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TWISTED ALEXANDER INVARIANTS, REIDEMEISTER TORSION, AND CASSON–GORDON INVARIANTS

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1. INTRODUCTION

1.1. Twisting classical knot invariants

In this paper we extend several results about classical knot invariants derived from the infinite cyclic cover to the twisted case. Let X be a finite complex with fundamental group π, let $\rho : \pi \to GL(V)$ be a linear representation where V is a finite dimensional vector space over a field F, and let $\varepsilon : \pi \to \mathbf{Z}$ be a homomorphism. Finally let X_∞ be the infinite cyclic cover of X corresponding to ε. The representation ρ restricts to give a representation of $\pi_1(X_\infty)$ to $GL(V)$ and one can define the twisted homology groups $H_i(X_\infty; V)$. These are $F[\mathbf{Z}]$ modules via the action of $\mathbf{Z}$ as the deck transformations of X_∞; a polynomial representing the order of the $F[\mathbf{Z}]$ torsion of this module is called the twisted Alexander polynomial, Δ_i, associated to the space and representations. In the case that X is a classical knot complement and ρ is a trivial one-dimensional representation, Δ_1 is the classical Alexander polynomial.

We develop properties of these twisted Alexander modules and polynomials and prove the analogues of some of the classical results about the ordinary Alexander modules and polynomials; in particular we prove a number of results relating to their application in classical knot theory and concordance. One merit of this approach is that it gives a method of organizing non-abelian invariants of knots in a framework similar to the classical approach to abelian invariants.

Of particular interest is that these invariants offer a 3-dimensional definition of certain Casson–Gordon invariants, they lead to an elementary proof that these Casson–Gordon invariants provide obstructions to knots being slice, and there are simple algorithms for their computation. We show that for certain representations of a cyclic cover of a knot complement an associated twisted polynomial must have a factorization of the form $f(t)f(t^{-1})$ if the knot is slice. In this form the obstruction is seen as a direct generalization of the well-known result concerning the factorization of the standard Alexander polynomial of a slice knot. In a second paper [10] we will apply the results presented here to show that particular knots, e.g. 8_{17}, are not concordant to their inverses and to show that positive mutation can change the concordance class of a knot, answering [12, 1.53].

1.2. Background

A twisted Alexander polynomial was first described in [13], where the polynomial is defined only for knots in S^3; the arguments and description are in terms of a presentation of the fundamental group. In [22] Wada generalized this work and showed how to define a twisted polynomial given only a presentation of a group and representations to $\mathbf{Z}$ and

$GL(V)$. Again the work takes place only on the level of the group. Related work appears in [9]. More recently, Kitano [11] showed that in the case of classical knot groups the twisted polynomial of [22] can be interpreted in terms of the Reidemeister torsion of an associated acyclic complex; this interpretation is used to prove a symmetry property of the twisted polynomial.

1.3. Summary

Section 2 is devoted to a careful exposition of infinite cyclic covers and twisted homology. We also define the *homology torsion*, τ, to be the product $\Pi_i \Delta_i^{(-1)^i}$, an element in the field of fractions, $F(t)$.

Section 3 presents a description of the Reidemeister torsion of a certain chain complex over $F(t)$ defined using X, ρ, and ε. In Theorem 3.4 we prove that the Reidemeister torsion of this complex equals τ. This yields an algorithm for computing τ and also gives us access to some of its basic properties, for instance a Mayer–Vietoris style theorem. We then give several computations of the twisted invariants. Theorem 3.7 gives a formula relating the torsion of a satellite knot and the torsion of the corresponding satellite of the unknot in the case when the representation ρ is abelian on the complement of the companion. An example is given of a knot in S^3 and a representation to $SO(3)$ so that the corresponding twisted Alexander module $H_1(X_\infty; \mathbf{R}^3)$ has a free $\mathbf{R}[\mathbf{Z}]$ summand. This is in stark contrast to the untwisted case, where the Alexander modules of a knot are always torsion over $F[\mathbf{Z}]$.

In Section 4 we describe the relationship of our definitions to previous work on the subject. Theorem 4.1 states that if $H_1(X_\infty; V)$ is torsion over $F[\mathbf{Z}]$, then

$$\Delta_1 = W \cdot \Delta_0$$

where W denotes Wada's invariant [22]. Since Δ_0 is easily computed, we see that Wada's twisted polynomial is equivalent to our Δ_1. This is especially useful, since in certain cases Δ_1 provides an obstruction to slicing knots. Moreover, Wada's algorithm to compute W is a straightforward generalization of the standard method of computing the Alexander polynomial from the matrix of Fox derivatives. We finish this section by relating the Δ_i to the invariants of [13] and [9].

