FILTRATION OF THE CLASSICAL KNOT CONCORDANCE GROUP AND CASSON-GORDON INVARIANTS

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Abstract. It is known that if any prime power branched cyclic cover of a knot in S^3 is a homology sphere, then the knot has vanishing Casson-Gordon invariants. We construct infinitely many examples of (topologically) non-slice knots in S^3 whose prime power branched cyclic covers are homology spheres. We show that these knots generate an infinite rank subgroup of $\mathcal{F}_{(1.0)}/\mathcal{F}_{(1.5)}$ for which Casson-Gordon invariants vanish in Cochran-Orr-Teichner's filtration of the classical knot concordance group . As a corollary, it follows that Casson-Gordon invariants are not a complete set of obstructions to a second layer of Whitney disks.

1. Introduction

A knot in the 3-sphere is (topologically) slice if it bounds a locally flat 2-disk in the 4-ball. Two knots are said to be (topologically) concordant if the connected sum of one and the mirror image of the other with reversed orientation is slice. (Equivalently, there is a locally flat embedding of an annulus $S^1 \times [0,1]$ into $S^3 \times [0,1]$ whose restrictions to the boundary components give the knots.) This concordance relation is an equivalence relation, and the concordance classes form an abelian group $\mathcal{C}$, the classical knot concordance group, under the connected sum operation. In $\mathcal{C}$, the identity element is the class of slice knots.

In [COT1], Cochran, Orr, and Teichner (henceforth COT) define a geometric filtration of the classical knot concordance group $\mathcal{C}$

$$0 \subset \cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1.5)} \subset \mathcal{F}_{(1.0)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C}$$

where $\mathcal{F}_{(m)}$ is the set of (m)-solvable knots. (See Definition 3.4.) They show that (1.5)-solvable knots have vanishing Casson-Gordon invariants and that $\mathcal{F}_{(2.0)}/\mathcal{F}_{(2.5)} \neq 0$, thus giving the first examples of knots with vanishing Casson-Gordon invariants which are not (topologically) slice. (Refer to [CG] for Casson-Gordon invariants.) In [COT2], they extend their results to show $\mathcal{F}_{(2.0)}/\mathcal{F}_{(2.5)}$ has infinite rank. We improve their results further and prove:

Theorem 1.1 (Main Theorem). *In the above filtration, $\mathcal{F}_{(1.0)}/\mathcal{F}_{(1.5)}$ has an infinite rank subgroup of knots for which Casson-Gordon invariants vanish.*

Theorem 1.1 implies that Casson-Gordon invariants are not a complete set of obstructions to (1.5)-solvability. By contrast to the above result, the examples of [COT1] are (2.0)-solvable.

To show Casson-Gordon invariants of our examples vanish, we use the following theorem of Livingston.

Theorem 1.2. ([Liv, Theorem 0.5]) *A knot K has a prime power branched cyclic cover with nontrivial homology if and only if its Alexander polynomial has a nontrivial factor that is not an n-cyclotomic polynomial with n divisible by three distinct primes.*

The group of examples in Theorem 1.1 have a spanning set of knots with a fixed Seifert form and Alexander polynomial. The shared Alexander polynomial of these generators is $(\Phi_{30})^2$,

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the square of the 30-cyclotomic polynomial. By Theorem 1.2, these generators have *prime power* branched cyclic covers which are homology spheres. (One might compare this to the fact that if every finite branched cyclic cover of a knot is a homology sphere, then its Alexander polynomial is 1, hence the knot is topologically slice by Freedman's work [F].) It follows from the definition of Casson-Gordon invariants that Casson-Gordon invariants vanish for these knots. (See Proposition 6.5.) In fact, since any prime power branched cyclic cover is a homology sphere all the concordance invariants known prior to Cochran-Orr-Teichner's $L^{(2)}$-signature invariants, such as Gilmer's extension of Casson-Gordon invariants ([G]), Kirk and Livingston's twisted Alexander invariants ([KL]), and Letsche's invariants ([Le]), vanish for these knots.

Theorem 1.1 has a significant geometric consequence. Freedman's disk embedding theorem ([F]), together with the Cappell-Shaneson homology surgery approach ([CS]) to classifying knot concordance group, suggest that the Casson-Gordon invariants obstruct the construction of a second layer of Whitney disks for a Cappell-Shaneson surgery kernel of an algebraically slice knot. That this is so was shown in [COT1, Section 8 and 9]. Indeed, in [COT1], they showed that a knot is (1.5)-solvable if and only if for zero surgery on the knot in S^3, there exists an H_1-bordism which contains a spherical Lagrangian admitting a Whitney tower of height (1.5). (See [COT1, Theorem 8.4] and Section 3 in this paper). Since (1.5)-solvable knots have vanishing Casson-Gordon invariants, it follows that Casson-Gordon invariants obstruct a Whitney tower of height (1.5) in the above sense. Precise definitions of a Whitney tower and other terminologies are given in [COT1, Section 8] and are reviewed in Section 3 in this paper.

We briefly discuss Whitney towers here. In 4-manifolds, Whitney disks may no longer be embedded, but may themselves have intersections, which might or might not occur in algebraically cancelling pairs. If these intersections occur in algebraically cancelling pairs, one can construct immersed Whitney disks for these cancelling pairs of points in the usual manner. Very roughly speaking, a Whitney tower is obtained by iterating this procedure. We have the following corollary of Theorem 1.1.

Corollary 1.3. *There is an algebraically slice knot with vanishing Casson-Gordon invariants such that zero surgery on the knot in S^3 does not bound an H_1-bordism which contains a spherical Lagrangian admitting a Whitney tower of height (1.5).*

Proof. It follows from Theorem 1.1 and [COT1, Theorem 8.4]. □

Corollary 1.3 says that Casson-Gordon invariants are not a complete set of obstructions to a second layer of Whitney disks.

To find the knots generating the subgroup in Theorem 1.1, we follow the method of COT. We begin by constructing a ribbon knot with the rational Alexander module $\mathbb{Q}[t,t^{-1}]/(\Phi_{30}(t))^2$. In particular, its Alexander polynomial is $(\Phi_{30}(t))^2$. Henceforth we refer to this ribbon knot as the *seed knot* to the examples of Theorem 1.1. (See Remark 2.3. See [K] for the definition of a ribbon knot. In particular, a ribbon knot is a slice knot.) We modify this seed knot using a family of Arf invariant zero knots in a way described in [COT2, Setion 3] and reviewed in Section 4 in this paper. The resulting knots are shown to have the same Seifert form with the seed knot, so their prime power branched cyclic covers are also homology spheres by Theorem 1.2. Another important fact, which will be used significantly in this paper, is that $\mathbb{Q}[t,t^{-1}]/(\Phi_{30}(t))^2$ has a unique nontrivial proper submodule. (See the proofs of Lemma 6.3, Proposition 6.4 and Theorem 1.1.)

