



# New obstructions to doubly slicing knots

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## Abstract

A knot in the 3-sphere is called doubly slice if it is a slice of an unknotted 2-sphere in the 4-sphere. We give a bi-sequence of new obstructions for a knot being doubly slice. We construct it following the idea of Cochran–Orr–Teichner’s filtration of the classical knot concordance group. This yields a bi-filtration of the monoid of knots (under the connected sum operation) indexed by pairs of half integers. Doubly slice knots lie in the intersection of this bi-filtration. We construct examples of knots which illustrate the non-triviality of this bi-filtration at all levels. In particular, these are new examples of algebraically doubly slice knots that are not doubly slice, and many of these knots are slice. Cheeger–Gromov’s von Neumann rho invariants play a key role to show non-triviality of this bi-filtration. We also show some classical invariants are reflected at the initial levels of this bi-filtration, and obtain a bi-filtration of the double concordance group.

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## 1. Introduction

We work in the topologically locally flat category. An  $n$ -knot in the  $(n + 2)$ -sphere is called doubly slice (or doubly null cobordant) if it is a slice of an unknotted  $(n + 1)$ -sphere in the  $(n + 3)$ -sphere. The notion of doubly slice knots was introduced by Fox [8] in the 1960s. For odd dimensional knots, Sumners [24] showed if a knot is doubly slice, then it has an associated Seifert form which is hyperbolic. We call

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the knots satisfying this Seifert form condition algebraically doubly slice (or algebraically doubly null cobordant). It was shown that for odd high-dimensional simple knots, this Seifert form obstruction is sufficient for being doubly slice [24,15]. This result was generalized to even high-dimensional knots by Stoltzfus [23,22] using the obstructions based on the linking form defined by Levine [17] and Farber [7]. In this paper, we work in the classical dimension. So by “knot” we mean a 1-knot in the 3-sphere unless mentioned otherwise.

In [11], Gilmer and Livingston showed that there exists a slice knot which is algebraically doubly slice but not doubly slice. (A knot is called slice if it bounds a locally flat 2-disk in the 4-ball.) One can see that if a knot is doubly slice then every finite branched cyclic cover of the knot is embedded in the 4-sphere. They applied their own obstructions to embedding 3-manifolds into the 4-sphere to show their example is not doubly slice. High-dimensional analogues of this result were obtained by Ruberman [19]. Recently, Friedl [10] found doubly slicing obstructions using eta invariants associated to finite dimensional unitary representations.

Meanwhile, Cochran, Orr, and Teichner (henceforth COT) established a filtration of the classical knot concordance group  $\mathcal{C}$  [4].

$$0 \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_{1.5} \subset \mathcal{F}_{1.0} \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C},$$

where  $\mathcal{F}_m$  is the set of all  $(m)$ -solvable knots. Roughly speaking, a 3-manifold is said to be  $(m)$ -solvable (via  $W$ ) if it bounds a spin 4-manifold  $W$  such that the inclusion map induces an isomorphism on first homology and satisfies a certain condition on the intersection form of the  $m$ th derived cover of  $W$ . A knot is called  $(m)$ -solvable (via  $W$ ) if zero surgery on the knot in the 3-sphere is  $(m)$ -solvable (via  $W$ ). If  $K$  is  $(n)$ -solvable via  $W$ , then  $W$  is called an  $(m)$ -solution for the knot (or for zero surgery on the knot in the 3-sphere). COT showed that if a knot is  $(1.5)$ -solvable, then all the previously known concordance invariants including Casson–Gordon invariants vanish for the knot [4, Theorem 9.11]. They also showed that  $\mathcal{F}_2/\mathcal{F}_{2.5}$  has infinite rank [4,5]. Later Cochran and Teichner showed that their filtration is highly non-trivial. That is,  $\mathcal{F}_n/\mathcal{F}_{n.5}$  is infinite for all  $n$  [6].

In this paper, we give new obstructions for knots being doubly slice using the ideas of COT. One easily sees that a knot is doubly slice if and only if there exist two slice disk and 4-ball pairs whose union along their boundary gives an unknotted 2-sphere in the 4-sphere. In this regard, for half-integers  $m$  and  $n$ , we define a knot to be  $(m, n)$ -solvable if the knot has an  $(m)$ -solution and an  $(n)$ -solution such that the union of these solutions along their boundary gives a closed 4-manifold whose fundamental group is isomorphic to an infinite cyclic group. (See Definition 2.2.) In particular, we define a knot to be doubly  $(m)$ -solvable if it is  $(m, m)$ -solvable. We remark that Freedman [9] showed that a 2-knot is unknotted in the 4-sphere if and only if the fundamental group of the knot exterior is isomorphic to an infinite cyclic group. We show that a doubly slice knot is  $(m, n)$ -solvable for all  $m, n$  (Proposition 2.5). For given half-integers  $k \geq m$  and  $\ell \geq n$ , if a knot is  $(k, \ell)$ -solvable then it is  $(m, n)$ -solvable. This is easily proven since a  $(k)$ -solution (respectively an  $(\ell)$ -solution) for a knot is an  $(m)$ -solution (respectively an  $(n)$ -solution) (refer to [4, Remark 1.1.3]). Moreover, we show that if two knots are  $(m, n)$ -solvable, then so is their connected sum (Proposition 2.6). This implies that if we denote by  $\mathcal{F}_{m,n}$  the set of  $(m, n)$ -solvable knots, then  $\{\mathcal{F}_{m,n}\}_{m,n \geq 0}$  becomes a bi-filtration of the monoid of knots (under the connected sum operation). We investigate this bi-filtration and construct examples of knots showing non-triviality of the bi-filtration at all levels. Our main theorem is as follows:

**Theorem 1.1.**

- (1) For a given integer  $m \geq 2$ , there exists a ribbon knot (hence slice)  $K$  such that  $K$  is algebraically doubly slice, doubly  $(m)$ -solvable, but not doubly  $(m.5)$ -solvable.
- (2) For given integers  $k, \ell \geq 2$ , there exists an algebraically doubly slice knot  $K$  such that  $K$  is  $(k, \ell)$ -solvable, but not  $(k.5, \ell)$ -solvable, nor  $(k, \ell.5)$ -solvable.

A knot is called a *ribbon knot* if it bounds an immersed 2-disk (called ribbon or ribbon disk) in the 3-sphere with only ribbon singularities. (We say an immersed 2-disk  $f(D^2)$  where  $f : D^2 \rightarrow S^3$  is an immersion has ribbon singularities if the inverse image of the singularities consists of pairs of arcs on  $D^2$  such that one arc of each pair is interior to  $D^2$ .) Note that a ribbon knot is a slice knot. To see this, push the singular parts of the ribbon disk into  $B^4$  to get a slice disk.

Classical invariants are reflected at the initial levels of the bi-filtration. In particular, we show that if a knot is doubly  $(1)$ -solvable, then its Blanchfield form is hyperbolic (Proposition 2.10). (It is unknown to the author if the converse is true.) We also show that a knot has vanishing Arf invariant if and only if it is doubly  $(0)$ -solvable, and algebraically slice if and only if it is doubly  $(0.5)$ -solvable (Corollary 2.9).

To prove the main theorem, we construct a fibred doubly slice knot of genus 2 which will be called the seed knot. We choose a trivial link in the 3-sphere that is disjoint from the seed knot and choose auxiliary Arf invariant zero knots. Then genetic modification is performed on the seed knot via the chosen trivial link and auxiliary knots to obtain the desired examples of knots. This genetic modification is the same as the one used in [5,6] and will be explained in Section 3 in this paper. In fact, in [6] Cochran and Teichner make use of genetic modification to construct the examples of knots which are  $(m)$ -solvable but not  $(m.5)$ -solvable in COT's filtration of the knot concordance group. In comparison with their examples, to prove Theorem 1.1(1), our examples need to be slice, hence  $(k)$ -solvable for all  $k$ . Hence a technical difficulty arises, and we perform genetic modification in a more sophisticated way than in [6]. To show a knot is not doubly  $(m.5)$ -solvable, we use von Neumann  $\rho$ -invariants defined by Cheeger and Gromov [3]. In particular, we make use of the fact that there is a universal bound for von Neumann  $\rho$ -invariants for a fixed 3-manifold [6, 18, Theorem 3.1.1]. More details about this can be found in Sections 4 and 5.

This bi-filtration of knots induces a bi-filtration of the double concordance group. Two knots  $K_1$  and  $K_2$  are called doubly concordant if  $K_1 \# J_1$  is isotopic to  $K_2 \# J_2$  for some doubly slice knots  $J_1$  and  $J_2$ . (Here ‘ $\#$ ’ means the connected sum.) This is an equivalence relation, and the equivalence classes with the connected sum operation form the double concordance group. We denote the set of the equivalence classes represented by  $(m, n)$ -solvable knots by  $\overline{\mathcal{F}}_{m,n}$ . We show that each  $\overline{\mathcal{F}}_{m,n}$  is a subgroup of the double concordance group and  $\{\overline{\mathcal{F}}_{m,n}\}_{m,n \geq 0}$  is a bi-filtration of the double concordance group (Corollary 6.4).

This paper is organized as follows. In Section 2, we define  $(m, n)$ -solvable knots and show that doubly slice knots are  $(m, n)$ -solvable for all  $m$  and  $n$ . We induce a bi-filtration of the monoid of knots and investigate properties of the bi-filtration at the initial levels. In Section 3, we explain how to construct  $(m, n)$ -solvable knots using genetic modification. In Section 4, we explain Cochran and Teichner's work in [6] and show when  $(m)$ -solutions are not  $(m.5)$ -solutions. In Section 5, we give a proof of Theorem 1.1. In Section 6, we construct a bi-filtration of the double concordance group. Finally, in Section 7 we give the examples of knots demonstrating non-triviality of the bi-filtration of the monoid of knots at lower levels.

**Notation.** Throughout this paper,  $M_K$  denotes 0-surgery on a knot  $K$  in  $S^3$  and  $\Lambda$  (respectively  $\Lambda'$ ) denotes the group ring  $\mathbb{Z}[t, t^{-1}]$  (respectively  $\mathbb{Q}[t, t^{-1}]$ ). The set of non-negative integers is denoted by  $\mathbb{N}_0$ . For

convenience, we use the same notations for a simple closed curve and the homotopy (or homology) class represented by the curve. The usual integer coefficient system is to be understood if coefficients are omitted for homology.

## 2. $(m, n)$ -solvable knots and the basic properties

For the reader's convenience, the definition of  $(n)$ -solvability is given below. For the related terminologies and more explanations about  $(n)$ -solvable knots, refer to [4].

**Definition 2.1** (Cochran et al. [4]). Let  $n \in \mathbb{N}_0$ . A 3-manifold  $M$  is  $(n)$ -solvable (resp.  $(n.5)$ -solvable) via  $W$  if there is an  $H_1$ -bordism  $W$  which contains an  $(n)$ -Lagrangian (resp.  $(n + 1)$ -Lagrangian) with  $(n)$ -duals. If  $M$  is zero surgery on a knot or a link then the corresponding knot or link is called  $(n)$ -solvable (resp.  $(n.5)$ -solvable).

Now we define  $(m, n)$ -solvable knots and doubly  $(m)$ -solvable knots as follows.

**Definition 2.2.** Let  $m, n \in \frac{1}{2}\mathbb{N}_0$ . A 3-manifold  $M$  is called  $(m, n)$ -solvable via  $(W_1, W_2)$  if  $M$  is  $(m)$ -solvable via  $W_1$  and  $(n)$ -solvable via  $W_2$  such that the fundamental group of the union of  $W_1$  and  $W_2$  along their boundary  $M$  is isomorphic to  $\mathbb{Z}$ . (i.e.,  $\pi_1(W_1 \cup_M W_2) \cong \mathbb{Z}$ .) A knot  $K$  is called  $(m, n)$ -solvable via  $(W_1, W_2)$  if  $M_K$  is  $(m, n)$ -solvable via  $(W_1, W_2)$ . The ordered pair  $(W_1, W_2)$  is called an  $(m, n)$ -solution for  $K$  (or  $M_K$ ). The set of all  $(m, n)$ -solvable knots is denoted by  $\mathcal{F}_{m,n}$ .

**Definition 2.3.** A knot  $K$  is doubly  $(m)$ -solvable if it is  $(m, m)$ -solvable. An  $(m, m)$ -solution for  $K$  is called a double  $(m)$ -solution for  $K$ .