In [15] Milnor proved a duality theorem for Reidemeister torsion and used it to show that the Alexander polynomial of a knot is symmetric. In Section 5 we indicate how Milnor's arguments apply in the present situation and give a duality theorem for the twisted polynomial when X is a manifold and summarize the duality properties of τ in Theorem 5.1. Corollary 5.2 states that if X is an odd-dimensional manifold and ρ is a unitary representation then $\tau = \bar{\tau}$, where $^{-}: F(t) \to F(t)$ is a conjugation taking t to t^{-1}. (In the special case of classical knot complements and orthogonal representations, this result appears in [11].) Corollary 5.3 states that if X is an odd-dimensional manifold and X is the boundary of a suitable W such that ρ and ε extend to $\pi_1 W$, then $\tau = f\bar{f}$ for some $f \in F(t)$. This result is the starting point for the applications to knot concordance.

In Section 6 the focus is on developing slicing obstructions for classical knots and on relating these obstructions to Casson–Gordon invariants. Let X denote the p^r-fold cover of a knot complement, $S^3 - K$, and suppose that there is a map, χ, of the homology of the associated branched cover onto $\mathbf{Z}/d$, $d = q^r$ a prime power. Since $\mathbf{Z}/d$ acts on $\mathbf{Q}(\zeta_d)$ by multiplication, where ζ_d is a primitive dth root of unity, this defines a 1-dimensional $\mathbf{Q}(\zeta_d)$ representation ρ_χ of $\pi_1(X)$. This along with the natural representation to $\mathbf{Z}$ yields a twisted polynomial $\Delta_1(X; \rho_\chi)$. Theorem 6.2 says that if K is slice then for a particular collection of such χ the twisted polynomial factors as $f(t)\bar{f}(t^{-1})(t-1)$. This result is applied in [10] to concordance questions.

In the situation just described, Casson and Gordon [2] defined a Witt class invariant $CG(X, \chi)$ that also obstructs slicing. One can define the determinant of this Witt class; it is an element of $\mathbf{Q}(\zeta)(t)^{\times}$ modulo elements of the form $\pm\zeta^a t^n f\bar{f}$. Theorem 6.5 states that this determinant is equal to τ. Thus computations of τ for this choice of X, ρ, and ε are in fact giving computations of Casson–Gordon invariants.

In his paper on infinite cyclic coverings [17], Milnor developed a duality theory for the homology of an infinite cyclic covering space. In Section 7 we briefly discuss the generalizations of these results to the twisted homology of an infinite cyclic cover. A more general approach appears in [18]. A consequence is a duality pairing of the twisted homology modules. A signature obstruction to slicing knots follows.

2. INFINITE CYCLIC COVERS AND THE TWISTED ALEXANDER POLYNOMIAL

2.1. Chain complexes of infinite cyclic covers

Let X be a connected finite CW complex. Suppose that $\varepsilon: \pi_1 X \to \mathbf{Z}$ is a surjective homomorphism defining an infinite cyclic cover $X_\infty \to X$. Let $\tilde{X} \to X$ be the universal covering of X.

For ease of notation, denote $\pi_1 X$ by π, and the kernel of ε by π'. Thus one has a short exact sequence

$$1 \to \pi' \to \pi \xrightarrow{\varepsilon} \mathbf{Z} \to 1.$$

Notice that $\tilde{X}$ is also the universal cover of X_∞ with covering group π'. Assume that π and π' act on the left on the universal cover $\tilde{X}$.

Let F denote a field. The group ring $F[\mathbf{Z}]$ will always be identified with the Laurent polynomial ring $F[t, t^{-1}]$. Let V be a finite-dimensional vector space over F.

Let $C_*(\tilde{X})$ denote the cellular chain complex with coefficients in F. This is a free left $F[\pi]$ module on (lifts of) the cells of X, and, by restriction, also a free left $F[\pi']$ module on the cells of X_∞.