This paper is organized in the following manner. In Section 2, we construct a ribbon knot whose rational Alexander module is cyclic of order $(\Phi_{30}(t))^2$, i.e., $\mathbb{Q}[t,t^{-1}]/(\Phi_{30}(t))^2$. This will be our seed knot. In Section 3, we explain the definition and properties of the Cochran-Orr-Teichner filtration of the classical knot concordance group and its relation to Whitney towers. In Section 4, we discuss how to construct a family of (n)-solvable knots from a given ribbon knot using a certain Arf invariant zero knot. This method, applied to the ribbon knot mentioned above, will be used to construct the generators of the desired subgroup. In Section 5, $L^{(2)}$-signatures and their properties are reviewed. Finally, in Section 6, we provide the construction of a set of generators of the subgroup in Theorem 1.1 and the proof of Theorem 1.1.

Remark 1.4. For any $n \in \mathbb{N}$ which is divisible by at least three distinct primes, we can also find an infinite rank subgroup of $\mathcal{F}_{(1.0)}/\mathcal{F}_{(1.5)}$ that has generators with the Alexander polynomial $(\Phi_n(t))^2$ (the square of the n-cyclotomic polynomial). The only difference in the proof will be finding a seed knot with the rational Alexander module $\mathbb{Q}[t,t^{-1}]/(\Phi_n(t))^2$ as we do for $n = 30$ in Section 2.

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2. Construction of the seed knot

In this section, we construct our seed knot. That is, we will construct a knot which is a ribbon knot and has the rational Alexander module $\mathbb{Q}[t,t^{-1}]/(\Phi_{30}(t))^2$. Then by Theorem 1.2 of C. Livingston, the seed knot will have prime power branched cyclic covers that are homology spheres.

First, we find a Seifert matrix whose Alexander polynomial is $\Phi_{30}(t)$. Recall that $\Phi_{30}(t) = t^8+t^7-t^5-t^4-t^3+t+1$. This can be done by applying Levine's arguments in [L, 14 on page 236] which originated from Seifert [S]. But we will need a Seifert surface that is a boundary connected sum of disks with two bands, and it's not clear how to find such a Seifert surface from the resulting Seifert matrix. So we modify Levine's arguments a little. The final matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -9 & 1 & 0 & 1 & 26 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 26 & 1 & 24 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $\det(A^T - tA) = \Phi_{30}(t)$. One can easily construct a knot, say K_1, and its Seifert surface whose Seifert matrix with respect to a certain choice of basis is the matrix A. The Seifert surface is obtained in the usual way as the boundary connected sum of disks with two bands under proper twists and intertwining among bands. Figure 1 is a part of K_1 and its Seifert surface. The rectangles containing integers symbolize full twists between the two strands which pass vertically through the rectangles. Thus the rectangle labelled $+24$ symbolizes 24 right-handed full twists. Let $u_i, 1 \le i \le 8$, be the simple closed curves on the Seifert surface each of which goes once around a band. With proper orientations, $\{u_i\}_{1\le i\le 8}$ is a basis with respect to which the Seifert matrix is the matrix A. It is known that $A^T - tA$ is a presentation matrix of the rational Alexander module of K_1. By column and row operations on $A^T - tA$ over $\mathbb{Q}[t,t^{-1}]$-coefficients, we determine that the rational Alexander module of K_1 is isomorphic to $\mathbb{Q}[t,t^{-1}]/\Phi_{30}(t)$, whose only generator is represented by a dual of u_8. (A dual of u_8 is a simple closed curve in the complement of the Seifert surface such that it has linking number one with u_8 and no linking with the other u_i's.)

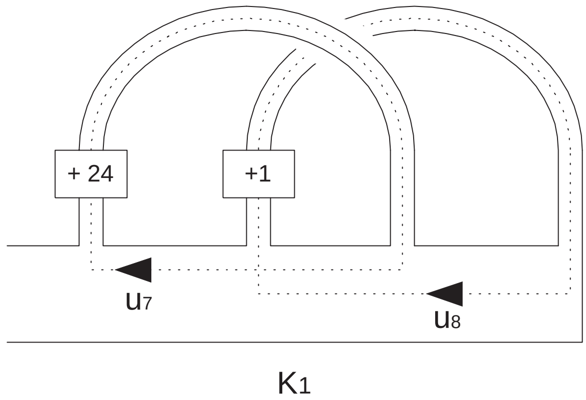

FIGURE 1.

Let $K_2 = K_1\#(-K_1)$, the connected sum of K_1 and its inverse. Then K_2 is a ribbon knot. (See, for instance, [K, Proposition 5.10 p.83].) Its rational Alexander module is $\mathbb{Q}[t,t^{-1}]/\Phi_{30}(t)\oplus \mathbb{Q}[t,t^{-1}]/\Phi_{30}(t)$, and its Seifert surface is obtained as the boundary connected sum of the Seifert surface of K_1 and that of $-K_1$ which is the mirror image of the Seifert surface of K_1. See Figure 2 below. Let M_2 denote zero surgery on K_2 in S^3. The rational Alexander module of K_2, $H_1(M_2;\mathbb{Q}[t,t^{-1}])$, is generated by $v_i, 1 \leq i \leq 16$, where for $1 \leq i \leq 8$, $v_i = u_i$, and for $9 \leq i \leq 16$, v_i is the mirror image of $-u_{(17-i)}$. With this choice of basis, the Seifert matrix of K_2 is the matrix $B = (b_{ij})\,, 1 \leq i,j \leq 16$, defined by

$$b_{ij} = \begin{cases} a_{ij} & : \quad 1 \leq i,j \leq 8 \\ -a_{(17-i)(17-j)} & : \quad 9 \leq i,j \leq 16 \\ 0 & : \quad \text{otherwise} \end{cases}$$

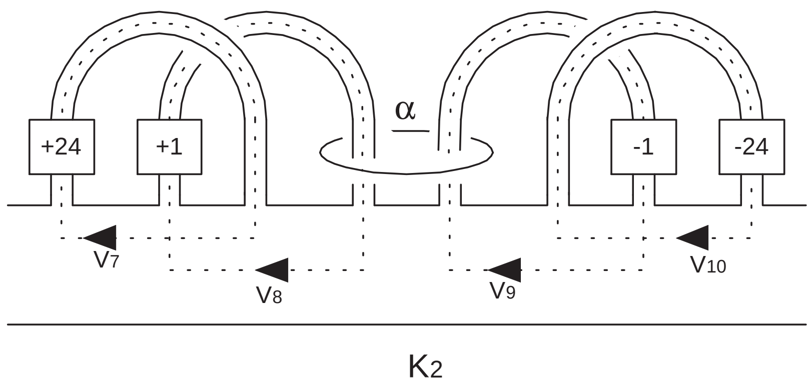

FIGURE 2.