### Remark 2.4.

(i) By van Kampen Theorem, the condition

$$\pi_1(W_1 \cup_M W_2) \cong \mathbb{Z}$$

is equivalent to the condition that the following diagram is a push-out diagram in the category of groups and homomorphisms. In the diagram,  $i_1$  and  $i_2$  are the homomorphisms induced from the inclusion maps from  $M_K$  into  $W_1$  and  $W_2$ , and  $j_1$  and  $j_2$  are the abelianization.

$$\begin{array}{ccc} & \pi_1(W_1) & \\ i_1 \nearrow & & \searrow j_1 \\ \pi_1(M) & & \mathbb{Z} \\ i_2 \searrow & & \nearrow j_2 \\ & \pi_1(W_2) & \end{array}$$

In other words, the condition is equivalent to the condition

$$\pi_1(W_1) *_{\pi_1(M)} \pi_1(W_2) \cong \mathbb{Z}.$$

- (ii) Let  $E_K$  be the exterior of  $K$  in  $S^3$  (i.e.,  $E_K \equiv S^3 \setminus N(K)$  where  $N(K)$  is an open tubular neighborhood of  $K$ ). Then  $\pi_1(W_1 \cup_{M_K} W_2) \cong \pi_1(W_1 \cup_{E_K} W_2)$ . This is easily proven using the fact that  $\pi_1(M_K) \cong \pi_1(E_K)/\langle \ell \rangle$  where  $\langle \ell \rangle$  is the subgroup normally generated by the longitude  $\ell$  of  $K$ .
- (iii) If a knot  $K$  is  $(m, n)$ -solvable, then one easily sees that  $K$  is  $(k)$ -solvable where  $k$  is the maximum of  $m$  and  $n$ .

The following proposition shows that doubly slice knots are contained in the intersection of all  $\mathcal{F}_{m,n}$ 's.

**Proposition 2.5.** *If a knot  $K$  is doubly slice, then it is  $(m, n)$ -solvable for all  $m$  and  $n$ .*

**Proof.** Since  $K$  is doubly slice, there are two slice disk and 4-ball pairs  $(B_1^4, D_1^2)$  and  $(B_2^4, D_2^2)$  such that  $(S^3, K) = \partial(B_1^4, D_1^2) = \partial(B_2^4, D_2^2)$  and  $D_1^2 \cup_K D_2^2$  is an unknotted 2-sphere in the 4-sphere. Since the second homology of a slice disk exterior is trivial, every slice disk exterior is an  $(m)$ -solution for the knot for all  $m$  (see [4, Remark 1.3.1]). So if we let  $W_i \equiv B_i^4 \setminus N(D_i^2)$  for  $i = 1, 2$ , then we may think that  $W_1$  is an  $(m)$ -solution and  $W_2$  is an  $(n)$ -solution for a given pair of half-integers  $m$  and  $n$ . Furthermore,  $W_1 \cup_{E_K} W_2$  is homeomorphic to the exterior of an unknotted 2-sphere in the 4-sphere (which is homeomorphic to  $S^1 \times D^3$ ), hence  $(W_1, W_2)$  satisfies the required fundamental group condition.  $\square$

The following proposition shows  $\mathcal{F}_{m,n}$  is a submonoid of the monoid of knots under the connected sum operation.

**Proposition 2.6.** *Suppose  $K$  and  $J$  are  $(m, n)$ -solvable knots. Then  $K \# J$  is  $(m, n)$ -solvable.*

**Proof.** Let  $(V_1, V_2)$  be an  $(m, n)$ -solution for  $K$  and  $(W_1, W_2)$  be an  $(m, n)$ -solution for  $J$ . We will construct a specific  $(m, n)$ -solution for  $K \# J$  using these solutions. We begin by constructing a standard cobordism  $C$  between  $M_K \amalg M_J$  and  $M_{K \# J}$ . Start with  $(M_K \amalg M_J) \times [0, 1]$  and add a 1-handle to  $(M_K \amalg M_J) \times \{1\}$  such that the upper boundary is a connected 3-manifold given by surgery on a split link  $K \amalg J$  with 0-framing. Next, add a 2-handle with 0-framing to the upper boundary along an unknotted circle which wraps around  $K$  and  $J$  once. (This equates the meridional generators of the first homology of  $M_K$  and  $M_J$ .) The resulting 4-manifold is  $C$ . That is,  $\partial_- C = M_K \amalg M_J$  and  $\partial_+ C = M_{K \# J}$ . See [5, Theorem 4.1] and its proof for more details.

Now let  $X_i$  be the union of  $C$ ,  $V_i$ , and  $W_i$  along the boundaries as shown in Fig. 1 for  $i = 1, 2$ . We claim that  $(X_1, X_2)$  is an  $(m, n)$ -solution for  $K \# J$ . First, we show that  $X_1$  is an  $(m)$ -solution for  $K \# J$ . (The proof that  $X_2$  is an  $(n)$ -solution for  $K \# J$  will follow similarly.) In the construction of the cobordism  $C$ , one can see that  $H_1(C) \cong \mathbb{Z}$  and the inclusion from any boundary component of  $C$  induces an isomorphism. It follows that the inclusion induced map  $H_1(M_{K \# J}) \rightarrow H_1(X_1)$  is an isomorphism. Since adding a 1-handle and a 2-handle has no effect on  $H_2$ ,  $H_2(C) \cong H_2(M_K) \oplus H_2(M_J)$ . Let  $Y_1 \equiv V_1 \amalg W_1$ . From the pair of spaces  $(C, Y_1)$ , we get the following Mayer–Vietoris sequence:

$$\cdots \rightarrow H_2(M_K \amalg M_J) \rightarrow H_2(C) \oplus H_2(Y_1) \rightarrow H_2(X_1) \rightarrow H_1(M_K \amalg M_J) \rightarrow \cdots$$

Since  $H_1(M_K \amalg M_J) \rightarrow H_1(Y_1)$  is an isomorphism,  $H_2(X_1) \rightarrow H_1(M_K \amalg M_J)$  is the zero map. By the above observation on  $H_2(C)$ ,  $H_2(M_K \amalg M_J) \rightarrow H_2(C)$  is an isomorphism, hence surjective. Since the boundary map  $H_3(V_1, M_K) \rightarrow H_2(M_K)$  is the dual of an isomorphism  $H^1(V_1) \rightarrow H^1(M_K)$ , it is an isomorphism. Hence  $H_2(M_K) \rightarrow H_2(V_1)$  is the zero map. Similarly,  $H_2(M_J) \rightarrow H_2(W_1)$  is the

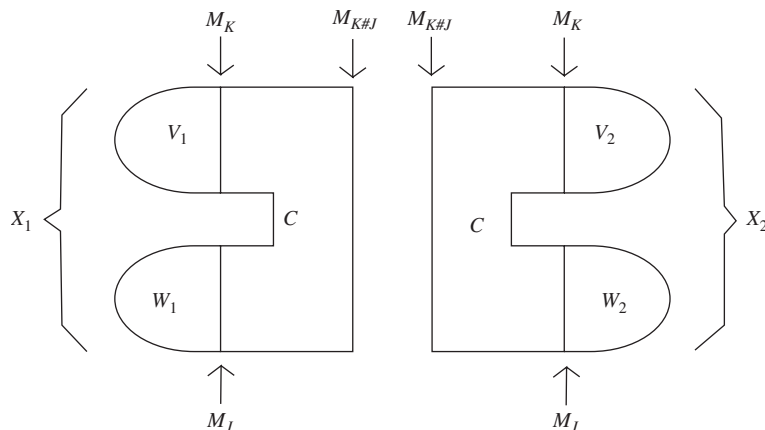


Fig. 1.

zero map, thus so is  $H_2(M_K \amalg M_J) \rightarrow H_2(Y_1)$ . So  $H_2(X_1) \cong H_2(Y_1) \cong H_2(V_1) \oplus H_2(W_1)$ . Since the intersection form on  $Y_1$  splits naturally on  $V_1$  and  $W_1$ , the “union” of the  $(m)$ -Lagrangians and  $(m)$ -duals for  $V_1$  and  $W_1$  forms the  $(m)$ -Lagrangian and  $(m)$ -dual for  $X_1$ . So  $X_1$  is an  $(m)$ -solution for  $K \# J$ . (For more details on  $(m)$ -solutions,  $(m)$ -Lagrangians, and  $(m)$ -duals, refer to [4, Sections 7,8].)

It remains to show  $\pi_1(X_1 \cup_{M_K \# J} X_2) \cong \mathbb{Z}$  or  $\pi_1(X_1 \cup_{E_K \# J} X_2) \cong \mathbb{Z}$  by Remark 2.4. Thus to prove the proposition, it is enough to show the following diagram is a push-out diagram.

$$\begin{array}{ccc}
 \pi_1(E_{K \# J}) & \xrightarrow{i_1} & \pi_1(X_1) \\
 \downarrow i_2 & & \downarrow j_1 \\
 \pi_1(X_2) & \xrightarrow{j_2} & \mathbb{Z}
 \end{array}$$

Since  $(V_1, V_2)$  and  $(W_1, W_2)$  are  $(m, n)$ -solutions for  $K$  and  $J$ , respectively, we have the following push-out diagrams.

$$\begin{array}{ccccccc}
 \pi_1(E_K) & \longrightarrow & \pi_1(V_1) & \pi_1(E_J) & \longrightarrow & \pi_1(W_1) \\
 \downarrow & & \downarrow & \downarrow & & \downarrow \\
 \pi_1(V_2) & \longrightarrow & \mathbb{Z} & \pi_1(W_2) & \longrightarrow & \mathbb{Z}
 \end{array}$$

By taking free products and factoring out each group by the normal subgroup  $\langle \mu_K \mu_J^{-1} \rangle$ , we have the following push-out diagram. (Here  $\mu_K$  and  $\mu_J$  are meridians of  $K$  and  $J$ , respectively.)

$$\begin{array}{ccc}
 \frac{\pi_1(E_K) * \pi_1(E_J)}{\langle \mu_K \mu_J^{-1} \rangle} & \longrightarrow & \frac{\pi_1(V_1) * \pi_1(W_1)}{\langle \mu_K \mu_J^{-1} \rangle} \\
 \downarrow & & \downarrow \\
 \frac{\pi_1(V_2) * \pi_1(W_2)}{\langle \mu_K \mu_J^{-1} \rangle} & \longrightarrow & \mathbb{Z}
 \end{array}$$

Note that

$$\frac{\pi_1(E_K) * \pi_1(E_J)}{\langle \mu_K \mu_J^{-1} \rangle} \cong \pi_1(E_{K \# J}).$$

By the construction of the cobordism  $C$ ,

$$\frac{\pi_1(V_i) * \pi_1(W_i)}{\langle \mu_K \mu_J^{-1} \rangle} \cong \pi_1(X_i)$$

for  $i = 1, 2$ .  $\square$

From Propositions 2.5 and 2.6, we can easily deduce the following corollary.

**Corollary 2.7.** *The family  $\{\mathcal{F}_{m,n}\}_{m,n \geq 0}$  is a bi-filtration of the monoid of knots under the connected sum operation where doubly slice knots lie in the intersection of all  $\mathcal{F}_{m,n}$ 's.*

Next, we study the properties of this bi-filtration at lower levels.

**Proposition 2.8.** *Suppose  $n=0$  or  $0.5$ . Then a knot  $K$  is doubly  $(n)$ -solvable if and only if it is  $(n)$ -solvable.*

**Proof.** One direction is clear by Remark 2.4(iii). For the other direction, suppose  $K$  is  $(n)$ -solvable via  $W$ . By doing surgery on the commutator subgroup of  $\pi_1(W)$  (note that the commutator subgroup is finitely normally generated), we may assume that  $\pi_1(W) \cong \mathbb{Z}$ . Let  $W_1$  and  $W_2$  be copies of  $W$ . Their fundamental groups are isomorphic to  $\mathbb{Z}$  and generated by the meridian of  $K$ . So using van Kampen Theorem, one sees that  $\pi_1(W_1 \cup_{M_K} W_2) \cong \mathbb{Z}$ , hence  $K$  is doubly  $(n)$ -solvable via  $(W_1, W_2)$ .  $\square$

It is known that a knot is  $(0)$ -solvable if and only if it has vanishing Arf invariant, and  $(0.5)$ -solvable if and only if it is algebraically slice (that is, its associated Seifert forms are metabolic). (See [4].) So we have the following corollary.

