) = a$, the *leading degree* (or just *degree*) of $f(t)$.
- $\star$ $\text{tdeg}(f(t)) = a - n$, the *trailing degree* of $f(t)$.

Note that length satisfies the following formula, let f, g be two non-zero polynomials in Λ then

$$\text{length}(f(t)g(t)) = \text{length}(f(t)) + \text{length}(g(t)) \tag{3.2}$$

Remark: Throughout this chapter we shall endeavour to use lower case Roman letters for polynomials and lower case Greek letters for their coefficients. When talking of a sequence of polynomials $g_1, \ldots, g_m$ in Λ we shall often use the following notation for the degrees and coefficients

$$g_i(t) = \gamma_{i,0}t^{a_i} + \gamma_{i,1}t^{a_i-1} + \cdots + \gamma_{i,n_i}t^{a_i-n_i} \tag{3.3}$$

We shall also drop the t in most cases and simply write f, g etc for elements of Λ.

3.3 Châtelet Bases for Ideals in Λ

In this section we extend the work of Albert Châtelet, in [6][1] on ideals of the ring $\mathbb{Z}[t]$, to ideals of the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$. The constructions are similar to those in [6], save that the length of Laurent polynomials takes the role played by the leading degree (of the non-Laurent polynomials).

Let $\mathcal{I}$ be an ideal of Λ, we will assume that $\mathcal{I} \neq \{0\}$, and let $n \in (\mathbb{N} - \{0\}) \cup \{\infty\}$.

Definition 3.3.1 Let $\mathcal{L}_n(\mathcal{I})$ denote the set of leading coefficients of elements of $\mathcal{I}$ of length strictly less than n along with the number 0.

$$\mathcal{L}_n(\mathcal{I}) = \{0\} \cup \{\text{lcoeff}(f) | f \in \mathcal{I} \text{ and length}(f) < n\} \tag{3.4}$$

Note that $\mathcal{L}_\infty(\mathcal{I})$ denotes the set of leading coefficients of *all* the polynomials in $\mathcal{I}$. In the rest of this section, to simplify notation we shall write $\mathcal{L}_n$ for $\mathcal{L}_n(\mathcal{I})$.

Lemma 3.3.2 $\mathcal{L}_n$ is an ideal of $\mathbb{Z}$.

Proof: By definition $0 \in \mathcal{L}_n$. Let $\phi_0, \gamma_0 \in \mathcal{L}_n$ with $\phi_0 \neq 0 \neq \gamma_0$ and let $f(t), g(t) \in \mathcal{I}$ be two polynomials, of length less than n, with leading coefficients ϕ_0 and γ_0 respectively. Suppose that

$$f(t) = \phi_0 t^a + \phi_1 t^{a-1} + \cdots + \phi_r t^{a-r} \tag{3.5}$$

$$g(t) = \gamma_0 t^b + \gamma_1 t^{b-1} + \cdots + \gamma_s t^{b-s} \tag{3.6}$$

The polynomial $f(t) + t^{a-b}g(t)$ belongs to $\mathcal{I}$, has length less than n and has leading coefficient $\phi_0 + \gamma_0$, so $\phi_0 + \gamma_0 \in \mathcal{L}_n$. Similarly, assuming $\phi_0 \neq \gamma_0$ the polynomial

[1] The work was published shortly after Albert Châtelet's death by François Châtelet.

$f(t)-t^{a-b}g(t)$ belongs to $\mathcal{I}$, has length less than n and has leading coefficient $\phi_0-\gamma_0$, so $\phi_0-\gamma_0 \in \mathcal{L}_n$.

If $m \in \mathbb{Z}$ then the polynomial $mf(t)$ belongs to $\mathcal{I}$, has length less than n and has leading coefficient $m\phi_0$, so $m\phi_0 \in \mathcal{L}_n$. ∎

Remark: The $\mathcal{L}_i$ form an ascending chain of ideals

$$\{0\} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots \subseteq \mathcal{L}_\infty \subseteq \mathbb{Z} \tag{3.7}$$

Note that $\mathbb{Z}$ is a principal ideal domain so the ideal $\mathcal{L}_n$ is generated by a single integer, λ_n say.

$$\mathcal{L}_n = \langle \lambda_n \rangle_{\mathbb{Z}} \tag{3.8}$$

So every $f(t) \in \mathcal{I}$ of length strictly less than n can be written as

$$f(t) = q\lambda_n t^a + \phi_1 t^{a-1} + \cdots + \phi_m t^{a-m}, \quad m < n \tag{3.9}$$

for some $q, a \in \mathbb{Z}$ and $m \in \mathbb{N}$. And since the ideals form an ascending chain we know that λ_i divides λ_{i-1} for each i.

The ring $\mathbb{Z}$ has the property that any ascending chain of ideals in $\mathbb{Z}$ must terminate after a finite number of steps, i.e. there exists $N \in \mathbb{N}$ such that $\mathcal{L}_n = \mathcal{L}_N$ for all $n \geq N$. We let n_0 be the smallest such N. Hence *every* polynomial in $\mathcal{I}$ has leading coefficient a multiple of λ_{n_0}. Some of the inclusions in the sequence

$$\{0\} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots \subsetneq \mathcal{L}_{n_0} \subseteq \mathbb{Z} \tag{3.10}$$

may not be strict. We are interested in the points in the sequence (3.10) where the inclusions are strict. So let $n_0, n_1, \ldots, n_m$ be the subsequence of $n_0, n_0-1, \ldots, 2, 1$ satisfying the properties

P1: $\mathcal{L}_{n_i} \neq \{0\}$ for $0 \leq i \leq m$.

P2: The number n_i is the smallest j for which $\mathcal{L}_{n_{i-1}-1} = \mathcal{L}_j$

So the $\mathcal{L}_{n_i}$ form a strictly increasing subsequence of the sequence in (3.10)

$$\{0\} \subsetneq \mathcal{L}_{n_m} \subsetneq \mathcal{L}_{n_{m-1}} \subsetneq \cdots \subsetneq \mathcal{L}_{n_1} \subsetneq \mathcal{L}_{n_0} \subseteq \mathbb{Z} \tag{3.11}$$

Lemma 3.3.3 For each $0 \leq i \leq m$ there exists a polynomial b_i in $\mathcal{I}$ of length $n_i - 1$ with leading coefficient λ_{n_i}, where $\mathcal{L}_{n_i} = \langle \lambda_{n_i} \rangle_{\mathbb{Z}}$.

Proof: Let $0 \leq i \leq m$. There are polynomials in $\mathcal{I}$ of length strictly less than n_i with leading coefficient λ_{n_i}. But $\lambda_{n_i} \notin \mathcal{L}_{n_i - 1}$ since $\mathcal{L}_{n_i-1} \subsetneq \mathcal{L}_{n_i}$, hence there are no polynomials in $\mathcal{I}$ of length strictly less than $n_i - 1$ with leading coefficient λ_{n_i}. Hence there must be some polynomial in $\mathcal{I}$ of length $n_i - 1$ with leading coefficient λ_{n_i}. ∎

Definition 3.3.4 We will refer to a polynomial b_i in $\mathcal{I}$ of length $n_i - 1$ and leading coefficient λ_{n_i} as an i^{th} *reduced polynomial* of $\mathcal{I}$. A *reduced set* $\mathcal{B}$ for $\mathcal{I}$ is a set of $m+1$ polynomials in $\mathcal{I}$

$$\mathcal{B} = \{b_0, b_1, \ldots, b_m\} \tag{3.12}$$

where the polynomial b_i is an i^{th} reduced polynomial of $\mathcal{I}$, whose existence is assured by lemma 3.3.3.

Lemma 3.3.5 If b_i is an i^{th} reduced polynomial for $\mathcal{I}$ and f is any non-zero polynomial in $\mathcal{I}$ which, when $i \neq 0$, satisfies

$$\text{length}(f) < n_{i-1} - 1 \tag{3.13}$$

then there exist polynomials $e \in \Lambda$ and $g \in \mathcal{I}$ such that

$$f = eb_i + g \tag{3.14}$$

and where g satisfies

$$g = 0 \qquad \text{if } i = m, \tag{3.15}$$

$$g = 0 \text{ or length}(g) < n_i - 1 \qquad \text{otherwise.} \tag{3.16}$$

Proof: Let $\text{length}(f) = n$ and $\text{ldeg}(b_i) = c$. We shall prove the result by induction on n.

If $n < n_i - 1$ then setting $e = 0$ and $g = f$ proves the lemma. So we will assume that $n_i - 1 \leq n$ and that the result is true for all $k < n$. Now if $i = 0$ then the leading coefficient of f is a multiple of λ_{n_0}. If $i > 0$ then since $n_i - 1 \leq n < n_{i-1} - 1$,

$$\mathcal{L}_{n_{i-1}-1} = \mathcal{L}_{n_i} = \langle \lambda_{n_i} \rangle \quad \text{from property P2} \tag{3.17}$$

So for all $i \geq 0$, f can be written as

$$f = q\lambda_{n_i} t^a + \phi_1 t^{a-1} + \cdots + \phi_n t^{a-n} \tag{3.18}$$

for some $q \in \mathbb{Z}$. So we set $f' = f - qt^{a-c}b_i \in \mathcal{I}$ and notice that

$$f = qt^{a-c}b_i + f' \tag{3.19}$$

Now if $f' = 0$ then the result is true with $e = qt^{a-c}$ and $g = 0$. If $f' \neq 0$ then length$(f') < n$ so by assumption $f' = eb_i + g$ for some $e \in \Lambda$ and $g \in \mathcal{I}$ such that $g = 0$ or length$(g) < n_i - 1$ so

$$f = (e + qt^{a-c})b_i + g \tag{3.20}$$

as required. ∎

Theorem 3.3.6 A reduced set $\mathcal{B}$ for $\mathcal{I}$ is actually a basis for $\mathcal{I}$ in that

$$\mathcal{I} = \langle \mathcal{B} \rangle_\Lambda \tag{3.21}$$

Moreover, if a reduced set $\mathcal{B}$ for $\mathcal{I}$ is known then given any polynomial f in Λ one can determine whether or not f is an element of $\mathcal{I}$

Proof: Of course $\langle \mathcal{B} \rangle_\Lambda \subseteq \mathcal{I}$ since $\mathcal{B} \subset \mathcal{I}$. Let f be any polynomial in $\mathcal{I}$. By lemma 3.3.5 there exists $e_0 \in \Lambda$ and $g_0 \in \mathcal{I}$ such that

$$f = e_0 b_0 + g_0 \tag{3.22}$$

and $g = 0$ or length$(g) < n_0 - 1$. And note that the inductive process in lemma 3.3.5 can easily be carried out to determine e_0 and g_0. If $g_0 \neq 0$ then applying lemma 3.3.5 again to g_0 yields polynomials $e_1 \in \Lambda$ and $g_1 \in \mathcal{I}$ such that

$$f = e_0 b_0 + e_1 b_1 + g_1 \tag{3.23}$$

and $g = 0$ or length$(g_1) < n_1 - 1$. We keep applying lemma 3.3.5 to the non-zero remainders g_i until we have polynomials $e_0, e_1, \ldots, e_m$ such that

$$f = e_0 b_0 + e_1 b_1 + \cdots + e_m b_m \in \langle \mathcal{B} \rangle_\Lambda \tag{3.24}$$

Hence $\mathcal{I} \subset \langle \mathcal{B} \rangle_\Lambda$. So $\mathcal{I} = \langle \mathcal{B} \rangle_\Lambda$.

Finally, given any polynomial $f \in \Lambda$ we can determine whether or not f is an element of $\mathcal{I}$ by carrying out the reduction process just described. Either we will end up with an expression like that in (3.24) which implies that $f \in \mathcal{I}$ or the process will fail at some point in that we arrive at a non-zero remainder g (or f to begin with)

$$f = \sum_{i=0}^{s} e_i b_i + g \quad , s \leq m \tag{3.25}$$

such that there is no polynomial $b \in \mathcal{B}$ for which length$(b) \leq$ length(g) and lcoeff(b) divides lcoeff(g). By definition of the reduced elements in $\mathcal{B}$ this implies that $g \notin \mathcal{I}$. But $\sum_{i=0}^{s} e_i b_i \in \mathcal{I}$, hence $f \notin \mathcal{I}$. ■

Definition 3.3.7 We will refer to a reduced set $\mathcal{B}$ for an ideal $\mathcal{I}$ of Λ as a *Châtelet basis* for $\mathcal{I}$.

Note that if we have Châtelet bases $\mathcal{B}(\mathcal{I})$ and $\mathcal{B}(\mathcal{J})$ for two ideals $\mathcal{I}$ and $\mathcal{J}$ then we can easily decide whether $\mathcal{I}$ and $\mathcal{J}$ are the same ideal or not by carrying out the reduction process in theorem 3.3.6 to see whether $\mathcal{I} \subseteq \mathcal{J}$ and vice versa. Indeed we can say that $\mathcal{I} \neq \mathcal{J}$ if by inspection we see that the bases do not consist of an equal number of polynomials with corresponding lengths and leading coefficients.

Given $f \in \mathcal{I}$ expressed as

$$f = e_0 b_0(t) + e_1 b_1 + \cdots + e_m b_m \tag{3.26}$$

we can say something about the Λ-coefficients as well. Consider the equation

$$f = e_0 b_0 + g_0 \tag{3.27}$$

When we divide g_0 by b_1 to get

$$g_0 = e_1 b_1 + g_1 \tag{3.28}$$

we see from the proof of lemma 3.3.5 that

$$\text{length}(e_1 b_1) \leq \text{length}(g_0) < n_0 - 1 \tag{3.29}$$

so that

$$\text{length}(e_1) < n_0 - n_1 - 2 \tag{3.30}$$

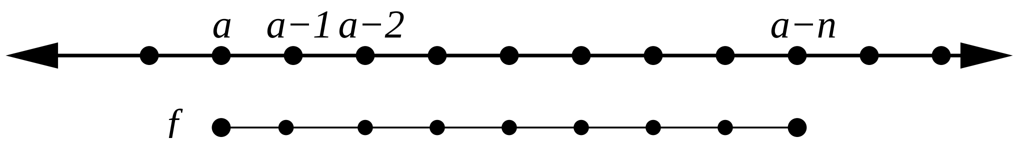


Figure 3.1: Polynomial as line segment

Similarly

$$\text{length}(e_i) < n_{i-1} - n_i - 2 \tag{3.31}$$

for all $1 \leq i \leq m$.

3.4 Line Segment Diagrams of Polynomials

For some of the discussion and proofs in the next section it will be useful to have a simple conceptual model for linear combinations of polynomials. Let f be a polynomial in Λ with leading degree a and of length n, where $a \in \mathbb{Z}$ and $n \in \mathbb{N}$. We can represent f as a line segment of length n against a number line representing powers of t as in figure 3.1 Note that due to the way we write polynomials with leading degree first the number line runs backward so to speak. The dots on the line segment represent the coefficients of f, some of which may be zero of course, but not the ones at the ends of the line segment which represent the leading and trailing coefficients of f. Suppose we have f expressed as a sum of polynomials

$$f = \sum_{h \in H} h \tag{3.32}$$

then we can represent this expression for f with the diagram D consisting of the line segments representing each of the polynomials $h \in H$ together with the line segment representing f at the bottom of the diagram, as shown in figure 3.2.

Definition 3.4.1 We shall say that D is a *diagram for* f and that $P(D)$ are the *polynomials in* D where $P(D) = H$. We define the *maximum leading degree of the diagram* D as the number

$$\max\{\text{ldeg}(p) | p \in P(D)\} \tag{3.33}$$

and the *minimum trailing degree of the diagram* D as the number

$$\min\{\text{tdeg}(p) | p \in P(D)\} \tag{3.34}$$

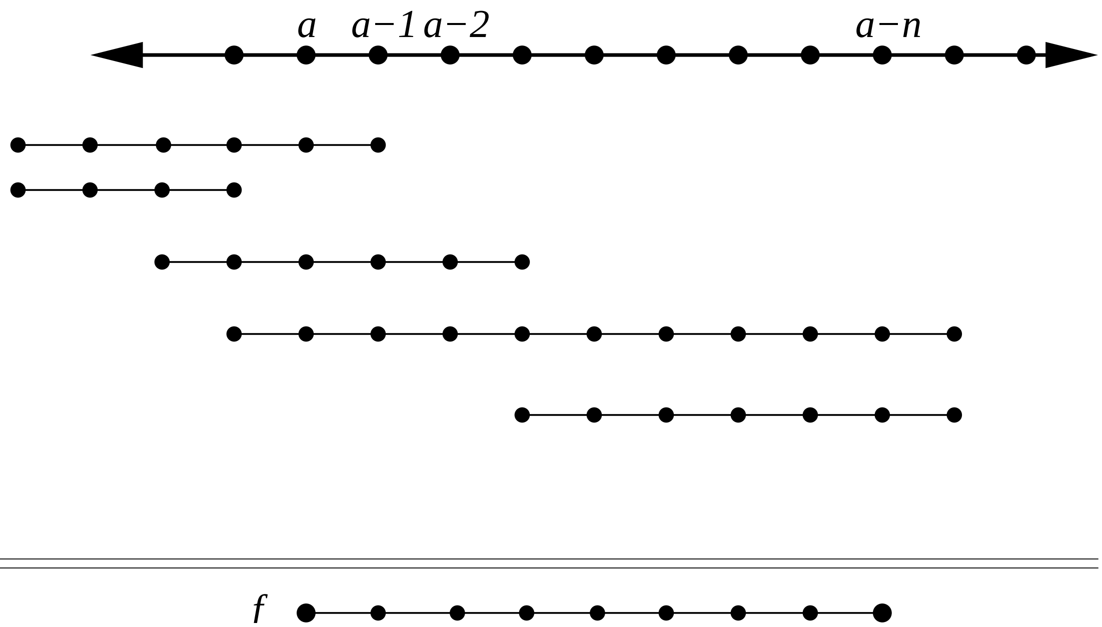


Figure 3.2: The diagram D for f.

In such a diagram we see how each coefficient ϕ of f is obtained by summing the coefficients of the $p \in P(D)$ that lie directly above ϕ. Of course there are many different diagrams for f according to how we express f as a sum of polynomials.

3.5 The First Algorithm

Let $G \subset \Lambda$ be a finite set of polynomials. In this section we show how to calculate a Châtelet basis, $\mathcal{B}$, for the ideal $\mathcal{G} = \langle G \rangle_\Lambda$.

Definition 3.5.1 Let $g_1, \ldots, g_m$, where $m \geq 1$, be polynomials in Λ. A polynomial g in Λ is a *leading degree monomial combination* of the $g_1, \ldots, g_m$ if

$$g = m_1 g_1 + m_2 g_2 + \cdots + m_m g_m \tag{3.35}$$

where the m_i are all monomials

$$m_i = \alpha_i t^{b_i} \quad , \alpha_i, b_i \in \mathbb{Z}, i = 1, 2, \ldots, m \tag{3.36}$$

and the polynomials $m_i g_i$ all have the same leading degree.

Similarly we can have polynomials lined up at their trailing coefficients

Definition 3.5.2 Let $g_1, \ldots, g_m$, $m \geq 1$ be polynomials in Λ. A polynomial g in Λ is a *trailing degree monomial combination* of the $g_1, \ldots, g_m$ if

$$g = m_1 g_1 + m_2 g_2 + \cdots + m_m g_m \tag{3.37}$$

where the m_i are all monomials

$$m_i = \alpha_i t^{b_i} \quad , \alpha_i, b_i \in \mathbb{Z}, i = 1, 2, \ldots, m \tag{3.38}$$

and the polynomials $m_i g_i$ all have the same trailing degree.

We note that if g is a leading degree or trailing degree monomial linear combination of $g_1, \ldots, g_m$ then

$$\text{length}(g) \leq \max\left\{\text{length}(g_i) | 1 \leq i \leq m\right\} \tag{3.39}$$

We now define four kinds of monomial combinations. Let $g_1, \ldots, g_m$ be polynomials in Λ with leading coefficients

$$\text{lcoeff}(g_i) = c_i, \quad i = 1, \ldots, m \tag{3.40}$$

and trailing coefficients

$$\text{tcoeff}(g_i) = d_i, \quad i = 1, \ldots, m \tag{3.41}$$

and let $c = \gcd(c_1, \ldots, c_m)$ and $d = \gcd(d_1, \ldots, d_m)$.

Definition 3.5.3 A polynomial g in Λ is of type $\text{GCD}_l(g_1, \ldots, g_m)$ if g is a leading degree monomial combination of $g_1, \ldots, g_m$

$$g = m_1 g_1 + m_2 g_2 + \cdots + m_m g_m \tag{3.42}$$

$$m_i = a_i t^{b_i}, \quad i = 1, \ldots, m \tag{3.43}$$

and

$$a_1 c_1 + a_2 c_2 + \cdots + a_m c_m = c \tag{3.44}$$

Definition 3.5.4 A polynomial g in Λ is of type $\text{GCD}_t(g_1, \ldots, g_m)$ if g is a trailing degree monomial combination of $g_1, \ldots, g_m$

$$g = m_1 g_1 + m_2 g_2 + \cdots + m_m g_m \tag{3.45}$$

$$m_i = a_i t^{b_i}, \quad i = 1, \ldots, m \tag{3.46}$$

and

$$a_1d_1 + a_2d_2 + \cdots + a_md_m = d \tag{3.47}$$

Definition 3.5.5 A polynomial g in Λ is of type $S_l(g_1, \ldots, g_m)$ if g is a leading degree monomial combination of $g_1, \ldots, g_m$

$$g = m_1g_1 + m_2g_2 + \cdots + m_mg_m \tag{3.48}$$

$$m_i = a_it^{b_i}, \quad i = 1, \ldots, m \tag{3.49}$$

and

$$a_1c_1 + a_2c_2 + \cdots + a_mc_m = 0 \tag{3.50}$$

Definition 3.5.6 A polynomial g in Λ is of type $S_t(g_1, \ldots, g_m)$ if g is a trailing degree monomial combination of $g_1, \ldots, g_m$

$$g = m_1g_1 + m_2g_2 + \cdots + m_mg_m \tag{3.51}$$

$$m_i = a_it^{b_i}, \quad i = 1, \ldots, m \tag{3.52}$$

and

$$a_1d_1 + a_2d_2 + \cdots + a_md_m = 0 \tag{3.53}$$

So of course for any polynomial g_1, the zero polynomial is the only polynomial of type $S_l(g_1)$ or $S_t(g_1)$. We note that it is possible for a polynomial g in Λ to be of type $\text{GCD}_l(g_1, \ldots, g_m)$ and $S_t(g_1, \ldots, g_m)$ or of type $\text{GCD}_t(g_1, \ldots, g_m)$ and $S_l(g_1, \ldots, g_m)$. Also g can be of type $S_l(g_1, \ldots, g_m)$ and $S_t(g_1, \ldots, g_m)$ or of type $\text{GCD}_l(g_1, \ldots, g_m)$ and $\text{GCD}_t(g_1, \ldots, g_m)$. But g can not be of type $\text{GCD}_l(g_1, \ldots, g_m)$ and $S_l(g_1, \ldots, g_m)$ or of type $\text{GCD}_t(g_1, \ldots, g_m)$ and $S_t(g_1, \ldots, g_m)$.