Even though K_2 is a ribbon knot, its rational Alexander module is generated by two elements. In particular, it's not cyclic. So we modify K_2 a little more. Choose an unknot α around the Seifert surface of K_2 as in Figure 2. After $+1$ surgery on α, K_2 will be modified to a new knot, say K_s, in S^3 since the resulting ambient manifold obtained by $+1$ surgery on an unknot in S^3 is homeomorphic with S^3. A part of K_s is illustrated in Figure 3. Let $w_i, 1 \leq i \leq 16$, denote the image of v_i under the surgery. $\{w_i\}_{1\leq i\leq 16}$ is a basis of the Seifert form of K_s. The Seifert matrix with respect to this basis is obtained by changing the matrix B such that only b_{ij} with $7 \leq i, j \leq 10$ are changed from

$$\begin{pmatrix} 24 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -24 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 24 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & -24 \end{pmatrix}.$$

Denote the resulting matrix by C. Let M_s denote zero surgery on K_s in S^3.

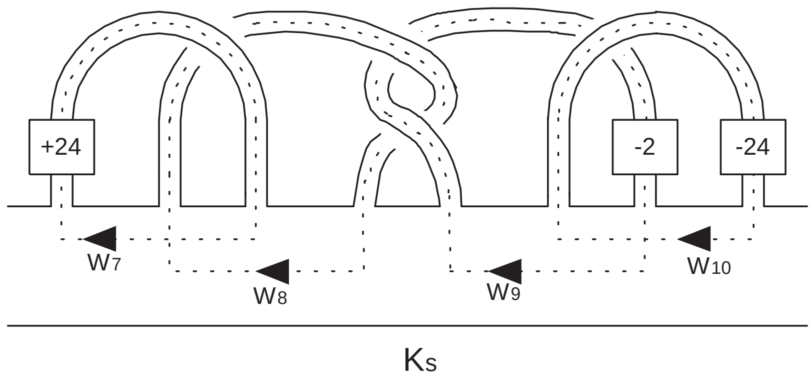

FIGURE 3.

Proposition 2.1. *The rational Alexander module of K_s is cyclic of order $(\Phi_{30}(t))^2$, i.e., $H_1(M_s; \mathbb{Q}[t, t^{-1}]) \cong \mathbb{Q}[t, t^{-1}]/(\Phi_{30}(t))^2$.*

Proof. $C^T - tC$ is a presentation matrix of $H_1(M_s; \mathbb{Q}[t, t^{-1}])$. By column and row operations on $C^T - tC$ over $\mathbb{Q}[t, t^{-1}]$-coefficients, we find out that $H_1(M_s; \mathbb{Q}[t, t^{-1}]) \cong \mathbb{Q}[t, t^{-1}]/(\Phi_{30}(t))^2$ whose only generator is a dual of w_9. □

Proposition 2.2. *K_s is a ribbon knot.*

Proof. One can construct a ribbon disk for the ribbon knot K_2 using the method in [K, Proposition 5.10]. In particular, a ribbon disk can be obtained such that α is disjoint from the ribbon disk and the spanning disk of α has no intersection with the singularities of the ribbon disk. After $+1$ surgery along α, the image of the ribbon disk of K_2 would be a ribbon disk of K_s. □

Remark 2.3. By Proposition 2.1 and 2.2, K_s has the rational Alexander module which is cyclic of order $(\Phi_{30}(t))^2$ and it is a ribbon knot. So K_s is our desired seed knot.

3. FILTERING THE KNOT CONCORDANCE GROUP AND WHITNEY TOWERS

This section and the next two sections are brief expositions of some of the work in [COT1] and [COT2]. These sections contain no new results but serve to clarify ideas and make this paper more self-contained.

In [COT1], Cochran, Orr, and Teichner established a geometric filtration of the knot concordance group $\mathcal{C}$

$$0 \subset \cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1.5)} \subset \mathcal{F}_{(1.0)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C}$$

where $\mathcal{F}_{(n)}$ is the subgroup of *(n)-solvable knots* for $n \in \{0, 0.5, 1.0, 1.5, \cdots\}$. The precise definition of this filtration and Whitney towers, and their relations will be discussed in this section.

Let $G^{(i)}$ denote the *i-th derived subgroup* of a group G, inductively defined by $G^{(0)} \equiv G$ and $G^{(i+1)} \equiv [G^{(i)}, G^{(i)}]$. For a CW-complex W, we denote the regular covering of W corresponding to the subgroup $\pi_1(W)^{(n)}$ by $W^{(n)}$. If W is a spin 4-manifold, then we have the usual intersection form

$$\lambda_n : H_2(W^{(n)}) \times H_2(W^{(n)}) \longrightarrow \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}]$$

A more detailed description of λ_n and the self-intersection invariant μ_n can be found in [W, Chapter 5] and [COT1, Section 7]. In particular, λ_0 is the ordinary intersection form on $H_2(W)$.

Now fix a closed oriented 3-manifold M.

Definition 3.1. An H_1-*bordism* is a 4-dimensional spin manifold W with boundary M such that the inclusion map induces an isomorphism $H_1(M) \xrightarrow{\cong} H_1(W)$.

An (n)-*surface* is a generic immersion of a closed oriented surface F, say $f : F \looparrowright X$, such that $f_*(\pi_1(F)) \leq \pi_1(X)^{(n)}$.

Definition 3.2. Let W be an H_1-bordism such that λ_0 is a hyperbolic form on $H_2(W)$.