**Corollary 2.9.** *A knot is doubly  $(0)$ -solvable if and only if it has vanishing Arf invariant, and doubly  $(0.5)$ -solvable if and only if it is algebraically slice.*

We investigate the relationship between the bi-filtration  $\{\mathcal{F}_{m,n}\}_{m,n \geq 0}$  and algebraically doubly slice knots. A  $2k$  by  $2k$  matrix is called *doubly null cobordant* if it is congruent by an integer unimodular matrix to a matrix of the form

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

in which  $0$  designates the  $k$  by  $k$  matrix with entries  $0$ . For a knot  $K$ , we have the (non-singular and sesquilinear) Blanchfield form  $B\ell : H_1(M_K; \Lambda) \times H_1(M_K; \Lambda) \rightarrow \mathbb{Q}(t)/\Lambda$  (see [1]). For a Seifert matrix of  $K$ , say  $A$ , the Blanchfield form is presented by  $(1-t)(tA - A^T)^{-1}$  (see [17]). Kearton showed that this presentation matrix is doubly null cobordant if and only if the Seifert matrix  $A$  is  $S$ -equivalent to a doubly null cobordant Seifert matrix (see [14,26]). This implies that the Blanchfield form of a knot  $K$  is hyperbolic (that is,  $H_1(M_K; \Lambda) = A \oplus B$  where  $A$  and  $B$  are  $\Lambda$ -submodules of  $H_1(M; \Lambda)$  and they are



self-annihilating with respect to  $B\ell$ ) if and only if  $K$  has a Seifert matrix which is  $S$ -equivalent to a doubly null cobordant matrix. Now we have the following proposition.

**Proposition 2.10.** *Suppose a knot  $K$  is doubly (1)-solvable via  $(W_1, W_2)$ . Let  $i_j : M_K \rightarrow W_j$  be the inclusion map for  $j = 1, 2$ . Then*

$$H_1(M_K; \Lambda) = \text{Ker}(i_1)_* \oplus \text{Ker}(i_2)_*.$$

Furthermore,  $\text{Ker}(i_1)_* \cong H_1(W_2; \Lambda)$  and  $\text{Ker}(i_2)_* \cong H_1(W_1; \Lambda)$ . Moreover, for each  $j$ ,  $\text{Ker}(i_j)_*$  is a self-annihilating submodule (that is,  $\text{Ker}(i_j)_* = \text{Ker}(i_j)_*^\perp$ ) with respect to the Blanchfield form

$$B\ell : H_1(M_K; \Lambda) \times H_1(M_K; \Lambda) \rightarrow \mathbb{Q}(t)/\Lambda.$$

Hence the Blanchfield form  $B\ell$  is hyperbolic.

**Proof.** Let  $W$  be  $W_1 \cup_{M_K} W_2$ . Recall that  $\Lambda' \equiv \mathbb{Q}[t, t^{-1}]$ . Since  $W_1 \cap W_2 = M_K$ , we have the following Mayer–Vietoris sequence:

$$\cdots \rightarrow H_2(W; \Lambda') \xrightarrow{\hat{\partial}} H_1(M_K; \Lambda') \xrightarrow{f} H_1(W_1; \Lambda') \oplus H_1(W_2; \Lambda') \xrightarrow{g} H_1(W; \Lambda') \rightarrow \cdots.$$

Since  $\pi_1(W) \cong \mathbb{Z}$ ,  $H_1(W; \Lambda') = \{0\}$ . We show that  $f$  is injective. Suppose  $x \in \text{Ker } f$ . We can consider  $f$  as  $((i_1)_*, (i_2)_*)$ . Therefore,  $x \in \text{Ker}(i_j)_*$  for  $j=1, 2$ . By [4, Theorems 3.5 and 3.6],  $x$  induces a representation  $\phi : \pi_1(M) \rightarrow \Gamma_1^U$  where  $\Gamma_1^U \equiv (\mathbb{Q}(t)/\Lambda') \rtimes \mathbb{Z}$  such that  $\phi$  can be extended to  $\Phi_1 : \pi_1(W_1) \rightarrow \Gamma_1^U$  and  $\Phi_2 : \pi_1(W_2) \rightarrow \Gamma_1^U$ , hence we have the following commutative (push-out) diagram.

$$\begin{array}{ccccc} & & \pi_1(W_1) & & \\ & \nearrow i_1 & & \searrow j_1 & \\ \pi_1(M_K) & & & & \mathbb{Z} \xrightarrow{\alpha} \Gamma_1^U \\ & \searrow i_2 & & \nearrow j_2 & \\ & & \pi_1(W_2) & & \end{array} \quad \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array}$$

In this diagram, we get the homomorphism  $\alpha$  by the universal property of the push-out diagram. Let  $\varepsilon : \pi_1(M_K) \rightarrow \mathbb{Z}$  be the abelianization (in fact,  $\varepsilon = j_1 \circ i_1$ ). For  $y \in \pi_1(M_K)$ ,  $\phi(y)$  is calculated as  $\phi(y) = (B\ell'(x, y\mu^{-\varepsilon(y)}), \varepsilon(y))$  for a meridian  $\mu$  of the knot  $K$  and the rational Blanchfield pairing

$$B\ell' : H_1(M_K; \Lambda') \times H_1(M_K; \Lambda') \rightarrow \mathbb{Q}(t)/\Lambda'.$$

Thus,  $\phi(\mu) = (0, 1) \in \Gamma_1^U$ . By the commutativity of the diagram, we have  $\alpha(1) = (0, 1) \in \Gamma_1^U$ . Thus for any meridian, say  $\mu'$ , of the knot,  $\phi(\mu') = \alpha(1) = (0, 1) \in \Gamma_1^U$ . Thus  $\text{Im } \alpha = \{0\} \rtimes \mathbb{Z} \subset \Gamma_1^U$ , hence  $\phi(y) \in \{0\} \rtimes \mathbb{Z}$  for all  $y \in \pi_1(M_K)$ . Therefore,  $B\ell'(x, x') = 0$  for all  $x' \in H_1(M_K; \Lambda')$ . Since the rational Blanchfield pairing is non-singular, this implies  $x = 0$ , hence  $f$  is injective. Hence  $H_1(M_K; \Lambda') = \text{Ker}(i_1)_* \oplus \text{Ker}(i_2)_*$  where  $\text{Ker}(i_1)_* \cong H_1(W_2; \Lambda')$  and  $\text{Ker}(i_2)_* \cong H_1(W_1; \Lambda')$ .

Now we replace the coefficients  $\Lambda'$  by  $\Lambda$ . One sees that  $H_1(W; \Lambda) = \{0\}$  because  $\pi_1(W) \cong \mathbb{Z}$ . The homomorphism  $f$  is still injective since  $H_1(M; \Lambda)$  is  $\mathbb{Z}$ -torsion free. Therefore,  $H_1(M_K; \Lambda) = \text{Ker}(i_1)_* \oplus \text{Ker}(i_2)_*$  where  $\text{Ker}(i_1)_* \cong H_1(W_2; \Lambda)$ ,  $\text{Ker}(i_2)_* \cong H_1(W_1; \Lambda)$ . We need to show that  $\text{Ker}(i_j)_*$  is



self-annihilating for each  $j$ . Since  $W_j$  is an (integral) (1)-solution for  $K$ ,

$$TH_2(W_j, M_K; \Lambda) \xrightarrow{\hat{\partial}} H_1(M_K; \Lambda) \xrightarrow{(i_j)_*} H_1(W_j; \Lambda)$$

is exact by [4, Lemma 4.5] where  $TH_2$  denotes the  $\Lambda$ -torsion submodule. Note that the Kronecker map

$$\kappa : H^1(W_j; \mathbb{Q}(t)/\Lambda) \rightarrow \text{Hom}_\Lambda(H_1(W_j; \Lambda), \mathbb{Q}(t)/\Lambda)$$

is an isomorphism from the universal coefficient spectral sequence and the map

$$(i_j)^\# : \text{Hom}_\Lambda(H_1(W_j; \Lambda), \mathbb{Q}(t)/\Lambda) \rightarrow \text{Hom}_\Lambda(H_1(M; \Lambda)/\text{Ker}(i_j)_*, \mathbb{Q}(t)/\Lambda)$$

is also an isomorphism since  $(i_j)_* : H_1(M; \Lambda) \rightarrow H_1(W_j; \Lambda)$  is onto. Now one follows the course of the proof of [4, Theorem 4.5] and obtains that  $\text{Ker}(i_j)_* = (\text{Ker}(i_j)_*)^\perp$ .  $\square$

By the observation preceding Proposition 2.10, we have the following corollary.

**Corollary 2.11.** *If a knot  $K$  is doubly (1)-solvable, then  $K$  has a Seifert matrix which is  $S$ -equivalent to a doubly null cobordant matrix.*

It is unknown to the author if a knot with the hyperbolic Blanchfield form is doubly (1)-solvable.

**Remark 2.12.** That a matrix is  $S$ -equivalent to a doubly null cobordant matrix does not imply that the matrix itself is doubly null cobordant. Thus, that a knot is algebraically doubly slice does not mean that all of its associated Seifert forms are hyperbolic (but at least there is one Seifert form that is hyperbolic) (see [14]).

### 3. Genetic modification

In this section we recall the notion of genetic modification and show when it preserves  $(m, n)$ -solvability of a knot. This modification of knots is the same as the one used in [5,6].

Let  $K$  be a knot in  $S^3$ . Let  $\eta$  be a trivial knot in  $S^3$  which is disjoint from  $K$ . Let  $J$  be another knot. Take the exterior of  $\eta$  (which is homeomorphic to a solid torus) and the exterior of  $J$ . Now identify them along their boundary such that the meridian of  $\eta$  (say  $\mu_\eta$ ) is identified with the longitude of  $J$  (say  $\ell_J$ ) and the longitude of  $\eta$  (say  $\ell_\eta$ ) is identified with the meridian of  $J$  (say  $\mu_J$ ). The resulting ambient manifold is homeomorphic to  $S^3$ , and we denote the image of  $K$  under this modification by  $K(J, \eta)$ . In fact,  $K(J, \eta)$  is a satellite of  $J$ . This construction can be generalized to the case that we have a trivial link  $\{\eta_1, \eta_2, \dots, \eta_n\}$  which misses  $K$  and a set of auxiliary knots  $\{J_1, J_2, \dots, J_n\}$  by repeating the construction. We denote the resulting knot by  $K(\{J_1, J_2, \dots, J_n\}, \{\eta_1, \eta_2, \dots, \eta_n\})$ . More details can be found in [5].

The following proposition is proved in the proof of [5, Proposition 3.1]. For a group  $G$ , we define  $G^{(0)} \equiv [G, G]$ , and inductively  $G^{(n+1)} \equiv [G^{(n)}, G^{(n)}]$  for  $n \geq 0$ . That is,  $G^{(n)}$  is the  $n$ th derived subgroup of  $G$ .

**Proposition 3.1** (Cochran et al. [5, Proposition 3.1]). *If  $K$  is  $(n)$ -solvable via  $W$ ,  $\eta \in \pi_1(W)^{(n)}$ , and  $J$  is a knot with vanishing Arf invariant, then  $K(J, \eta)$  is  $(n)$ -solvable.*

We give a brief explanation as to how to construct an  $(n)$ -solution for  $K(J, \eta)$  from  $W$  in the above proposition. This will also serve to set the notations that will be used later in this paper. Since Arf invariant vanishes for  $J$ ,  $J$  is  $(0)$ -solvable. Let  $W_J$  be a  $(0)$ -solution for  $J$ . By doing surgery on the commutator subgroup of  $\pi_1(W_J)$ , we may assume that  $\pi_1(W_J) \cong \mathbb{Z}$ . Note that  $\partial W = M_K$  and  $\partial W_J = M_J = E_J \cup S^1 \times D^2$  where  $E_J$  is the exterior of  $J$ ,  $\{*\} \times \partial D^2$  is the longitude  $\ell_J$ , and  $S^1 \times \{*\}$  is the meridian  $\mu_J$ . Let  $\eta \times D^2$  be a tubular neighborhood of  $\eta$  in  $M_K$ . Then the  $(n)$ -solution for  $K(J, \eta)$ , say  $W'$ , is obtained from  $W$  and  $W_J$  by identifying  $\eta \times D^2 \subset \partial W$  and  $S^1 \times D^2 \subset \partial W_J$ .

The next proposition shows that we have a similar result for  $(m, n)$ -solvable knots. In the statement,  $W'_1$  and  $W'_2$  denote the  $(m)$ -solution and the  $(n)$ -solution for  $K(J, \eta)$  obtained from  $W_1$  and  $W_2$  by the above construction in the previous paragraph.