Suppose that $\rho: \pi' \to GL(V)$ is given. Then ρ induces a $F[\pi']$ module structure on V. For notational convenience, we will write the action of $GL(V)$ on V on the *right* and therefore take V to be a right $F[\pi']$-module. Form the chain complex

$$C_*(X_\infty; V_\rho) = V \otimes_\rho C_*(\tilde{X}). \tag{2.1}$$

We denote the homology of this complex by $H_*(X_\infty; V_\rho)$, or just $H_*(X_\infty; V)$ if ρ is understood.

A case of special interest to us occurs when ρ is the restriction of a representation of π. Then in addition to the right $F[\pi']$ module structure on V one can form a right $F[\pi']$ module $F[\mathbf{Z}] \otimes_F V$ by taking the tensor product $\varepsilon \otimes \rho$; the action is given by

$$(p \otimes v) \cdot \gamma = (p\ t^{\varepsilon(\gamma)}) \otimes (v\rho(\gamma)) \ \text{ for } \ \gamma \in \pi. \tag{2.2}$$

This action is used to construct the chain complex

$$C_*(X; V[\mathbf{Z}]_\rho) = (F[\mathbf{Z}] \otimes_F V) \otimes_\rho C_*(\tilde{X}). \tag{2.3}$$

Since $F[\mathbf{Z}]$ is a PID, X is a finite complex, and V is finite dimensional over F, $H_i(X; V[\mathbf{Z}])$ is a finitely generated module over $F[\mathbf{Z}]$ for each i. Thus it has a direct sum decomposition into cyclic modules.

The following theorem states that the chain complexes (2.1) and (2.3) are isomorphic as $F[\mathbf{Z}]$ complexes. We omit the routine proof (Ch. [1], Chapter III, Proposition 6.2 and Corollary 8.2.)

THEOREM 2.1. *Fix an element γ in π with $\varepsilon(\gamma) = t$. Define a map*

$$\Phi : C_*(X; V[\mathbf{Z}]_\rho) \to C_*(X_\infty; V)$$

by

$$\Phi\left(\left(\sum_n f_n t^n \otimes v\right) \otimes z\right) = \sum_n f_n v \gamma^{-n} \otimes \gamma^n z,$$

where $\sum_n f_n t^n$ is an element of $F[\mathbf{Z}]$. Then Φ is a well-defined chain isomorphism, independent of the choice of γ. Moreover, Φ is equivariant with respect to the well-defined left $\mathbf{Z}$ actions on these two complexes defined by $t^n \cdot ((p \otimes v) \otimes z) = (t^n p \otimes v) \otimes z$ on $C_*(X; V[\mathbf{Z}]_\rho)$ and $t^n \cdot (v \otimes z) = v\gamma^{-n} \otimes \gamma^n z$ on $C_*(X_\infty; V)$.

One can make similar constructions for cohomology. The following set up will be used throughout this article. Assume the field F is equipped with a conjugation $^- : F \to F$. Extend the conjugation to $F[\mathbf{Z}]$ by taking $\bar{t} = t^{-1}$. Suppose that W is another representation of π and a non-degenerate inner product $\{\ ,\ \} : V \times W \to F$ is given satisfying

$$\{rv, w\} = r\{v, w\} = \{v, \bar{r}w\} \quad \text{for } r \in F \tag{2.4}$$

and

$$\{v \cdot \gamma, w\} = \{v, w \cdot \gamma\} \quad \text{for } \gamma \in \pi. \tag{2.5}$$

The main examples to keep in mind are:

1. $(V, \{\ ,\ \})$ is a real orthogonal representation of π, $W = V$, and $\bar{r} = r$,
2. $(V, \{\ ,\ \})$ is a unitary representation of π, $W = V$, and $\bar{r}$ is the complex conjugate of r, and
3. $W = \mathrm{Hom}_F(V, F)$ with the dual representation $(w \cdot \gamma)(v) = w(v\gamma^{-1})$, $\{v, w\} = w(v)$, and $\bar{r} = r$.