Of course there can be infinitely many polynomials of any of these types so we collect them together as follows.

Definition 3.5.7 Given a subset $H \subset \Lambda$ we define $\mathcal{S}_l(H)$ to be the collection of all leading coefficient cancelling polynomials, i.e.

$$\mathcal{S}_l(H) = \{f \in \Lambda | f \text{ is of type } S_l(h_1, \ldots, h_m) \text{ where } h_1, \ldots, h_m \in H\} \tag{3.54}$$

Similarly we define $\mathcal{S}_t(H)$ to be the set of all trailing coefficient cancelling polynomials, i.e.

$$\mathcal{S}_t(H) = \{f \in \Lambda | f \text{ is of type } S_t(h_1, \ldots, h_m) \text{ where } h_1, \ldots, h_m \in H\} \tag{3.55}$$

Note that if H is a subset of the ideal $\mathcal{G}$ then $\mathcal{S}_l(H)$ and $\mathcal{S}_t(H)$ are also subsets of $\mathcal{G}$.

Definition 3.5.8 Let $\mathcal{P}(\mathcal{G})$ denote the power set of $\mathcal{G}$. We define the set map $\mathcal{S} : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G})$ as follows

$$\mathcal{S}(H) = \mathcal{S}_l(H) \cup \mathcal{S}_t(H) \cup H \tag{3.56}$$

We shall denote the iterates H, $\mathcal{S}(H)$, $\mathcal{S}(\mathcal{S}(H))$, ... etc of this map by $\mathcal{S}^0(H)$, $\mathcal{S}^1(H)$, $\mathcal{S}^2(H)$, ... etc.

Note that the iterates form an ascending chain of subsets of $\mathcal{G}$

$$H \subset \mathcal{S}(H) \subset \mathcal{S}^2(H) \subset \ldots \tag{3.57}$$

Also if the lengths of elements of H is bounded above, say

$$L = \max\{\text{length}(h) | h \in H\} \tag{3.58}$$

then from (3.39) we see that

$$L = \max\{\text{length}(h) | h \in \mathcal{S}^n(H)\} \tag{3.59}$$

for all n.

Suppose we have a non-empty set H of polynomials. We will show below how to construct a finite set of polynomials $\text{GCDset}_l(H) \subset \langle H \rangle_\Lambda$ which is, in a sense that will be made precise later, an approximation to a Châtelet basis for the ideal $\langle H \rangle_\Lambda$. The construction attempts to mimic the definition of the reduced polynomials in section 3.3.

For each $n \geq 1$ we define c_n as

$$c_n(H) = \gcd\{\text{lcoeff}(h) | h \in H \text{ and length}(h) < n\} \tag{3.60}$$

where we assume that $\gcd\{\} = 0$. In the rest of this section we will write c_n for $c_n(H)$.

Remark 3.5.9 Note that λ_n divides c_n, where $\mathcal{L}_n(\langle H\rangle_\Lambda) = \langle\lambda_n\rangle_{\mathbb{Z}}$, (see (3.8)).

Now each c_n divides c_{n-1} so there must exist $N \in \mathbb{N}$ such that $c_n = c_N$ for all $n \geq N$. Let n_0 be the least such N and let $n_0, n_1, \ldots, n_m$ be the subsequence of $n_0, n_0 - 1, \ldots, 1$ satisfying

1. $c_{n_i} \neq 0 \quad, 0 \leq i \leq m$.

2. n_i is the smallest j for which $c_{n_{i-1}-1} = c_j$, $1 \leq i \leq m$.

Now for each $0 \leq i \leq m$ there will exist a finite number of polynomials $h_{i,1}, h_{i,2}, \ldots, h_{i,r_i} \in H$ such that $\text{length}(h_{i,j}) < n_i$ for $1 \leq j \leq r_i$ and $\gcd\{\text{lcoeff}(h_i) | 1 \leq i \leq r_i\} = c_{n_i}$. Let f_i be any polynomial satisfying

$$f_i \text{ is of type } \mathrm{GCD}_l(h_{i,1}, h_{i,2}, \ldots, h_{i,r_i}) \tag{3.61}$$

Now the sequence of polynomials $F = (f_0, \ldots, f_m)$ has strictly increasing leading coefficients like the polynomials in a Châtelet Basis however the lengths of F might not be strictly decreasing like the polynomials in a Châtelet basis because some of our f_i might be of type $S_t(h_{i,1}, h_{i,2}, \ldots, h_{i,r_i})$.

So if there exists an i with $0 \leq i \leq m-1$ such that $\text{length}(f_i) \leq \text{length}(f_{i+1})$ then let F' be the sequence $F' = (f'_0, \ldots, f'_{m-1})$ obtained from F by omitting the polynomial f_{i+1}. Make this F' our new F and keep repeating this procedure until we have a sequence of polynomials $F = (f_0, \ldots, f_r)$ with strictly increasing leading coefficients and strictly decreasing lengths, i.e. one that *looks like* it might be a Châtelet basis for the ideal $\langle H\rangle_\Lambda$.

Of course we note that if the set H is finite then it is a straightforward manner to compute such a sequence F from H.

Definition 3.5.10 We shall refer to such a sequence F as $\mathrm{GCDset}_l(H)$. It is not uniquely defined as there are many choices for the polynomials h_j appearing in (3.61). However, the ambiguity in $\mathrm{GCDset}_l(H)$ will ultimately be no greater than the ambiguity in the definition of the Châtelet basis itself as will be made clear by theorem 3.5.12.

Lemma 3.5.11 If $b \in \mathcal{B}$ is one of the polynomials in a Châtelet basis $\mathcal{B}$ for $\mathcal{G} = \langle G \rangle_\Lambda$ then there exists an $N \in \mathbb{N}$ such that for each $m \geq N$ there is a polynomial

$$b' \in \mathrm{GCDset}_l(\mathcal{S}^m(G)) \tag{3.62}$$

that has the same length and leading coefficient as b.

Proof: Let b be one of the polynomials in the Châtelet basis $\mathcal{B}$ for $\mathcal{G}$. Let $\mathrm{lcoeff}(b) = \lambda_{n+1}$ and $\mathrm{length}(b) = n$. Suppose that b is given by

$$b = \sum_{g \in G} f_g g \tag{3.63}$$

and that each of the polynomials f_g has the form

$$f_g = \sum_{i=0}^{n_g} \phi_{g,i} t^{a_g - i} \tag{3.64}$$

We will consider the following linear combination representation for b

$$b = \sum_{g \in G} \sum_{i=0}^{n_g} \phi_{g,i} t^{a_g - i} g \tag{3.65}$$

We consider the diagram, D, for b with respect to the $\phi_{g,i} t^{a_g - i} g$. We will denote the polynomials appearing in D as

$$P(D) = \left\{ \phi_{g,i} t^{a_g - i} g \quad | \quad g \in G \quad 0 \leq i \leq n_g \right\} \tag{3.66}$$

First focus on the the portion of this diagram at and to the left of the leading degree of b, which will look something like figure 3.3. The diagram D has the following properties (with $n(D) = 0$):

1. There exists $n(D) \in \mathbb{N}$ such that all the polynomials appearing in D are monomial multiples of elements of $\mathcal{S}^n(G)$ for all $n \geq n(D)$.

2. The maximum leading degree of the diagram u satisfies $u \geq a$

If $u > a$ we can change this diagram D for b to a new diagram D' for b that also satisfies these two properties. The maximum leading degree u' of D' however satisfies $u > u'$. So assume $u > a$. Now let $P_u(D) \subset P(D)$ be the set of polynomials of D

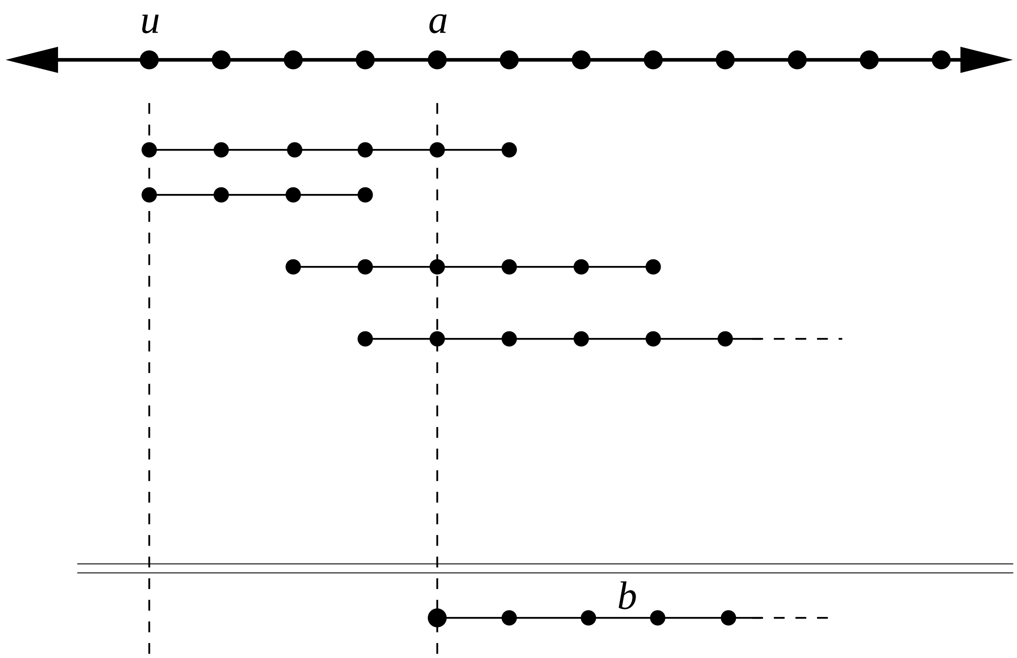


Figure 3.3: The diagram D.

with leading degree u. Now since $u > a$ it must be that the leading coefficients of the polynomials in $P_u(D)$ cancel each other out, i.e. that

$$\sum_{p \in P_u(D)} p \in \mathcal{S}_l(\mathcal{S}^{n(D)}(G)) \subset \mathcal{S}^{n(D)+1}(G) \tag{3.67}$$

Let D' be the diagram consisting of the line segments representing all the polynomials in $P(D) - P_u(D)$ together with the polynomial $\sum_{p \in P_u(D)} p$. Clearly this new diagram D' satisfies property 1 with $n(D') = n(D) + 1$ and also property 2 for which the maximum leading degree u' of D' satisfies $u > u'$.

We repeat this process until we arrive at a diagram, which we denote D_l, for b satisfying properties 1 and 2 whose maximum leading degree is equal to a, the leading degree of b. The diagram D_l will look something like that appearing in figure 3.4. The diagram D_l satisfies three properties

1. There exists $n(D_l) \in \mathbb{N}$ such that all the polynomials appearing in D are monomial multiples of elements of $\mathcal{S}^n(G)$ for all $n \geq n(D_l)$.

2′. The maximum leading degree of D_l is equal to a, the leading degree of b.

3. The minimum trailing degree u of the diagram satisfies $u \leq a - n$, where $a - n$ is the trailing degree of b

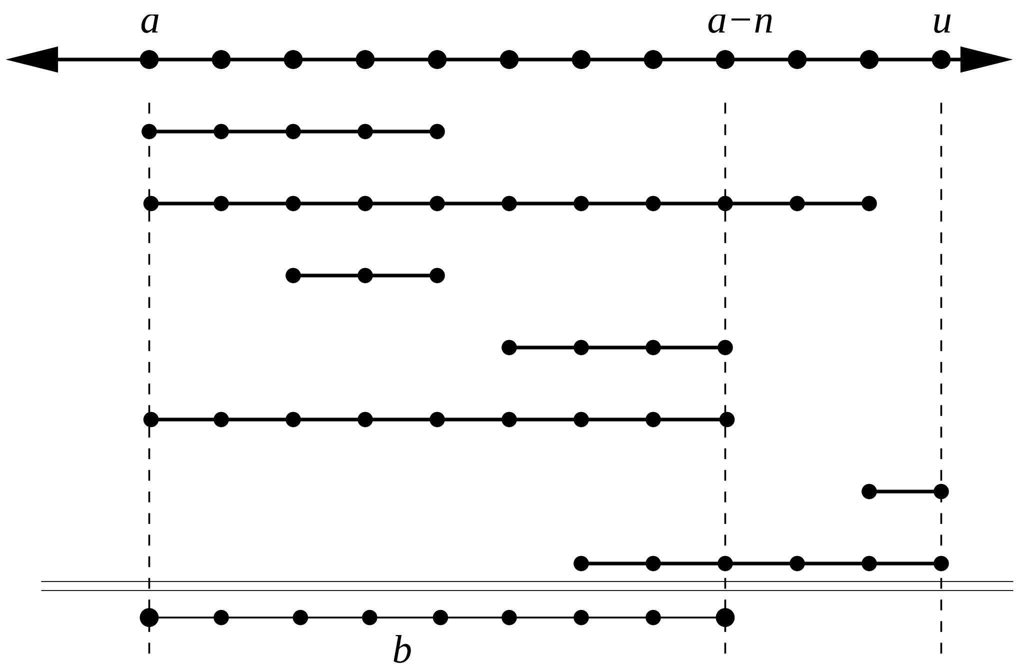


Figure 3.4: The diagram D_l

If $a-n > u$ then we can change this diagram to a diagram D_l' for b which still satisfies properties 1, $2'$ and 3 but whose minimum trailing degree u' satisfies $a-n \geq u' > u$. So assume that $a-n > u$ and let $P_u(D_l) \subset P(D_l)$ be the set of polynomials in D_l of trailing degree equal to u. But as before, since $a-n > u$ it must be that

$$\sum_{p \in P_u(D_l)} p \in \mathcal{S}_t(\mathcal{S}^{n(D_l)}(G)) \subset \mathcal{S}^{n(D_l)+1}(G) \tag{3.68}$$

So let D_l' be the diagram with line segments representing all the polynomials in $P(D_l) - P_u(D_l)$ together with the line segment representing $\sum_{p \in P_u(D_l)} p$. Clearly D_l' also satisfies property 1 with $n(D_l') = n(D_l)+1$, property $2'$ and property 3 with the minimum trailing degree u' of D_l' also satisfying $a-n \geq u' > u$.

We repeat this process until we arrive at a diagram D_t for b satisfying properties 1,$2'$ and 3 whose minimum trailing degree is equal to $a-n$ the trailing degree of b. So D_t looks something like the diagram in figure 3.5. Let $N = n(D_t)$ and let

$$P = \{p \in P(D_t) | \mathrm{ldeg}(p) = a\} \tag{3.69}$$

Now for each $p \in P$, $\mathrm{length}(p) \leq n$ and

$$\mathrm{lcoeff}(b) = \sum_{p \in P} \mathrm{lcoeff}(p) \tag{3.70}$$

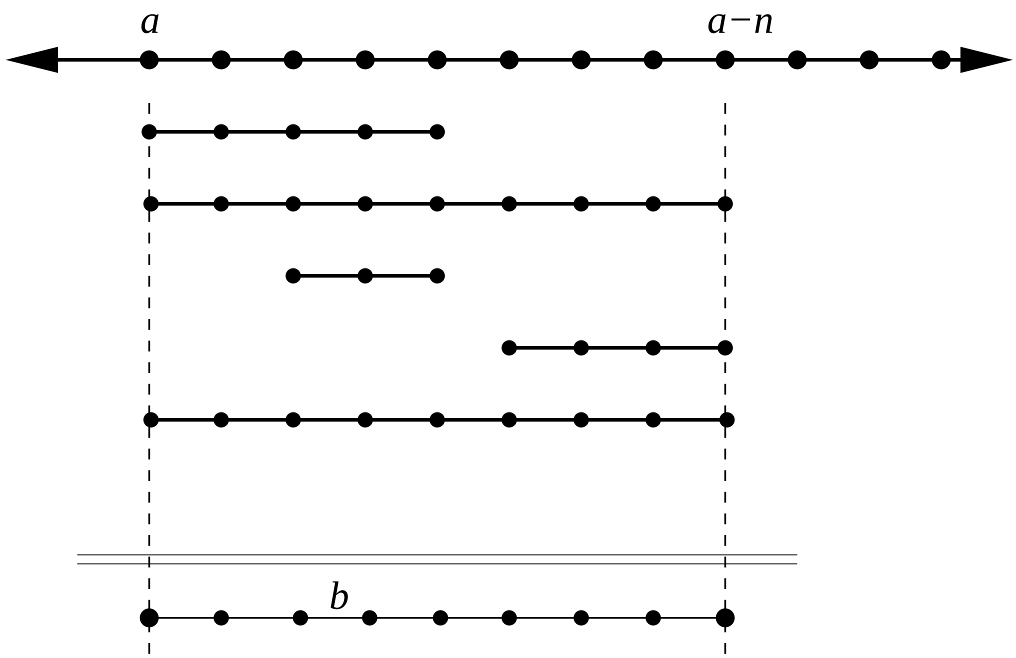


Figure 3.5: The diagram D_t

By property 1 each of the polynomials $p \in P$ is a monomial multiple of a polynomial s_p say with $s_p \in \mathcal{S}^m(G)$ for all $m \geq N$. Hence for each $p \in P$, $\mathrm{length}(s_p) = \mathrm{length}(p) \leq n$. Now $c = \gcd\{\mathrm{lcoeff}(s_p)|p \in P\}$ must divide $\lambda_{n+1} = \mathrm{lcoeff}(b)$. But of course, for $m \geq N$, $c_{n+1}(\mathcal{S}^m(G))$ (from (3.60)) divides c. Hence $c_{n+1}(\mathcal{S}^m(G))$ divides λ_{n+1}. But from remark 3.5.9, λ_{n+1} divides $c_{n+1}(\mathcal{S}^m(G))$, so

$$\lambda_{n+1} = c = c_{n+1}(\mathcal{S}^m(G)) \quad \text{, for all } m \geq N \tag{3.71}$$

So in the construction of $\mathrm{GCDset}_l(\mathcal{S}^m(G))$ there will be a polynomial $b' \in \mathrm{GCDset}_l(\mathcal{S}^m(G))$ with leading coefficient λ_{n+1} and of length $\leq n$. However $\mathrm{length}(b') = n$ since b, being an element of the Châtelet basis for $\mathcal{G}$ has minimal length among those polynomials in $\mathcal{G}$ with leading coefficient λ_{n+1}. So the result is proved. ■

By repeated applications of this lemma we can establish the following theorem.

Theorem 3.5.12 There exists an $M \in \mathbb{N}$ such that for all $m \geq M$, $\mathrm{GCDset}_l(\mathcal{S}^m(G))$ is a Châtelet basis for $\mathcal{G}$.

Proof: Let $\mathcal{B} = \{b_0, \ldots, b_s\}$ be a Châtelet basis of $\mathcal{G}$, where the b_i are arranged in order of decreasing length. By lemma 3.5.11, for each $0 \leq i \leq s$ there exists N_i such that for all $m \geq N_i$ there exits a polynomial $b'_i \in \mathrm{GCDset}_l(\mathcal{S}^m(G))$ with the same leading coefficient and length as b_i. If we let $M = \max\{N_i | 0 \leq i \leq s\}$ then for each $m \geq M$ there is a Châtelet basis contained in $\mathrm{GCDset}_l(\mathcal{S}^m(G))$.

Fix an $m \geq M$. Let $\mathcal{B}' = \{b'_0, \ldots, b'_s\}$ be the Châtelet basis contained in $\mathrm{GCDset}_l(\mathcal{S}^m(G))$, with the b'_i in order of decreasing length. By construction, $\mathrm{GCDset}_l(\mathcal{S}^m(G))$ is a set of polynomials

$$F = \{f_0, \ldots, f_r\} \subset \mathcal{G} \tag{3.72}$$

where $s \leq r$, of strictly decreasing length and with $\mathrm{lcoeff}(f_i)$ a strict multiple of $\mathrm{lcoeff}(f_{i-1})$ for each $1 \leq i \leq r$.

Now b'_0 is one of these polynomials. In fact $b'_0 = f_0$, for if $b'_0 = f_i$, where $i > 0$, then $\mathrm{lcoeff}(b'_0)$ would be a strict multiple of $\mathrm{lcoeff}(f_0)$ which is a contradiction of the fact that the leading coefficient of every polynomial in $\mathcal{G}$ is a multiple of $\mathrm{lcoeff}(b'_0)$.

Similarly, suppose we have established that $b'_i = f_i$ for each $0 \leq i \leq k < s$. If $b'_{k+1} = f_j$ for some $j > k+1$ then $\mathrm{lcoeff}(b'_{k+1})$ is a strict multiple of $\mathrm{lcoeff}(f_{k+1})$. This contradicts the fact the every polynomial in $\mathcal{G}$ of length strictly less than $\mathrm{length}(b'_k)$ has leading coefficient a multiple of $\mathrm{lcoeff}(b'_{k+1})$. Hence $b'_{k+1} = f_{k+1}$.