**Proposition 3.2.** *Suppose  $K$  is  $(m, n)$ -solvable via  $(W_1, W_2)$ ,  $\eta \in \pi_1(W_1)^{(m)} \cap \pi_1(W_2)^{(n)}$ , and  $J$  is a knot with vanishing Arf invariant. Then  $K' = K(J, \eta)$  is  $(m, n)$ -solvable via  $(W'_1, W'_2)$ .*

**Proof.** By Proposition 3.1,  $W'_1$  and  $W'_2$  are an  $(m)$ -solution and an  $(n)$ -solution for  $K'$ , respectively. Let  $W \equiv W'_1 \cup_{M_{K'}} W'_2$ . We need to show that  $\pi_1(W) \cong \mathbb{Z}$ . For convenience, let  $M \equiv M_K$  and  $M' \equiv M_{K'}$ . Since  $K$  is  $(m, n)$ -solvable via  $(W_1, W_2)$ , we have the following push-out diagram in the category of groups and homomorphisms.

$$\begin{array}{ccc} & \pi_1(W_1) & \\ i_1 \nearrow & & \searrow j_1 \\ \pi_1(M) & & \mathbb{Z} \\ i_2 \searrow & & \nearrow j_2 \\ & \pi_1(W_2) & \end{array}$$

We will show that the following diagram is also a push-out diagram, then this will complete the proof. In the diagram,  $i'_1$  and  $i'_2$  are the homomorphisms induced from the inclusions and  $j'_1$  and  $j'_2$  are the abelianization.

$$\begin{array}{ccc} & \pi_1(W'_1) & \\ i'_1 \nearrow & & \searrow j'_1 \\ \pi_1(M') & & \mathbb{Z} \\ i'_2 \searrow & & \nearrow j'_2 \\ & \pi_1(W'_2) & \end{array}$$

Suppose we are given a commutative diagram as below where  $\Gamma$  is a group.

$$\begin{array}{ccc} & \pi_1(W'_1) & \\ i'_1 \nearrow & & \searrow \alpha_1 \\ \pi_1(M') & & \Gamma \\ i'_2 \searrow & & \nearrow \alpha_2 \\ & \pi_1(W'_2) & \end{array}$$

We study relationship among the fundamental groups of the spaces. Observe that  $M' = (M \setminus \text{int}(\eta \times D^2)) \cup_{\eta \times S^1} E_J$  where  $\eta \times S^1 = \partial(\eta \times D^2)$ . Let  $X = M \setminus \text{int}(\eta \times D^2)$ . By van Kampen Theorem,  $\pi_1(M) \cong \pi_1(X) / \langle \mu_\eta \rangle$  where  $\langle \mu_\eta \rangle$  is the subgroup normally generated by  $\mu_\eta$  in  $\pi_1(X)$ , and

$$\pi_1(M') \cong \frac{\pi_1(X) * \pi_1(E_J)}{\langle \ell_\eta \mu_J^{-1}, \mu_\eta \ell_J^{-1} \rangle}.$$

For  $W'_1$  and  $W'_2$ , van Kampen Theorem shows that for  $i = 1, 2$ ,

$$\pi_1(W'_i) \cong \frac{\pi_1(W_i) * \pi_1(W_J)}{\langle \ell_\eta \mu_J^{-1} \rangle} \cong \frac{\pi_1(W_i) * \langle \mu_J \rangle}{\langle \ell_\eta \mu_J^{-1} \rangle} \cong \pi_1(W_i).$$

For simplicity, let

$$G \equiv \frac{\pi_1(X) * \pi_1(E_J)}{\langle \ell_\eta \mu_J^{-1}, \mu_\eta \ell_J^{-1} \rangle}$$

and  $f : G \rightarrow \pi_1(M')$  be the isomorphism given by van Kampen Theorem. Consider the following commutative diagram:

$$\begin{array}{ccc} & \pi_1(W'_1) & \\ i'_1 \circ f \nearrow & & \searrow \alpha_1 \\ G & & \Gamma \\ i'_2 \circ f \searrow & & \nearrow \alpha_2 \\ & \pi_1(W'_2) & \end{array}$$

Since  $\ell_J = e$  in  $\pi_1(W_J)$ ,  $\ell_J = e$  in  $\pi_1(W'_1)$  and  $\pi_1(W'_2)$ . Furthermore,  $\pi_1(E_J)$  is mapped into  $\langle \mu_J \rangle (= \pi_1(W_J))$  in  $\pi_1(W'_i)$ . Thus,  $i'_1 \circ f$  and  $i'_2 \circ f$  factor through  $\pi_1(M) * \langle \mu_J \rangle / \langle \ell_\eta \mu_J^{-1} \rangle$  which is isomorphic to  $\pi_1(M)$ . So we have the following commutative diagram:

$$\begin{array}{ccccc} & & \pi_1(W'_1) & & \\ & i'_1 \circ f \nearrow & & \searrow \alpha_1 & \\ G & \longrightarrow & \pi_1(M) & & \Gamma \\ & i'_2 \circ f \searrow & & \nearrow \alpha_2 & \\ & & \pi_1(W'_2) & & \end{array}$$

Let  $p_1 : \pi_1(W'_1) \rightarrow \pi_1(W_1)$  be the inverse of the isomorphism  $\pi_1(W_1) \rightarrow \pi_1(W'_1)$  induced from the inclusion. Define  $p_2 : \pi_1(W'_2) \rightarrow \pi_1(W_2)$  similarly. Then the above diagram induces the following commutative diagram:

$$\begin{array}{ccccc} & & \pi_1(W_1) & & \\ & p_1 \circ k_1 \nearrow & & \searrow \alpha_1 \circ p_1^{-1} & \\ \pi_1(M) & \longrightarrow & & & \Gamma \\ & p_2 \circ k_2 \searrow & & \nearrow \alpha_2 \circ p_2^{-1} & \\ & & \pi_1(W_2) & & \end{array}$$

One sees that  $i_1 = p_1 \circ k_1$  and  $i_2 = p_2 \circ k_2$ . By the universal property of the push-out diagram, we have a unique homomorphism  $\beta : \mathbb{Z} \rightarrow \Gamma$  that makes the following diagram commutative:

$$\begin{array}{ccccc}
 & \pi_1(W_1) & & & \\
 i_1 \nearrow & & \searrow j_1 & \alpha_1 \circ p_1^{-1} & \\
 \pi_1(M) & & \mathbb{Z} & \xrightarrow{\beta} & \Gamma \\
 i_2 \searrow & & \nearrow j_2 & \alpha_2 \circ p_2^{-1} & \\
 & \pi_1(W_2) & & &
 \end{array}$$

Thus, the following diagram is also commutative:

$$\begin{array}{ccccc}
 & \pi_1(W'_1) & & & \\
 i'_1 \nearrow & & \searrow j'_1 & \alpha_1 & \\
 \pi_1(M') & & \mathbb{Z} & \xrightarrow{\beta} & \Gamma \\
 i'_2 \searrow & & \nearrow j'_2 & \alpha_2 & \\
 & \pi_1(W'_2) & & &
 \end{array}$$

where  $j'_1 \equiv j_1 \circ p_1$ ,  $j'_2 \equiv j_2 \circ p_2$ . The choice of  $\beta : \mathbb{Z} \rightarrow \Gamma$  is unique because it is unique in the previous diagram involving  $\pi_1(M)$ ,  $\pi_1(W_1)$ , and  $\pi_1(W_2)$ .  $\square$

Note that  $\pi_1(W'_i) \cong \pi_1(W_i)$  in the above proof. Therefore by applying Proposition 3.2 repeatedly, we obtain the following corollary.

**Corollary 3.3.** Suppose  $K$  is  $(m, n)$ -solvable via  $(W_1, W_2)$ . Suppose  $\eta_i \in \pi_1(W_1)^{(m)} \cap \pi_1(W_2)^{(n)}$ , and the Arf invariant vanishes for  $J_i$  for  $1 \leq i \leq n$ . Then

$$K(\{J_1, J_2, \dots, J_n\}, \{\eta_1, \eta_2, \dots, \eta_n\})$$

is  $(m, n)$ -solvable via  $(W'_1, W'_2)$ .

The following lemma and proposition give conditions under which the knot resulting from genetic modification performed on a ribbon knot is still a ribbon knot. Let  $f_i : D^2 \rightarrow S^3$  be immersions,  $1 \leq i \leq n$ , where each immersed disk  $f_i(D^2)$  has only ribbon singularities. We say  $f_i(D^2)$  have *ribbon intersections* if for  $i \neq j$ ,  $f_i(D^2)$  misses the singularities of  $f_j(D^2)$  and  $f_i^{-1}(f_i(D^2) \cap f_j(D^2))$  consists of arcs on  $D^2$  either having endpoints on  $\partial D^2$  or interior to  $D^2$ . Recall that  $\eta_i$ ,  $1 \leq i \leq n$ , denotes a trivial link which misses a knot  $K$ . Let  $J_i$ ,  $1 \leq i \leq n$ , denote knots in  $S^3$  (not necessarily with vanishing Arf invariant), and  $B^4$  denote the standard 4-ball.

**Lemma 3.4.** Suppose  $K$  is a ribbon knot bounding a ribbon disk  $B$ . Let  $\eta_i$ ,  $1 \leq i \leq n$ , bound disjoint embedded disks  $D_i$  in  $S^3$  such that  $D_i$  and  $B$  have ribbon intersections. Let  $J_i$ ,  $1 \leq i \leq n$ , be knots in  $S^3$ . Then  $K' \equiv K(\{J_i\}_{1 \leq i \leq n}, \{\eta_i\}_{1 \leq i \leq n})$  is a ribbon knot.

**Proof.** Since  $D_i$  and  $B$  have ribbon intersections, a component of the intersection of  $D_i$  with  $B$  is an arc on  $D_i$  either having end points on  $\partial D_i$ , say a *type I arc*, or interior to  $D_i$ , say a *type II arc*. We claim that we may assume the intersection of  $D_i$  with  $B$  is only type II arcs. We use an “outermost arc argument” to

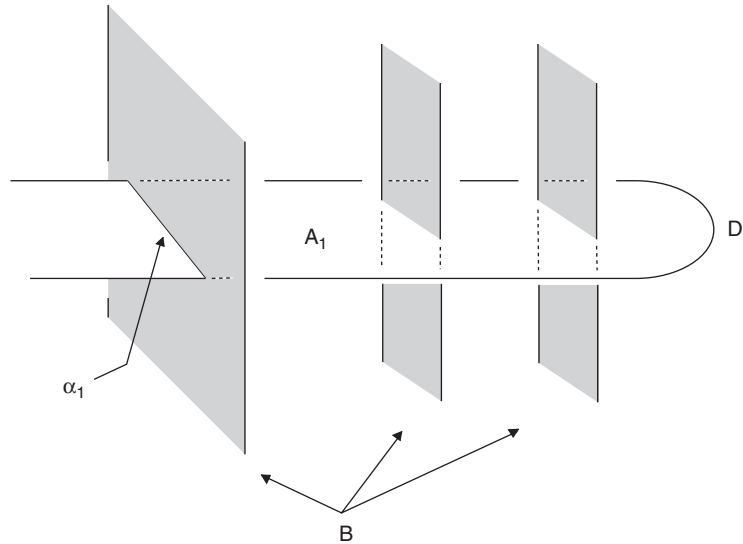


Fig. 2.

show this. Denote type I intersection arcs of  $D_i$  with  $B$  by  $\alpha_j$ ,  $1 \leq j \leq \ell$ . Suppose  $\alpha_1$  is an outermost arc. That is,  $\alpha_1$  splits  $D_i$  into two disks, say  $A_1$  and  $A_2$ , such that  $A_1$  intersects the ribbon disk  $B$  in only type II arcs. See Fig. 2.

Now deform the interior of  $B$  along  $A_1$  using a finger move and remove the intersection arc  $\alpha_1$ . This may introduce new self-intersections for  $B$ . But since the intersection of  $A_1$  with  $B$  consists of only type II arcs, the new self-intersections for  $B$  are ribbon singularities. Hence the deformed (immersed) disk is a ribbon disk and it has the same boundary  $K$  as  $B$ . We repeat this process until we remove all type I intersection arcs on  $D_i$  and this proves the claim.

Observe that  $K'$  is indeed the result of cutting open  $K$  along  $D_i$  and tying all the strands that pass through  $D_i$  into  $J_i$  with 0-frame. By the claim the intersection of  $D_i$  with  $B$  is arcs interior to  $D_i$ , hence  $B$  passes through the interior of  $D_i$  like bands. (See the two bands on the right in Fig. 2.) Thus, by cutting open  $B$  along  $D_i$  and tying the bands into  $J_i$ , we obtain an immersed disk, say  $B'$ , which is bounded by  $K'$ . One easily sees that tying  $B$  into  $J_i$  does not introduce new self-intersections. Hence  $B'$  is still a ribbon disk.  $\square$

The following proposition is due to Peter Teichner.