Construct the cochain complexes

$$\mathrm{Hom}_{F[\mathbf{Z}]}(C_*(X; V[\mathbf{Z}]), F[\mathbf{Z}])$$

and

$$\mathrm{Hom}_{F[\pi]}(C_*(\tilde{X}), W[\mathbf{Z}]).$$

For the second complex we mean the complex of F-linear maps which satisfy $h(\gamma \cdot z) = h(z) \cdot \gamma^{-1}$ for $\gamma \in \pi$. These chain complexes are anti-isomorphic, that is isomorphic as $F[\mathbf{Z}]$ cochain complexes provided one of them is given the conjugate $F[\mathbf{Z}]$ module structure $(p \cdot h)(z) = \bar{p} \cdot h(z)$. Denote the cohomology of the second complex by $H^*(X; W[\mathbf{Z}])$ and let $\bar{H}^*(X; W[\mathbf{Z}])$ denote the same group with the conjugate $F[\mathbf{Z}]$ module structure, so that

$$\overline{H}^*(X; W[\mathbf{Z}]) = H^*(\mathrm{Hom}_{F[\mathbf{Z}]}(C_*(X; V[\mathbf{Z}]), F[\mathbf{Z}])).$$

The universal coefficient theorem applied to the PID $F[\mathbf{Z}]$ implies that

$$\begin{aligned} &H^q(\mathrm{Hom}_{F[\mathbf{Z}]}(C_*(X; V[\mathbf{Z}]), F[\mathbf{Z}])) \\ &\quad = \mathrm{Hom}_{F[\mathbf{Z}]}(H_q(X; V[\mathbf{Z}]), F[\mathbf{Z}]) \oplus \mathrm{Ext}_{F[\mathbf{Z}]}(H_{q-1}(X; V[\mathbf{Z}]), F[\mathbf{Z}]) \end{aligned}$$

and so

$$\overline{H}^*(X; W[\mathbf{Z}]) = \mathrm{Hom}_{F[\mathbf{Z}]}(H_q(X; V[\mathbf{Z}]), F[\mathbf{Z}]) \oplus \mathrm{Ext}_{F[\mathbf{Z}]}(H_{q-1}(X; V[\mathbf{Z}]), F[\mathbf{Z}]). \quad (2.6)$$

We will also need to consider cases when $\varepsilon : \pi_1(X) \to \mathbf{Z}$ is not onto. In that case take $X_\infty \to X$ to be the (disconnected) infinite cyclic cover induced by ε, so that the path components of X_∞ correspond to the cokernel of ε. One can pull back a local coefficient system from X to X_∞ to define $H_*(X_\infty; V) = H_*(X; V[\mathbf{Z}])$. A concrete way to realize the chain complexes in this context is to assume X is a subspace of a connected complex Y and that ε and ρ extend to Y so that $\varepsilon : \pi_1(Y) \to \mathbf{Z}$ is onto. (For example, take Y to be the wedge of X and S^1 and extend ε and ρ by sending the extra generator to $t \in \mathbf{Z}$ and the identity in $GL(V)$.) Then one can construct the chain complex $C_*(X; V[\mathbf{Z}])$ by substituting $\pi_1(Y)$ for π and $p^{-1}(X)$ for $\tilde{X}$, where $p : \tilde{Y} \to Y$ is the universal cover of Y.

2.2. Twisted Alexander polynomials and the homology torsion

Recall that any finitely generated module M over a PID R can be decomposed as the direct sum of cyclic modules:

$$M \cong R/\langle a_1 \rangle \oplus \cdots \oplus R/\langle a_k \rangle.$$

The elements a_i are well defined modulo units in R under the added condition that a_i divides a_{i+1} for all $i < k$. For details see, for instance, [8].

The *order* (of the torsion) of a finitely generated module M over a PID is defined to be the product of all the ideals appearing in the torsion part of the direct sum decomposition of M. We will confuse the ideal with any of its generators; thus for the ring $F[\mathbf{Z}]$ the order will be a Laurent polynomial defined up to multiplication by ut^n for $u \in F$. If the module is free, then we take the order to be 1. Moreover, for the rest of this paper, the terms "polynomial" and "Laurent polynomial" will be synonymous. We now make the following definitions.

Definition 2.2. Given a representation $\rho : \pi \to GL(V)$ and infinite cyclic cover $X_\infty \to X$, let $\Delta_i = \Delta_i(X, \varepsilon, \rho, V) \in F[\mathbf{Z}]$ denote the order of the torsion of $H_i(X; V[\mathbf{Z}]_\rho)$ viewed as a $F[\mathbf{Z}]$ module. We call Δ_i the ith twisted Alexander polynomial of X twisted by ρ.

Definition 2.3. In the situation above, define the *homology torsion*

$$\tau = \tau(X, \varepsilon, \rho, V) = \frac{\Pi_i \Delta_{2i+1}}{\Pi_i \Delta_{2i}} \in F(t)$$

Where $F(t)$ denotes the field of rational functions over F.