So we conclude that $b'_i = f_i$ for $0 \leq i \leq s$. It must be that $r = s$ for by definition of the Châtelet basis there are no polynomials in $\mathcal{G}$ of length less than $\mathrm{length}(b'_s)$. Hence $\mathrm{GCDset}_l(\mathcal{S}^m(G))$ is a Châtelet basis for $\mathcal{G}$. ■

So now we can describe the first algorithm, algorithm 1. The algorithm consists of a number of statements which are executed sequentially. We'll briefly describe the notation used. A statement $A \leftarrow B$ is an assignment of the value B to the variable A. A **while** loop is a statement group of the form

while *condition* **do**

statements

end while

If the *condition* is false then we jump to the first statement following the corresponding **end while**. If the *condition* is true then the *statements* are executed. When the **end while** statement is reached we return to the corresponding **while** statement and execute it again. In algorithm 1 the $1 = 1$ condition just means that the **while** loop will execute repeatedly without end. In a later algorithm we shall use an **if** loop which is a statement of the form

if *condition* **then**

statements

end if

If the *condition* is false then we jump to the first statement following the corresponding **end if**. If the *condition* is true then the *statements* are executed until the statement **end if** is reached where execution shall continue with the first statement following **end if**.

```
n ← 0
H ← G
while 1 = 1 do
    H ← S^n(H)
    B ← GCDset_l(H)
    n ← n + 1
end while
```

Algorithm 1: Calculate a Châtelet basis for $\mathcal{G} = \langle G \rangle_\Lambda$

By theorem 3.5.12 after a finite number of steps and for all steps thereafter algorithm 1 will have $B = \mathcal{B}$ a Châtelet basis for $\mathcal{G} = \langle G \rangle_\Lambda$. However this algorithm is far from satisfactory or implementable as it involves constructing infinite sets and notwithstanding this, we don't have any way of telling when a Châtelet basis has been reached. In the rest of this chapter we will settle these matters. First by altering the algorithm to one that is implementable, in that it no longer requires any infinite constructions. And then by altering this second algorithm to one that terminates after a finite number of steps with a Châtelet basis.

3.6 The Second Algorithm

In this section we show how we can alter the definition of $\mathcal{S}_l(H)$ and $\mathcal{S}_t(H)$ used in the definition of the set map $\mathcal{S}(H)$, see definitions 3.5.7 and 3.5.8, so that they only consist of certain polynomials of type $S_l(h_1, h_2)$ and $S_t(h_1, h_2)$ and in particular, if

H is a finite set of polynomials then all the iterates of the set map $\mathcal{S}$ will also be finite and hence implementable.

Consider two polynomials g_1, g_2 in Λ

$$g_i(t) = \gamma_{i,0}t^{a_i} + \gamma_{i,1}t^{a_i-1} + \cdots + \gamma_{i,n_i}t^{a_i-n_i} \quad , i = 1, 2 \tag{3.73}$$

Let

$$c = \gcd(\gamma_{1,0}, \gamma_{2,0}) \tag{3.74}$$

$$d = \gcd(\gamma_{1,n_1}, \gamma_{2,n_2}) \tag{3.75}$$

so that their leading and trailing coefficients factorize as

$$\gamma_{i,0} = \frac{\gamma_{i,0}}{c}c \quad i = 1, 2 \tag{3.76}$$

$$\gamma_{i,n_i} = \frac{\gamma_{i,n_i}}{d}d \quad i = 1, 2 \tag{3.77}$$

Definition 3.6.1 We define the *minimal* polynomials of type $S_l(g_1, g_2)$ and $S_t(g_1, g_2)$ as

$$S_l^{\min}(g_1, g_2) = \frac{\gamma_{2,0}}{c}g_1 - \frac{\gamma_{1,0}}{c}t^{a_1-a_2}g_2 \tag{3.78}$$

$$S_t^{\min}(g_1, g_2) = \frac{\gamma_{2,n_2}}{d}g_1 - \frac{\gamma_{1,n_1}}{d}t^{a_1-n_1-(a_2-n_2)}g_2 \tag{3.79}$$

Definition 3.6.2 Given a subset H of Λ we define the sets $\mathcal{S}'_l(H)$ and $\mathcal{S}'_t(H)$ as

$$\mathcal{S}'_l(H) = \left\{S_l^{\min}(h_1, h_2) | h_1, h_2 \in H\right\} \tag{3.80}$$

$$\mathcal{S}'_t(H) = \left\{S_t^{\min}(h_1, h_2) | h_1, h_2 \in H\right\} \tag{3.81}$$

Similarly we define the set map $\mathcal{S}' : \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{G})$ as

$$\mathcal{S}'(H) = \mathcal{S}'_l(H) \cup \mathcal{S}'_t(H) \cup H \tag{3.82}$$

Now we can define the second algorithm. The map $\mathcal{S}'$ simply takes the place of the map $\mathcal{S}$.

Lemma 3.6.3 Lemma 3.5.11 also holds for algorithm 2. That is, for each polynomial b in a Châtelet basis $\mathcal{B}$ for $\mathcal{G}$ there exists an $N \in \mathbb{N}$ such that for all $m \geq N$ there is a polynomial

$$b' \in \mathrm{GCDset}_l(\mathcal{S}'^m(G)) \tag{3.83}$$

that has the same length and leading coefficient as b.

$n \leftarrow 0$

$H \leftarrow G$

while $1 = 1$ **do**

  $H \leftarrow \mathcal{S}'^n(H)$

  $B \leftarrow \mathrm{GCDset}_l(H)$

  $n \leftarrow n + 1$

end while

Algorithm 2: Calculate a Châtelet basis for $\mathcal{G} = \langle G \rangle_\Lambda$

The proof of lemma 3.6.3 is similar to that of lemma 3.5.11. First we will need some lemmas showing how the general S_l and S_t polynomials can be expressed in terms of the minimum pairwise ones.

Lemma 3.6.4 If f is a polynomial of type $S_l(g_1, g_2)$ then f is a monomial multiple of $S_l^{\min}(g_1, g_2)$ and if f is a polynomial of type $S_t(g_1, g_2)$ then f is a monomial multiple of $S_t^{\min}(g_1, g_2)$.

Proof: Suppose f is of type $S_l(g_1, g_2)$, so that f can be written as

$$f(t) = t^a(ug_1 + vt^{a_1 - a_2}g_2) \tag{3.84}$$

for some integers a, u and v. Now since f is of type $S_l(g_1, g_2)$ we must have

$$u\gamma_{1,0} + v\gamma_{2,0} = 0 \tag{3.85}$$

$$\implies u\frac{\gamma_{1,0}}{c} + v\frac{\gamma_{2,0}}{c} = 0 \tag{3.86}$$

But since $\frac{\gamma_{1,0}}{c}$ and $\frac{\gamma_{2,0}}{c}$ are co-prime $\frac{\gamma_{1,0}}{c}$ must divide v and $\frac{\gamma_{2,0}}{c}$ must divide u. Suppose u and v factorize as

$$u = u'\frac{\gamma_{2,0}}{c} \tag{3.87}$$

$$v = v'\frac{\gamma_{1,0}}{c} \tag{3.88}$$

for some integers u' and v'. So we can rewrite equation 3.86 as

$$u'\frac{\gamma_{2,0}}{c}\frac{\gamma_{1,0}}{c} + v'\frac{\gamma_{1,0}}{c}\frac{\gamma_{2,0}}{c} = 0 \tag{3.89}$$

Which implies that

$$u' = -v' \tag{3.90}$$

So from equation 3.84 we see that

$$f(t) = t^a(u'\frac{\gamma_{2,0}}{c}g_1 - u'\frac{\gamma_{1,0}}{c}t^{a_1-a_2}g_2) \tag{3.91}$$

$$= u't^a S_l^{\min}(g_1, g_2) \tag{3.92}$$

On the other hand, suppose f is of type $S_t(g_1, g_2)$, so that f can be written as

$$f(t) = t^a(ug_1 + vt^{a_{n_1}-a_{n_2}}g_2) \tag{3.93}$$

for some integers a, u and v. Now since f is of type $S_t(g_1, g_2)$ we must have

$$u\gamma_{1,n_1} + v\gamma_{2,n_2} = 0 \tag{3.94}$$

$$\Longrightarrow u\frac{\gamma_{1,n_1}}{d} + v\frac{\gamma_{2,n_2}}{d} = 0 \tag{3.95}$$

But since $\frac{\gamma_{1,n_1}}{d}$ and $\frac{\gamma_{2,n_2}}{d}$ are co-prime $\frac{\gamma_{1,n_1}}{d}$ must divide v and $\frac{\gamma_{2,n_2}}{d}$ must divide u. Suppose u and v factorize as

$$u = u'\frac{\gamma_{2,n_2}}{d} \tag{3.96}$$

$$v = v'\frac{\gamma_{1,n_1}}{d} \tag{3.97}$$

for some integers u' and v'. So we can rewrite equation 3.95 as

$$u'\frac{\gamma_{2,n_2}}{d}\frac{\gamma_{1,n_1}}{d} + v'\frac{\gamma_{1,n_1}}{d}\frac{\gamma_{2,n_2}}{d} = 0 \tag{3.98}$$

Which implies that

$$u' = -v' \tag{3.99}$$

So from equation 3.93 we see that

$$f(t) = t^a(u'\frac{\gamma_{2,n_2}}{d}g_1 - u'\frac{\gamma_{1,n_1}}{d}t^{a_{n_1}-a_{n_2}}g_2) \tag{3.100}$$

$$= u't^a S_t^{\min}(g_1, g_2) \tag{3.101}$$

■

Throughout the proofs of the next lemmas we shall use the hat notation to denote omission from products and sequences, i.e.

$$c_1c_2\ldots\hat{c_i}\ldots c_n := c_1c_2\ldots c_{i-1}c_{i+1}\ldots c_n \tag{3.102}$$

$$(c_1, c_2, \ldots, \hat{c_i}, \ldots, c_n) := (c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \tag{3.103}$$

Lemma 3.6.5 If f is any polynomial of type $\mathrm{GCD}_l(g_2, g_3, \ldots, g_n)$ then $S_l^{\min}(g_1, f)$ is a linear combination of the polynomials in

$$\left\{S_l^{\min}(g_1, g_i) : 2 \leq i \leq n\right\} \tag{3.104}$$

Similarly if f is any polynomial of type $\mathrm{GCD}_t(g_2, g_3, \ldots, g_n)$ then $S_t^{\min}(g_1, f)$ is a linear combination of the polynomials in

$$\left\{S_t^{\min}(g_1, g_i) : 2 \leq i \leq n\right\} \tag{3.105}$$

Proof: Recall the notation from 3.3 for the polynomials $g_1, \ldots, g_m$

$$g_i(t) = \gamma_{i,0} t^{a_i} + \gamma_{i,1} t^{a_i - 1} + \cdots + \gamma_{i,n_i} t^{a_i - n_i} \tag{3.106}$$

The leading coefficients of the g_i will factorize as follows

$$\gamma_{i,0} = \gamma'_{i,0} c_1 c_2 \ldots \hat{c}_i \ldots c_n c \quad , i = 1, \ldots, n \tag{3.107}$$

where

$$c = \mathrm{gcd}(\gamma_{1,0}, \gamma_{2,0}, .., \gamma_{n,0}) \tag{3.108}$$

$$c c_i = \mathrm{gcd}(\gamma_{1,0}, \gamma_{2,0}, .., \hat{\gamma}_{i,0}, \ldots, \gamma_{n,0}) \tag{3.109}$$

These factors satisfy the following

$$\mathrm{gcd}(c_i, c_j) = 1 \qquad 1 \leq i, j \leq n, \quad i \neq j \tag{3.110}$$

$$\mathrm{gcd}(\gamma'_{i,0}, c_i) = 1 \qquad 1 \leq i \leq n \tag{3.111}$$

Let f be a polynomial of type $\mathrm{GCD}_l(g_2, g_3, \ldots, g_m)$

$$f = b_2 t^{v_2} g_2 + b_3 t^{v_3} g_3 + \cdots + b_m t^{v_m} g_m \tag{3.112}$$

From 3.109 we have

$$c c_1 = \mathrm{gcd}(\gamma_{2,0}, \gamma_{3,0}, \ldots, \gamma_{m,0}) \tag{3.113}$$

so

$$\mathrm{lcoeff}(f) = b_2 \gamma_{2,0} + b_3 \gamma_{3,0} + \cdots + b_m \gamma_{m,0} = c c_1 \tag{3.114}$$

Now $\gcd(\text{lcoeff}(g_1), \text{lcoeff}(f)) = c$ and $\text{ldeg}(f) = v_2 + a_2$, hence

$$S_l^{\min}(g_1, f) = c_1 g_1 - \frac{\gamma_{1,0}}{c} t^{a_1 - v_2 - a_2} f \tag{3.115}$$

$$= c_1 g_1 - \frac{\gamma_{1,0}}{c} t^{a_1 - v_2 - a_2} \sum_{i=2}^{m} b_i t^{v_i} g_i \tag{3.116}$$

$$= \sum_{i=2}^{m} b_i \frac{\gamma_{i,0}}{c} g_1 - \frac{\gamma_{1,0}}{c} t^{a_1 - v_2 - a_2} \sum_{i=2}^{m} b_i t^{v_i} g_i \quad \text{from (3.114)} \tag{3.117}$$

$$= \sum_{i=2}^{m} b_i \frac{\gamma_{i,0}}{c} g_1 - \frac{\gamma_{1,0}}{c} t^{a_1 - v_2 - a_2} \sum_{i=2}^{m} b_i t^{v_i} g_i \tag{3.118}$$

$$= \sum_{i=2}^{m} b_i \left(\frac{\gamma_{i,0}}{c} g_1 - \frac{\gamma_{1,0}}{c} t^{a_1 - v_2 - a_2 + v_i} g_i \right) \tag{3.119}$$

$$= \sum_{i=2}^{m} b_i \left(\gamma'_{i,0} c_1 c_2 \ldots \hat{c}_i \ldots c_m g_1 - \gamma'_{1,0} c_2 c_3 \ldots c_m t^{a_1 - a_i} g_i \right) \tag{3.120}$$

$$= \sum_{i=2}^{m} b_i c_2 c_3 \ldots \hat{c}_i \ldots c_m \left(\gamma'_{i,0} c_1 g_1 - \gamma'_{1,0} c_i t^{a_1 - a_i} g_i \right) \tag{3.121}$$

$$= \sum_{i=2}^{m} b_i c_2 c_3 \ldots \hat{c}_i \ldots c_m S_l^{\min}(g_1, g_i) \tag{3.122}$$

Similarly if f is a polynomial of type $\text{GCD}_t(g_2, g_3, \ldots, g_m)$ then a similar proof focusing on trailing coefficients shows that

$$S_t^{\min}(g_1, f) = \sum_{i=2}^{m} b_i c_2 c_3 \ldots \hat{c}_i \ldots c_m S_t^{\min}(g_1, g_i) \tag{3.123}$$

■

Lemma 3.6.6 Any polynomial of type $S_l(g_1, \ldots, g_m)$ is a monomial linear combination of the polynomials

$$\left\{ S_l^{\min}(g_i, g_j) : 1 \le i, j \le m, \quad i \ne j \right\} \tag{3.124}$$

and any polynomial of type $S_t(g_1, \ldots, g_m)$ is a monomial linear combination of the polynomials

$$\left\{ S_t^{\min}(g_i, g_j) : 1 \le i, j \le m, \quad i \ne j \right\} \tag{3.125}$$

Proof: We shall proceed by induction on m. The case $m = 2$ is true by lemma 3.6.4. So we assume that the result is true for $m = k$ and let f be a polynomial of type $S_l(g_1, g_2, \ldots, g_{k+1})$. So f can be written as

$$f = u_1 t^{v_1} g_1 + u_2 t^{v_2} g_2 + \cdots + u_{k+1} t^{v_{k+1}} g_{k+1} \tag{3.126}$$

for some integers u_i and v_i and since f is of type $S_l(g_1, g_2, \ldots, g_{k+1})$ we have that

$$\sum_{i=1}^{k+1} u_i \gamma_{i,0} = 0 \tag{3.127}$$

Hence

$$\sum_{i=1}^{k+1} u_i \gamma'_{i,0} c_1 c_2 \ldots \hat{c_i} \ldots c_{k+1} c = 0 \quad \text{from } 3.107 \tag{3.128}$$

Let $1 \leq j \leq k+1$. Now since cc_j divides 0 and cc_j divides the i^{th} term of the sum in equation (3.128) for each $i \neq j$ it must be that cc_j divides the j^{th} term of the sum also. Considering equations 3.110 and 3.111 we see that c_j must divide u_j. So we can factorize the u_i as

$$u_i = u'_i c_i \quad 1 \leq i \leq k+1 \tag{3.129}$$

for some integers u'_i.

Let h be any polynomial of type $\mathrm{GCD}_l(g_2, g_3, \ldots, g_{k+1})$. By multiplying h by a power of t we can assume that $\mathrm{ldeg}(h) = a_1$. Consider the polynomial

$$S = f - u'_1 t^{v_1} S_l^{\min}(g_1, h) \tag{3.130}$$

$$= f - u'_1 t^{v_1} (c_1 g_1 - \frac{\gamma_{1,0}}{c} h) \tag{3.131}$$

This has the effect of cancelling the contribution of g_1 to f. So S is a polynomial of type $S_l(g_2, \ldots, g_{k+1})$ and hence by assumption a linear combination of polynomials in

$$\{S_l^{\min}(g_i, g_j) : 2 \leq i, j \leq k+1, \quad i \neq j\} \tag{3.132}$$

And by lemma 3.6.5 $S_l^{\min}(g_1, h)$ is a linear combination of polynomials in

$$\{S_l^{\min}(g_1, g_j) : 2 \leq j \leq k+1, \} \tag{3.133}$$

Therefore since $f = S + u'_1 t^{v_1} S_l^{\min}(g_1, h)$, f is a linear combination of polynomials in

$$\{S_l^{\min}(g_i, g_j) : 1 \leq i, j \leq k+1, \quad i \neq j\} \tag{3.134}$$

■

Proof of lemma 3.6.3. The proof is based on the proof of lemma 3.5.11. The difference is how we construct the diagram D' from D. So assume that we have a diagram D for b (as in figure 3.3) satisfying the properties

1. There exists $n(D) \in \mathbb{N}$ such that all the polynomials appearing in D are monomial multiples of elements of $\mathcal{S}'^m(G)$ for all $m \geq n(D)$.

2. The maximum leading degree, u, of the diagram satisfies $u \geq a$

If $u > a$ then as before we let $P_u(D) \subseteq P(D)$ be the set of all the polynomials in D with leading degree u. Since $u > a$ then there must be cancellation of the coefficients at level u, i.e.

$$\sum_{p \in P_u(D)} p \in \mathcal{S}_l(\mathcal{S}'^{n(D)}(G)) \tag{3.135}$$

But by lemma 3.6.6 $\sum_{p \in P_u(D)} p$ is a monomial linear combination of the polynomials in

$$\mathcal{S}'_l(\mathcal{S}'^{n(D)}(G)) = \left\{S_l^{\min}(h_1, h_2) | h_1, h_2 \in \mathcal{S}'^{n(D)}(G)\right\} \tag{3.136}$$

We can write this monomial linear combination as

$$\sum_{p \in P_u(D)} p = \sum_{h \in H} m_h h \tag{3.137}$$

for some subset $H \subseteq \mathcal{S}'_l(\mathcal{S}'^{n(D)}(G)) \subset \mathcal{S}'(\mathcal{S}'^{n(D)}(G))$, where the m_h are monomials. Then the diagram D' for b consisting of the polynomials

$$P(D') = (P(D) - P_u(D)) \cup \{m_h h | h \in H\} \tag{3.138}$$

will also satisfy properties 1 and 2 with $n(D') = n(D) + 1$ and the maximum leading degree, u' of D' satisfying $u > u'$.

As in the proof of lemma 3.5.11 we repeat this procedure to arrive at a diagram D_l for b satisfying property 1 and with the maximum leading coefficient of D_l equal to a the maximum leading coefficient of b. Then turning to the trailing degree end of D_l we construct new diagrams D'_l in a similar way to that described above until we arrive at a diagram D_t as in figure 3.5.

The remainder of the proof now proceeds exactly as in lemma 3.5.11. ■

Theorem 3.6.7 Hence there exists an $M \in \mathbb{N}$ such that for all $m \geq M$, $\text{GCDset}_l(\mathcal{S}'^m(G))$ is a Châtelet basis for $\mathcal{G}$.

Proof: The proof is exactly the same as the proof of theorem 3.5.12 ■

So algorithm 2 also succeeds in constructing a Châtelet basis for $\mathcal{G}$ and it is an implementable algorithm, in that if G is a finite set then all the iterates of the $\mathcal{S}'$ map are also finite. However at any point during the running of the algorithm we have no way of telling whether a Châtelet basis has yet been reached. The third and final algorithm, described in the next section, addresses this.

3.7 The Third Algorithm

For the third algorithm we once again modify the map $\mathcal{S}'$. The idea is that of all the extra polynomials created by the $\mathcal{S}'$ map only some of them will actually play a role in improving the approximation B to a Châtelet basis $\mathcal{B}$. We start by formalising the division process described in theorem 3.3.6.

Definition 3.7.1 Let B be a finite set of polynomials in Λ. Let f be a polynomial in Λ with $\text{lcoeff}(f) = \phi$, $\text{ldeg}(f) = a_f$ and $\text{length}(f) = n_f$. We say that f *reduces to* f' *with respect to* B if there exists a polynomial $b \in B$ with $\text{lcoeff}(b) = \beta$, $\text{ldeg}(b) = a_b$ and $\text{length}(b) = n_b$ such that $n_b \leq n_f$, $\phi = q\beta$ for some $q \in \mathbb{Z}$ and

$$f' = f - qt^{a_f - a_b}b \tag{3.139}$$

We say that f *is minimal with respect to* B if it cannot be reduced, i.e. if $f = 0$ or if there is no $b \in B$ with $\text{length}(b) \leq \text{length}(f)$ and $\text{lcoeff}(b)$ dividing $\text{lcoeff}(f)$. We use the notation $f \longrightarrow_B f'$ to indicate that f reduces to f' with respect to B and $f \longrightarrow_B^+ f'$ to indicate that f' is obtained from f by a chain of reductions with respect to B.