**Proposition 3.5** (Teichner). *Suppose  $K$  is a ribbon knot bounding a ribbon disk  $B$ . Let  $\tilde{B}$  be a slice disk for  $K$  obtained by deforming the ribbon disk  $B$  into  $B^4$ . Suppose  $\eta_i$  ( $1 \leq i \leq n$ ) are knots in  $S^3 \setminus K$  that are homotopically trivial in  $B^4 \setminus \tilde{B}$ . Then there exists a trivial link  $\tau_i$  ( $1 \leq i \leq n$ ) in  $S^3$  which is disjoint from  $K$  such that each  $\tau_i$  is homotopic to  $\eta_i$  in  $S^3 \setminus K$  and  $K(\{J_i\}_{1 \leq i \leq n}, \{\tau_i\}_{1 \leq i \leq n})$  is a ribbon knot.*

**Proof.** We may think of  $B$  as an (immersed) band sum of embedded disks in  $S^3$ . Note that the kernel of the homomorphism  $\pi_1(S^3 \setminus K) \rightarrow \pi_1(B^4 \setminus \tilde{B})$  induced by inclusion is normally generated by the meridians to the bands of  $B$ . Hence there is a trivial link  $\tau_i$  in  $S^3$  which is disjoint from  $K$  such that each  $\tau_i$  is homotopic to  $\eta_i$  in  $S^3 \setminus K$  and  $\tau_i$  ( $1 \leq i \leq n$ ) bound mutually disjoint embedded disks, say  $D_i$ , in  $S^3$  where each  $D_i$  is

obtained by taking a band sum of copies of the meridional disks to the bands of  $B$ . One sees that  $D_i$  and  $B$  have ribbon intersection. Now the proposition follows from Lemma 3.4.  $\square$

#### 4. $(m)$ -Solutions that are not $(m.5)$ -solutions

Throughout this section, we assume  $K$  is a genus 2 fibred knot that is  $(m)$ -solvable. Let

$$K' \equiv K(\{J_i\}_{1 \leq i \leq n}, \{\eta_i\}_{1 \leq i \leq n}),$$

the knot resulting from genetic modification. We assume that all  $J_i$  are  $(0)$ -solvable and  $\eta_i$  are lying in  $\pi_1(M_K)^{(m)}$ . By Proposition 3.1,  $K'$  is  $(m)$ -solvable. Let  $V$  be an  $(m)$ -solution for  $K'$ . In this section we investigate conditions under which it is guaranteed that  $V$  is not an  $(m.5)$ -solution for  $K'$ . The key result is Proposition 4.4.

We briefly explain the strategy for proving Theorem 1.1(1) to clarify why this investigation will play an important role for the proof of the main theorem. To prove the main theorem we construct a fibred genus 2 doubly slice knot  $K$  and perform genetic modification via  $\eta_i$  with  $\eta_i \in \pi_1(M_K)^{(m)}$  for all  $i$ . The resulting knot  $K'$  is doubly  $(m)$ -solvable by Corollary 3.3. Then we show that with a suitable choice of  $\eta_i$  and  $J_i$ , for any given double  $(m)$ -solution  $(V_1, V_2)$  for  $K'$ , at least one of  $V_1$  and  $V_2$  is not an  $(m.5)$ -solution. This will show that  $K'$  is not doubly  $(m.5)$ -solvable.

In fact, what we investigate was studied by Cochran and Teichner in [6] in which they create the examples of knots that are  $(m)$ -solvable but not  $(m.5)$ -solvable. In [6], they show that there is a trivial link  $L \equiv \{\eta_i\}_{1 \leq i \leq n}$  which lies in  $\pi_1(M_K)^{(m)} \setminus \pi_1(M_K)^{(m+1)}$  such that  $K'$  is not  $(m.5)$ -solvable. However, note that to prove Theorem 1.1(1) we need  $K'$  to be  $(n)$ -solvable for all  $n$ . Thus, we use not the whole link  $L$  but its sublinks for genetic modification to construct our examples, and we need to find out how to choose those sublinks.

We follow arguments in [6]. Any result in this section can be obtained from [6], with a little investigation if needed.

Throughout this section  $M$  and  $M'$  denote zero surgeries on  $K$  and  $K'$ , respectively. We assume  $\eta_i$ ,  $J_i$ , and  $V$  as in the first paragraph of this section. We begin by giving a “standard” method which gives us an  $(m)$ -solution  $W$  for  $K$  from a given  $(m)$ -solution  $V$  for  $K'$ . We construct a standard cobordism  $C$  between  $M$  and  $M'$  as follows. For each  $(0)$ -solvable knot  $J_i$ , choose a  $(0)$ -solution  $W_i$  such that  $\pi_1(W_i) \cong \mathbb{Z}$ . We form  $C$  from  $M \times [0, 1]$  and  $W_i$  by identifying  $\eta_i \times D^2$  in  $M \times \{1\}$  and the solid torus  $S^1 \times D^2$  in  $\partial W_i = (S^3 \setminus N(J_i)) \cup S^1 \times D^2$  in such a way that the meridian of  $\eta_i$  is glued with the longitude of  $J_i$  and the longitude of  $\eta_i$  is glued with the meridian of  $J_i$  for  $1 \leq i \leq n$ . ( $N(J_i)$  is an open tubular neighborhood of  $J_i$  in  $S^3$ .) One sees that  $\partial_- C = M$  and  $\partial_+ C = M'$ . Now we define  $W$  to be the union of the cobordism  $C$  and the  $(m)$ -solution  $V$  for  $K'$  along  $M'$ . Then  $\partial W = M$  and  $W$  is an  $(m)$ -solution for  $K$ . To see  $W$  is an  $(m)$ -solution for  $K$ , the readers are referred to [6].

Since  $M$  fibers over  $S^1$  with a fiber genus 2 closed surface  $\Sigma$ ,  $\pi_1(M) \cong \pi_1(\Sigma) \rtimes \mathbb{Z}$  where  $\pi_1(\Sigma) \cong \pi_1(M)^{(1)}$ . Let  $S$  denote  $\pi_1(\Sigma)$ . The group  $S$  has a presentation  $\langle x_1, x_2, x_3, x_4 \mid [x_1, x_2][x_3, x_4] \rangle$ . Let  $(a, b)$  and  $(c, d)$  be orderings of the sets  $\{1, 2\}$  and  $\{3, 4\}$ , respectively. We define the set  $P_n^{a,c}$  whose elements are pairs of elements in  $S^{(n)} (= \pi_1(M)^{(n+1)})$  for each  $n$  inductively as follows. (Therefore, we define the four sets  $P_n^{1,3}$ ,  $P_n^{1,4}$ ,  $P_n^{2,3}$ , and  $P_n^{2,4}$ .) Define  $P_1^{a,c} = \{([x_a, x_b], [x_a, x_c])_{a,c}\}$ . The subscript  $a, c$  for the pair is used to designate that this pair is an element of  $P_n^{a,c}$  to prevent possible confusion in the future use. Assume

$P_n^{a,c}$  has been defined. We define  $P_{n+1}^{a,c}$  as follows. For each  $(y, z)_{a,c} \in P_n^{a,c}$ ,  $P_{n+1}^{a,c}$  contains the following three pairs:

$$([y, y^{x_a}], [z, z^{x_a}])_{a,c}, ([y, z], [z, z^{x_a}])_{a,c}, ([y, y^{x_a}], [y, z])_{a,c},$$

where  $y^x = x^{-1}yx$ . Thus,  $P_{n+1}^{a,c}$  has  $3^n$  pairs.

Next, we introduce the notion of *algebraic solutions*. For a group  $G$ , let  $G_k \equiv G/G_{tf}^{(k)}$  where  $G_{tf}^{(k)}$  is the  $k$ th rational derived group of  $G$  of Harvey [12]. The following definition and propositions can be found in [6].

**Definition 4.1** (Cochran and Teichner [6, Definition 6.1]). A homomorphism  $r : S \rightarrow G$  is called an *algebraic (n)-solution* ( $n \geq 1$ ) if the following hold:

- (1)  $r_* : H_1(S; \mathbb{Q}) \rightarrow H_1(G; \mathbb{Q})$  has two-dimensional image and there exists an ordering  $(a, b)$  of the set  $\{1, 2\}$  and an ordering  $(c, d)$  of the set  $\{3, 4\}$  such that  $r_*(x_a)$  and  $r_*(x_c)$  are non-trivial.
- (2) For each  $0 \leq k \leq n-1$ , the following composition is non-trivial even after tensoring with the quotient field  $\mathcal{K}(G_k)$  of  $\mathbb{Z}G_k$ :

$$H_1(S; \mathbb{Z}G_k) \xrightarrow{r_*} H_1(G; \mathbb{Z}G_k) \cong G_{tf}^{(k)} / [G_{tf}^{(k)}, G_{tf}^{(k)}] \rightarrow G_{tf}^{(k)} / G_{tf}^{(k+1)}.$$

We remark that if  $r : S \rightarrow G$  is an algebraic  $(n)$ -solution, then for any  $k < n$  it is an algebraic  $(k)$ -solution. The following proposition is proved in the proof of [6, Lemma 6.7].

**Proposition 4.2** (Cochran and Teichner [6]). For any algebraic  $(n)$ -solution  $r : S \rightarrow G$  such that  $r_*(x_a)$  and  $r_*(x_c)$  are non-trivial, there exists a pair in  $P_n^{a,c}$  (which is called a special pair) which maps to a  $\mathbb{Z}G_n$ -linearly independent set under the composition:

$$S^{(n)} \rightarrow S^{(n)} / S^{(n+1)} \cong H_1(S; \mathbb{Z}S_n) \xrightarrow{r_*} H_1(S; \mathbb{Z}G_n).$$

Let  $W$  be the  $(m)$ -solution for  $K$  obtained from an  $(m)$ -solution  $V$  for  $K'$  by the “standard” method explained as above in this section. Let  $\mathcal{G} \equiv \pi_1(W)^{(1)}$ . The inclusion  $i : M \rightarrow W$  induces a homomorphism  $h : S \rightarrow \mathcal{G}$ .

**Proposition 4.3** (Cochran and Teichner [6, Proposition 6.2]). The homomorphism  $h : S \rightarrow \mathcal{G}$  is an algebraic  $(m)$ -solution.

By Propositions 4.2 and 4.3, there exists an ordering  $(a, b)$  of the set  $\{1, 2\}$  and an ordering  $(c, d)$  of the set  $\{3, 4\}$  such that  $h_*(x_a)$  and  $h_*(x_c)$  are non-trivial. Now we have the following proposition. We remind the reader that  $J_i$  are  $(0)$ -solvable and  $\{\eta_i\}_{1 \leq i \leq n}$  is a trivial link which misses  $K$ . In the following proposition,  $\rho_{\mathbb{Z}}(J_i)$  denotes the von Neumann  $\rho$ -invariant  $\rho(M_{J_i}, \phi)$  where  $\phi : \pi_1(M_{J_i}) \rightarrow \mathbb{Z}$  is the abelianization. It is known that for  $M$ , there is an upper bound for von Neumann  $\rho$ -invariants. More precisely, there exists a constant  $c_M$  such that  $|\rho(M, \phi)| \leq c_M$  for every representation  $\phi : \pi_1(M) \rightarrow \Gamma$  where  $\Gamma$  is a group (see [3,18, Theorem 3.1.1]). For von Neumann  $\rho$ -invariants, refer to [3–5].

**Proposition 4.4.** Suppose  $\rho_{\mathbb{Z}}(J_i) > c_M$  for  $1 \leq i \leq n$ . Suppose  $(a, b)$  and  $(c, d)$  are orderings of the sets  $\{1, 2\}$  and  $\{3, 4\}$ , respectively, such that  $h_*(x_a)$  and  $h_*(x_c)$  are non-trivial in  $\mathcal{G} (= \pi_1(W)^{(1)})$ . If  $\{\eta_i\}_{1 \leq i \leq n}$



is a link in  $S^3 \setminus \Sigma$  such that the set of all homotopy classes represented by  $\eta_i$  contains all homotopy classes in the pairs in  $P_{m-1}^{a,c}$ , then the  $(m)$ -solution  $V$  for  $K'$  is not an  $(m.5)$ -solution for  $K'$ .