1. A *Lagrangian* for λ_0 is a direct summand of $H_2(W)$ of half rank on which λ_0 vanishes.
2. An (n)-*Lagrangian* is a submodule of $H_2(W^{(n)})$ on which λ_n and μ_n vanish and which maps onto a Lagrangian of λ_0 on $H_2(W)$.
3. A *spherical Lagrangian* is a submodule of $\pi_2(W)$ on which λ_n, μ_n $(n \geq 0)$ vanish and which maps onto a Lagrangian of λ_0.
4. For $k \leq n$, (k)-*duals* of an (n)-Lagrangian generated by (n)-surfaces $\ell_1, \cdots, \ell_g$ are (k)-surfaces $d_1, \cdots, d_g$ such that $H_2(W)$ has rank $2g$ and
$$\lambda_k(\ell_i, d_j) = \delta_{i,j}.$$

Before giving the definition of (n)-solvability, we discuss Whitney towers. Let W be a 4-manifold with boundary M and γ be a framed circle in M. A *Whitney disk* is an immersed disk Δ in W which bounds γ and such that the unique framing on the normal bundle of Δ restricts to the given framing on γ. γ is called its *Whitney circle*.

Definition 3.3.
1. A *Whitney tower of height* (0) is a collection $\mathcal{C}_0$ of 2-spheres $S_i \looparrowright W^4$.
2. For $n \in \mathbb{N}$, a *Whitney tower of height* (n) *on* $\mathcal{C}_0$ is a sequence $\mathcal{C}_j = \{\Delta_{j,k}\}_k, j = 1, \ldots, n$, of collections of framed immersed Whitney disks $\Delta_{j,k}$ in general position such that for $j = 2, \ldots, n$, the collection $\mathcal{C}_j$ pairs up all $\mathcal{C}_{j-1}$-(self)-intersections and has interiors disjoint from $\mathcal{C}_1, \ldots, \mathcal{C}_{j-1}$.
3. For $n \in \mathbb{N}$, a *Whitney tower of height* $(n.5)$ *on* $\mathcal{C}_0$ is a sequence $\mathcal{C}_j = \{\Delta_{j,k}\}_k, j = 1, \ldots, n+1$ of collections of framed immersed Whitney disks such that $\mathcal{C}_1, \ldots, \mathcal{C}_n$ consist of a Whitney tower of height n on $\mathcal{C}_0$ and $\mathcal{C}_{n+1}$ pairs up all $\mathcal{C}_n$-(self)-intersections and has interiors disjoint from $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}$ (but $\mathcal{C}_{n+1}$ is allowed to intersect the previous collection $\mathcal{C}_n$).

Refer to [COT1, Section 7] for more details about Whitney towers.

Definition 3.4. A 3-manifold M is (n)*-solvable* (resp. $(n.5)$*-solvable*) if there is an H_1-bordism W which contains an (n)-Lagrangian (resp. $(n+1)$-Lagrangian) with (n)-duals. If M is zero surgery on a knot or a link then the corresponding knot or link is called (n)-solvable (resp. $(n.5)$-solvable).

In Definition 3.4, M is said to be *(n)-solvable via* (resp. *(n.5)-solvable via*) W, and W is called an *(n)-solution* (resp. *(n.5)-solution*) for M.

Theorem 3.5. *(*[COT1, Theorem 8.4, 8.8]*) Let M be a closed oriented 3-manifold and $n \in \{0, 0.5, 1.0, 1.5, \cdots\}$. Then M is (n)-solvable if and only if there is an H_1-bordism which contains a spherical Lagrangian admitting a Whitney tower of height (n).*

Remark 3.6. The exterior of a slice disk is an (n)-solution for the slice knot (and for its zero surgery M) for all n.

4. CONSTRUCTING (n)-SOLVABLE KNOTS

In this section, we obtain an (n)-solvable knot by modifying a given ribbon knot K. For this purpose, we make use of a grafting construction, which produces a satellite knot of K. For further details on this construction and more general cases, the reader should consult [COT2, Section 3].

Simply speaking, seize a collection of parallel strands of K in one hand and tie these into a knot, say J. More precisely, choose a circle, say η, in $S^3 \setminus K$ which bounds an embedded disk in S^3. Now cut open K along this disk and tie all the strands passing through this disk into J, or more exactly, through a tubular neighborhood of J with 0-framing. Then the resulting ambient manifold is still homeomorphic with S^3, and under this identification, we obtain a new knot K' which is the image of K. We denote the resulting knot K' by $K(J, \eta)$. Moreover, this construction has another very useful description. $K(J, \eta)$ is obtained by taking the union of the exterior of η and that of J along the boundary in such a way that the resulting ambient manifold is homeomorphic with S^3.

Making use of the above construction, we get the following proposition due to COT. We outline a proof here for completeness and to establish notation for what follows. M (resp. M_J) denotes zero surgery on K (resp. J) in S^3. Note that a knot is (0)-solvable if and only if it has Arf invariant zero. ([COT1, Remark 8.2].)

Proposition 4.1. *If $\eta \in \pi_1(M)^{(n)}$ and J has Arf invariant zero, then $K(J, \eta)$ is (n)-solvable.*

Proof. This is a special case of [COT2, Proposition 3.1]. Let W be the exterior of a ribbon disk for K in B^4. (Note that W may be viewed as an (n)-solution.) Let W_J be the (0)-solution for J such that a canonical epimorphism $\pi_1(M_J) \longrightarrow \mathbb{Z}$ extends to $\pi_1(W_J)$. By doing surgery on elements in $\pi_1(W_J)^{(1)}$, we can assume that $\pi_1(W_J) \cong \mathbb{Z}$. Let μ_J denote the meridian of a tubular neighborhood of J and let ℓ_J be the 0-framed longitude. Then $\partial W_J = M_J = E_J \cup (S^1 \times D^2)$ where $S^1 \times \{*\}$ is μ_J, and $\{*\} \times \partial D^2$ is ℓ_J. Let W' be the 4-manifold obtained from W_J and W by identifying the solid torus $S^1 \times D^2 \subset \partial W_J$ with $\eta \times D^2 \subset \partial W$. Observe that $\partial W' = M'$, zero surgery on $K' = K(J, \eta)$. Then W' is an (n)-solution for K'. See [COT2, Proposition 3.1] for more details. □

5. Detecting (n)-solvability using $L^{(2)}$-signatures

A group Γ is called *poly-torsion-free-abelian (PTFA)* if it admits a normal series $\langle 1\rangle = G_0 \lhd G_1 \lhd \cdots \lhd G_n = \Gamma$ such that the factors G_{i+1}/G_i are torsion-free abelian. If Γ is PTFA, then the group ring $\mathbb{Q}\Gamma$ is a right Ore domain, hence $\mathbb{Q}\Gamma$ embeds in its classical right ring of quotients $\mathcal{K}_\Gamma$. ([COT1, Proposition 2.5].) Let M be an oriented closed 3-manifold. Suppose $\phi : \pi_1(M) \longrightarrow \Gamma$ is a homomorphism where Γ is a PTFA group and suppose there are an oriented compact 4-manifold W bounded by M and a homomorphism $\psi : \pi_1(W) \longrightarrow \Gamma$ which extends ϕ, *i.e.*, $(M,\phi) = \partial(W,\psi)$. Then the *(reduced) $L^{(2)}$-signature* or *von Neumann ρ-invariant* $\rho(M,\phi) \in \mathbb{R}$ is defined to be $\rho(M,\phi) = \sigma^{(2)}_\Gamma(W,\psi) - \sigma_0(W)$ where $\sigma^{(2)}_\Gamma$ is the $L^{(2)}$-signature of the intersection form on $H_2(W;\mathcal{K}_\Gamma)$ and σ_0 is the ordinary signature. We refer the reader to [COT1, Section 5] for more discussion of $L^{(2)}$-signatures. The following theorem, due to COT, gives an obstruction for a knot being $(n.5)$-solvable.