Of course, if $\mathcal{B}$ is a Châtelet basis for the ideal $\mathcal{G}$ then every polynomial $f \in \mathcal{G}$ satisfies $f \longrightarrow_{\mathcal{B}}^+ 0$, see lemma 3.3.5.

We can now give the definition of algorithm 3. First we establish that algorithm 3 actually terminates after a finite number of steps.

Lemma 3.7.2 Algorithm 3 terminates.

$B \leftarrow \mathrm{GCDset}_l(G)$

$F \leftarrow G \cup B$

$S \leftarrow \mathcal{S}'_l(F) \cup \mathcal{S}'_t(F)$

while $S \neq \{\}$ **do**

  choose any $f \in S$

  $S \leftarrow S - \{f\}$

  $f \longrightarrow^+_B f'$, where f' is minimal w.r.t. B

  if $f' \neq 0$ **then**

    $B \leftarrow \mathrm{GCDset}_l(B \cup \{f'\})$

    $S \leftarrow S \cup \left\{S_l^{\min}(b, f),\ S_l^{\min}(f', f),\ S_l^{\min}(b, f') | b \in B,\ f \in F\right\} \cup$
    $\left\{S_t^{\min}(b, f),\ S_t^{\min}(f', f),\ S_t^{\min}(b, f') | b \in B,\ f \in F\right\}$

    $F \leftarrow F \cup B \cup \{f'\}$

  end if

end while

Algorithm 3: Calculate a Châtelet basis for $\mathcal{G} = \langle G \rangle_\Lambda$

Proof: It is not immediately apparent why it should stop since the algorithm terminates if and when the set S is empty. Now every run through the **while** loop results in a polynomial being subtracted from S. However, each time the **if** loop is activated polynomials are added to S. So if the **if** loop is activated often enough then the algorithm might never terminate. However we will show below, using the ascending chain condition for ideals in $\mathbb{Z}$, that the **if** loop can only be activated a finite number of times. And therefore the set S can only increase in size a finite number of times, and after it has done so for the last time the **while** loop will simply repeat until S is empty and the algorithm terminates.

Consider the set B during the running of the algorithm. Recall from (3.60) page 68, the definition of the numbers $c_n(B)$, and consider the $\mathbb{Z}$-ideals that they generate. These satisfy

$$\{0\} \subseteq \langle c_1(B) \rangle_{\mathbb{Z}} \subseteq \langle c_2(B) \rangle_{\mathbb{Z}} \subseteq \dots \tag{3.140}$$

When the **if** loop executes the set B is recalculated to a new value which we denote as B'. Now $B' = \mathrm{GCDset}_l(B \cup \{f'\})$ which implies that

$$\langle c_n(B)\rangle_{\mathbb{Z}} \subseteq \langle c_n(B')\rangle_{\mathbb{Z}} \quad \text{for all } n \tag{3.141}$$

But f' is a polynomial that is minimal with respect to B. That means that the inclusion is strict for $n' = \mathrm{length}(f') + 1$ (and maybe for some other $n > n'$)

$$\langle c_{n'}(B)\rangle_{\mathbb{Z}} \subsetneq \langle c_{n'}(B')\rangle_{\mathbb{Z}} \tag{3.142}$$

Let L be the maximum of the lengths of polynomials in G, i.e.

$$L = \max\{\mathrm{length}(g) | g \in G\} \tag{3.143}$$

Now the f' that are involved in the recalculation of B all satisfy $\mathrm{length}(f') \leq L$. Hence every time the **if** loop executes at least one of the ideals $\langle c_n(B)\rangle_{\mathbb{Z}}$ with $1 \leq n \leq L+1$ gets strictly larger, and by the ascending chain condition for ideals in $\mathbb{Z}$ the **if** loop executes only a finite number of times. ■

The relationship between the sets F and B in algorithm 3 is not as clear as in the previous algorithms. But the following is still true.

Lemma 3.7.3 Before the first and after every execution of the **while** loop, $B = \mathrm{GCDset}_l(F)$.

Proof: Before the first execution of the **while** loop $B = \mathrm{GCDset}_l(G)$ and $F = G \cup B$. Let $B = \{b_0, \ldots, b_r\}$ where the polynomials are arranged in order of decreasing length and increasing leading coefficient. Let the lengths of these polynomials be $\mathrm{length}(b_i) = n_i$, for $0 \leq i \leq r$. Consider the numbers $c_n(G)$ and $c_n(G \cup B)$ defined on page 68. From the definition of GCDset_l we see that $c_n(G \cup B)$ divides $c_n(G)$ for all $n \geq 1$. Moreover the subsequence $c_{n_j}(G \cup B)$, $j = 0, \ldots, s$ that satisfies properties 1 and 2 on page 68 is given by $n_j = \mathrm{length}(b_j)$ for $j = 1, \ldots, s = r$. Hence when forming $\mathrm{GCDset}_l(G \cup B)$ with the f_i of equation (3.61) we can take f_i to be a polynomial of type $\mathrm{GCD}_l(b_i)$. So we can take $f_i = b_i$ and hence $B = \mathrm{GCDset}_l(F)$.

So assume that before an execution of the **while** $B = \mathrm{GCDset}_l(F)$. Let B^* and F^* denote the values of B and F after the execution of the **while** loop. Now if the **if** loop did not execute then $B^* = B$ and $F^* = F$ so there is nothing to prove. However if the **if** loop does execute then

$$B^* = \mathrm{GCDset}_l(B \cup \{f'\}) \tag{3.144}$$

$$F^* = F \cup B^* \cup \{f'\} \tag{3.145}$$

where f' is minimal with respect to B. By considering the numbers $c_n(F \cup \{f'\})$ and $c_n(B \cup \{f'\})$ it is easy to see that

$$B^* = \mathrm{GCDset}_l(F \cup \{f'\}) \tag{3.146}$$

and then applying the reasoning of the first paragraph of this proof establishes that

$$B^* = \mathrm{GCDset}_l(F \cup B^* \cup \{f'\}) \tag{3.147}$$

Therefore after an execution of the **while** loop the statement $B = \mathrm{GCDset}_l(F)$ remains true. ■

Now we can proceed to proving that algorithm 3 also succeeds in producing a Châtelet basis. Let B_{term} and F_{term} to denote the values of B and F upon termination of algorithm 3.

Lemma 3.7.4 For each polynomial b in a Châtelet basis for $\mathcal{G}$ there exists a polynomial $b' \in B_{term}$ having the same length and leading coefficient as b.

Proof: Note that $F \subset F_{term}$ is always true throughout the running of the algorithm.

Let $b \in \mathcal{B}$ be one of the polynomials in a Châtelet basis for $\mathcal{G}$. Let D be the initial diagram constructed for b in the proof of lemma 3.5.11. That is D is a diagram for b satisfying the following properties.

1. Each polynomial $p \in P(D)$ is a monomial multiple of an element of F_{term}.

2. The maximum leading degree u of D satisfies $u \geq a$, where a is the leading degree of b.

As in the proof of lemma 3.5.11 we form the set $P_u(D)$ of polynomials in D with leading degree u. Recall from (3.137) that

$$\sum_{p \in P_u(D)} p = \sum_{h \in H} m_h h \tag{3.148}$$

where H is a subset of $\mathcal{S}_l'(F_{term})$. Now each element of $\mathcal{S}_l'(F_{term})$ will have been an element of the set S at some point in the running of the algorithm and so will have been reduced with respect to B at some point, (the same polynomial may have been reduced and then added to S again at a later point). So for each $h \in H$ let a reduction calculated for it in the algorithm be

$$h = \sum_{i=0}^{r^h} e_i^h b_i^h + h' \tag{3.149}$$

Note that the superscript h on the b_i is needed as the set B may have had different values when the various elements of H were reduced.

For each $h \in H$ and $0 \leq i \leq m$ we can write the polynomial e_i^h as

$$e_i^h = \sum_{j=0}^{n_i^h} \epsilon_{i,j}^h t^{a_i^h - j} \tag{3.150}$$

where $\epsilon_{i,j}^h, a_i^h \in \mathbb{Z}$ and $n_i^h \in \mathbb{N}$.

So we can write $\sum_{p \in P_u(D)} p$ as

$$\sum_{p \in P_u(D)} p = \left(\sum_{h \in H} \sum_{i=0}^{r^h} \sum_{j=0}^{n_i^h} m_h \epsilon_{i,j}^h t^{a_i^h - j} b_i^h \right) + h' \tag{3.151}$$

which is an expression for $\sum_{p \in P_u(D)} p$ as a monomial linear combination of elements of F_{term}. Now we can construct the next diagram D' for b. The polynomials $P(D')$ in D are

$$\begin{aligned} P(D') = {} & (P(D) - P_u(D)) \cup \\ & \left\{ m_h \epsilon_{i,j}^h t^{a_i^h - j} b_i^h | h \in H,\, 0 \leq i \leq r^h,\, 0 \leq j \leq n_i^h \right\} \cup \{f'\} \end{aligned} \tag{3.152}$$

Of course all these polynomials added to $P(D) - P_u(D)$ have leading degree strictly less than u. So the new diagram D' for b does indeed satisfy properties 1 and 2 with the maximum leading degree, u', of D' satisfying $u > u'$.

As before, we continue constructing new diagrams for b until we arrive at a diagram D_l satisfying property 1 and whose maximum leading degree is equal to a the leading degree of b. The diagram D_l will look something like that in figure 3.4 on page 72. Then, focusing on the trailing degree end of D_l we see that D_l satisfies properties 1 (from above) and

2'. The maximum leading degree of D_l is equal to a, the leading degree of b.

3. The minimum trailing degree, u, of D_l satisfies $a - n \geq u$.

If $a-n > u$ then we can construct a new diagram D_l' for b using the technique above. This new diagram will also satisfy properties 1, 2' and 3 however the minimum trailing degree, u' of D_l' will also satisfy $u' > u$. We continue in this way until we arrive at a diagram D_t for b satisfying property 1 and with maximum trailing degree and minimum trailing degree equal to the leading and trailing degree of b respectively. Then the argument at the end of lemma 3.5.11 is used to show that there is a polynomial $b' \in B_{term} = \text{GCDset}_l(F_{term})$ such that b' has the same leading coefficient and length as b. ■

Theorem 3.7.5 B_{term} is a Châtelet basis for the ideal $\mathcal{G}$.

Proof: By lemma 3.7.4 B_{term} contains a Châtelet basis for $\mathcal{G}$ and the argument in the proof of theorem 3.5.12 shows that B_{term} is a Châtelet basis for $\mathcal{G}$. ■

Chapter 4

A Conjecture on the Châtelet Bases of Alexander Ideals

We know from [22] that a knot ideal $\mathcal{I}$ is characterised by the following two conditions

1. $\overline{\mathcal{I}} = \mathcal{I}$

2. $\epsilon(\mathcal{I}) = \mathbb{Z}$

where is the linear extension to Λ of the conjugation map $\bar{t} = t^{-1}$, and ϵ is the augmentation map that evaluates a polynomial at 1.

We might conjecture the following statement

Conjecture. If an ideal $\mathcal{G}$ of Λ satisfies the two conditions above then $\mathcal{G}$ has a Châtelet basis consisting of symmetric polynomials, i.e. $\mathcal{G}$ has a Châtelet basis $\mathcal{B}$ for which each $b \in \mathcal{B}$ satisfies

$$b = t^{2a-n}\bar{b} \tag{4.1}$$

where a is the leading degree of b and n the length of b.

We have not been able to find a counter example to this among the Alexander ideals of prime knots of up to 14 crossings. As a step toward this conjecture we can prove the following result.

Lemma 4.0.6 Let $\mathcal{B} = \{b_0, \ldots, b_m\}$ be a Châtelet basis for the ideal $\mathcal{G}$ of Λ, with the polynomials arranged in order of decreasing length (i.e. b_0 is the longest and

b_m the shortest). If $\mathcal{G}$ satisfies conditions 1 and 2 above then the polynomial b_m is symmetric, i.e. it will satisfy

$$b_m = t^{2a_m - n_m}\overline{b_m} \tag{4.2}$$

where a_m and n_m are the leading degree and lengths respectively of b_m.

Proof: We will use the notation

$$b_i = \beta_{i,0}t^{a_i} + \cdots + \beta_{i,n_i}t^{a_i - n_i} \tag{4.3}$$

for the polynomials b_i, $0 \leq i \leq m$. Now each $\beta_{i,0}$ is a strict multiple of $\beta_{i-1,0}$ for each $1 \leq i \leq m$. Let $\beta_{i,0} = p_{i-1}\beta_{i-1,0}$, $1 \leq i \leq m$.

By condition 1 we know that $\overline{b_m} \in \mathcal{G}$. Hence by theorem 3.3.6

$$\overline{b_m} \longrightarrow^{+}_{\mathcal{B}} 0 \tag{4.4}$$

But since length$(\overline{b_m})$ = length(b_m) this means that

$$\overline{b_m} = qt^{n_m - 2a_m}b_m \tag{4.5}$$

for some $q \in \mathbb{Z}$, which implies that

$$b_m = q^2 b_m \tag{4.6}$$

Hence $q^2 = 1$ so q must equal 1 or -1, i.e. b_m must be symmetric or anti-symmetric. Let us assume that $q = -1$ so that $b_m = -t^{2a_m - n_m}\overline{b_m}$. A consequence of this is that $\epsilon(b_m) = 0$. We shall show that this is false which will prove the lemma.

Now by condition 2, $\epsilon(\mathcal{G}) = \mathbb{Z}$. Since $\mathcal{B}$ is a basis for $\mathcal{G}$, $\epsilon(\mathcal{G}) = \langle\epsilon(\mathcal{B})\rangle_{\mathbb{Z}}$. So it is not true that $\epsilon(b_i) = 0$ for $0 \leq i \leq m$. Consider the polynomial $\mathcal{S}_l^{\min}(b_0, b_1) \in \mathcal{G}$. This has length strictly less than n_0 so reducing it with respect to $\mathcal{B}$ must lead to an equation

$$\mathcal{S}_l^{\min}(b_0, b_1) = p_0 b_0 - t^{a_0 - a_1}b_1 = \sum_{i=1}^{m} e_i b_i \tag{4.7}$$

for some polynomials $e_i \in \Lambda$. If $\epsilon(b_i) = 0$ for each $1 \leq i \leq m$, then from equation (4.7), $\epsilon(b_0) = 0$ also. This cannot be so we conclude that it is not the case that $\epsilon(b_i) = 0$ for each $1 \leq i \leq m$. Then in the same way we consider the reduction of $\mathcal{S}_l^{\min}(b_1, b_2)$ giving

$$\mathcal{S}_l^{\min}(b_1, b_2) = p_1 b_1 - t^{a_1 - a_2}b_2 = \sum_{i=2}^{m} f_i b_i \tag{4.8}$$

for some polynomials $f_i \in \Lambda$. From this we conclude that it is not true that $\epsilon(b_i) = 0$ for each $2 \leq i \leq m$. Continuing in this way we can conclude that $\epsilon(b_m) \neq 0$ contradicting the assumption made earlier. Hence the polynomial b_m is symmetric. ■

Chapter 5

Some results on the Alexander ideals of prime knots

5.1 Châtelet Bases for the Alexander Ideals

The algorithms described in chapters 2 and 3 were used to calculate Châtelet bases for Alexander ideals of all prime knots of up to 14 crossings. These ideals are included on the CD-ROM forming part of appendix B. This was done using the computer packages *Maple* and *Knotscape*, see appendix B for more details.

In appendix A we have reproduced the Alexander ideals for all prime knots of up to 12 crossings with length of the chain of Alexander ideals greater than or equal to 3, and the Alexander ideals for the 13 and 14 crossing prime knots with length of the chain of Alexander ideals equal to 3. For an explanation of the format of these tables we refer to the beginning of appendix B.

5.2 Some Statistics and the Number of Distinct Ideals

In table 5.1 we can see how many prime knots there are of crossing number up to 14 and how many have chains of Alexander ideals of length 1,2 or 3. The last column of the table shows for each crossing number the proportion of knots having

Crossing no.	# of knots	# chain length 1	# chain length 2	# chain length 3	approx.	proportions	
3	1	1	0	0	1.000	0.000	0.000
4	1	1	0	0	1.000	0.000	0.000
5	2	2	0	0	1.000	0.000	0.000
6	3	3	0	0	1.000	0.000	0.000
7	7	7	0	0	1.000	0.000	0.000
8	21	20	1	0	0.952	0.048	0.000
9	49	41	8	0	0.837	0.163	0.000
10	165	148	17	0	0.897	0.103	0.000
11	552	491	61	0	0.889	0.111	0.000
12	2176	1896	273	7	0.871	0.125	0.003
13	9988	8968	1001	19	0.898	0.100	0.001
14	46972	41823	5032	117	0.890	0.107	0.002

Table 5.1: The number of prime knots of various chain length.

chain length 1,2 or 3. We could remark that these proportions seem to be roughly conserved as the crossing number increases.

It is also interesting to see how many distinct ideals arise among the Alexander ideals of these knots. As mentioned in chapter 3 when we have Châtelet bases for two ideals we can decide whether the two ideals are the same or not. Doing this for the prime knots of up to 14 crossings yield the following numbers

- ⋆ The are 20196 distinct Alexander polynomials, i.e. 1^{st} Alexander ideals.
- ⋆ There are 155 distinct 2^{nd} Alexander ideals.
- ⋆ There are 5 distinct 3^{rd} Alexander ideals, all of which also arise as 2^{nd} Alexander ideals.

Interestingly all 30 of the principal 2^{nd} Alexander ideals and both of the two principal 3^{rd} Alexander ideals appear as 1^{st} Alexander ideals.

In tables 5.2, 5.3 and 5.4 we give Châtelet bases for the 155 distinct ideals arising as the 2^{nd} Alexander ideals of prime knots of up to 14 crossings.

Châtelet bases for the five ideals arising as the 3^{rd} Alexander ideals of prime knots of up to 14 crossings are $[1]$, $[3, t-2]$, $[2, t^2-t+1]$, $[t^2-t+1]$ and $[5, t-4]$.

5.2.1 Nakanishi Index

The Nakanishi Index of an Alexander module is defined as the smallest n for which the module is presented by an $n \times n$ matrix. From the definition of Alexander ideals we see that the length of the chain of Alexander ideals is a lower bound on the Nakanishi index. In Kawauchi's book [21] there are presentation matrices given for the prime knots of up to 10 crossings for which the Nakanishi index is greater than 1. These were take from Nakanishi's Masters thesis. However there is an omission from this table as the 42^{nd} alternating 10 crossing knot from *Knotscape's* table (= 65^{th} alternating 10 crossing knot in Kawauchi's table) is not included in the list and has non-trivial second Alexander ideal, a Châtelet basis for it being $\{2, t^2-t+1\}$. So it's Nakanishi index must be ≥ 2.