**Proof.** By Proposition 4.3, the homomorphism  $h$  is an algebraic  $(m)$ -solution, hence an algebraic  $(m-1)$ -solution. By Proposition 4.2, for the homomorphism  $h$  there exists a special pair in  $P_{m-1}^{a,c}$  which maps to a  $\mathbb{Z}\mathcal{G}_{m-1}$ -linearly independent set under the composition

$$S^{(m-1)} \rightarrow S^{(m-1)}/S^{(m)} \cong H_1(S; \mathbb{Z}[S/S^{(m-1)}]) \xrightarrow{h_*} H_1(S; \mathbb{Z}\mathcal{G}_{m-1}),$$

where  $\mathcal{G}_{m-1} = \mathcal{G}/\mathcal{G}_{tf}^{(m-1)}$ . So there is at least one pair, say  $(y, z)$ , in  $P_{m-1}^{a,c}$  which maps to a basis of  $H_1(S; \mathcal{K}(\mathcal{G}_{m-1}))$  where  $\mathcal{K}(\mathcal{G}_{m-1})$  is the (skew) quotient field of  $\mathbb{Z}\mathcal{G}_{m-1}$ . By part (2) of Definition 4.1, at least one of  $y$  and  $z$  maps non-trivially under the composition

$$S^{(m-1)} \rightarrow H_1(S; \mathbb{Z}\mathcal{G}_{m-1}) \xrightarrow{h_*} H_1(H; \mathbb{Z}\mathcal{G}_{m-1}) \rightarrow \mathcal{G}_{tf}^{(m-1)}/[\mathcal{G}_{tf}^{(m-1)}, \mathcal{G}_{tf}^{(m-1)}] \rightarrow \mathcal{G}_{tf}^{(m-1)}/\mathcal{G}_{tf}^{(m)}.$$

By our choice of  $\eta_i$ , this tells us that there exists  $\eta_j$  for some  $j$  which maps non-trivially to  $\mathcal{G}_{tf}^{(m-1)}/\mathcal{G}_{tf}^{(m)}$ , hence  $i_*(\eta_j) \notin \mathcal{G}_{tf}^{(m)} = \pi_1(W)_{tf}^{(m+1)}$ .  $((\pi_1(W))_{tf}^{(1)})_{tf}^{(m)} = \pi_1(W)_{tf}^{(m+1)}$  since  $H_1(W) \cong \pi_1(W)/[\pi_1(W), \pi_1(W)] \cong \mathbb{Z}$  which is torsion free.)

Let  $\Gamma \equiv \pi_1(W)/\pi_1(W)_{tf}^{(m+1)}$ . Then  $\Gamma$  is an  $(m)$ -solvable poly-torsion-free-abelian group by [12, Corollary 3.6]. Let  $\psi : \pi_1(W) \rightarrow \Gamma$  be the projection. By [6, Proposition 4.5],

$$\rho(M, \psi|_{\pi_1(M)}) - \rho(M', \psi|_{\pi_1(M')}) = \sum_{i=1}^n \varepsilon_i \rho_{\mathbb{Z}}(J_i),$$

where  $\varepsilon_i = 0$  if  $\psi(\eta_i) = e$ , and  $\varepsilon_i = 1$  otherwise.

If  $V$  were an  $(m.5)$ -solution for  $K'$ ,  $\rho(M', \psi|_{\pi_1(M')}) = 0$  by [4, Theorem 4.2]. Since  $\psi(\eta_j) \neq e$ , it follows that  $\rho(M, \psi|_{\pi_1(M)}) > c_M$ , which is a contradiction. Therefore,  $V$  is not an  $(m.5)$ -solution for  $K'$ .  $\square$

## 5. The proof of the main theorem

We use the same notations as in Section 4. In particular,  $M \equiv M_K$  and  $M' \equiv M_{K'}$ . Before giving the proof, we start with our choice for the seed knot  $K$  and a little lemma for  $M$ . Let  $T$  be the right-handed trefoil. We define  $K$  to be  $T \# (-T)$ . See Fig. 3. The rectangles containing integers symbolize full twists. Thus, the rectangle labelled  $+1$  symbolizes 1 right-handed full twist. Then  $K$  is doubly slice by the following theorem and its corollary due to Zeeman and Sumners, respectively.

**Theorem 5.1** (Zeeman [27, Corollary 2, p. 487]). *Every 1-twist-spun knot is unknotted.*

**Corollary 5.2** (Sumners [24]).  *$J \# (-J)$  is doubly slice for every knot  $J$ .*

More generally, in [27] Zeeman proves that the complement of a  $k$ -twist-spun knot in  $S^4$  fibers with fiber the punctured  $k$ -fold cyclic cover of  $S^3$  branched along the knot we are spinning. Also it is

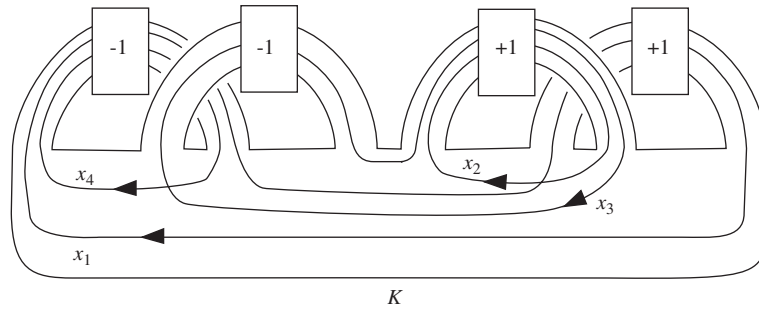


Fig. 3.

well-known that  $J\#(-J)$  is a ribbon knot for every knot  $J$ . (For instance, see [13, Proposition 5.10, p. 83].) Moreover, since  $T$  is a genus 1 fibred knot,  $K$  is a genus 2 fibred knot. Combining all these, one sees that  $K$  is a genus 2 fibred doubly slice ribbon knot.

The knot  $K$  bounds the obvious Seifert surface  $F$  that is the boundary connected sum of disks with bands as one sees in Fig. 3. Since  $K$  is fibred,  $M$  fibers over  $S^1$  with a fiber  $\Sigma$  which is obtained by taking the union of  $F$  and a 2-disk (surgery disk) along the boundary. Let  $x_1, x_2, x_3$ , and  $x_4$  denote the simple closed curves on  $\Sigma$  as shown in Fig. 3 whose homology classes form a symplectic basis for  $H_1(\Sigma)$ . Recall that  $S = \pi_1(\Sigma) \cong \pi_1(M)^{(1)}$ . Thus, the group  $S$  has a presentation  $\langle x_1, x_2, x_3, x_4 \mid [x_1, x_2][x_3, x_4] \rangle$  as in Section 4 where we abuse notations for convenience so that each  $x_i$  in the presentation is identified with the homotopy class represented by the simple closed curve  $x_i$  on  $\Sigma$ . Recall that  $\Lambda \equiv \mathbb{Z}[t, t^{-1}]$ .

**Lemma 5.3.** Any pair of  $x_i$ 's except for the pair  $(x_1, x_3)$  generates  $H_1(M; \Lambda)$ .

**Proof.** Denote by  $y_1$  the simple closed curve which traverses once clockwise the leftmost band on  $\Sigma$  in Fig. 3. Similarly, denote by  $y_2, y_3, y_4$  the simple closed curves traversing once clockwise the remaining bands on  $\Sigma$ , respectively. (We number  $y_i$  from left to right.) Then, in  $H_1(M; \Lambda)$ , with suitable orientations for  $x_i$  and  $y_i$ , we have relations  $x_1 = y_1 + y_4$ ,  $x_2 = y_3$ ,  $x_3 = y_2 + y_3$ , and  $x_4 = y_1$ . With the choice of basis  $\{y_1, y_2, y_3, y_4\}$ , the Seifert matrix of  $K$  is

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then  $H_1(M; \Lambda)$  is presented by the matrix  $tA^t - A$  with respect to the basis  $\{y_1^*, y_2^*, y_3^*, y_4^*\}$  where  $A^t$  denotes the transpose of  $A$  and  $y_i^*$  denotes an Alexander dual of  $y_i$  in  $S^3 \setminus \Sigma$ . Since  $A$  is invertible,  $t - A(A^t)^{-1}$  is a presentation matrix of  $H_1(M; \Lambda)$  with respect to the basis  $\{y_1, y_2, y_3, y_4\}$ . Thus,  $H_1(M; \Lambda) \cong \Lambda/(t^2 - t + 1) \oplus \Lambda/(t^2 - t + 1)$  where  $y_2$  and  $y_3$  are identified with  $(1, 0)$  and  $(0, 1)$ , respectively. Also  $y_1$  and  $y_4$  are identified with  $(t, 0)$  and  $(0, t)$ . Using the relations among  $x_i$  and  $y_i$  and noting that  $t^2 - t + 1 = 0$ , one easily deduces the lemma.  $\square$

Now we give the proof of Theorem 1.1.

**Proof of Theorem 1.1(1).** Let  $n = 2 \cdot |P_{m-1}^{1,3} \cup P_{m-1}^{1,4} \cup P_{m-1}^{2,3}| = 2 \cdot 3 \cdot 3^{m-2} = 2 \cdot 3^{m-1}$ . (Recall that  $P_{m-1}^{a,c}$  were defined in Section 4.) Let  $c_M$  be a positive number given by [3,18, Theorem 3.1.1] such that  $|\rho(M, \phi)| \leq c_M$  for every representation  $\phi : \pi_1(M) \rightarrow \Gamma$  where  $\Gamma$  is a group. For  $1 \leq i \leq n$ , let  $J_i$  be an Arf invariant zero knot such that  $\rho_{\mathbb{Z}}(J_i) > c_M$ . (For example, one can choose  $J_i$  to be the connected sum of suitably many even number of left-handed trefoils.) Since  $S = \pi_1(\Sigma) \cong \pi_1(M \setminus \Sigma) \cong \pi_1(S^3 \setminus F)$ , we can choose  $n$  simple closed curves in  $S^3 \setminus F$  which represent all of the homotopy classes in the pairs in  $P_{m-1}^{1,3} \cup P_{m-1}^{1,4} \cup P_{m-1}^{2,3}$ . Label these simple closed curves by  $\eta'_i$ ,  $1 \leq i \leq n$ .

Recall that  $K$  is a ribbon knot. We claim that there is a slice disk  $D$  for  $K$  obtained by deforming a ribbon disk for  $K$  into  $B^4$  such that  $\eta'_i$  are homotopically trivial in  $B^4 \setminus D$ . For the proof of this claim and later use, we give two slice disk and 4-ball pairs  $(B^4, D_1)$  and  $(B^4, D_2)$  for  $K$  (not  $K'$ ) such that their union along the boundary gives an unknotted  $S^2$  in  $S^4$ : from [27] and Corollary 5.2,  $(B^4, D_1)$  is obtained by half-spinning  $T$  without twist and  $(B^4, D_2)$  is obtained by half-spinning  $-T$  with a 1-twist. Let  $W_1$  be the exterior of  $D_1$  in  $B^4$  and  $W_2$  the exterior of  $D_2$  in  $B^4$ . We show  $\eta'_i$  represent the trivial element in  $\pi_1(W_1)$ . This will show the claim since  $D_1$  can be obtained by deforming a ribbon disk for  $K$  into  $B^4$ . (To see this, refer to [13, Proposition 5.10, p. 83].) Let  $(a, b)$  be any of the ordered pairs  $(1, 3)$ ,  $(1, 4)$ , and  $(2, 3)$ . Observe that the simple closed curves  $x_1$  and  $x_3$  bound embedded disks in  $W_1$ . (These disks are easily obtained by half-spinning without twist the half of  $x_1$  and the half of  $x_2$  in  $B^4$ .) Therefore,  $x_1 = x_3 = e$  in  $\pi_1(W_1)$ , hence  $[x_a, x_b] = [x_a, x_c] = e$  in  $\pi_1(W_1)$ . Suppose  $(y, z)_{a,c}$  be an element of  $P_j^{a,c}$  ( $1 \leq j \leq m-2$ ) such that  $y = z = e$  in  $\pi_1(W_1)$ . Then  $[y, y^{x_a}] = [z, z^{x_a}] = [y, z] = e$  in  $\pi_1(W_1)$ . Now using an induction argument, one sees that every homotopy class in the pairs in  $P_{m-1}^{a,c}$  represents the trivial element in  $\pi_1(W_1)$ , hence  $\eta'_i = e$  in  $\pi_1(W_1)$  for all  $i$ .

Now by Proposition 3.5 there is a trivial link  $\eta_i$  ( $1 \leq i \leq n$ ) such that each  $\eta_i$  is homotopic to  $\eta'_i$  in  $S^3 \setminus K$  and  $K' \equiv K(\{J_i\}_{1 \leq i \leq n}, \{\eta_i\}_{1 \leq i \leq n})$  is a ribbon knot. In particular,  $\eta_i$  represent all of the homotopy classes in the pairs in  $P_{m-1}^{1,3} \cup P_{m-1}^{1,4} \cup P_{m-1}^{2,3}$ . Observe that a homotopy in  $S^3 \setminus K$  between  $\eta'_i$  and  $\eta_i$  can be constructed by using crossing change in  $S^3 \setminus F$  and the isotopy (which can be extended to the ambient isotopy). Hence we may assume that  $\eta_i$  are disjoint from  $F$ .