Theorem 5.1. *([COT1, Theorem 4.2]) Suppose Γ is an (n)-solvable group and M is (n)-solvable. If $\phi : \pi_1(M) \longrightarrow \Gamma$ extends over some $(n.5)$-solution W for M, then $\rho(M,\phi) = 0$.*

Corollary 5.2. *If K is a slice knot and ϕ extends over the exterior of a slice disk, then $\rho(M,\phi) = 0$ for any PTFA group Γ where M is zero surgery on K in S^3.*

As to calculating ρ-invariants, if $\Gamma = \mathbb{Z}$ and ϕ is not trivial , then $\rho(M,\phi)$ is easily calculated as a certain integral over S^1. (See [COT2, Property 2.4].) In particular, if K has Arf invariant zero (i.e., K is (0)-solvable), then we can assign a real value $\rho(K)$ to K that is "canonically" induced from ρ-invariants as follows. Let M be zero surgery on K in S^3. Choose a (0)-solution W of M such that a canonical epimorphism $\phi : \pi_1(M) \longrightarrow \mathbb{Z}$ extends to $\psi : \pi_1(W) \longrightarrow \mathbb{Z}$ and $\pi_1(W) \cong \mathbb{Z}$ as we did in the proof of Proposition 4.1. Then we can calculate $\rho(M,\phi)$ via (W,ψ) and we define $\rho(K)$ to be $\rho(M,\phi)$. These "canonical" real numbers will play an important role in our work. (See Proposition 4.1 and the paragraph preceding Proposition 6.4.)

Now we investigate how ρ-invariants change under the grafting construction described in Section 4. Though there is no general additive property of ρ-invariants, if the representations of the fundamental groups of the relevant manifolds are matched up nicely under the grafting construction, we can derive an additive property. In particular, to prove the main theorem, we only need to look into ρ-invariants of $K(J,\eta)$ where K is a ribbon knot and J has Arf invariant zero.

Suppose K is a ribbon knot and J has Arf invariant zero. Let W be the exterior of a ribbon disk for K. W_J, W', M_J, M', η are defined as in Proposition 4.1. Suppose we are given homomorphisms $\phi : \pi_1(M) \longrightarrow \Gamma$ and $\phi_J : \pi_1(M_J) \longrightarrow \Gamma$ such that $\phi([\eta]) = \phi_J([\mu_J])$ where Γ is a PTFA group. Then ϕ and ϕ_J produce a unique homomorphism $\phi' : \pi_1(M') \longrightarrow \Gamma$ (For this, observe that $M' = (M \setminus (\eta \times D^2)) \bigcup_{S^1\times S^1} E_J$ where $M \setminus (\eta \times D^2) \subset M$ and $E_J \subset M_J$. Use Van Kampen Theorem noticing that for $\{*\} \times \partial D^2 \subset \eta \times D^2$, $\phi([\{*\} \times \partial D^2]) = \phi_J([\ell_J]) = 0$). Then we have the following proposition due to COT.

Proposition 5.3. *Suppose ϕ' extends to $\psi' : \pi_1(W') \longrightarrow \Gamma$. Then $\rho(M',\phi') = \rho(J)$ if $\phi(\eta) \neq 1$, and $\rho(M',\phi') = 0$ if $\phi(\eta) = 1$.*

Proof. By [COT2, Proposition 3.2], $\rho(M',\phi') = \rho(M,\phi) + \rho(M_J,\phi_J)$. $\rho(M,\phi) = 0$ by Corollary 5.2. Since η generates $\pi_1(W_J) \cong \mathbb{Z}$, $\rho(M_J,\phi_J) = \rho(J)$ if $\phi(\eta) \neq 1$ and $\rho(M_J,\phi_J) = 0$ if $\phi(\eta) = 1$ by Property 2.3 and 2.5 in [COT2]. □

6. Proof of main theorem

We begin this section by briefly reviewing some very useful machinery for the proof of the main theorem. This originates from [COT1, Section 2, 3, and 4], so for all the detailed arguments and more generalized facts, the readers should consult [COT1].

Definition 6.1. The family of *rationally universal* groups $\{\Gamma_n^U\}$ is defined inductively by $\Gamma_0^U = \mathbb{Z}$, $\mathcal{R}_0^U = \mathbb{Q}[t,t^{-1}]$ and for $n \geq 0$, setting

$$S_n = \mathbb{Q}[\Gamma_n^U, \Gamma_n^U] - \{0\}, \qquad \mathcal{R}_n^U = (\mathbb{Q}\Gamma_n^U)S_n^{-1}$$

and

$$\Gamma_{n+1}^U = \mathcal{K}_n/\mathcal{R}_n^U \rtimes \Gamma_n^U.$$

Here $\mathcal{K}_n$ is the right ring of quotients of $\mathbb{Q}[\Gamma_n^U]$.

It is shown in [COT1, Proposition 2.5] that $\mathbb{Q}[\Gamma_n^U]$ is an Ore domain, i.e., $\mathbb{Q}[\Gamma_n^U]$ has a right ring of quotients. (Note that inductively Γ_n^U is PTFA.) The semi-direct product is defined via the left multiplication of Γ_n^U on $\mathcal{K}_n/\mathcal{R}_n^U$. One can show that Γ_n^U is an (n)-solvable group for all $n \geq 0$. Observe that $\mathcal{K}_0 = \mathbb{Q}(t)$ and $\Gamma_1^U = \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}] \rtimes \mathbb{Z}$.