$[1]$	$[t^2-t+1]$	$[3,t-2]$
$[7,t+1]$	$[t^2-3t+1]$	$[5,t-4]$
$[2,t^2-t+1]$	$[3,t^2-2]$	$[t^4-2t^3+3t^2-2t+1]$
$[t^4-3t^3+3t^2-3t+1]$	$[2,t^4+t^2+1]$	$[11,t+1]$
$[3,t^2-t+1]$	$[5,t^2+2t-4]$	$[4,t^2-3t+1]$
$[9,3t+3,t^2-t+4]$	$[t^4-4t^3+5t^2-4t+1]$	$[15,t^2-6t+1]$
$[9,3t-6,t^2-t-2]$	$[3t^2-3t+3,t^3+1]$	$[3,t^3+1]$
$[13,t-12]$	$[2,t^4-t^3-t^2-t+1]$	$[7,t^2+2t+1]$
$[5,t^3+t^2-4t+1]$	$[t^4-5t^3+7t^2-5t+1]$	$[9,t+1]$
$[7,t^2+4t+1]$	$[8,t^2+t+1]$	$[t^4-4t^3+7t^2-4t+1]$
$[4,t^2-t+1]$	$[3t^2-7t+3]$	$[2t^2-3t+2]$
$[2t^2-5t+2]$	$[5,t^2+t-4]$	$[t^4-2t^3+t^2-2t+1]$
$[3,t^3+t^2-2t-2]$	$[3,t^4-t^3-2t^2+2t+1]$	$[3t^2-5t+3]$
$[4t^2-4t+4,t^4-3t^2+4t-3]$	$[11,t^2+4t-10]$	$[3,t^4-t^3-t^2-t+1]$
$[17,t+1]$	$[7,t^2+1]$	$[5,t^2-4]$
$[9,t^2+5t+1]$	$[7,t^2-6t-6]$	$[7,t^2+3t-6]$
$[5,t^2+4t-4]$	$[3,t^4-2t^3+t-2]$	$[15,3t+3,t^2+4]$
$[33,3t-30,t^2-23]$	$[8,t^2-5t+1]$	$[19,t+1]$
$[t^4-3t^3+5t^2-3t+1]$	$[5t^2-5t+5,t^3+1]$	$[15,5t-10,t^2-4]$
$[15,t-14]$	$[14,2t+2,t^2-t+5]$	$[13,t^2+9t+1]$
$[6,2t+2,t^4-t^3-t^2+t-3]$	$[8,t^2-3t+1]$	$[13,t^2+10t+1]$
$[16,t^2+5t+1]$	$[21,3t+3,t^2-8]$	$[15,t^2+2t+1]$
$[3t^2-3t+3,t^4-t^3+2t^2-t+1]$	$[4,2t^2-2t+2,t^4-t^2+2t-1]$	$[27,3t+3,t^2-t+7]$
$[5t^2-15t+5,t^3-2t^2-2t+1]$	$[9,t^2-6t+1]$	$[23,t+1]$
$[4,2t^2+2t-2,t^4+t^2+2t-1]$	$[11,t^2+3t+1]$	$[35,5t-30,t^2-t+26]$
$[2t^2-6t+2,t^4-t^2-18t+7]$	$[t^4-6t^3+11t^2-6t+1]$	$[t^4-6t^3+9t^2-6t+1]$
$[9,t^2-7t-8]$	$[21,7t+7,t^2+t-6]$	$[21,t+1]$
$[3,t^5-t^4-2t^3+t^2-t+1]$	$[6t^2-9t+6,2t^3-t^2-t+2]$	$[11,t^2-10t-10]$

Table 5.2: The ideals arising as 2$^{\text{nd}}$ Alexander ideals of prime knots of up to 14 crossings

$[2t^4-5t^3+7t^2-5t+2]$	$[2t^4-7t^3+9t^2-7t+2]$
$[13,t^2+7t-12]$	$[3,t^4+1]$
$[9t^2-9t+9,t^3+1]$	$[13,t^2-7t-12]$
$[9t^2-27t+9,3t^3-6t^2-6t+3,t^4-t^3+5t^2-28t+10]$	$[33,11t-22,t^2-8t-21]$
$[4t^2-4t+4,t^4+2t^3-t^2+2t+1]$	$[5t^2-5t+5,t^4-4t^3+t-4]$
$[3t^2-3t+3,t^4-2t^3+t-2]$	$[27,t-26]$
$[3,t^4-t^2+1]$	$[27,9t-18,t^2+5t+10]$
$[11,t^2+8t+1]$	$[14,t^2-13t+1]$
$[10,t^2+t+1]$	$[11,t^2+7t-10]$
$[2t^4-6t^3+9t^2-6t+2]$	$[2t^4-4t^3+5t^2-4t+2]$
$[t^4-7t^3+11t^2-7t+1]$	$[9t^2-9t+9,t^4-4t^3+5t^2-4t+1]$
$[9,t^2-3t-8]$	$[8t^2-8t+8,t^4+4t^3+5t^2-4t+9]$
$[10,2t+2,t^2-t-7]$	$[25,t-24]$
$[10,2t-8,t^4-t^3-t^2+t+5]$	$[2t^4-7t^3+11t^2-7t+2]$
$[25,5t-20,t^2-3t-4]$	$[17,t^2+t+1]$
$[3,t^4-2t^3-2t^2-2t+1]$	$[17,t^2-13t+1]$
$[11,t^2+2t+1]$	$[6,2t+2,t^2+t-3]$
$[11,t^2+6t-10]$	$[t^6-2t^5+4t^4-5t^3+4t^2-2t+1]$
$[3t^4-3t^3+3t^2-3t+3,t^5+1]$	$[t^6-2t^5+3t^4-3t^3+3t^2-2t+1]$
$[2t^4-6t^3+7t^2-6t+2]$	$[35,7t+7,t^2+t-20]$
$[6,t^2-3t+1]$	$[7t^2-7t+7,t^3+1]$
$[t^6-4t^5+6t^4-7t^3+6t^2-4t+1]$	$[t^6-3t^5+4t^4-5t^3+4t^2-3t+1]$
$[3,t^5-2t^4+t+1]$	$[t^6-3t^5+5t^4-5t^3+5t^2-3t+1]$
$[15,5t+5,t^2-3t+11]$	$[19,t^2-15t-18]$
$[t^4-t^3+t^2-t+1]$	$[4,t^4-t^3+t^2-t+1]$
$[11,t^2-6t+1]$	$[t^4-t^2+1]$
$[27,9t-18,t^2+2t+19]$	$[15,3t-12,t^2+2t+1]$

Table 5.3: The ideals arising as 2^{nd} Alexander ideals of prime knots of up to 14 crossings, contd.

$[17, t^2 + 2t + 1]$	$[9, 3t + 3, t^4 - t^3 - t^2 + 2t - 8]$
$[12, t^2 - 3t + 1]$	$[10, t^2 - 5t + 1]$
$[26, t^2 + 3t + 1]$	$[17, t^2 + 11t + 1]$
$[19, t^2 - 5t + 1]$	$[26, 2t - 24, t^2 - t - 15]$
$[41, t^2 + 30t + 1]$	$[6t^2 - 6t + 6, t^4 - 4t^3 + 5t^2 - 4t + 1]$
$[76, 4t - 72, t^2 - 3t - 23]$	$[2t^2 - 2t + 2, t^4 - t^2 + 2t - 1]$
$[8, 2t^2 + 2t - 6, t^4 - t^2 - 2t + 7]$	$[27, 3t - 24, t^2 + 2t + 19]$
$[25, 5t - 20, t^2 + 2t + 11]$	$[22, 2t - 20, t^4 + t^3 - t^2 - t + 11]$
$[55, 5t - 50, t^2 - t - 24]$	

Table 5.4: The ideals arising as 2^{nd} Alexander ideals of prime knots of up to 14 crossings, contd.

5.3 Comparison with the Jones and other polynomials

The well known invariant of knots and links, the *Jones polynomial* was found by Jones in [15]. It has the following characterisation in terms of so called *skein relations* found by Kauffman, see [20].

Proposition 5.3.1 The Jones polynomial is characterised as being the unique function on the collection of oriented links, taking values in $\mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$ satisfying the conditions below. Let $\bigcirc^n$ denote the link diagram of n unlinked unknotted components.

1. $V(\bigcirc^n) = (-t^{-\frac{1}{2}} - t^{\frac{1}{2}})^n$

2. If the three links L_+, L^-, L_0 have diagrams that are identical outside of a small region, in which they appear as shown in figure 5.1 then

$$t^{-1}V(L_+) - tV(L_-) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(L_0) = 0. \tag{5.1}$$

This allows the calculation of the Jones polynomial of a knot from a diagram, one decomposes the knot diagram by altering the crossings as in figure 5.1 and keeping

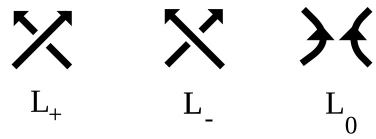


Figure 5.1: Defining the Jones polynomial.

track of the linear combinations from (5.1) until one reaches diagrams of trivial links. In fact it can be shown that the Jones polynomial of a knot is an element of Λ. The discovery of the Jones polynomial initiated lots of developments in knot theory and others subsequently found other related multi-variable polynomials such as the Kauffman and HOMFLY polynomials, also definable by *skein relations*. The Jones polynomial would be regarded as more powerful than the first Alexander polynomial but it is not strictly so. Both the Jones and Alexander polynomial are strictly weaker than the HOMFLY, both being obtainable from the HOMFLY polynomial by certain substitutions of the variables. We should remark as well that the Alexander polynomial also satisfies a skein relation.

Although these invariants are very powerful, using the skein relation defining them it is possible to construct families of knots that are indistinguishable by them, see for instance [18].

Also the operation of *knot mutation* can generate inequivalent knots with the same polynomial invariants. But the higher Alexander ideals can distinguish some of these knots as we will see in the next section.

5.3.1 Knot mutation

Mutation is a local operation on knots first introduced by J.H. Conway in [7]. It is well known that mutation preserves the Alexander polynomial of a knot and the Jones, HOMFLY and Kauffman polynomials.

Definition 5.3.1 Let $d \subset S^2$ be a diagram for a knot k and let $B \subset S^2$ be a closed disk intersecting k in four points as shown in figure 5.2 so that rotations of $180°$ about the vertical, horizontal or perpendicular axis of the page preserve set-wise these four points. We can form a new knot diagram d' by rotating the disk B by $180°$ in one of the three senses mentioned. We orient d' so that the orientation on

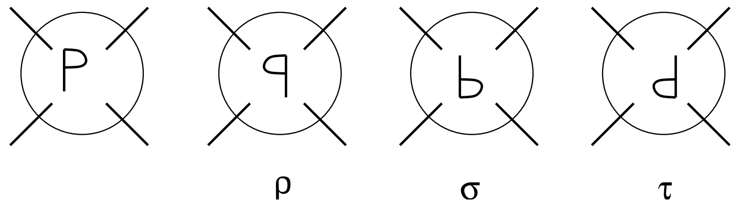


Figure 5.2: The disk B and the three rotations of it.

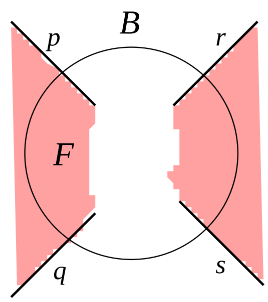


Figure 5.3:

Figure 5.4: The mutating 3-ball.

the part of d' in $S^2 - B$ agrees with that of d. A knot k' with diagram d' is called a *mutant of* k. A pair of knots related by a finite sequence of mutations will be said to be *related by mutation*.

We have defined mutation in terms of the rotation of a disk intersecting a knot diagram. Equivalently it can be defined in terms of a 3-ball B intersecting the knot in S^3, and we shall use this approach below.

We now show how mutation can affect the Alexander module of a knot. Let k and k' be a pair of mutant knots. We can choose a Seifert surface F for k so that it intersects the mutating 3-ball B as shown in figure 5.3. Now $H_1(F)$ has a basis $\{a_1, \ldots, a_r, b_0, b_1, \ldots, b_s\}$ where the a_i are represented by cycles in $F - (F \cap B)$, the b_i, for $i > 0$ by cycles in $F \cap B$ and b_0 is represented by a cycle in F that goes through B.

An important point is that we can assume that the segment of k entering B at p emerges at r or s. Since k is a knot this must be so for either B or the complementary ball $S^3 - B$. And any mutation of k done by rotating B can be realised by rotating $S^3 - B$ (modulo orientation which does not affect the Alexander invariants). So the cycle representing b_0 can be chosen to be disjoint from the b_i, for $i > 0$ by having it *stick close* to the segment of k in B that goes through p. Let β_i be cycles representing b_i, $0 \leq i \leq s$, and recall equation (1.1) relating the linking and intersection numbers

$$\mathrm{Link}(\beta_0^+, \beta_i) = \mathrm{Link}(\beta_i^+, \beta_0) + \mathrm{Int}(\beta_0, \beta_i) \tag{5.2}$$

for all $0 \leq i \leq s$. But since the β_i can be chosen so that β_0 is disjoint from β_i, $\mathrm{Int}(\beta_0, \beta_i) = 0$ for all $0 < i \leq s$. Hence

$$\phi(b_0, b_i) = \phi(b_i, b_0) \text{ for } i > 0 \tag{5.3}$$

So k will have a Seifert matrix C

$$C = \begin{pmatrix} A & u & 0 \\ v^T & m & w^T \\ 0 & w & B \end{pmatrix} \tag{5.4}$$

where A is the $r \times r$ matrix representing the Seifert form ϕ on $V - B$ with respect to the (a_i), B the $s \times s$ matrix representing ϕ on $V \cap B$ with respect to the (b_i), u, v, w are column vectors and $m = \phi(b_0, b_0)$.

Of the three mutations ρ, σ and τ we can identify one of them as the *positive* mutation and the other two as *negative* mutations.

Definition 5.3.2 A mutation given by a rotation μ is *positive* if the orientation on the the part of the Seifert surface $\mu(F)$ inside B is the same as that on the part of the original Seifert surface F in B. A mutation given by μ is *negative* if this orientation is reversed.

We will now show that a positive mutation preserves the Alexander module, and hence the Alexander ideals and polynomials. However this is not necessarily true for negative mutations and we shall present some examples later in this section. This argument comes from [8].

The mutant knot k' will have a Seifert surface F' which is one of $(\rho(F), \sigma(F), \tau(F))$ where $\rho(F)$ etc. denotes the surface obtained by cutting out $F \cap B$ from F and gluing it back after the appropriate rotation, the orientation on this new surface coming from the orientation on k'. It is clear that $\{a_1, \ldots, a_r, b'_0, b'_1, \ldots, b'_s\}$ will be a basis for $H_1(F')$, where b'_i is the equivalence class of the image of the cycle β_i under the rotation.

If k' is a positive mutant of k then k' will have Seifert matrix C' given by

$$C' = \begin{pmatrix} A & u & 0 \\ v^T & m & -w^T \\ 0 & -w & B \end{pmatrix} \tag{5.5}$$

with respect to the basis $\{a_1, \ldots, a_r, b'_0, b'_1, \ldots, b'_s\}$. But by changing the basis slightly to $\{a_1, \ldots, a_r, b'_0, -b'_1, \ldots, -b'_s\}$ we see that k' also has C as a Seifert matrix and hence the Alexander module of k' is the same as that of k.

For the other two mutations, k' will have Seifert matrix C' given by

$$C' = \begin{pmatrix} A & u & 0 \\ v^T & m & w^T \\ 0 & w & B^T \end{pmatrix} \tag{5.6}$$

with respect to the basis $\{a_1, \ldots, a_r, b'_0, b'_1, \ldots, b'_s\}$. So k' will have Alexander matrix

$$tC' - C'^T = \begin{pmatrix} tA - A^T & tu - v & 0 \\ tv^T - u^T & tm - m & tw^T - w^T \\ 0 & tw - w & tB^T - B \end{pmatrix} \tag{5.7}$$

and in general this does not present the same Λ-module as $tC - C^T$.

However in [8] it is shown by a closer examination of the presentation matrices for the Alexander modules that all types of mutation do preserve the first Alexander polynomial.

Mutation also preserves the Jones polynomial. Using proposition 5.3.1 one can express the Jones polynomial of a knot k as a Λ-linear combination of Jones polynomials of links that outside of B are identical with k and inside of B look like one of the three in figure 5.5. Now note that each of these is invariant under mutation so when the Jones polynomial of the mutant of k is calculated one gets the same answer. Using the skein relation defining the Kauffman and HOMFLY polynomials one can show that these too are unchanged by mutation.

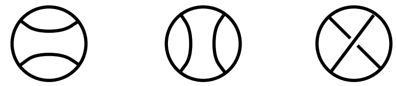


Figure 5.5: Mutation and the Jones polynomial.

Of course recognising mutants is a non-trivial matter unless one is presented with their actual *mutated diagrams.* However we adopted the strategy that if two inequivalent knots have the same Jones, HOMFLY and Kauffman polynomials then there is a good chance that they are related by mutation. Hoste and Thistlethwaite's program *Knotscape* also contains tables of these polynomials for the prime knots of up to 14 crossings. By searching these and cross referencing the results with our tables of Châtelet bases for the higher Alexander ideals we found the following three pairs of knots related by a negative mutation. Each pair have the same Jones, Kauffman and HOMFLY polynomial but they are distinguished by their Alexander modules as can be seen from the Châtelet bases for their Alexander ideals.

The notations n^a_m and n^{na}_m refer to the m^{th} alternating n crossing knot and the m^{th} non-alternating n crossing knot in the *Knotscape* table. The knot diagrams in the figures 5.6, 5.7 and 5.8 in which the mutations of the pairs can be easily seen, were produced using *Knotscape.* Pairs **1** and **2** are pairs of mutant knots while pair **3** are related by a sequence of two mutations.

Mutant pair 1. Figure 5.6 shows 13^a_{2720} on the left and 13^a_{2727} on the right. Châtelet bases for the Alexander ideals are

$$\mathcal{I}_1(13^a_{2720}) : \{4t^6 - 24t^5 + 57t^4 - 73t^3 + 57t^2 - 24t + 4\} \tag{5.8}$$

$$\mathcal{I}_2(13^a_{2720}) : \{9, 3t - 6, t^2 - t - 2\} \tag{5.9}$$

$$\mathcal{I}_3(13^a_{2720}) : \{3, t - 2\} \tag{5.10}$$

$$\mathcal{I}_1(13^a_{2727}) : \{4t^6 - 24t^5 + 57t^4 - 73t^3 + 57t^2 - 24t + 4\} \tag{5.11}$$

$$\mathcal{I}_2(13^a_{2727}) : \{2t^2 - 5t + 2\} \tag{5.12}$$

$$\mathcal{I}_2(13^a_{2727}) : \{3, t - 2\} \tag{5.13}$$

Mutant pair 2. Figure 5.7 shows 13^{na}_{2937} on the left and 13^{na}_{2955} on the right. Châtelet bases for the Alexander ideals are

$$\mathcal{I}_1(13^{na}_{2937}) : \{2t^6 - 15t^5 + 41t^4 - 55t^3 + 41t^2 - 15t + 2\} \tag{5.14}$$

$$\mathcal{I}_2(13^{na}_{2937}) : \{15, 5t - 10, t^2 - 4\} \tag{5.15}$$

$$\mathcal{I}_1(13^{na}_{2955}) : \{2t^6 - 15t^5 + 41t^4 - 55t^3 + 41t^2 - 15t + 2\} \tag{5.16}$$

$$\mathcal{I}_2(13^{na}_{2955}) : \{3, t + 1\} \tag{5.17}$$

$$\tag{5.18}$$

Mutant pair 3. Figure 5.8 shows 14^{a}_{10405} on the left and 14^{a}_{10410} on the right. Châtelet bases for the Alexander ideals are

$$\mathcal{I}_1(14^{a}_{10405}) : \{2t^8 - 15t^7 + 49t^6 - 91t^5 + 109t^4 - 91t^3 + 49t^2 - 15t + 2\} \tag{5.19}$$

$$\mathcal{I}_2(14^{a}_{10405}) : \{3, t + 1\} \tag{5.20}$$

$$\mathcal{I}_1(14^{a}_{10410}) : \{2t^8 - 15t^7 + 49t^6 - 91t^5 + 109t^4 - 91t^3 + 49t^2 - 15t + 2\} \tag{5.21}$$

$$\mathcal{I}_2(14^{a}_{10410}) : \{33, 11t - 22, t^2 - 8t - 21\} \tag{5.22}$$

$$\tag{5.23}$$

We remark that in all of these examples the Alexander polynomials of the higher ideals would not have detected the inequivalences of the knots. There are many more pairs of knots that the higher Alexander ideals distinguish but the Jones, Kauffman and HOMFLY polynomials do not. A closer investigation of their diagrams as drawn by *Knotscape* might show them to be mutant pairs or related by a sequence of mutations.

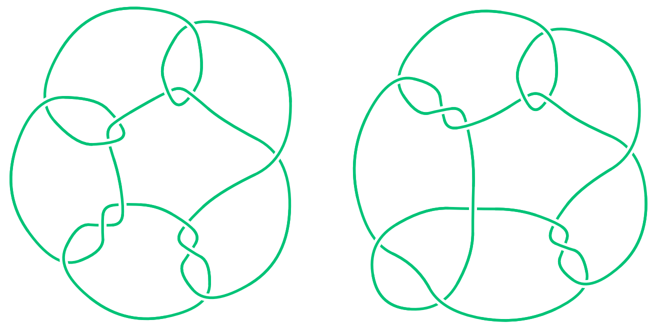


Figure 5.6: Pair 1: 13^a_{2720} and 13^a_{2727}.

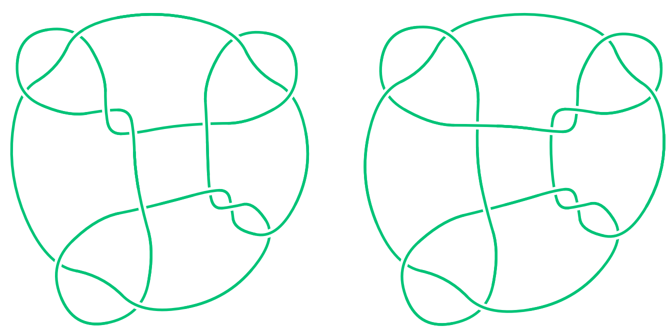


Figure 5.7: Pair 2: 13^{na}_{2937} and 13^{na}_{2955}.

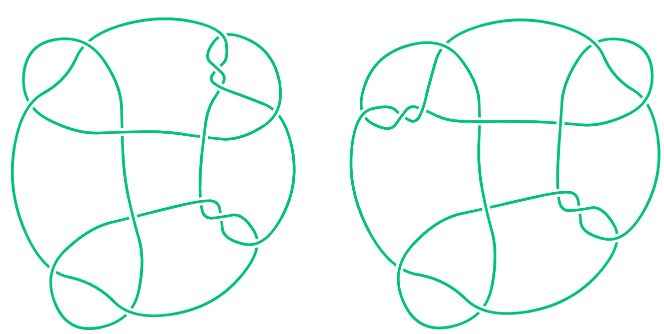


Figure 5.8: Pair 3: 14^a_{10405} and 14^a_{10410}.

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Appendix A

Table of Alexander Ideals

The entries in these tables have the same form as the entries in the `.txt` files as explained in appendix B. However to save space we have represented polynomials by their coefficient lists and ommited the final trivial ideal [1] from each line. So for example the second line in the table below represents the 18^{th} alternating 9 crossing knot which has a chain of Alexander ideals of length 2. The first being generated by the Alexander polynomial $2t^4 - 11t^3 + 19t^2 - 11t + 2$ and whose second Alexander ideal has a Châtelet basis $\{3, t-2\}$.

A.1 Prime knots up to 12 crossings with chain length 2 and 3.