Remark.

1. Since each Δ_i is only defined up to multiplication by rt^n for $r \in F, n \in \mathbf{Z}$ (i.e. the units in $F[\mathbf{Z}]$) it follows that $\tau \in F(t)$ is defined only up to multiplication by rt^n.
2. In contrast to the definition give above, some articles define the order to be zero if the free part of the module is non-trivial. This makes some formulas easier, especially those involving the multiplicative Euler characteristic of an exact sequence.
3. Equation (2.6) shows that if $\Delta^i = \Delta^i(X, \varepsilon, \rho_W, W)$ is defined to be the torsion of $H^i(X; W[\mathbf{Z}])$, then $\Delta^i = \overline{\Delta}_{i-1}$ and similarly for the homology torsion τ. Thus the Δ^i give the same information as the Δ_i. Notice however that we are not defining the Alexander polynomials in terms of the cohomology $H^*(X_\infty; W)$ since this can be infinitely generated as a $F[\mathbf{Z}]$ module.

3. REIDEMEISTER TORSION AND BASIC PROPERTIES OF Δ_i AND τ

3.1. Reidemeister torsion and homology torsion

We begin by interpreting τ as the Reidemeister torsion of an associated complex. This relationship is well-known. We include some arguments to establish to what extent the torsion we study is well-defined and to give a computational algorithm which is important in the applications of [10].

Fix a representation $\rho:\pi\to GL(V)$. Closely related to the complex $C(X;V[\mathbf{Z}]_\rho)$ is the complex

$$C(X;V(t)_\rho)=F(t)\otimes_{F[\mathbf{Z}]}C(X;V[\mathbf{Z}]_\rho)=(F(t)\otimes_F V)\otimes_\rho C_*(\tilde{X})$$

(recall $F(t)$ denotes the field of rational functions over F) with the action as in Eq. (2.2). Let $H_*(X;V(t)_\rho)$ denote its homology. We will usually suppress the subscript ρ from the notation. There is a natural inclusion $C_*(X;V[\mathbf{Z}])\subset C_*(X;V(t))$.

The universal coefficient theorem implies that $H_k(X;V(t))=F(t)\otimes_{F[\mathbf{Z}]}H_k(X;V[\mathbf{Z}])$. In particular, $H_k(X;V[\mathbf{Z}])$ is torsion over $F[\mathbf{Z}]$ if and only if $H_k(X;V(t))=0$.

Now suppose that a basis $\{e_i\}$ for the F-vector space V is given. Give the chain complex $C_*(\tilde{X})$ a basis $\{z_i\}$ by choosing a lift of each cell of X to $\tilde{X}$. Then $C_*(X;V[\mathbf{Z}])$ becomes a free based $F[\mathbf{Z}]$ complex with basis $1\otimes e_i\otimes z_j$. Similarly $C_*(X;V(t))$ becomes a free based $F(t)$ complex with the same basis.

The *Reidemeister torsion* τ' of the based chain complex $C_*(X;V(t))$ is a function from the set of bases of the homology $H_*(X;V(t))$ to the units in $F(t)$. The definition we take for τ' is the one in [16]: if C_* is a based chain complex, c_i is a basis for C_i, b_i a basis for the boundaries B_i, h_i a basis for the homology H_i, then

$$\tau'(h_*)=\frac{\Pi_i\det(b_{2i},\tilde{h}_{2i},\tilde{b}_{2i-1}|c_{2i})}{\Pi_i\det(b_{2i+1},\tilde{h}_{2i+1},\tilde{b}_{2i}|c_{2i+1})}. \tag{3.1}$$

In this expression $\det(x|y)$ means the determinant of the change of basis matrix from x to y, $\tilde{h}_i$ is a choice of lift of h_i to C_i, and $\tilde{b}_i$ is a choice of lift of b_i to C_{i+1} using the differential $\partial:C_{i+1}\to C_i$. Since we have not oriented our bases it is clear that there is a sign ambiguity in this definition. (We denote this function τ' temporarily; we will explain shortly why $\tau'=\tau$.) We refer to [3] and [16] for basic properties of τ'.

The function τ' depends on the choice of basis for $H_*(X;V(t))$ in the following way. If $\{h_i\}$, $\{h_i'\}$ are two choices of basis for $H_i(X;V(t))$ with corresponding change of basis matrix A, then the values of τ' differ by $\det(A)^{(-1)^i}$.