Suppose M is a closed 3-manifold with $\beta_1(M) = 1$ and we have a homomorphism $\phi_0 : \pi_1(M) \longrightarrow \Gamma_0^U$. Then we can define the rational Alexander module $\mathcal{A}_0(M) \equiv H_1(M;\mathcal{R}_0^U)$ and the (non-singular) Blanchfield form $B\ell_0 : \mathcal{A}_0(M) \times \mathcal{A}_0(M) \longrightarrow \mathcal{K}_0/\mathcal{R}_0^U$. Then,

$$\mathcal{A}_0(M) \equiv H_1(M;\mathcal{R}_0^U) \cong H^2(M;\mathcal{R}_0^U) \cong H^1(M;\mathcal{K}_0/\mathcal{R}_0^U).$$

and there is a bijection $f : H^1(M;\mathcal{K}_0/\mathcal{R}_0^U) \longleftrightarrow \mathrm{Rep}^*_{\Gamma_0^U}(\pi_1(M),\Gamma_1^U)$ ($\mathrm{Rep}^*_{\Gamma_0^U}(\pi_1(M),\Gamma_1^U)$ is defined to be the representations from $\pi_1(M)$ to Γ_1^U which agree with ϕ_0 after composing with the projection $\Gamma_1^U \longrightarrow \Gamma_0^U$ modulo $\mathcal{K}_0/\mathcal{R}_0^U$-conjugations.) So any choice $x_0 \in \mathcal{A}_0(M)$ will (together with ϕ_0) induce $\phi_1 : \pi_1(M) \longrightarrow \Gamma_1^U$. We refer to this as *the coefficient system corresponding to x_0 (and ϕ_0)*. One can think of (the image of) this element x_0 as an element of $\mathrm{Hom}_{\mathcal{R}_0^U}(\mathcal{A}_0(M),\mathcal{K}_0/\mathcal{R}_0^U)$ under the Kronecker map from $H^1(M;\mathcal{K}_0/\mathcal{R}_0^U)$. This image is called the *character induced by* x_0. Now we obtain some very useful facts which are summarized in the following remark.

Remark 6.2. ([COT1, Theorem 3.5, 3.6, and 4.4]) Suppose $M = \partial W$ is a compact 3-manifold with $\beta_1(M) = 1$ and $\phi_0 : \pi_1(M) \longrightarrow \Gamma_0^U$ is given.

(i) The isomorphism $H_1(M;\mathcal{R}_0^U) \cong H^1(M;\mathcal{K}_0/\mathcal{R}_0^U)$ with f gives a natural bijection $\tilde{f}$: $\mathcal{A}_0(M) \longleftrightarrow \mathrm{Rep}^*_{\Gamma_0^U}\left(\pi_1(M),\Gamma_1^U\right)$.

(ii) If $x \in \mathcal{A}_0(M)$, then the character induced by x is given by $y \mapsto B\ell_0(x,y)$.

(iii) Assume that the non-trivial map $\phi_0 : \pi_1(M) \longrightarrow \Gamma_0^U$ extends to a map $\psi_0 : \pi_1(W) \longrightarrow \Gamma_0^U$ and that ϕ_1 is a representative of a class in $\mathrm{Rep}^*_{\Gamma_0^U}\left(\pi_1(M),\Gamma_1^U\right)$ corresponding to $x \in H_1(M;\mathcal{R}_0^U)$. Let

$$P_0 \equiv \mathrm{Ker}\{j_* : H_1(M;\mathcal{R}_0^U) \longrightarrow H_1(W;\mathcal{R}_0^U)\}.$$

Then if M is (1)-solvable via W, then ϕ_1 extends to $\pi_1(W)$ if and only if $x \in P_0$.

(iv) Suppose M is (1)-solvable via W and ϕ_0 is a non-trivial coefficient system that extends to $\pi_1(W)$. Then the Blanchfield form $B\ell_0$ is hyperbolic, and in fact the kernel of j_* : $H_1(M;\mathcal{R}_0^U) \longrightarrow H_1(W;\mathcal{R}_0^U)$ is self-annihilating. (i.e., $\ker j_* = (\ker j_*)^\perp$.)

From Theorem 5.1 and Remark 6.2, to prove that a knot K is not (1.5)-solvable, basically we need to investigate the representations of the fundamental group induced from all self-annihilating submodules of $\mathcal{A}_0(M) \equiv H_1(M;\mathbb{Q}[t,t^{-1}])$ where M is zero surgery on K in S^3. But in case $\mathcal{A}_0(M)$ has a unique proper submodule, we have the following useful lemma.

Lemma 6.3. *Suppose M is* (1)*-solvable and $\mathcal{A}_0(M)$ has a unique proper submodule P. If there exists $p \in P$ such that $\rho(M,\phi) \neq 0$ for $\phi : \pi_1(M) \longrightarrow \Gamma_1^U$ induced from p, then M is not* (1.5)*-solvable.*

Proof. Suppose M is (1.5)-solvable via W. Let $\mathcal{A}_0(W) \equiv H_1(W;\mathbb{Q}[t,t^{-1}])$. Since W is also a (1)-solution of M, the kernel of the inclusion-induced map $i_* : \mathcal{A}_0(M) \longrightarrow \mathcal{A}_0(W)$ is self-annihilating with respect to $B\ell_0$ by Remark 6.2 (iv). Since $\mathcal{A}_0(M)$ has a unique proper submodule P, $\operatorname{Ker} i_* = P$. By Remark 6.2 (i) and (iii) , $\phi : \pi_1(M) \longrightarrow \Gamma_1^U$ induced from $p(\in P)$ extends to $\pi_1(W)$. Then since W is assumed to be a (1.5)-solution of M, by Theorem 5.1, $\rho(M,\phi) = 0$. This leads us to a contradiction. □

Through this section, K_s denotes our seed ribbon knot which was constructed in Section 2 and η is the designated circle in the complement of the Seifert surface of K_s in S^3 as in Figure 4. Notice that η is a dual of w_9, so it represents the homology class which generates the rational Alexander module of K_s.

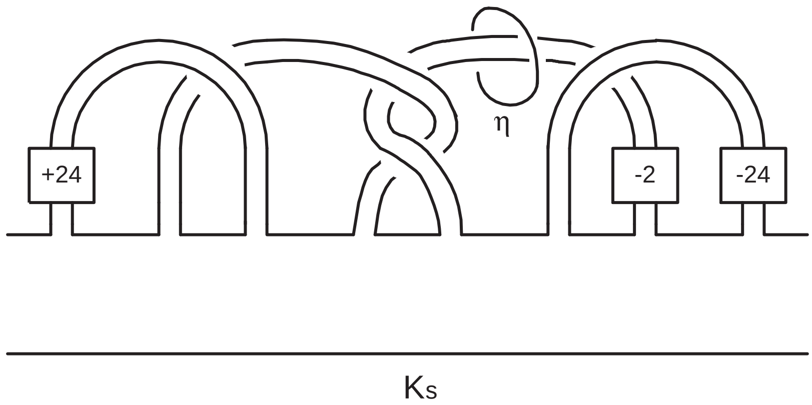

FIGURE 4.