$[8, 0, 12, 2, [[1, -5, 10, -13, 10, -5, 1]], [[1, -1, 1]]]$
$[9, 0, 18, 2, [[2, -11, 19, -11, 2]], [[3], [1, -2]]]$
$[9, 0, 29, 2, [[3, -12, 19, -12, 3]], [[7], [1, 1]]]$
$[9, 0, 37, 2, [[1, -7, 18, -23, 18, -7, 1]], [[1, -3, 1]]]$
$[9, 0, 40, 2, [[7, -13, 7]], [[3], [1, -2]]]$
$[9, 1, 5, 2, [[2, -5, 2]], [[3], [1, -2]]]$
$[9, 1, 6, 2, [[1, -7, 11, -7, 1]], [[3], [1, -2]]]$
$[9, 1, 7, 2, [[1, -4, 6, -5, 6, -4, 1]], [[3], [1, 1]]]$
$[9, 1, 8, 2, [[3, -6, 7, -6, 3]], [[5], [1, -4]]]$
$[10, 0, 27, 2, [[1, -7, 19, -27, 19, -7, 1]], [[3], [1, -2]]]$
$[10, 0, 42, 2, [[2, -7, 14, -17, 14, -7, 2]], [[2], [1, -1, 1]]]$
$[10, 0, 51, 2, [[5, -14, 19, -14, 5]], [[2], [1, -1, 1]]]$
$[10, 0, 62, 2, [[4, -16, 23, -16, 4]], [[3], [1, -2]]]$
$[10, 0, 89, 2, [[2, -11, 24, -31, 24, -11, 2]], [[3], [1, 0, -2]]]$
$[10, 0, 94, 2, [[1, -9, 26, -37, 26, -9, 1]], [[2], [1, -1, 1]]]$
$[10, 0, 96, 2, [[2, -9, 18, -23, 18, -9, 2]], [[1, -1, 1]]]$
$[10, 0, 103, 2, [[1, -4, 10, -16, 19, -16, 10, -4, 1]], [[1, -2, 3, -2, 1]]]$
$[10, 0, 105, 2, [[2, -8, 17, -21, 17, -8, 2]], [[5], [1, 1]]]$
$[10, 0, 121, 2, [[1, -6, 15, -24, 29, -24, 15, -6, 1]], [[1, -3, 3, -3, 1]]]$
$[10, 0, 123, 2, [[2, -5, 6, -7, 6, -5, 2]], [[2], [1, -1, 1]]]$
$[10, 1, 28, 2, [[3, -10, 13, -10, 3]], [[2], [1, -1, 1]]]$
$[10, 1, 29, 2, [[1, -2, 3, -2, 1]], [[2], [1, -1, 1]]]$

$[10,1,30,2,[[2,-3,2,-1,2,-3,2]],[[2],[1,-1,1]]]$
$[10,1,35,2,[[1,-5,12,-15,12,-5,1]],[[2],[1,1,-1]]]$
$[10,1,39,2,[[1,-3,5,-7,5,-3,1]],[[5],[1,1]]]$
$[10,1,42,2,[[1,-6,11,-13,11,-6,1]],[[7],[1,1]]]$
$[11,0,43,2,[[4,-15,30,-37,30,-15,4]],[[1,-1,1]]]$
$[11,0,44,2,[[1,-5,14,-24,29,-24,14,-5,1]],[[1,-1,1]]]$
$[11,0,47,2,[[1,-5,14,-24,29,-24,14,-5,1]],[[1,-1,1]]]$
$[11,0,57,2,[[1,-5,12,-20,23,-20,12,-5,1]],[[1,-1,1]]]$
$[11,0,87,2,[[2,-11,28,-39,28,-11,2]],[[2],[1,1,-1]]]$
$[11,0,97,2,[[2,-9,16,-17,16,-9,2]],[[2],[1,-1,1]]]$
$[11,0,107,2,[[2,-11,26,-33,26,-11,2]],[[2],[1,-1,1]]]$
$[11,0,123,2,[[9,-29,41,-29,9]],[[3],[1,-2]]]$
$[11,0,132,2,[[2,-13,32,-41,32,-13,2]],[[2],[1,1,-1]]]$
$[11,0,133,2,[[5,-20,29,-20,5]],[[2],[1,-1,-1]]]$
$[11,0,135,2,[[2,-13,36,-51,36,-13,2]],[[3],[1,1]]]$
$[11,0,143,2,[[2,-11,20,-23,20,-11,2]],[[2],[1,-1,1]]]$
$[11,0,155,2,[[3,-16,40,-53,40,-16,3]],[[3],[1,-2]]]$
$[11,0,157,2,[[1,-6,16,-28,33,-28,16,-6,1]],[[2],[1,0,1,0,1]]]$
$[11,0,165,2,[[2,-9,18,-23,18,-9,2]],[[2],[1,-1,1]]]$
$[11,0,173,2,[[2,-12,32,-43,32,-12,2]],[[3],[1,-2]]]$
$[11,0,181,2,[[2,-11,23,-27,23,-11,2]],[[3],[1,1]]]$
$[11,0,196,2,[[1,-6,17,-31,37,-31,17,-6,1]],[[7],[1,-6]]]$
$[11,0,231,2,[[1,-5,12,-20,23,-20,12,-5,1]],[[1,-1,1]]]$
$[11,0,239,2,[[1,-7,22,-42,51,-42,22,-7,1]],[[3],[1,0,1]]]$
$[11,0,249,2,[[2,-11,27,-37,27,-11,2]],[[3],[1,-2]]]$
$[11,0,263,2,[[2,-6,11,-14,15,-14,11,-6,2]],[[1,-1,1]]]$
$[11,0,277,2,[[1,-6,17,-28,31,-28,17,-6,1]],[[3],[1,1]]]$
$[11,0,291,2,[[5,-14,20,-21,20,-14,5]],[[3],[1,-2]]]$
$[11,0,293,2,[[1,-5,12,-15,15,-15,12,-5,1]],[[3],[1,-2]]]$
$[11,0,297,2,[[2,-15,42,-57,42,-15,2]],[[1,-3,1]]]$
$[11,0,314,2,[[1,-7,21,-36,41,-36,21,-7,1]],[[3],[1,1]]]$
$[11,0,317,2,[[3,-14,28,-35,28,-14,3]],[[5],[1,-4]]]$
$[11,0,321,2,[[3,-15,27,-31,27,-15,3]],[[11],[1,1]]]$
$[11,0,322,2,[[2,-13,36,-49,36,-13,2]],[[2],[1,-1,1]]]$
$[11,0,329,2,[[11,-36,51,-36,11]],[[2],[1,-1,1]]]$
$[11,0,332,2,[[1,-7,22,-40,49,-40,22,-7,1]],[[1,-1,1]]]$
$[11,0,340,2,[[4,-11,18,-21,18,-11,4]],[[2],[1,-1,1]]]$
$[11,0,347,2,[[2,-11,26,-33,26,-11,2]],[[2],[1,-1,1]]]$
$[11,0,352,2,[[2,-13,32,-41,32,-13,2]],[[3],[1,-1,1]]]$
$[11,0,354,2,[[9,-26,35,-26,9]],[[2],[1,-1,1]]]$
$[11,0,366,2,[[8,-20,25,-20,8]],[[3],[1,-2]]]$
$[11,1,49,2,[[1,0,-3,0,1]],[[2],[1,-1,1]]]$
$[11,1,71,2,[[2,-7,14,-17,14,-7,2]],[[1,-1,1]]]$
$[11,1,72,2,[[2,-9,18,-23,18,-9,2]],[[1,-1,1]]]$
$[11,1,73,2,[[1,-2,3,-2,1]],[[1,-1,1]]]$
$[11,1,74,2,[[1,-2,3,-2,1]],[[1,-1,1]]]$
$[11,1,75,2,[[2,-7,14,-17,14,-7,2]],[[1,-1,1]]]$
$[11,1,76,2,[[1,-3,6,-8,9,-8,6,-3,1]],[[1,-1,1]]]$
$[11,1,77,2,[[1,-1,-2,8,-11,8,-2,-1,1]],[[1,-1,1]]]$
$[11,1,78,2,[[1,-3,6,-8,9,-8,6,-3,1]],[[1,-1,1]]]$
$[11,1,81,2,[[1,-3,4,-4,3,-4,4,-3,1]],[[1,-1,1]]]$
$[11,1,83,2,[[3,-12,19,-12,3]],[[2],[1,-1,-1]]]$
$[11,1,90,2,[[2,-7,8,-7,8,-7,2]],[[2],[1,-1,1]]]$
$[11,1,91,2,[[1,-8,13,-8,1]],[[2],[1,-1,1]]]$
$[11,1,126,2,[[3,-6,4,-1,4,-6,3]],[[3],[1,1]]]$
$[11,1,133,2,[[1,-4,6,-2,-1,-2,6,-4,1]],[[5],[1,-4]]]$

$[11, 1, 148, 2, [[1, -5, 10, -14, 15, -14, 10, -5, 1]], [[5], [1, 2, -4]]]$
$[11, 1, 157, 2, [[1, -6, 15, -21, 15, -6, 1]], [[3], [1, 0, -2]]]$
$[11, 1, 162, 2, [[3, -14, 21, -14, 3]], [[2], [1, -1, 1]]]$
$[11, 1, 164, 2, [[1, -5, 10, -13, 10, -5, 1]], [[1, -1, 1]]]$
$[11, 1, 165, 2, [[1, -7, 20, -29, 20, -7, 1]], [[2], [1, -1, 1]]]$
$[11, 1, 167, 2, [[1, -5, 15, -21, 15, -5, 1]], [[3], [1, -2]]]$
$[11, 1, 175, 2, [[2, -9, 14, -15, 14, -9, 2]], [[2], [1, -1, -1]]]$
$[11, 1, 183, 2, [[1, 1, -6, 9, -6, 1, 1]], [[4], [1, -3, 1]]]$
$[11, 1, 185, 2, [[2, -11, 24, -31, 24, -11, 2]], [[3], [1, 0, 1]]]$
$[12, 0, 29, 2, [[1, -7, 24, -48, 59, -48, 24, -7, 1]], [[2], [1, -1, -1]]]$
$[12, 0, 30, 2, [[1, -7, 22, -40, 49, -40, 22, -7, 1]], [[2], [1, -1, 1]]]$
$[12, 0, 33, 2, [[1, -7, 22, -40, 49, -40, 22, -7, 1]], [[2], [1, -1, 1]]]$
$[12, 0, 36, 2, [[2, -10, 21, -30, 33, -30, 21, -10, 2]], [[2], [1, 1, -1]]]$
$[12, 0, 100, 2, [[3, -21, 53, -71, 53, -21, 3]], [[5], [1, -4]]]$
$[12, 0, 113, 2, [[1, -7, 24, -48, 59, -48, 24, -7, 1]], [[2], [1, -1, 1]]]$
$[12, 0, 114, 2, [[4, -21, 46, -59, 46, -21, 4]], [[2], [1, -1, 1]]]$
$[12, 0, 116, 2, [[1, -7, 20, -36, 43, -36, 20, -7, 1]], [[2], [1, 1, -1]]]$
$[12, 0, 117, 2, [[4, -21, 46, -59, 46, -21, 4]], [[2], [1, 1, -1]]]$
$[12, 0, 119, 2, [[1, -7, 22, -40, 49, -40, 22, -7, 1]], [[1, -1, 1]]]$
$[12, 0, 122, 2, [[1, -7, 20, -36, 43, -36, 20, -7, 1]], [[2], [1, -1, 1]]]$
$[12, 0, 157, 2, [[1, -7, 22, -40, 49, -40, 22, -7, 1]], [[2], [1, -1, 1]]]$
$[12, 0, 164, 2, [[1, -7, 20, -36, 43, -36, 20, -7, 1]], [[1, -1, 1]]]$
$[12, 0, 166, 2, [[1, -7, 20, -36, 43, -36, 20, -7, 1]], [[1, -1, 1]]]$
$[12, 0, 167, 2, [[2, -8, 19, -30, 35, -30, 19, -8, 2]], [[1, -1, 1]]]$
$[12, 0, 177, 2, [[2, -15, 41, -55, 41, -15, 2]], [[3], [1, 1]]]$
$[12, 0, 182, 2, [[1, -7, 20, -36, 43, -36, 20, -7, 1]], [[2], [1, -1, 1]]]$
$[12, 0, 195, 2, [[1, -7, 18, -28, 33, -28, 18, -7, 1]], [[2], [1, -1, -1]]]$
$[12, 0, 215, 2, [[1, -7, 22, -42, 51, -42, 22, -7, 1]], [[3], [1, 0, 1]]]$
$[12, 0, 216, 2, [[1, -7, 18, -24, 25, -24, 18, -7, 1]], [[3], [1, 0, -2]]]$
$[12, 0, 218, 2, [[2, -15, 42, -57, 42, -15, 2]], [[2], [1, 1, -1]]]$
$[12, 0, 244, 2, [[5, -24, 56, -73, 56, -24, 5]], [[3], [1, -2]]]$
$[12, 0, 245, 2, [[1, -7, 24, -49, 63, -49, 24, -7, 1]], [[3], [1, -2]]]$
$[12, 0, 248, 2, [[2, -13, 34, -47, 34, -13, 2]], [[2], [1, -1, -1]]]$
$[12, 0, 249, 2, [[2, -13, 36, -49, 36, -13, 2]], [[2], [1, -1, 1]]]$
$[12, 0, 253, 2, [[4, -19, 36, -43, 36, -19, 4]], [[2], [1, -1, 1]]]$
$[12, 0, 265, 2, [[1, -8, 30, -66, 87, -66, 30, -8, 1]], [[3], [1, -2]]]$
$[12, 0, 270, 2, [[6, -25, 37, -25, 6]], [[3], [1, 1]]]$
$[12, 0, 279, 2, [[2, -15, 40, -55, 40, -15, 2]], [[2], [1, -1, -1]]]$
$[12, 0, 291, 2, [[4, -17, 34, -43, 34, -17, 4]], [[2], [1, -1, 1]]]$
$[12, 0, 295, 2, [[6, -23, 46, -57, 46, -23, 6]], [[1, -1, 1]]]$
$[12, 0, 297, 2, [[2, -8, 19, -30, 35, -30, 19, -8, 2]], [[1, -1, 1]]]$
$[12, 0, 298, 2, [[1, -7, 20, -36, 43, -36, 20, -7, 1]], [[3], [1, -1, 1]]]$
$[12, 0, 311, 2, [[5, -23, 42, -49, 42, -23, 5]], [[3], [1, 1]]]$
$[12, 0, 312, 2, [[9, -40, 61, -40, 9]], [[2], [1, -1, 1]]]$
$[12, 0, 327, 2, [[4, -18, 40, -51, 40, -18, 4]], [[5], [1, 1]]]$
$[12, 0, 332, 2, [[2, -14, 41, -57, 41, -14, 2]], [[3], [1, -2]]]$
$[12, 0, 347, 2, [[4, -23, 54, -69, 54, -23, 4]], [[2], [1, -1, 1]]]$
$[12, 0, 348, 2, [[2, -17, 54, -79, 54, -17, 2]], [[1, -3, 1]]]$
$[12, 0, 376, 2, [[4, -15, 30, -37, 30, -15, 4]], [[2], [1, -1, 1]]]$
$[12, 0, 381, 2, [[3, -17, 36, -43, 36, -17, 3]], [[3], [1, 0, 1]]]$
$[12, 0, 386, 2, [[1, -8, 25, -39, 43, -39, 25, -8, 1]], [[3], [1, -2]]]$
$[12, 0, 396, 2, [[1, -7, 22, -40, 49, -40, 22, -7, 1]], [[9], [3, 3], [1, -1, 4]]]$
$[12, 0, 408, 2, [[5, -29, 66, -85, 66, -29, 5]], [[3], [1, 0, 1]]]$
$[12, 0, 413, 2, [[1, -8, 29, -62, 79, -62, 29, -8, 1]], [[3], [1, 1]]]$
$[12, 0, 427, 2, [[1, -8, 26, -48, 59, -48, 26, -8, 1]], [[1, -4, 5, -4, 1]]]$
$[12, 0, 429, 2, [[4, -19, 42, -53, 42, -19, 4]], [[2], [1, -1, 1]]]$

$[12,0,433,2,[[3,-19,44,-57,44,-19,3]],[[3],[1,-1,1]]]$
$[12,0,435,2,[[1,-8,26,-48,59,-48,26,-8,1]],[[2,-2,2],[1,0,-1,2,-1]]]$
$[12,0,444,2,[[2,-13,30,-39,30,-13,2]],[[2],[1,-1,1]]]$
$[12,0,448,2,[[5,-26,43,-26,5]],[[2],[1,-1,1]]]$
$[12,0,465,2,[[1,-8,26,-52,67,-52,26,-8,1]],[[2],[1,0,-1,0,1]]]$
$[12,0,466,2,[[1,-8,26,-44,51,-44,26,-8,1]],[[2],[1,0,1,0,1]]]$
$[12,0,475,2,[[1,-8,28,-56,69,-56,28,-8,1]],[[2],[1,0,1,0,-1]]]$
$[12,0,481,2,[[2,-13,28,-33,28,-13,2]],[[2],[1,-1,1]]]$
$[12,0,493,2,[[1,-7,22,-36,39,-36,22,-7,1]],[[3],[1,-2]]]$
$[12,0,494,2,[[2,-15,48,-71,48,-15,2]],[[2],[1,-1,1]]]$
$[12,0,503,2,[[1,-7,19,-27,27,-27,19,-7,1]],[[3],[1,-2]]]$
$[12,0,554,3,[[2,-15,45,-65,45,-15,2]],[[9],[3,-6],[1,-1,-2]],[[3],[1,-2]]]$
$[12,0,561,2,[[1,-7,25,-49,61,-49,25,-7,1]],[[5],[1,1]]]$
$[12,0,563,2,[[4,-16,25,-27,25,-16,4]],[[3],[1,-2]]]$
$[12,0,569,2,[[1,-7,21,-31,33,-31,21,-7,1]],[[3],[1,-2]]]$
$[12,0,574,2,[[3,-9,16,-20,21,-20,16,-9,3]],[[1,-1,1]]]$
$[12,0,576,2,[[1,-3,6,-8,9,-9,9,-8,6,-3,1]],[[1,-1,1]]]$
$[12,0,594,2,[[3,-11,22,-27,22,-11,3]],[[3],[1,-1,1]]]$
$[12,0,615,2,[[7,-27,54,-67,54,-27,7]],[[3,-3,3],[1,0,0,1]]]$
$[12,0,634,2,[[4,-15,30,-37,30,-15,4]],[[3],[1,0,0,1]]]$
$[12,0,647,2,[[3,-10,20,-28,31,-28,20,-10,3]],[[2,-2,2],[1,0,-1,2,-1]]]$
$[12,0,664,2,[[1,-8,22,-34,39,-34,22,-8,1]],[[13],[1,-12]]]$
$[12,0,679,2,[[7,-20,27,-20,7]],[[3],[1,-1,1]]]$
$[12,0,683,2,[[1,-7,16,-22,25,-22,16,-7,1]],[[3],[1,-1,1]]]$
$[12,0,692,2,[[2,-8,19,-30,35,-30,19,-8,2]],[[1,-1,1]]]$
$[12,0,693,2,[[1,-7,18,-28,33,-28,18,-7,1]],[[2],[1,-1,1]]]$
$[12,0,694,2,[[2,-10,21,-30,33,-30,21,-10,2]],[[2],[1,-1,-1]]]$
$[12,0,701,2,[[2,-10,27,-46,55,-46,27,-10,2]],[[1,-1,1]]]$
$[12,0,703,2,[[1,-9,32,-64,79,-64,32,-9,1]],[[2],[1,-1,1]]]$
$[12,0,712,2,[[1,-9,34,-70,87,-70,34,-9,1]],[[3],[1,2,-2]]]$
$[12,0,725,2,[[4,-16,25,-27,25,-16,4]],[[3],[1,-2]]]$
$[12,0,742,2,[[2,-8,17,-26,29,-26,17,-8,2]],[[1,-1,1]]]$
$[12,0,750,3,[[8,-34,51,-34,8]],[[9],[3,-6],[1,-1,-2]],[[3],[1,-2]]]$
$[12,0,769,2,[[2,-16,45,-63,45,-16,2]],[[3],[1,-2]]]$
$[12,0,780,2,[[5,-24,51,-65,51,-24,5]],[[5],[1,1]]]$
$[12,0,787,2,[[8,-29,43,-29,8]],[[3],[1,-2]]]$
$[12,0,801,2,[[2,-8,17,-26,29,-26,17,-8,2]],[[1,-1,1]]]$
$[12,0,806,2,[[2,-11,27,-44,51,-44,27,-11,2]],[[2],[1,-1,1]]]$
$[12,0,808,2,[[3,-15,32,-41,32,-15,3]],[[2],[1,-1,1]]]$
$[12,0,810,2,[[4,-23,57,-75,57,-23,4]],[[3],[1,-2]]]$
$[12,0,868,2,[[1,-10,39,-84,109,-84,39,-10,1]],[[2],[1,-1,-1,-1,1]]]$
$[12,0,873,2,[[5,-22,42,-51,42,-22,5]],[[3],[1,1]]]$
$[12,0,886,2,[[1,-7,24,-45,53,-45,24,-7,1]],[[3],[1,-2]]]$
$[12,0,895,2,[[2,-10,27,-51,63,-51,27,-10,2]],[[3],[1,1]]]$
$[12,0,904,2,[[2,-10,23,-36,41,-36,23,-10,2]],[[2],[1,-1,1]]]$
$[12,0,905,2,[[4,-20,43,-55,43,-20,4]],[[3],[1,-1,1]]]$
$[12,0,906,2,[[1,-8,26,-53,69,-53,26,-8,1]],[[7],[1,2,1]]]$
$[12,0,907,2,[[2,-11,29,-45,51,-45,29,-11,2]],[[5],[1,-4]]]$
$[12,0,921,2,[[2,-9,21,-35,41,-35,21,-9,2]],[[5],[1,-4]]]$
$[12,0,941,2,[[2,-8,17,-26,29,-26,17,-8,2]],[[2],[1,-1,1]]]$
$[12,0,949,2,[[2,-10,25,-42,49,-42,25,-10,2]],[[2],[1,-1,-1]]]$
$[12,0,960,2,[[4,-25,66,-91,66,-25,4]],[[2],[1,-1,1]]]$
$[12,0,970,2,[[2,-8,15,-18,19,-18,15,-8,2]],[[2],[1,-1,-1]]]$
$[12,0,973,2,[[8,-29,53,-63,53,-29,8]],[[3],[1,1]]]$
$[12,0,975,2,[[4,-22,52,-69,52,-22,4]],[[5],[1,1,-4,1]]]$
$[12,0,987,2,[[2,-10,27,-46,55,-46,27,-10,2]],[[1,-1,1]]]$