PROPOSITION 3.1. *Fix a basis of the homology $H_*(X;V(t))$ and let τ' denote the Reidemeister torsion of $C_*(X;V(t))$ with respect to this basis.*

(i). *If one of the lifts z_i of the* n-cells of X *is replaced by another lift, say $\gamma\cdot z_i$ where $\gamma\in\pi$, then τ' changes to $\pm t^{\delta\varepsilon(\gamma)b}\det(\rho(\gamma))^\delta\tau'$ where b denotes the dimension of V and $\delta=(-1)^{n-1}$.*

(ii) *If the basis for V is changed from $\{e_i\}$ to $\{Ae_i\}$ for some matrix A, then τ' changes to $\pm(\det(A))^{E(X)}\tau'$ where $E(X)$ denotes the Euler characteristic of X.*

Proof. From its definition one sees that if the basis $\{c_i\}$ of the i-chains C_i is changed to c_i', then the torsion changes according to the rule

$$\tau'_{c'}=\tau'_c\cdot\frac{\Pi_i\det(c_{2i}|c'_{2i})}{\Pi_i\det(c_{2i+1}|c'_{2i+1})}.$$

Thus we must examine the effect of changing bases. The overall sign ambiguity in both cases (i) and (ii) comes from the fact that we have not oriented our bases.

We start with case (i). Suppose a single n-cell z_j is replaced by γz_j for some $\gamma \in \pi$. Assume by re-indexing that $j = 1$. Order the original basis for $C_n(X; V(t))$ by taking

$$(1 \otimes e_1 \otimes z_1, \cdots, 1 \otimes e_b \otimes z_1, 1 \otimes e_1 \otimes z_2, \cdots, 1 \otimes e_b \otimes z_k).$$

Since

$$1 \otimes e_i \otimes \gamma z_1 = t^{\varepsilon(\gamma)} \otimes e_i \rho(\gamma) \otimes z_1 = t^{\varepsilon(\gamma)}(1 \otimes e_i \rho(\gamma) \otimes z_1),$$

the change of basis matrix M_n will be the block sum of the matrix $t^{\varepsilon(\gamma)}\rho(\gamma)$ and $k-1$ copies of the identity matrix, where k denotes the number of n-cells of X. Its determinant is $t^{a(\gamma)b} \det \rho(\gamma)$. This establishes the first assertion.

For the second assertion, suppose that the basis $\{e_i\}$ is replaced by the basis $\{Ae_i\}$. Order the original basis for $C_*(X; V(t))$ as before. This time the change of basis matrix M_n will be the block sum of the matrices A, one for each n-cell of X. The determinant of M_n is therefore $\det(A)^{k_n}$ where k_n denotes the number of n-cells. Therefore, τ' changes to $\pm \tau' \det(A)^{E(X)}$. Q.E.D.

We tabulate the information this proposition gives us for some special cases in the following corollary.

COROLLARY 3.2. *Let τ' be defined as above for the complex $C_*(X; V(t))$ with respect to some fixed basis for homology. Then*:

(i) *The torsion τ' is well-defined in $F(t)^\times$ modulo $\{rt^n \mid r \in F, n \in \mathbf{Z}\}$.*
(ii) *If F is a subfield of $\mathbf{C}$, V is a unitary vector space over F, and $\rho : \pi \to U(V)$, then τ' is well-defined in $F(t)^\times$ modulo $\{zt^n \mid z \in F, |z| = 1, n \in \mathbf{Z}\}$.*
(iii) *If $F = \mathbf{C}$, V is a unitary vector space over $\mathbf{C}$, and $\rho : \pi \to SU(V)$, then τ' is well-defined in $\mathbf{C}(t)^\times$ modulo $\{\pm t^n \mid n \in \mathbf{Z}\}$.*
(iv) *If $V = F$ and $\rho : \pi \to F^\times$, then τ' is well-defined in $F(t)^\times$ modulo $\{\pm t^n \rho(\gamma) \mid \gamma \in \pi, n \in \mathbf{Z}\}$.*

Having introduced and examined the Reidemeister torsion for $C_*(X; V(t))$, we now show how to compute it. As a consequence, we will show that $\tau' = \tau$ modulo $\{rt^n \mid r \in F\}$. Corollary 3.2 asserts that τ' can have a smaller indeterminacy than τ, but note that τ is defined without reference to any basis of the chain complex. One can view τ' as a refinement of τ, or, alternatively, view τ as a basis-free definition of τ'.