We show  $K'$  satisfies the other required conditions. To see  $K'$  is doubly  $(m)$ -solvable, just observe that  $\eta_i$  lie in  $\pi_1(M)^{(m)}$  which is mapped into  $\pi_1(W_1)^{(m)}$  and  $\pi_1(W_2)^{(m)}$ . Now it follows from Proposition 3.2 that  $K'$  is doubly  $(m)$ -solvable.

Let us assume  $(V'_1, V'_2)$  is a double  $(m)$ -solution for  $K'$ . We show that at least one of  $V'_1$  and  $V'_2$  is not an  $(m.5)$ -solution for  $K'$ . Since  $m \geq 2$ ,  $(V'_1, V'_2)$  is a double  $(1)$ -solution for  $K'$ . By Proposition 2.10 and its proof, we have  $H_1(M'; A') \cong H_1(V'_1; A') \oplus H_1(V'_2; A')$  where  $A' = \mathbb{Q}[t, t^{-1}]$ . For  $i = 1, 2$ , let  $V_i$  be the  $(m)$ -solution for  $K$  obtained from the cobordism  $C$  and  $V'_i$  as in Section 4. Using the Mayer–Vietoris sequence, one verifies that  $H_1(M'; A') \cong H_1(M; A')$  and  $H_1(V'_i; A') \cong H_1(V_i; A')$  for  $i = 1, 2$ . So the inclusions  $i_1 : M \rightarrow V_1$  and  $i_2 : M \rightarrow V_2$  induce the isomorphism  $H_1(M; A') \cong H_1(V_1; A') \oplus H_1(V_2; A')$ . We will take care of three cases: in  $H_1(V_1; A')$ , (1)  $(i_1)_*(x_1) \neq 0$ , (2)  $(i_1)_*(x_3) \neq 0$ , and (3)  $(i_1)_*(x_1) = (i_1)_*(x_3) = 0$ .

*Case 1:* Suppose  $(i_1)_*(x_1) \neq 0$  in  $H_1(V_1; A')$ . Since  $(i_1)_*$  is not a zero homomorphism (see [4, Theorem 4.4]), by Lemma 5.3  $(i_1)_*(x_3) \neq 0$  or  $(i_1)_*(x_4) \neq 0$ . Suppose  $(i_1)_*(x_3) \neq 0$ . Note the homotopy classes in the pairs in  $P_{m-1}^{1,3}$  are represented by some of  $\eta_i$ . Thus, Proposition 4.4 implies that  $V'_1$  is not an  $(m.5)$ -solution for  $K'$ . In case  $(i_1)_*(x_4) \neq 0$ , one proves  $V'_1$  is not an  $(m.5)$ -solution for  $K'$  using  $P_{m-1}^{1,4}$  with a similar argument.

*Case 2:* If  $(i_1)_*(x_3) \neq 0$ , again  $V'_1$  is not an  $(m.5)$ -solution for  $K'$  by a reason similar to Case 1. One should use  $x_1$  and  $x_2$  instead of  $x_3$  and  $x_4$  noting the homotopy classes in the pairs in  $P_{m-1}^{1,3}$  and  $P_{m-1}^{2,3}$  are represented by  $\eta_i$ .

*Case 3:* Suppose  $(i_1)_*(x_1) = (i_1)_*(x_3) = 0$ . Note  $x_1 \neq 0$  and  $x_3 \neq 0$  in  $H_1(M; A')$ . Then  $(i_2)_*(x_1) \neq 0$  and  $(i_2)_*(x_3) \neq 0$  in  $H_1(V_2; A')$  since  $H_1(M; A') \cong H_1(V_1; A') \oplus H_1(V_2; A')$ . By Proposition 4.4, since the homotopy classes in the pairs in  $P_{m-1}^{1,3}$  are represented by some of  $\eta_i$ ,  $V'_2$  is not an  $(m.5)$ -solution for  $K'$ .

It remains to show that  $K'$  is algebraically doubly slice. Using the basis  $\{x_1, x_3, x_2 - x_1, x_4 - x_3\}$  of  $H_1(F)$ , one easily sees that the associated Seifert form of  $K$  is hyperbolic. Since  $\eta_i$  are disjoint from  $F$ , this hyperbolic Seifert form does not change under the above genetic modification. Hence  $K'$  is algebraically doubly slice.  $\square$

**Proof of Theorem 1.1(2).** Since a knot is  $(k, \ell)$ -solvable if and only if it is  $(\ell, k)$ -solvable, without loss of generality we may assume  $\ell \geq k \geq 2$ . Let  $n = 2 \cdot |P_{k-1}^{1,3} \cup P_{k-1}^{1,4} \cup P_{k-1}^{2,3} \cup P_{\ell-1}^{2,4}| = 2 \cdot (3 \cdot 3^{k-2} + 3^{\ell-2})$ . Let  $c_M$  be the constant as in the proof of Theorem 1.1(1), that is, such that  $|\rho(M, \phi)| \leq c_M$  for every representation  $\phi : \pi_1(M) \rightarrow \Gamma$  where  $\Gamma$  is a group. For  $1 \leq i \leq n$ , let  $J_i$  be an Arf invariant zero knot such that  $\rho_{\mathbb{Z}}(J_i) > c_M$ . Let  $\{\eta_i\}_{1 \leq i \leq n}$  be a trivial link in  $S^3 \setminus \Sigma$  which represents all homotopy classes in the pairs in  $P_{k-1}^{1,3} \cup P_{k-1}^{1,4} \cup P_{k-1}^{2,3} \cup P_{\ell-1}^{2,4}$ . Using genetic modification, construct  $K' \equiv K(\{J_i\}_{1 \leq i \leq n}, \{\eta_i\}_{1 \leq i \leq n})$ .

One sees that  $K'$  is algebraically doubly slice using the same reasoning as in the proof of Theorem 1.1(1). Let  $W_1$  and  $W_2$  be the slice disk exteriors for  $K$  as in the proof of Theorem 1.1(1). Then  $K$  is  $(k, \ell)$ -solvable via  $(W_1, W_2)$ . Since  $x_1$  and  $x_3$  map to the trivial element in  $\pi_1(W_1)$ , the elements in the pairs in  $P_{k-1}^{1,3} \cup P_{k-1}^{1,4} \cup P_{k-1}^{2,3}$  are the trivial element in  $\pi_1(W_1)$  and in particular in  $\pi_1(W_1)^{(\ell)}$ . Since  $\pi_1(M)^{(\ell)}$  maps into  $\pi_1(W_1)^{(\ell)}$ , the elements of  $P_{\ell-1}^{2,4}$  also lie in  $\pi_1(W_1)^{(\ell)}$ . Regarding  $W_2$ , since  $\ell \geq k$ , the elements of  $P_{k-1}^{1,3} \cup P_{k-1}^{1,4} \cup P_{k-1}^{2,3} \cup P_{\ell-1}^{2,4}$  lie in  $\pi_1(W_2)^{(k)}$ . By Proposition 3.2,  $K'$  is  $(k, \ell)$ -solvable (via  $(W'_2, W'_1)$  following the notation in Section 3).

Suppose  $(V'_1, V'_2)$  is a  $(k, \ell)$ -solution for  $K'$ . We show that  $V'_2$  is not an  $(\ell.5)$ -solution for  $K'$ . Let  $V_1$  be the  $(k)$ -solution for  $K$  obtained from  $V'_1$  and the cobordism  $C$  between  $M$  and  $M'$  as in Section 4. Let  $V_2$  be the  $(\ell)$ -solution for  $K$  obtained from  $V'_2$  and  $C$ . As in the proof of Theorem 1.1(1), the inclusions  $i_1 : M \rightarrow V_1$  and  $i_2 : M \rightarrow V_2$  induce the isomorphism  $((i_1)_*, (i_2)_*) : H_1(M; A') \xrightarrow{\cong} H_1(V_1; A') \oplus H_1(V_2; A')$ . We consider the case  $k = \ell$  first. Then  $P_{\ell-1}^{2,4} = P_{k-1}^{2,4}$ . Since  $V_2$  is a  $(k)$ -solution for  $K$ , the inclusion  $i_2 : M \rightarrow V_2$  induces an algebraic  $(k)$ -solution  $r_2 : \pi_1(M)^{(1)} \rightarrow \pi_1(V_2)^{(1)}$  by Proposition 4.3. So there are orderings  $(a, b)$  and  $(c, d)$  of the sets  $\{1, 2\}$  and  $\{3, 4\}$  such that  $(i_2)_*(x_a) \neq 0$  and  $(i_2)_*(x_c) \neq 0$ . Since all homotopy classes in the pairs in  $P_{k-1}^{a,c}$  are represented by  $\eta_i$ ,  $V'_2$  is not a  $(k.5)$ -solution (i.e., not an  $(\ell.5)$ -solution) for  $K'$  by Proposition 4.4.

We assume  $\ell > k$ . Since  $V'_2$  is an  $(\ell)$ -solution for  $K'$ , it is a  $(k+1)$ -solution for  $K'$ , so  $V_2$  is a  $(k+1)$ -solution for  $K$ . Thus,  $V_2$  is an algebraic  $(k+1)$ -solution by Proposition 4.3, hence there are orderings  $(a, b)$  and  $(c, d)$  of  $\{1, 2\}$  and  $\{3, 4\}$ , respectively, such that  $(i_2)_*(x_a) \neq 0$  and  $(i_2)_*(x_c) \neq 0$ . For these  $a, b, c$ , and  $d$ , if  $(a, c)$  is one of  $(1, 3)$ ,  $(1, 4)$ , and  $(2, 3)$ , then since all homotopy classes in the pairs in  $P_{k-1}^{1,3} \cup P_{k-1}^{1,4} \cup P_{k-1}^{2,3}$  are represented by  $\eta_i$ , by Proposition 4.4  $V'_2$  is not a  $(k+1)$ -solution for  $K$ , which is a contradiction. So we deduce that  $(i_2)_*(x_2) \neq 0$ ,  $(i_2)_*(x_4) \neq 0$ , and  $(i_2)_*(x_1) = (i_2)_*(x_3) = 0$ . Since all homotopy classes in the pairs in  $P_{\ell-1}^{2,4}$  are represented by  $\eta_i$ , by Proposition 4.4  $V'_2$  is not an  $(\ell.5)$ -solution for  $K'$ .

Finally, we show that  $V'_1$  is not a  $(k.5)$ -solution for  $K'$ . If  $k = \ell$ ,  $V'_1$  is not a  $(k.5)$ -solution for  $K$  with the same reason that  $V'_2$  was not a  $(k.5)$ -solution (when  $k = \ell$ ). If  $\ell > k$ , as we showed in the previous paragraph,  $(i_2)_*(x_1) = (i_2)_*(x_3) = 0$ . Since we have the isomorphism

$$((i_1)_*, (i_2)_*) : H_1(M; A') \xrightarrow{\cong} H_1(V_1; A') \oplus H_1(V_2; A'),$$

it implies that  $(i_1)_*(x_1) \neq 0$  and  $(i_1)_*(x_3) \neq 0$ . Since all homotopy classes in the pairs in  $P_{k-1}^{1,3}$  are represented by  $\eta_i$ , Proposition 4.4 tells us that  $V'_1$  is not a  $(k.5)$ -solution for  $K'$ .  $\square$

## 6. Bi-filtration of the double concordance group

We denote the double concordance group by  $\mathcal{DC}$  and the double concordance class of  $K$  by  $[K]$ . Since connected sum is an abelian operation,  $\mathcal{DC}$  is an abelian group.  $[-K]$  is the inverse of  $[K]$  in  $\mathcal{DC}$  by Corollary 5.2. Recall that  $K_1$  and  $K_2$  are concordant if  $K_1 \# (-K_2)$  is slice. Similarly, it is known that if  $K_1 \# (-K_2)$  is doubly slice then  $K_1$  and  $K_2$  are doubly concordant. But little is known about the double concordance group because we have the following unanswered conjecture.

**Conjecture 6.1.** *If knots  $J$  and  $K \# J$  are doubly slice, then  $K$  is doubly slice.*

In this section, we construct a bi-filtration of the double concordance group using the notion of bi-solvability.

**Definition 6.2.** For  $m, n \geq 0$ ,  $\overline{\mathcal{F}}_{m,n}$  is defined to be the set of the double concordance classes represented by  $(m, n)$ -solvable knots.