$[12, 0, 990, 2, [[1, -8, 26, -48, 59, -48, 26, -8, 1]], [[2, -2, 2], [1, 0, -1, 2, -1]]]$
$[12, 0, 1019, 2, [[1, -10, 39, -80, 101, -80, 39, -10, 1]], [[1, -5, 7, -5, 1]]]$
$[12, 0, 1022, 2, [[3, -21, 58, -79, 58, -21, 3]], [[9], [1, 1]]]$
$[12, 0, 1025, 2, [[2, -12, 31, -52, 61, -52, 31, -12, 2]], [[7], [1, 4, 1]]]$
$[12, 0, 1026, 2, [[3, -17, 38, -49, 38, -17, 3]], [[8], [1, 1, 1]]]$
$[12, 0, 1053, 2, [[2, -10, 27, -48, 59, -48, 27, -10, 2]], [[3], [1, 0, -2]]]$
$[12, 0, 1079, 2, [[2, -11, 31, -56, 67, -56, 31, -11, 2]], [[2], [1, 1, -1]]]$
$[12, 0, 1092, 2, [[1, -9, 29, -52, 61, -52, 29, -9, 1]], [[3], [1, 1]]]$
$[12, 0, 1093, 2, [[1, -8, 25, -44, 51, -44, 25, -8, 1]], [[3], [1, -2]]]$
$[12, 0, 1097, 2, [[16, -54, 77, -54, 16]], [[3], [1, 0, 1]]]$
$[12, 0, 1102, 2, [[1, -8, 30, -68, 91, -68, 30, -8, 1]], [[2], [1, 0, -1, 0, -1]]]$
$[12, 0, 1105, 2, [[1, -8, 30, -64, 83, -64, 30, -8, 1]], [[1, -4, 7, -4, 1]]]$
$[12, 0, 1123, 2, [[1, -9, 30, -56, 69, -56, 30, -9, 1]], [[1, -1, 1]]]$
$[12, 0, 1124, 2, [[1, -13, 50, -77, 50, -13, 1]], [[4], [1, -1, 1]]]$
$[12, 0, 1142, 2, [[4, -14, 21, -21, 21, -14, 4]], [[3], [1, -2]]]$
$[12, 0, 1152, 2, [[1, -9, 34, -72, 93, -72, 34, -9, 1]], [[1, -3, 1]]]$
$[12, 0, 1164, 2, [[4, -13, 22, -27, 22, -13, 4]], [[2], [1, -1, 1]]]$
$[12, 0, 1167, 2, [[1, -10, 35, -68, 85, -68, 35, -10, 1]], [[2], [1, -1, 1, -1, 1]]]$
$[12, 0, 1181, 2, [[4, -16, 30, -35, 30, -16, 4]], [[3], [1, -2]]]$
$[12, 0, 1183, 2, [[4, -15, 26, -31, 26, -15, 4]], [[11], [1, -10]]]$
$[12, 0, 1194, 2, [[2, -10, 23, -34, 37, -34, 23, -10, 2]], [[5], [1, -4]]]$
$[12, 0, 1202, 2, [[9, -42, 67, -42, 9]], [[3, -7, 3]]]$
$[12, 0, 1205, 2, [[4, -15, 26, -29, 26, -15, 4]], [[4], [1, -3, 1]]]$
$[12, 0, 1206, 2, [[4, -24, 57, -75, 57, -24, 4]], [[2, -3, 2]]]$
$[12, 0, 1225, 2, [[1, -5, 14, -28, 41, -47, 41, -28, 14, -5, 1]], [[1, -1, 1]]]$
$[12, 0, 1229, 2, [[1, -5, 14, -27, 40, -47, 40, -27, 14, -5, 1]], [[2], [1, -1, -1, 1, -1]]]$
$[12, 0, 1251, 2, [[4, -25, 62, -83, 62, -25, 4]], [[4], [1, -3, 1]]]$
$[12, 0, 1260, 2, [[1, -4, 10, -19, 27, -31, 27, -19, 10, -4, 1]], [[1, -1, 1]]]$
$[12, 0, 1269, 2, [[4, -17, 38, -51, 38, -17, 4]], [[4], [1, -3, 1]]]$
$[12, 0, 1280, 2, [[1, -9, 28, -51, 63, -51, 28, -9, 1]], [[3], [1, 0, -2]]]$
$[12, 0, 1283, 2, [[1, -3, 6, -10, 13, -15, 13, -10, 6, -3, 1]], [[1, -1, 1]]]$
$[12, 0, 1286, 2, [[3, -7, 8, -9, 8, -7, 3]], [[3], [1, -1, 1]]]$
$[12, 0, 1288, 2, [[1, -3, 7, -14, 21, -25, 21, -14, 7, -3, 1]], [[2, -2, 2], [1, 0, -1, 2, -1]]]$
$[12, 1, 55, 2, [[2, -11, 26, -33, 26, -11, 2]], [[2], [1, -1, 1]]]$
$[12, 1, 56, 2, [[1, -2, 3, -2, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 57, 2, [[1, -2, 3, -2, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 58, 2, [[2, -13, 30, -39, 30, -13, 2]], [[2], [1, -1, -1]]]$
$[12, 1, 59, 2, [[1, -3, 0, 8, -13, 8, 0, -3, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 60, 2, [[1, -5, 12, -20, 23, -20, 12, -5, 1]], [[2], [1, 1, 1]]]$
$[12, 1, 61, 2, [[1, -5, 12, -20, 23, -20, 12, -5, 1]], [[2], [1, -1, -1]]]$
$[12, 1, 62, 2, [[2, -9, 18, -23, 18, -9, 2]], [[2], [1, -1, 1]]]$
$[12, 1, 63, 2, [[3, -10, 13, -10, 3]], [[2], [1, 1, -1]]]$
$[12, 1, 64, 2, [[1, -5, 10, -12, 13, -12, 10, -5, 1]], [[2], [1, 1, 1]]]$
$[12, 1, 66, 2, [[2, -9, 18, -23, 18, -9, 2]], [[2], [1, -1, 1]]]$
$[12, 1, 67, 2, [[1, -5, 8, -8, 7, -8, 8, -5, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 144, 2, [[1, -7, 18, -23, 18, -7, 1]], [[3], [1, 0, -2]]]$
$[12, 1, 145, 2, [[1, -6, 11, -6, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 147, 2, [[2, -8, 17, -21, 17, -8, 2]], [[5], [1, 1]]]$
$[12, 1, 219, 2, [[1, -5, 12, -20, 23, -20, 12, -5, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 220, 2, [[1, -3, 0, 8, -13, 8, 0, -3, 1]], [[2], [1, -1, -1]]]$
$[12, 1, 221, 2, [[1, -2, 3, -2, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 222, 2, [[2, -13, 30, -39, 30, -13, 2]], [[2], [1, -1, 1]]]$
$[12, 1, 223, 2, [[2, -11, 26, -33, 26, -11, 2]], [[2], [1, -1, 1]]]$
$[12, 1, 224, 2, [[2, -9, 18, -23, 18, -9, 2]], [[2], [1, -1, 1]]]$
$[12, 1, 225, 2, [[3, -10, 13, -10, 3]], [[2], [1, -1, 1]]]$
$[12, 1, 229, 2, [[1, -5, 8, -8, 7, -8, 8, -5, 1]], [[2], [1, -1, 1]]]$

$[12, 1, 257, 2, [[2, -6, 9, -6, 2]], [[5], [1, -4]]]$
$[12, 1, 261, 2, [[1, -5, 10, -12, 13, -12, 10, -5, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 268, 2, [[2, -5, 2]], [[3], [1, 1]]]$
$[12, 1, 269, 2, [[1, -9, 28, -41, 28, -9, 1]], [[3], [1, -2]]]$
$[12, 1, 270, 2, [[3, -16, 25, -16, 3]], [[3], [1, 1]]]$
$[12, 1, 273, 2, [[1, -3, 0, 3, 0, -3, 1]], [[3], [1, 0, -2]]]$
$[12, 1, 274, 2, [[3, -14, 21, -14, 3]], [[2], [1, -1, 1]]]$
$[12, 1, 276, 2, [[1, 1, -7, 11, -7, 1, 1]], [[5], [1, 1]]]$
$[12, 1, 294, 2, [[1, -5, 8, -6, 5, -6, 8, -5, 1]], [[3], [1, 0, -2]]]$
$[12, 1, 297, 2, [[2, -11, 22, -25, 22, -11, 2]], [[2], [1, -1, 1]]]$
$[12, 1, 332, 2, [[2, -2, 1, -2, 2]], [[3], [1, -2]]]$
$[12, 1, 333, 2, [[3, -11, 17, -11, 3]], [[3], [1, -2]]]$
$[12, 1, 334, 2, [[4, -16, 23, -16, 4]], [[3], [1, 1]]]$
$[12, 1, 355, 2, [[1, -3, 2, 1, 2, -3, 1]], [[2], [1, -1, -1]]]$
$[12, 1, 356, 2, [[1, -5, 14, -21, 14, -5, 1]], [[2], [1, -1, -1]]]$
$[12, 1, 357, 2, [[1, -9, 24, -31, 24, -9, 1]], [[2], [1, -1, -1]]]$
$[12, 1, 379, 2, [[2, -7, 14, -17, 14, -7, 2]], [[1, -1, 1]]]$
$[12, 1, 380, 2, [[2, -9, 18, -23, 18, -9, 2]], [[1, -1, 1]]]$
$[12, 1, 386, 2, [[2, -4, 3, 2, -5, 2, 3, -4, 2]], [[1, -1, 1]]]$
$[12, 1, 387, 2, [[1, -5, 6, -2, -1, -2, 6, -5, 1]], [[3], [1, -1, 1]]]$
$[12, 1, 388, 2, [[1, -5, 10, -13, 10, -5, 1]], [[3], [1, -1, 1]]]$
$[12, 1, 389, 2, [[1, -9, 24, -31, 24, -9, 1]], [[3], [1, -1, 1]]]$
$[12, 1, 393, 2, [[3, -12, 19, -12, 3]], [[2], [1, -1, -1]]]$
$[12, 1, 394, 2, [[1, -6, 11, -6, 1]], [[2], [1, 1, -1]]]$
$[12, 1, 397, 2, [[1, -5, 11, -15, 11, -5, 1]], [[7], [1, -6]]]$
$[12, 1, 402, 2, [[1, 1, -3, 3, -3, 1, 1]], [[3], [1, -2]]]$
$[12, 1, 403, 2, [[1, -3, 1, 1, 1, -3, 1]], [[3], [1, 1]]]$
$[12, 1, 414, 2, [[2, -6, 9, -6, 2]], [[5], [1, 1]]]$
$[12, 1, 420, 2, [[1, -7, 19, -27, 19, -7, 1]], [[3], [1, -2]]]$
$[12, 1, 436, 2, [[1, -5, 6, -5, 6, -5, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 440, 2, [[2, -9, 18, -23, 18, -9, 2]], [[3, -3, 3], [1, 0, 0, 1]]]$
$[12, 1, 442, 2, [[3, -10, 13, -10, 3]], [[2], [1, -1, 1]]]$
$[12, 1, 460, 2, [[1, -6, 15, -19, 15, -6, 1]], [[3], [1, -1, 1]]]$
$[12, 1, 462, 2, [[1, -6, 11, -6, 1]], [[2], [1, -1, -1]]]$
$[12, 1, 480, 2, [[1, -6, 19, -29, 19, -6, 1]], [[9], [1, -8]]]$
$[12, 1, 494, 2, [[1, -5, 8, -6, 5, -6, 8, -5, 1]], [[3], [1, -1, -2]]]$
$[12, 1, 495, 2, [[1, -3, 6, -7, 6, -3, 1]], [[9], [3, -6], [1, -1, -5]]]$
$[12, 1, 496, 2, [[1, -11, 28, -37, 28, -11, 1]], [[3], [1, 2, -2]]]$
$[12, 1, 498, 2, [[5, -20, 29, -20, 5]], [[2], [1, -1, 1]]]$
$[12, 1, 505, 2, [[1, -8, 24, -33, 24, -8, 1]], [[3], [1, -2]]]$
$[12, 1, 508, 2, [[1, -4, 10, -16, 19, -16, 10, -4, 1]], [[2, -2, 2], [1, 0, -1, 2, -1]]]$
$[12, 1, 509, 2, [[1, -5, 8, -7, 7, -7, 8, -5, 1]], [[7], [1, 1]]]$
$[12, 1, 510, 2, [[1, -11, 29, -39, 29, -11, 1]], [[11], [1, 1]]]$
$[12, 1, 518, 2, [[1, 0, -6, 16, -21, 16, -6, 0, 1]], [[2, -2, 2], [1, 0, -1, 2, -1]]]$
$[12, 1, 526, 2, [[1, -4, 6, -3, 1, -3, 6, -4, 1]], [[3], [1, 0, 1]]]$
$[12, 1, 533, 2, [[1, -4, 8, -12, 13, -12, 8, -4, 1]], [[2], [1, 0, -1, 0, 1]]]$
$[12, 1, 546, 2, [[2, -8, 14, -15, 14, -8, 2]], [[3], [1, -2]]]$
$[12, 1, 549, 2, [[1, -3, 6, -7, 6, -3, 1]], [[1, -1, 1]]]$
$[12, 1, 553, 3, [[4, -20, 33, -20, 4]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[12, 1, 554, 3, [[2, -7, 9, -7, 2]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[12, 1, 555, 3, [[1, -9, 33, -49, 33, -9, 1]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[12, 1, 556, 3, [[4, -20, 33, -20, 4]], [[2, -5, 2]], [[3], [1, -2]]]$
$[12, 1, 565, 2, [[2, -7, 9, -9, 9, -7, 2]], [[3], [1, 1]]]$
$[12, 1, 567, 2, [[3, -13, 26, -33, 26, -13, 3]], [[1, -1, 1]]]$
$[12, 1, 570, 2, [[1, -3, 6, -8, 9, -8, 6, -3, 1]], [[1, -1, 1]]]$
$[12, 1, 571, 2, [[1, -3, 4, -4, 3, -4, 4, -3, 1]], [[1, -1, 1]]]$

$[12, 1, 574, 2, [[1, -1, 0, 2, -3, 3, -3, 2, 0, -1, 1]], [[1, -1, 1]]]$
$[12, 1, 581, 2, [[3, -5, 4, -3, 4, -5, 3]], [[3], [1, -1, 1]]]$
$[12, 1, 582, 2, [[1, -2, 3, -2, 1]], [[3], [1, -1, 1]]]$
$[12, 1, 583, 2, [[5, -16, 21, -16, 5]], [[3], [1, -1, 1]]]$
$[12, 1, 592, 2, [[1, -4, 8, -8, 7, -8, 8, -4, 1]], [[7], [1, 1]]]$
$[12, 1, 598, 2, [[1, -8, 24, -33, 24, -8, 1]], [[3], [1, 1]]]$
$[12, 1, 600, 2, [[4, -9, 12, -13, 12, -9, 4]], [[3], [1, 2, 1]]]$
$[12, 1, 601, 2, [[1, -3, 6, -7, 6, -3, 1]], [[3], [1, 0, 0, -2]]]$
$[12, 1, 602, 2, [[1, -9, 24, -31, 24, -9, 1]], [[3], [1, 2, 1]]]$
$[12, 1, 604, 2, [[1, -4, 10, -16, 19, -16, 10, -4, 1]], [[2, -2, 2], [1, 0, -1, 2, -1]]]$
$[12, 1, 605, 2, [[1, -2, 0, 4, -7, 4, 0, -2, 1]], [[2, -2, 2], [1, 0, -1, 2, -1]]]$
$[12, 1, 611, 2, [[2, -9, 17, -19, 17, -9, 2]], [[5], [1, -4]]]$
$[12, 1, 617, 2, [[1, -3, 0, 3, 0, -3, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 622, 2, [[1, -6, 14, -19, 19, -19, 14, -6, 1]], [[3], [1, 1]]]$
$[12, 1, 626, 2, [[2, -15, 36, -47, 36, -15, 2]], [[3], [1, -1, 1]]]$
$[12, 1, 630, 2, [[1, -4, 3, 1, 3, -4, 1]], [[3], [1, 0, -2]]]$
$[12, 1, 636, 2, [[1, -7, 19, -27, 19, -7, 1]], [[9], [1, -8]]]$
$[12, 1, 637, 2, [[2, -13, 32, -41, 32, -13, 2]], [[3], [1, -1, -2]]]$
$[12, 1, 642, 3, [[1, 7, -15, 7, 1]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[12, 1, 643, 2, [[2, -8, 10, -9, 10, -8, 2]], [[7], [1, -6]]]$
$[12, 1, 651, 2, [[2, -11, 28, -37, 28, -11, 2]], [[5], [1, 1, -4]]]$
$[12, 1, 652, 2, [[1, -5, 15, -30, 37, -30, 15, -5, 1]], [[3], [1, 0, -2]]]$
$[12, 1, 654, 2, [[2, -8, 9, -7, 9, -8, 2]], [[3], [1, -2]]]$
$[12, 1, 660, 2, [[1, 3, -12, 17, -12, 3, 1]], [[2], [1, 1, -1]]]$
$[12, 1, 666, 2, [[1, -4, 10, -16, 19, -16, 10, -4, 1]], [[1, -1, 1]]]$
$[12, 1, 669, 2, [[2, -10, 23, -29, 23, -10, 2]], [[3], [1, 2, -2]]]$
$[12, 1, 701, 2, [[3, -16, 25, -16, 3]], [[3], [1, -2]]]$
$[12, 1, 706, 2, [[1, -4, 6, -8, 11, -8, 6, -4, 1]], [[1, -2, 1, -2, 1]]]$
$[12, 1, 714, 2, [[3, -14, 30, -37, 30, -14, 3]], [[3], [1, 0, -2]]]$
$[12, 1, 717, 2, [[1, -4, 12, -17, 12, -4, 1]], [[3], [1, 0, -2]]]$
$[12, 1, 737, 2, [[2, -7, 9, -9, 9, -7, 2]], [[3], [1, 1]]]$
$[12, 1, 742, 2, [[2, -9, 21, -27, 21, -9, 2]], [[3], [1, 0, -2]]]$
$[12, 1, 745, 2, [[1, -10, 30, -43, 30, -10, 1]], [[5], [1, 1]]]$
$[12, 1, 746, 2, [[2, -11, 28, -37, 28, -11, 2]], [[5], [1, -4, -4]]]$
$[12, 1, 752, 2, [[2, -9, 20, -25, 20, -9, 2]], [[2], [1, -1, -1]]]$
$[12, 1, 756, 2, [[2, -10, 23, -29, 23, -10, 2]], [[3], [1, -1, 1]]]$
$[12, 1, 757, 2, [[1, -9, 28, -39, 28, -9, 1]], [[2], [1, -1, -1]]]$
$[12, 1, 760, 2, [[2, -11, 29, -41, 29, -11, 2]], [[5], [1, 1]]]$
$[12, 1, 764, 2, [[3, -5, 3, -1, 3, -5, 3]], [[3], [1, 0, -2]]]$
$[12, 1, 779, 2, [[2, -11, 26, -33, 26, -11, 2]], [[2], [1, -1, -1]]]$
$[12, 1, 781, 2, [[1, -4, 10, -19, 25, -19, 10, -4, 1]], [[5], [1, -4, 1]]]$
$[12, 1, 798, 2, [[1, -7, 22, -31, 22, -7, 1]], [[2], [1, -1, -1]]]$
$[12, 1, 806, 2, [[4, -10, 12, -11, 12, -10, 4]], [[3], [1, 1, -2, -2]]]$
$[12, 1, 813, 2, [[2, -9, 18, -23, 18, -9, 2]], [[9], [3, -6], [1, 2, 1]]]$
$[12, 1, 817, 2, [[2, -6, 10, -13, 10, -6, 2]], [[7], [1, -6]]]$
$[12, 1, 837, 2, [[1, -7, 21, -35, 41, -35, 21, -7, 1]], [[13], [1, 1]]]$
$[12, 1, 838, 2, [[1, -6, 11, -6, 1]], [[1, -3, 1]]]$
$[12, 1, 839, 2, [[1, -7, 18, -23, 23, -23, 18, -7, 1]], [[11], [1, 1]]]$
$[12, 1, 840, 2, [[1, -6, 16, -24, 25, -24, 16, -6, 1]], [[2], [1, 0, -1, 0, 1]]]$
$[12, 1, 843, 2, [[3, -15, 34, -43, 34, -15, 3]], [[7], [1, 1]]]$
$[12, 1, 844, 2, [[3, -12, 16, -13, 16, -12, 3]], [[5], [1, 1]]]$
$[12, 1, 846, 2, [[5, -20, 31, -20, 5]], [[9], [1, 1]]]$
$[12, 1, 847, 2, [[1, -7, 18, -23, 18, -7, 1]], [[1, -3, 1]]]$
$[12, 1, 848, 2, [[1, -5, 12, -16, 17, -16, 12, -5, 1]], [[3], [1, -1, -2, 2, 1]]]$
$[12, 1, 869, 2, [[2, -12, 32, -43, 32, -12, 2]], [[3], [1, -2]]]$
$[12, 1, 873, 2, [[1, -5, 10, -16, 21, -16, 10, -5, 1]], [[4], [1, -3, 1]]]$

$[12, 1, 874, 2, [[1, -5, 14, -26, 31, -26, 14, -5, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 876, 2, [[2, -8, 18, -25, 18, -8, 2]], [[9], [1, -8]]]$
$[12, 1, 877, 2, [[1, -7, 20, -32, 35, -32, 20, -7, 1]], [[2], [1, -1, 1]]]$
$[12, 1, 878, 2, [[1, -11, 34, -47, 34, -11, 1]], [[2], [1, 1, -1]]]$
$[12, 1, 879, 2, [[1, -6, 17, -30, 35, -30, 17, -6, 1]], [[2], [1, -1, 1, 1, -1]]]$
$[12, 1, 881, 2, [[9, -30, 43, -30, 9]], [[3, -5, 3]]]$
$[12, 1, 883, 2, [[3, -12, 22, -25, 22, -12, 3]], [[3], [1, -2]]]$
$[12, 1, 887, 2, [[1, -6, 16, -25, 29, -25, 16, -6, 1]], [[5], [1, -4]]]$
$[12, 1, 888, 2, [[1, -1, -1, 6, -11, 13, -11, 6, -1, -1, 1]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[1]]]$

A.2 Prime 13 and 14 crossing knots with chain length = 3.