The lifts of cells of X and a basis of V determine bases of the chain complexes $C_*(X; V[\mathbf{Z}])$ and $C_*(X; V(t)) = F(t) \otimes_{F[\mathbf{Z}]} C_*(X; V[\mathbf{Z}])$; if $\{e_i\}$ are lifts to $\tilde{X}$ of the cells of X and $\{v_i\}$ is a basis for V, then $\{v_i \otimes e_j\}$ is a basis for $C_*(X; V[\mathbf{Z}])$ and $\{1 \otimes v_i \otimes e_j\}$ is a basis for $C_*(X; V(t))$.

With respect to this choice of basis, the differentials in the chain complex $C_*(X; V[\mathbf{Z}])$ are obtained as follows. Represent the differentials in the $\mathbf{Z}[\pi]$ complex $C_*(\tilde{X})$ as matrices over $\mathbf{Z}[\pi]$ by using the basis $\{e_i\}$. Then the differentials in $C_*(X; V[\mathbf{Z}])$ are represented as matrices obtained by replacing each $\mathbf{Z}[\pi]$ entry by its image under the homomorphism $\mathbf{Z}[\pi] \to M_{n \times n}(F[\mathbf{Z}])$ defined by

$$\sum_{\gamma \in \pi} n_\gamma \gamma \mapsto \sum_{\gamma \in \pi} n_\gamma t^{\varepsilon(\gamma)} \rho(\gamma).$$

In this expression we use the basis $\{v_i\}$ of V to express ρ as a representation on F^n. Note that each entry of the $\mathbf{Z}[\pi]$ matrix has been replaced by an $n \times n$ matrix. This same matrix clearly represents the differentials in $C_*(X; V(t))$ with respect to the basis $\{1 \otimes v_i \otimes e_j\}$.

Since $F[\mathbf{Z}]$ is a Euclidean domain and has a Euclidean valuation given by the degree of a Laurent polynomial, any matrix over $F[\mathbf{Z}]$ can be put into the form

$$(A \quad 0) \quad \text{or} \quad \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where A is diagonal by elementary row and column operations *with determinant equal to* ± 1. The next lemma follows by induction from this fact and the fact that $\partial^2 = 0$. We omit the routine proof. Notice that the boundaries $B_k =$ Image ∂_{k+1} form a free module since $F[\mathbf{Z}]$ is a PID. Again, the reader is refered to [8] for the necessary algebra background.

LEMMA 3.3. *Suppose that the free part of $H_k(X; V[\mathbf{Z}])$ has rank β_k, the boundaries B_k have rank α_k and that $C_k(X; V[\mathbf{Z}])$ has rank $\alpha_k + \beta_k + \gamma_k$ over $F[\mathbf{Z}]$. (Notice that $\gamma_k = \alpha_{k-1}$.) Then for each k there exist a basis $\{c_{k,l}\}_{l=1}^{\alpha_k+\beta_k+\gamma_k}$ for $C_k(X; V[\mathbf{Z}])$ so that:*

1. *The change of basis matrix from the geometric basis $\{v_i \otimes e_j \mid e_j \text{ is a } k \text{ cell}\}$ to $\{c_{k,l}\}$ has determinant equal to ± 1.*
2. *There exist non-zero elements $a_{k,l} \in F[\mathbf{Z}]$ for $l = 1, \ldots, \alpha_{k-1}$ so that $\partial c_{k,l+\alpha_k+\beta_k} = a_{k,l} c_{k-1,l}$.*
3. *$\partial c_{k,l} = 0$ for $l = 1, \ldots, \alpha_k + \beta_k$.*
4. *The cycles $c_{k,l+\alpha_k}$, $l = 1, \ldots, \beta_k$ give a basis for free part of $H_k(X; V[\mathbf{Z}])$.*

The second and third assertions say that in the bases $\{c_{k,l}\}$ and $\{c_{k-1,l}\}$, ∂_k has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_k & 0 & 0 \end{pmatrix}$$

where A_k is the diagonal $\alpha_k \times \alpha_k$ matrix with entries $a_{k,l}$.

The following theorem asserts that the Reidemeister torsion and the homology torsion coincide, and shows how to compute it. For X the complement of a knot in S^3, this result was also obtained by Kitano in [11].