By [COT2, Proposition 2.6], there are infinitely many Arf invariant zero knots $J_i (i \in \mathbb{N})$ such that $\{\rho(J_i)\}_{i\in\mathbb{N}}$ is linearly independent over integers. In particular, $\rho(J_i) \neq 0$. Let $K_i \equiv K_s(J_i,\eta)$ be the family of knots resulting from the grafting construction as described in Section 4. In the following propositions, we exploit important properties of K_i.

Proposition 6.4. *K_i ($i \in \mathbb{N}$) are* (1)*-solvable but not* (1.5)*-solvable.*

Proof. η lifts to a closed circle in the infinite cyclic cover of $S^3 \setminus K_s$, hence $\eta \in \pi_1(M)^{(1)}$ where M is zero surgery on K_s in S^3. Now it is clear from Proposition 4.1 that K_i are (1)-solvable.

We need to show that K_i are not (1.5)-solvable. Fix i. Let W' denote the (1)-solution for K_i formed as in the proof of Proposition 4.1 and let M' denote zero surgery on K_i in S^3. Recall that $\Gamma_0^U = \mathbb{Z}$. Let $\pi_1(M') \longrightarrow \Gamma_0^U$ be the canonical epimorphism which extends uniquely to an epimorphism $\pi_1(W')$. Looking into the grafting construction more closely, one can see that K_i has the same Seifert form as that of K_s, so the rational Alexander module $\mathcal{A}_0(M')$ is isomorphic to $\mathbb{Q}[t,t^{-1}]/(\Phi_{30}(t))^2$. Let $\mathcal{A}_0(W') \equiv H_1(W';\mathbb{Q}[t,t^{-1}])$. By Remark 6.2 (iv), since W' is a (1)-solution for M', the kernel of the inclusion-induced map $i_* : \mathcal{A}_0(M') \longrightarrow \mathcal{A}_0(W')$ is self-annihilating with respect to the (non-singular) Blanchfield form $B\ell_0$. Since $\mathcal{A}_0(M')$ has a unique proper submodule, say P_0, which is generated by $\Phi_{30}(t)$, $\operatorname{Ker} i_* = P_0$. Choose a non-zero $p_0 \in P_0$ such that $B\ell_0(\eta, p_0) \neq 0$. Such a p_0 exists since η generates $\mathcal{A}_0(M')$ and $B\ell_0$ is non-singular for which P is self-annihilating. Then p_0 induces $\phi : \pi_1(M') \longrightarrow \Gamma_1^U$ by Remark 6.2 (i). By Remark 6.2 (iii), ϕ extends to $\psi : \pi_1(W') \longrightarrow \Gamma_1^U$. Now we compute $\rho(M', \phi)$ using (W', ψ). Since $B\ell_0(\eta, p_0) \neq 0$, $\phi(\eta) \neq 1$ by Remark 6.2 (ii). By Proposition 5.3, $\rho(M', \phi) = \rho(J_i)$, which is nonzero by our choice of J_i. By Lemma 6.3, K_i is not (1.5)-solvable. □

Proposition 6.5. *K_i have vanishing Casson-Gordon invariants.*

Proof. Because K_i and K_s have the same Seifert form, they have the same Alexander polynomial which is $(\Phi_{30}(t))^2$. By Theorem 1.2, any prime power branched cyclic cover of K_i is a homology sphere. Hence all Casson-Gordon invariants vanish on K_i by [Lit, Corollary B2]. □

Now we are ready to prove the main theorem.

Proof of Theorem 1.1. First, we show that no non-trivial linear combination of $K_i, i \in \mathbb{N}$ is (1.5)-solvable. We follow [COT2]. Refer to the proof of [COT2, Theorem 4.1]. The only crucial difference in this proof is that we deal with (1.5)-solvability instead of (2.5)-solvability, so we use the second order invariants instead of the third order invariants. Since the proof will follow almost the same course of COT's proof for [COT2, Theorem 4.1], some details will be omitted. For convenience, we follow the notations used in [COT2].

Suppose that a non-trivial linear combination $\#_{i=1}^m n_i' K_i, n_i' \neq 0$, is (1.5)-solvable. We may assume all $n_i' > 0$ by replacing K_i by $-K_i$ if $n_i' < 0$ and $n_1' > 1$ if $m = 1$. Let M_i denote M_{K_i}, and note that $-M_i = M_{-K_i}$. Let M_0 denote 0-surgery on $\#_{i=1}^m n_i' K_i$. Let W_0 be a (1.5)-solution of M_0. Let W_i $(i > 0)$ denote the specific (1)-solution for M_i constructed as in Proposition 4.1 with the exterior of a ribbon disk for K. Let $n_1 = n_1' - 1$ and $n_i = n_i'$ if $i > 1$. Let W be the union of W_0, C (C is defined in the next paragraph), and all the copies of W_i $(i = 1, 2, \ldots, m)$ where there are n_i copies of W_i below C. Refer to Figure 5 below. Later we will show that W is a (1)-solution of M_1.

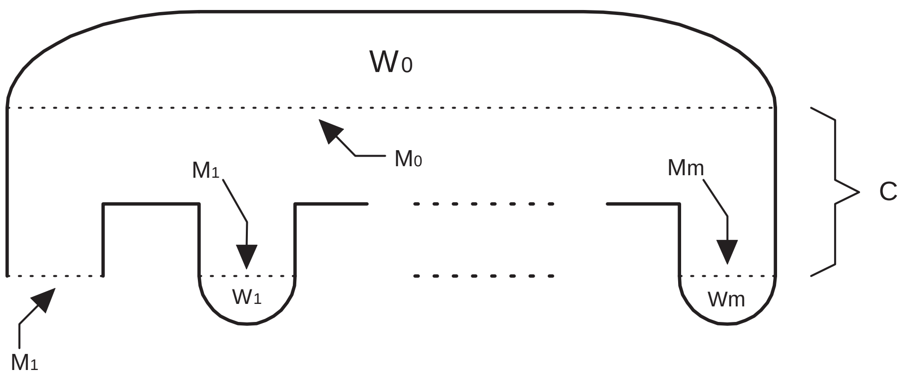

FIGURE 5.