$[13, 0, 1232, 3, [[2, -11, 31, -54, 65, -54, 31, -11, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 0, 1238, 3, [[2, -11, 31, -54, 65, -54, 31, -11, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 0, 1436, 3, [[7, -27, 54, -67, 54, -27, 7]], [[2, -2, 2], [1, 0, 1, 0, 1]], [[2], [1, -1, 1]]]$
$[13, 0, 1638, 3, [[2, -9, 21, -34, 39, -34, 21, -9, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 0, 1786, 3, [[1, -9, 33, -65, 81, -65, 33, -9, 1]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[13, 0, 2720, 3, [[4, -24, 57, -73, 57, -24, 4]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[13, 0, 2727, 3, [[4, -24, 57, -73, 57, -24, 4]], [[2, -5, 2]], [[3], [1, -2]]]$
$[13, 0, 3072, 3, [[2, -9, 21, -34, 39, -34, 21, -9, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 0, 4740, 3, [[4, -13, 25, -34, 37, -34, 25, -13, 4]], [[2, -2, 2], [1, 0, 1, 0, 1]], [[2], [1, -1, 1]]]$
$[13, 0, 4877, 3, [[16, -47, 63, -47, 16]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[13, 1, 2407, 3, [[2, -7, 15, -22, 25, -22, 15, -7, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 1, 2408, 3, [[1, -5, 10, -13, 10, -5, 1]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 1, 2409, 3, [[2, -3, -1, 10, -15, 10, -1, -3, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 1, 2410, 3, [[3, -11, 22, -27, 22, -11, 3]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 1, 2411, 3, [[1, -5, 10, -13, 10, -5, 1]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 1, 2412, 3, [[2, -7, 15, -22, 25, -22, 15, -7, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 1, 2413, 3, [[5, -21, 42, -53, 42, -21, 5]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 1, 2414, 3, [[3, -11, 22, -27, 22, -11, 3]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[13, 1, 2790, 3, [[2, -5, 5, -2, -1, -2, 5, -5, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 1975, 3, [[5, -24, 62, -104, 123, -104, 62, -24, 5]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 1977, 3, [[1, -7, 25, -56, 89, -103, 89, -56, 25, -7, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 1983, 3, [[1, -7, 25, -56, 89, -103, 89, -56, 25, -7, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 2438, 3, [[1, -7, 23, -50, 77, -89, 77, -50, 23, -7, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 2455, 3, [[1, -7, 23, -50, 77, -89, 77, -50, 23, -7, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 2456, 3, [[1, -7, 23, -50, 77, -89, 77, -50, 23, -7, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 2457, 3, [[2, -9, 24, -44, 62, -69, 62, -44, 24, -9, 2]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 3107, 3, [[2, -15, 53, -106, 131, -106, 53, -15, 2]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 3275, 3, [[2, -13, 39, -70, 85, -70, 39, -13, 2]], [[4], [2, 2, -2], [1, 0, 1, 0, -3]], [[2], [1, -1, -1]]]$
$[14, 0, 3277, 3, [[2, -13, 39, -70, 85, -70, 39, -13, 2]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 3288, 3, [[4, -21, 47, -70, 79, -70, 47, -21, 4]], [[4], [2, -2, -2], [1, 0, 1, 0, -3]], [[2], [1, -1, 1]]]$
$[14, 0, 3734, 3, [[5, -24, 62, -104, 123, -104, 62, -24, 5]], [[9], [3, 3], [1, -1, -2]], [[3], [1, 1]]]$
$[14, 0, 3735, 3, [[1, -7, 25, -56, 89, -103, 89, -56, 25, -7, 1]], [[9], [3, -6], [1, -1, 7]], [[3], [1, 1]]]$
$[14, 0, 4337, 3, [[2, -15, 53, -106, 131, -106, 53, -15, 2]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 4346, 3, [[2, -15, 45, -82, 99, -82, 45, -15, 2]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 4353, 3, [[2, -15, 47, -86, 105, -86, 47, -15, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 4363, 3, [[2, -15, 45, -82, 99, -82, 45, -15, 2]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 4637, 3, [[14, -61, 128, -161, 128, -61, 14]], [[9], [3, -6], [1, -1, -2]], [[3], [1, 1]]]$
$[14, 0, 5139, 3, [[7, -37, 82, -105, 82, -37, 7]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 5141, 3, [[7, -37, 82, -105, 82, -37, 7]], [[4], [2, -2, 2], [1, 0, -1, 2, 3]], [[2], [1, -1, 1]]]$
$[14, 0, 5241, 3, [[4, -17, 41, -66, 77, -66, 41, -17, 4]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 5282, 3, [[2, -15, 45, -82, 99, -82, 45, -15, 2]], [[4], [2, 2, 2], [1, 0, -1, 2, 3]], [[2], [1, -1, 1]]]$

$[14, 0, 5456, 3, [[1, -8, 31, -75, 123, -145, 123, -75, 31, -8, 1]], [[9], [3, 3], [1, -1, 7]], [[3], [1, 1]]]$
$[14, 0, 5681, 3, [[3, -24, 82, -158, 195, -158, 82, -24, 3]], [[9], [3, -6], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 0, 5816, 3, [[2, -13, 39, -70, 85, -70, 39, -13, 2]], [[4], [2, -2, -2], [1, 0, -1, -2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 5825, 3, [[2, -13, 37, -66, 79, -66, 37, -13, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 5828, 3, [[2, -13, 37, -66, 79, -66, 37, -13, 2]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 6088, 3, [[3, -21, 70, -134, 165, -134, 70, -21, 3]], [[9], [3, 3], [1, -1, 7]], [[3], [1, 1]]]$
$[14, 0, 6209, 3, [[2, -13, 31, -46, 53, -46, 31, -13, 2]], [[4], [2, -2, 2], [1, 0, -1, -2, 3]], [[2], [1, -1, 1]]]$
$[14, 0, 7282, 3, [[2, -17, 64, -138, 179, -138, 64, -17, 2]], [[9], [3, 3], [1, 2, -8]], [[3], [1, 1]]]$
$[14, 0, 7690, 3, [[2, -19, 75, -163, 211, -163, 75, -19, 2]], [[9], [3, 3], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 0, 9687, 3, [[1, -7, 23, -50, 77, -89, 77, -50, 23, -7, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 9694, 3, [[2, -9, 24, -44, 62, -69, 62, -44, 24, -9, 2]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 10043, 3, [[2, -15, 45, -73, 81, -73, 45, -15, 2]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[14, 0, 11983, 3, [[1, -8, 31, -75, 123, -145, 123, -75, 31, -8, 1]], [[9], [3, 3], [1, 2, 1]], [[3], [1, 1]]]$
$[14, 0, 11986, 3, [[5, -24, 62, -104, 123, -104, 62, -24, 5]], [[9], [3, -6], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 0, 11988, 3, [[4, -17, 41, -66, 77, -66, 41, -17, 4]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 11991, 3, [[4, -21, 47, -70, 79, -70, 47, -21, 4]], [[4], [2, -2, -2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 11993, 3, [[1, -7, 25, -56, 89, -103, 89, -56, 25, -7, 1]], [[9], [3, 3], [1, -1, 7]], [[3], [1, 1]]]$
$[14, 0, 11995, 3, [[2, -13, 31, -46, 53, -46, 31, -13, 2]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 0, 12213, 3, [[8, -36, 66, -77, 66, -36, 8]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[14, 0, 12326, 3, [[2, -19, 73, -150, 187, -150, 73, -19, 2]], [[9, -27, 9], [3, -6, -6, 3], [1, -1, 5, -28, 10]],$
$[[3], [1, 1]]]$
$[14, 0, 12787, 3, [[7, -42, 108, -145, 108, -42, 7]], [[9], [3, -6], [1, 2, 1]], [[3], [1, -2]]]$
$[14, 0, 13401, 3, [[2, -9, 22, -38, 50, -55, 50, -38, 22, -9, 2]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 0, 14268, 3, [[5, -22, 46, -66, 73, -66, 46, -22, 5]], [[9], [3, 3], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 0, 14270, 3, [[1, -7, 25, -52, 75, -85, 75, -52, 25, -7, 1]], [[9], [3, 3], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 0, 14496, 3, [[8, -32, 67, -96, 107, -96, 67, -32, 8]], [[9], [3, -6], [1, -1, 7]], [[3], [1, 1]]]$
$[14, 0, 17247, 3, [[9, -58, 147, -197, 147, -58, 9]], [[25], [5, -20], [1, -3, -4]], [[5], [1, -4]]]$
$[14, 0, 17728, 3, [[4, -15, 31, -46, 51, -46, 31, -15, 4]], [[2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 1, 5630, 3, [[3, -14, 36, -60, 71, -60, 36, -14, 3]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5631, 3, [[1, -4, 10, -16, 19, -16, 10, -4, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5632, 3, [[1, -4, 10, -16, 19, -16, 10, -4, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5633, 3, [[1, -6, 16, -28, 33, -28, 16, -6, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5634, 3, [[1, -4, 10, -16, 19, -16, 10, -4, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5635, 3, [[1, -6, 16, -28, 33, -28, 16, -6, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5636, 3, [[3, -16, 42, -72, 85, -72, 42, -16, 3]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5637, 3, [[1, -4, 10, -16, 19, -16, 10, -4, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5638, 3, [[3, -14, 36, -60, 71, -60, 36, -14, 3]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5642, 3, [[1, -5, 15, -30, 45, -51, 45, -30, 15, -5, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5643, 3, [[1, -3, 5, -4, 1, 1, 1, -4, 5, -3, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5644, 3, [[1, -1, -5, 22, -43, 53, -43, 22, -5, -1, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5645, 3, [[1, -3, 5, -4, 1, 1, 1, -4, 5, -3, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5646, 3, [[1, -5, 15, -30, 45, -51, 45, -30, 15, -5, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5647, 3, [[1, -5, 15, -30, 45, -51, 45, -30, 15, -5, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 5654, 3, [[1, -5, 15, -30, 45, -51, 45, -30, 15, -5, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 6284, 3, [[1, -5, 13, -24, 33, -37, 33, -24, 13, -5, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 6285, 3, [[1, -3, 3, 2, -11, 15, -11, 2, 3, -3, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 6286, 3, [[1, -5, 13, -24, 33, -37, 33, -24, 13, -5, 1]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 6302, 3, [[2, -7, 14, -18, 18, -17, 18, -18, 14, -7, 2]], [[1, -2, 3, -2, 1]], [[1, -1, 1]]]$
$[14, 1, 7211, 3, [[5, -27, 62, -79, 62, -27, 5]], [[4], [2, 2, -2], [1, 0, 1, 0, 1]], [[2], [1, -1, 1]]]$
$[14, 1, 7212, 3, [[1, -3, 6, -7, 6, -3, 1]], [[4], [2, -2, -2], [1, 0, -1, -2, -1]], [[2], [1, -1, -1]]]$
$[14, 1, 7213, 3, [[1, -3, 6, -7, 6, -3, 1]], [[4], [2, 2, -2], [1, 0, 1, 0, 1]], [[2], [1, -1, 1]]]$
$[14, 1, 7214, 3, [[3, -21, 50, -65, 50, -21, 3]], [[4], [2, 2, 2], [1, 0, 1, 0, -3]], [[2], [1, -1, -1]]]$
$[14, 1, 7215, 3, [[2, -7, 5, 6, -13, 6, 5, -7, 2]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 1, 7216, 3, [[2, -11, 29, -50, 59, -50, 29, -11, 2]], [[4], [2, -2, -2], [1, 0, -1, -2, -1]], [[2], [1, -1, -1]]]$
$[14, 1, 7217, 3, [[2, -11, 29, -50, 59, -50, 29, -11, 2]], [[4], [2, -2, 2], [1, 0, 1, 0, -3]], [[2], [1, -1, 1]]]$
$[14, 1, 7381, 3, [[3, -13, 26, -33, 26, -13, 3]], [[4], [2, 2, 2], [1, 0, -1, 2, 3]], [[2], [1, 1, -1]]]$

$[14, 1, 7382, 3, [[1, -11, 30, -39, 30, -11, 1]], [[4], [2, -2, -2], [1, 0, -1, -2, -1]], [[2], [1, -1, -1]]]$
$[14, 1, 7383, 3, [[2, -9, 15, -14, 13, -14, 15, -9, 2]], [[4], [2, -2, 2], [1, 0, 1, 0, 1]], [[2], [1, -1, 1]]]$
$[14, 1, 7384, 3, [[3, -13, 26, -33, 26, -13, 3]], [[4], [2, -2, 2], [1, 0, -1, 2, 3]], [[2], [1, -1, 1]]]$
$[14, 1, 7393, 3, [[2, -11, 21, -26, 27, -26, 21, -11, 2]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 1, 8968, 3, [[5, -14, 16, -6, -1, -6, 16, -14, 5]], [[9], [3, 3], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 1, 8969, 3, [[4, -15, 30, -37, 30, -15, 4]], [[9], [3, -6], [1, -1, -2]], [[3], [1, 1]]]$
$[14, 1, 8970, 3, [[6, -31, 68, -87, 68, -31, 6]], [[9], [3, 3], [1, -1, -2]], [[3], [1, -2]]]$
$[14, 1, 8971, 3, [[1, -5, 11, -8, -5, 13, -5, -8, 11, -5, 1]], [[9], [3, 3], [1, 2, -8]], [[3], [1, 1]]]$
$[14, 1, 8972, 3, [[1, -7, 22, -40, 49, -40, 22, -7, 1]], [[9], [3, 3], [1, -1, 7]], [[3], [1, 1]]]$
$[14, 1, 8973, 3, [[1, -7, 26, -54, 67, -54, 26, -7, 1]], [[9], [3, 3], [1, 2, -8]], [[3], [1, 1]]]$
$[14, 1, 10654, 3, [[2, -11, 29, -50, 59, -50, 29, -11, 2]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 1, 10655, 3, [[2, -7, 5, 6, -13, 6, 5, -7, 2]], [[4], [2, 2, -2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 1, 10656, 3, [[1, -3, 6, -7, 6, -3, 1]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 1, 10657, 3, [[3, -21, 50, -65, 50, -21, 3]], [[4], [2, -2, -2], [1, 0, -1, -2, 3]], [[2], [1, -1, 1]]]$
$[14, 1, 10658, 3, [[5, -27, 62, -79, 62, -27, 5]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 1, 10664, 3, [[2, -11, 21, -26, 27, -26, 21, -11, 2]], [[4], [2, -2, 2], [1, 0, 1, 0, 1]], [[2], [1, -1, 1]]]$
$[14, 1, 10665, 3, [[3, -13, 26, -33, 26, -13, 3]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, 1]]]$
$[14, 1, 10666, 3, [[1, -11, 30, -39, 30, -11, 1]], [[4], [2, -2, 2], [1, 0, -1, 2, -1]], [[2], [1, -1, -1]]]$
$[14, 1, 11342, 3, [[4, -15, 30, -37, 30, -15, 4]], [[9], [3, -6], [1, 2, 1]], [[3], [1, -2]]]$
$[14, 1, 12491, 3, [[1, -6, 17, -27, 29, -29, 29, -27, 17, -6, 1]], [[9], [3, -6], [1, 2, 1]], [[3], [1, 1]]]$
$[14, 1, 12492, 3, [[1, -3, 6, -7, 6, -3, 1]], [[9], [3, 3], [1, -1, 7]], [[3], [1, 1]]]$
$[14, 1, 12493, 3, [[2, -13, 45, -88, 109, -88, 45, -13, 2]], [[9], [3, 3], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 1, 12776, 3, [[3, -14, 36, -60, 71, -60, 36, -14, 3]], [[9], [3, 3], [1, -1, -2]], [[3], [1, 1]]]$
$[14, 1, 12777, 3, [[3, -4, -10, 38, -53, 38, -10, -4, 3]], [[9], [3, 3], [1, 2, -8]], [[3], [1, 1]]]$
$[14, 1, 12778, 3, [[3, -14, 36, -60, 71, -60, 36, -14, 3]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[14, 1, 12954, 3, [[2, -9, 15, -14, 13, -14, 15, -9, 2]], [[4], [2, -2, 2], [1, 0, -1, -2, -1]], [[2], [1, -1, 1]]]$
$[14, 1, 13279, 3, [[1, -7, 22, -40, 49, -40, 22, -7, 1]], [[9], [3, 3], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 1, 14347, 3, [[2, -9, 18, -23, 18, -9, 2]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[14, 1, 14348, 3, [[1, -10, 38, -78, 97, -78, 38, -10, 1]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[14, 1, 14349, 3, [[2, -15, 45, -65, 45, -15, 2]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[14, 1, 19386, 3, [[1, -3, 6, -7, 6, -3, 1]], [[9], [3, 3], [1, -1, 7]], [[3], [1, 1]]]$
$[14, 1, 19734, 3, [[2, -13, 32, -41, 32, -13, 2]], [[9], [3, 3], [1, -1, -2]], [[3], [1, 1]]]$
$[14, 1, 19735, 3, [[1, -11, 44, -90, 113, -90, 44, -11, 1]], [[9], [3, 3], [1, -1, -1, 2, -8]], [[3], [1, -2]]]$
$[14, 1, 19736, 3, [[2, -13, 32, -41, 32, -13, 2]], [[9], [3, -6], [1, 2, 1]], [[3], [1, -2]]]$
$[14, 1, 20057, 3, [[1, -6, 21, -41, 51, -41, 21, -6, 1]], [[9], [3, 3], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 1, 20058, 3, [[1, -5, 11, -15, 17, -15, 11, -5, 1]], [[9], [3, -6], [1, -1, -2]], [[3], [1, -2]]]$
$[14, 1, 20076, 3, [[1, -7, 20, -27, 13, 1, 13, -27, 20, -7, 1]], [[9], [3, 3], [1, 2, -8]], [[3], [1, 1]]]$
$[14, 1, 23439, 3, [[2, -4, -5, 24, -35, 24, -5, -4, 2]], [[9], [3, 3], [1, -1, 7]], [[3], [1, -2]]]$
$[14, 1, 24690, 3, [[2, -12, 29, -39, 29, -12, 2]], [[25], [5, -20], [1, -3, -4]], [[5], [1, 1]]]$
$[14, 1, 26223, 3, [[1, -6, 15, -25, 31, -25, 15, -6, 1]], [[25], [5, 5], [1, -3, -4]], [[5], [1, 1]]]$

Appendix B

The CD-ROM

The CDROM contains the files

`README`	`3cross.txt`	`4cross.txt`
`5cross.txt`	`6cross.txt`	`7cross.txt`
`8cross.txt`	`9cross.txt`	`10cross.txt`
`11cross.txt`	`12cross.txt`	`13cross.txt`
`14cross.txt`	`knots1-14.txt`	`Algorithms.mws`
`Process.mws`		

The .txt files contain Chatelet Bases for the Alexander ideals of prime knots of up to 14 crossings. It is intended that these files be read and used with the *Maple* worksheet `Process.mws` but they can also be browsed with any text file viewer. In the worksheet there is an explanation of the contents of the .txt files which is reproduced below. The contents of the .txt files were calculated using the program *Maple* running on the Sun Ultra-80 computers of the High Performance Computing service of the ITS at the University of Durham. We should mention that these calculations used the tables of knot diagrams contained in the program *Knotscape* by J. Hoste and M.B. Thistlethwaite.

The other *Maple* worksheet, `Algorithms.mws`, contains implementations of the algorithms described in the thesis. These are the algorithms that were used to calculate the .txt files.

Explanation of the `.txt` files:

They were obtained by running the Seifert matrix and Châtelet basis algorithms described in the thesis on the tables of diagram encodings of diagrams of prime knots from *Knotscape*. There is one file for each of the crossing numbers $3, \ldots, 14$ and then a file `knots1-14.txt` containing all of the knots of up to 14 crossings.

The first line of each file is a *Maple* list $[n]$, where n is the number of knots in that file. Then each subsequent line contains a maple list K, containing the Châtelet bases of the Alexander ideals of a knot. K is structures as follows. (n.b. The entries of a maple list are seperated by commas and enclosed in square brackets. The i-th entry of the list K is referred to by $K[i]$).

$K[1]$ is the crossing number of the knot, $K[2]$ is 0 or 1 depending on whether the knot is alternating or non-alternating respectively, $K[3]$ is the position of the knot in the Knotscape table, $K[4]$ is the length of the chain of non-trivial Alexander ideals. Then $K[4], K[5], ..$etc is the chain of Alexander ideals, each one represented as a list of the Châtelet generators for the ideal. So $K[4]$ always consists of a single element, the Alexander polynomial itself. The sequence $K[4], K[5], \ldots$ etc. terminates with the list [1] representing the first trivial ideal in the chain, all the ideals after that one being trivial as well of course.

Example. The last line of the file `10cross.txt` is the list

$$[\mathtt{10}, \mathtt{1}, \mathtt{42}, \mathtt{2}, [\mathtt{t}^6 - \mathtt{6} * \mathtt{t}^5 + \mathtt{11} * \mathtt{t}^4 - \mathtt{13} * \mathtt{t}^3 + \mathtt{11} * \mathtt{t}^2 - \mathtt{6} * \mathtt{t} + \mathtt{1}], [\mathtt{7}, \mathtt{t} + \mathtt{1}], [\mathtt{1}]]$$

which represents the 42nd non-alternating 10 crossing prime knot. This knot has two non-trivial Alexander ideals the first of which is generated by the Alexander polynomial

$$t^6 - 6t^5 + 11t^4 - 13t^3 + 11t^2 - 6t + 1$$

and the next ideal has Châtelet basis $\{7, t+1\}$ and the third and subsequent ideals are the whole ring Λ.