**Proposition 6.3.**  $\overline{\mathcal{F}}_{m,n}$  is a subgroup of  $\mathcal{DC}$ .

**Proof.** We show  $\overline{\mathcal{F}}_{m,n}$  is closed under addition. Let  $[K_1]$  and  $[K_2]$  be in  $\overline{\mathcal{F}}_{m,n}$ . Then  $K_1 \# J_1 = K'_1 \# J'_1$  and  $K_2 \# J_2 = K'_2 \# J'_2$  for some doubly slice knots  $J_1, J_2, J'_1, J'_2$  and  $(m, n)$ -solvable knots  $K'_1, K'_2$ . Thus, we get  $(K_1 \# K_2) \# (J_1 \# J_2) = (K'_1 \# K'_2) \# (J'_1 \# J'_2)$ . By Proposition 2.6  $K'_1 \# K'_2$  is  $(m, n)$ -solvable. Since the connected sum of doubly slice knots is doubly slice, it follows that  $K_1 \# K_2$  is doubly concordant to an  $(m, n)$ -solvable knot, hence  $[K_1] + [K_2] = [K_1 \# K_2] \in \overline{\mathcal{F}}_{m,n}$ .

Let  $[K] \in \overline{\mathcal{F}}_{m,n}$ . Then  $K$  is doubly concordant to some  $(m, n)$ -solvable knot  $J$ . Since  $-K$  is doubly concordant to  $-J$  and  $-J$  is  $(m, n)$ -solvable,  $[-K] \in \overline{\mathcal{F}}_{m,n}$ . So the inverse of  $K$  is in  $\overline{\mathcal{F}}_{m,n}$  since  $-[K] = [-K]$ .  $\square$

**Corollary 6.4.**  $\{\overline{\mathcal{F}}_{m,n}\}_{m,n \geq 0}$  is a bi-filtration of  $\mathcal{DC}$ .

Unfortunately, in spite of Theorem 1.1, it is not known if the bi-filtration of  $\mathcal{DC}$  is non-trivial because we have a difficulty similar to Conjecture 6.1. More precisely, it is unknown if the following is true: if  $J$  and  $K \# J$  are  $(m, n)$ -solvable, then  $K$  is  $(m, n)$ -solvable.

## 7. Doubly (1)-solvable knots that are not doubly (1.5)-solvable

In [11], Gilmer and Livingston give a slice knot that is algebraically doubly slice but not doubly slice. Their example is obtained from the knot  $K$  in Fig. 4 by tying the right band into a left-handed trefoil

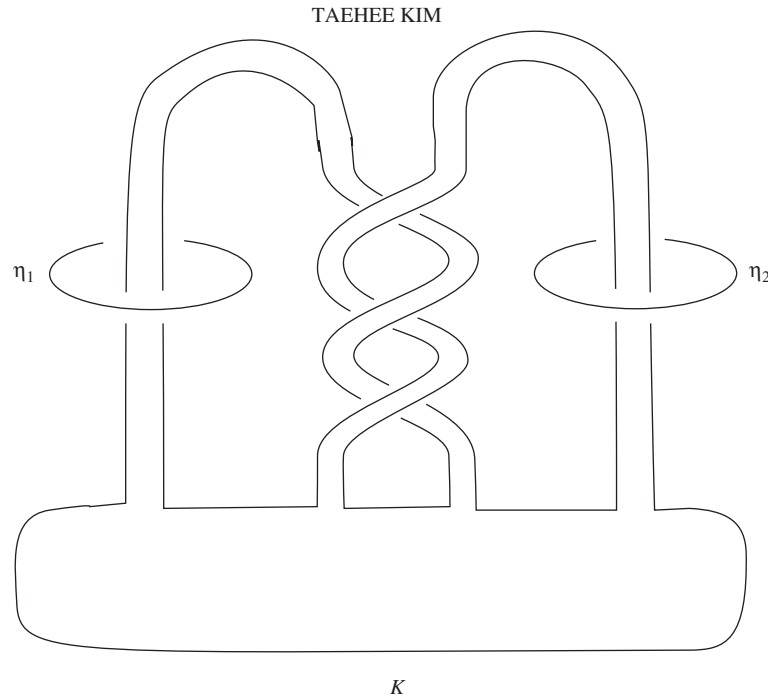


Fig. 4.

with 0-framing. In fact, by investigating the double branched cyclic covers of knots, they obtained an obstruction for a knot being doubly slice in terms of the signatures of specific simple closed curves on a Seifert surface of a knot. For more details, refer to [11, Theorem 4.2] and Section 5 in [11]. We prove the following theorem.

**Theorem 7.1.** *There exists an algebraically doubly slice knot  $K$  that is slice and doubly (1)-solvable but not doubly (1.5)-solvable (hence not doubly slice). Furthermore, the above Gilmer and Livingston's obstruction vanishes for  $K$ .*

We note that  $K$  in the above theorem can be shown not being doubly slice by applying Gilmer and Livingston's method to higher-fold finite branched cyclic covers instead of the double branched cyclic cover (also refer to [20,21]).

Before proving Theorem 7.2 we give useful properties of the knot  $K$  in Fig. 4. Let  $a$  and  $b$  be the simple closed curves on the obvious Seifert surface  $F$  which run around the left band and the right band, respectively. With respect to  $\{a, b\}$  with a suitable choice of orientation,  $K$  has the Seifert form represented by

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

hence it is algebraically doubly slice. Since both bands are unknotted and untwisted,  $K$  is doubly slice (hence slice). Since  $tA - A^t$  is a presentation matrix of  $H_1(M_K; \mathbb{A}')$ , one sees that  $H_1(M_K; \mathbb{A}') \cong \mathbb{A}'/(t-2) \oplus \mathbb{A}'/(2t-1)$ . That is, there are submodules  $P$  and  $Q$  such that  $H_1(M_K; \mathbb{A}') = P \oplus Q$ , and  $P \cong \mathbb{A}'/(t-2)$ ,



$Q \cong \Lambda'/(2t-1)$ . Here,  $P$  is generated by  $\eta_1$  and  $Q$  by  $\eta_2$  where  $\eta_1$  and  $\eta_2$ , indicated in Fig. 4, represent the Alexander duals of  $a$  and  $b$  in  $H_1(S^3 \setminus F)$ .

Moreover, the rational Blanchfield form  $B\ell'$  has exactly two self-annihilating submodules, which are  $P$  and  $Q$ . This can be shown easily using the presentation matrix  $(1-t)(tA-A')^{-1}$  of  $B\ell'$ . Now we give the proof of Theorem 7.1. In the proof,  $\sigma_\omega$  where  $\omega$  is a unit complex number is the Levine–Tristram signature function [25]. For convenience, we define  $\sigma_r$  ( $r \in \mathbb{Q}$ ) to be  $\sigma_\omega$  where  $\omega = e^{2\pi ir}$ .

**Proof of Theorem 7.1.** Let  $s$  be a number such that  $\frac{1}{3} < s < \frac{1}{2}$ . By the proof of [2, Theorem 1], there exists a knot  $J$  such that  $\sigma_r(J) = 0$  if  $0 < r < s$  or  $1-s < r < 1$  and  $\sigma_r(J) = 2$  if  $s < r < 1-s$ . Furthermore, the Alexander polynomial of  $J$ , say  $\Delta_J(t)$ , has the property that  $\Delta_J(-1) = \pm 1 \pmod{8}$ . By [16]  $J$  has vanishing Arf invariant, and in particular (0)-solvable.

Let  $K' \equiv K(J, \eta_2)$ , the knot resulting from genetic modification. For simplicity, let  $M' \equiv M_{K'}$ . Since  $\eta_2$  lies in  $\pi_1(M_K)^{(1)}$ ,  $K'$  is doubly (1)-solvable by Proposition 3.2. The associated Seifert form of  $F$  is hyperbolic and this Seifert form does not change under the above genetic modification, hence  $K'$  is algebraically doubly slice. The Seifert surface  $F'$  of  $K'$  can be obtained from  $F$  by tying the right band along  $J$  with 0-framing. Since the left band in  $F'$  remains unknotted and untwisted,  $K'$  is slice. Since  $\sigma_{\frac{1}{3}}(J) = \sigma_{\frac{2}{3}}(J) = 0$ ,  $\sigma_{\frac{1}{3}}(J\#J) = \sigma_{\frac{2}{3}}(J\#J) = 0$ . Hence the above Gilmer and Livingston's obstruction vanishes for  $K'$  (see Theorem 4.2 and Section 5 in [11] for more details). We need to show  $K'$  is not doubly (1.5)-solvable.

Since we use 0-framing when we tie a band of  $F$  into  $J$  to get  $F'$ ,  $K'$  has the same Seifert matrix  $A$  with respect to the images of  $a$  and  $b$  under genetic modification. So  $H_1(M'; \Lambda') \cong H_1(M_K; \Lambda')$ , and  $K$  and  $K'$  have isomorphic Blanchfield forms. Thus, the Blanchfield form of  $K'$  also has exactly two self-annihilating submodules. For convenience we abuse notations so that the images of  $a, b, \eta_1, \eta_2, P$ , and  $Q$  under genetic modification are denoted by the same letters. So  $H_1(M'; \Lambda') = P \oplus Q$ , and  $P \cong \Lambda'/(t-2)$ ,  $Q \cong \Lambda'/(2t-1)$ . Now since  $K$  is doubly slice, we have a double (1)-solution  $(W_1, W_2)$  for  $K$  where  $W_1$  and  $W_2$  are the slice disk exteriors. Let  $W'_1$  and  $W'_2$  be the (1)-solutions for  $K'$  constructed as in Section 3. Then  $(W'_1, W'_2)$  is a double (1)-solution for  $K'$  by Proposition 3.2. Let  $i_j$  be the inclusion map from  $M'$  into  $W'_j$  for  $j = 1, 2$ . Since the (rational) Blanchfield form of  $K'$  has exactly two self-annihilating submodules (which are  $P$  and  $Q$ ), Proposition 2.10 implies that we may assume  $\text{Ker}(i_1)_* = Q$  and  $\text{Ker}(i_2)_* = P$ . Since  $P = P^\perp$ ,  $\eta_2 \notin P$ , and the Blanchfield form is non-singular, there exists a non-zero  $p \in P$  such that  $B\ell(p, \eta_2) \neq 0$ . By [4, Theorem 3.5]  $p$  induces a representation  $\phi : \pi_1(M') \rightarrow \Gamma_1^U$  where  $\Gamma_1^U \equiv (\mathbb{Q}(t)/\Lambda') \rtimes \mathbb{Z}$ . By [4, Theorem 3.6],  $\phi$  extends to  $\psi : \pi_1(W'_2) \rightarrow \Gamma_1^U$ . So the von Neumann  $\rho$ -invariant  $\rho(M', \phi)$  can be computed using  $(W'_2, \psi)$ . Since  $B\ell(p, \eta_2) \neq 0$ , by [4, Theorem 3.5]  $\phi(\eta_2) \neq e$ . By [5, Proposition 3.2] and Property (2.2), (2.3), and (2.4) in [5],

$$\rho(M', \phi) = \rho(M_K, \psi|_{\pi_1(M_K)}) + \rho(M_J, \psi|_{\pi_1(M_J)}) = \rho(M_J, \psi|_{\pi_1(M_J)}) = 2(1-2s) \neq 0.$$

Now suppose  $(V_1, V_2)$  is a double (1)-solution for  $K'$ . Let  $j_1$  be the inclusion map from  $M'$  into  $V_1$ . Define  $j_2$  similarly. Since  $\text{Ker}(j_1)_*$  and  $\text{Ker}(j_2)_*$  are self-annihilating with respect to the rational Blanchfield form  $B\ell'$  by Proposition 2.10, without loss of generality we may assume  $\text{Ker}(j_1)_* = Q$  and  $\text{Ker}(j_2)_* = P$ . Let  $p \in P$  be as in the previous paragraph inducing the homomorphism  $\phi : \pi_1(M') \rightarrow \Gamma_1^U$ . By [4, Proposition 3.6]  $\phi$  extends to  $\psi' : \pi_1(V_2) \rightarrow \Gamma_1^U$ . So  $\rho(M', \phi)$  can be computed via  $(V_2, \psi')$ . If  $(V_1, V_2)$  were a double (1.5)-solution for  $K'$ ,  $V_2$  is a (1.5)-solution for  $K'$ . Therefore,  $\rho(M', \phi) = 0$  by [4, Theorem 4.2], which contradicts the above computation that  $\rho(M', \phi) \neq 0$ .  $\square$



In fact, one can show that  $K'$  as above is  $(1, n)$ -solvable for all  $n \in \mathbb{N}$ . We give another interesting example.

**Theorem 7.2.** *There exists a knot that is doubly  $(1)$ -solvable but not  $(1, 1.5)$ -solvable.*

**Proof.** Let  $J$  be the same as in the proof of Theorem 7.1. Let  $J_1 \equiv J$  and  $J_2 \equiv J$ . Define  $K' \equiv K(\{J_1, J_2\}, \{\eta_1, \eta_2\})$ , the knot resulting from genetic modification. Then  $K'$  is doubly  $(1)$ -solvable but not  $(1, 1.5)$ -solvable. The proof follows the same course as in Theorem 7.1, and details are omitted.  $\square$

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