The 4-manifold C is a standard cobordism between 0-surgery on $\#_{i=1}^m n_i' K_i$ and the disjoint union of 0-surgeries on the summands of $\#_{i=1}^m n_i' K_i$. Briefly, start with a collar on the disjoint union of 0-surgeries, and add 1-handles to get a connected 4-manifold whose upper boundary is given by surgery on the link consisting of the split union of K_i's, each with 0-framing. Next add 0-framed 2-handles to get a 0-surgery on a connected sum of K_i's on the upper boundary. See [COT2, Theorem 4.1] for more details, and note that C has a handlebody decomposition, relative to $\coprod_{i=1}^m n_i' M_i$, consisting of $(\sum_{i=1}^m |n_i|)$ 1-handles and the same number of 2-handles. Moreover, $H_1(C;\mathbb{Z}) \cong \mathbb{Z}$ and the inclusion from any of its boundary components induces an isomorphism on H_1. One can also see that $H_2(C) \cong H_2(\coprod n_i' M_i)$.

We prove that W is a (1)-solution of M_1. Since the inclusion-induced homomorphisms $H_1(M_i) \longrightarrow H_1(W_i)$ are isomorphisms for $i \geq 0$, the inclusion-induced $H_1(M_1) \longrightarrow H_1(W)$ is also an isomorphism. For $i \geq 0$, $H_2(M_i) \longrightarrow H_2(W_i)$ is the zero map since the boundary map $H_3(W_i, M_i) \longrightarrow H_2(M_i)$, the dual map of the inclusion induced $H^1(W_i) \longrightarrow H^1(M_i)$, is an isomorphism. Using this and Mayer-Vietoris sequence we can prove that $H_2(W) \cong H_2(W_0) \bigoplus_{i=1}^m n_i H_2(W_i)$. Now if one looks carefully at (1)-Lagrangians and their duals for W_0 and the W_i's, one can see that they form (1)-Lagrangian and its dual for W. So W is a (1)-solution for M_1.

We repeat the argument in Proposition 6.4. Let $\pi_1(M_1) \longrightarrow \Gamma_0^U$ be the canonical epimorphism which extends uniquely to an epimorphism $\pi_1(W)$. Recall that the rational Alexander module $\mathcal{A}_0(M_1) = H_1(M_1;\mathbb{Q}[t,t^{-1}])$ is isomorphic to $\mathbb{Q}[t,t^{-1}]/(\Phi_{30}(t))^2$. Let $\mathcal{A}_0(W) = H_1(W;\mathbb{Q}[t,t^{-1}])$. By Remark 6.2 (iv), since W is a (1)-solution for M_1, the kernel of the inclusion-induced map $j_* : \mathcal{A}_0(M_1) \longrightarrow \mathcal{A}_0(W)$ is self-annihilating with respect to the Blanchfield form $B\ell_0$. Since $\mathcal{A}_0(M_1)$ has a unique proper submodule, say P_0, the latter is this kernel. Choose a non-zero $p_0 \in P_0$, inducing $\phi_1 : \pi_1(M_1) \longrightarrow \Gamma_1^U$ by Remark 6.2 (i). By Remark 6.2 (iii), ϕ_1 extends to $\psi_1 : \pi_1(W) \longrightarrow \Gamma_1^U$. Therefore $\rho(M_1,\phi_1)$ can be computed using (W,ψ_1).

We compute $\rho(M_1,\phi_1)$ using (W,ψ_1). Let $\phi_{(i,j)}$ denote the restriction of ψ_1 to the j^{th} copy of $\pi_1(M_i), 1 \leq j \leq n_i$. Let ϕ_0 denote the restriction of ψ_1 to $\pi_1(M_0)$. Let $\mathcal{K}_1$ denote the classical right ring of quotients of $\mathbb{Z}\Gamma_1^U$. $H_*(M_i;\mathcal{K}_1) = 0$ for $i \geq 0$ ([COT1, Propositions 2.9 and 2.11]), so a Mayer-Vietoris sequence shows that $H_2(W;\mathcal{K}_1) \cong H_2(W_0;\mathcal{K}_1) \oplus H_2(C;\mathcal{K}_1) \oplus H_2(W_1;\mathcal{K}_1) \oplus \cdots \oplus H_2(W_m;\mathcal{K}_1)$ where W_i occurs n_i times. Here the coefficient systems on W_i $(i \geq 0)$ and C are induced by inclusions into W. By [COT2, Lemma 4.2] $H_2(C;\mathcal{K}_1) = 0$. And the intersection form on $H_2(W;\mathcal{K}_1)$ splits along the direct sum. From [COT1, Section 5], $\sigma^{(2)}_{\Gamma_1^U}$ can be viewed as a homomorphism from the Witt group of non-singular hermitian forms on finitely generated $\mathcal{K}_1$ modules, so we have

$$\rho(M_1,\phi_1) = \rho(M_0,\phi_0) + \sum_{i=1}^{m}\sum_{j=1}^{n_i} \rho(-M_i,\phi_{(i,j)})$$

Here $\rho(M_0,\phi_0) = 0$ by Theorem 5.1 because ϕ_0 extends to (1.5)-solution W_0. By Proposition 5.3, $\rho(-M_i,\phi_{(i,j)}) = -\rho(M_i,\phi_{(i,j)}) = -\rho(J_i)$ or 0. So we deduce that

$$\rho(M_1,\phi_1) + \sum_{i=1}^{m} c_i \rho(J_i) = 0$$

for some non-negative constants c_i's.

Now as in Proposition 6.4, pick $p_0 \in P_0$ such that $\phi_1(\eta) \neq 1$. Note that P_0 is equal to the kernel of the inclusion-induced map $i_* : \mathcal{A}_0(M_1) \longrightarrow \mathcal{A}_0(W_1)$. Then by Proposition 5.3, $\rho(M_1,\phi_1) = \rho(J_1)$, so we have

$$\rho(J_1) + \sum_{i=1}^{m} c_i \rho(J_i) = 0$$

which contradicts that $\{\rho(J_i)\}_{i\in\mathbb{N}}$ is linearly independent over integers.

Now it remains to show that Casson-Gordon invariants vanish on the subgroup generated by K_i. Recall that every prime power branched cyclic cover of K_i is a homology 3-sphere. One can show that the connected sum of two homology 3-spheres is a homology 3-sphere. Since a finite branched cyclic cover of the connected sum of two knots over S^3 is homeomorphic with the connected sum of the finite branched cyclic covers of the knots, the assertion follows from [Lit, Corollary B2]. □

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