THEOREM 3.4. *Let $\boldsymbol{h}_k$ denote a basis of the free part of $H_k(X; V[\mathbf{Z}])$ obtained by "diagonalizing" the differentials in the chain complex as in the previous lemma, so $\boldsymbol{h}_k = [c_{k,l+\alpha_k}]$, $l = 1 \ldots \beta_k$. Notice that $\boldsymbol{h}_k$ defines a basis for $H_k(X; V(t))$. Let g_k be the greatest common divisor of the $\alpha_k \times \alpha_k$ subdeterminants of ∂_k expressed in the geometric basis $\{v_i \otimes e_j \mid e_j \text{ is a } k \text{ cell}\}$.*

Then the Reidemeister torsion τ' of $C_*(X, V(t))$ with respect to the geometric basis $1 \otimes v_i \otimes e_j$ and the basis $\boldsymbol{h}_k$ is equal to the alternating product

$$\frac{\Pi g_{2k}}{\Pi g_{2k+1}}.$$

Moreover

$$\tau' = \tau \operatorname{modulo} \{rt^n \mid r \in F, n \in \mathbf{Z}\}.$$

Proof. From the previous lemma and definition of Reidemeister torsion one sees that with this choice of bases τ' is equal to the alternating product of the $a_{k,l}$. Clearly g_k is equal to the product of the $a_{k,l}$ over l. This shows that $\tau' = \Pi g_{2k} / \Pi g_{2k+1}$. Moreover, A_k is a diagonal presentation matrix for the torsion part of $H_{k-1}(X; V[\mathbf{Z}])$ and so $\Delta_k = \langle g_k \rangle$. Hence $\tau' = \tau$ modulo $\{rt^n \mid r \in F, n \in \mathbf{Z}\}$. Q.E.D.

This corollary shows how to compute τ' if one knows the differentials of $C_*(\tilde{X})$. The method produces a homology basis if $C_*(X; V(t))$ is not acyclic but this is somewhat impractical; it involves diagonalizing the matrices representing the differentials. (To make matters worse it is not clear that the equivalence class of homology basis obtained this way is invariant under subdivision.) However, if $C_*(X; V(t))$ is acyclic (which holds in many cases of interest) then one need only compute greatest common divisors of subdeterminants, it is not necessary to diagonalize. Moreover, computer implementation of an algorithm to compute the torsion for a 3-manifold by calculating subdeterminants is relatively straightforward.

If we need to distinguish τ from τ', we will call τ the *homology torsion* and τ' the *Reidemeister torsion*. In light of the previous lemma we use τ to denote both of these, with the indeterminacy depending on context. The distinction is that the homology torsion is defined purely in terms of the homology with $V[\mathbf{Z}]$ coefficients, whereas the Reidemeister torsion depends in general on the cell structure and the homology basis. The Reidemeister torsion has the advantage of sometimes having a smaller indeterminacy.

3.2. *Basic facts about* τ *and* Δ_i

For the rest of Section 3, we take τ to be well-defined up to $\{rt^n \mid r \in F, n \in \mathbf{Z}\}$. We do not use the refinements provided by Corollary 3.2.

PROPOSITION 3.5. *If ε is non-trivial, then $H_0(X; V[\mathbf{Z}])$ is torsion over $F[\mathbf{Z}]$.*

Proof. Recall from the first section that $H_0(X; V[\mathbf{Z}]) = H_0(X_\infty; V)$. Since $\varepsilon : \pi \to \mathbf{Z}$ is non-trivial, its cokernel is finite. Thus X_∞ is the union of finitely many path components. Thus as an F-module $H_0(X_\infty; V)$ is a finite direct sum $\oplus_k H_0(Y_k; V)$ where Y_k denotes the components of X_∞. Since $H_0(Y_k; V)$ can be identified with a quotient of V, $H_0(X_\infty; V)$ is a finitely generated F module. This implies that $H_0(X_\infty; V)$ cannot have a free $F[\mathbf{Z}]$ summand. Q.E.D.

Next, we state and prove a lemma which describes the dependence of the homology groups we are considering on the homomorphism $\varepsilon : \pi_1 X \to \mathbf{Z}$.

PROPOSITION 3.6. *Let (X, ε, ρ) be given.*

1. *If $\varepsilon : \pi_1 X \to \mathbf{Z}$ is trivial, then $H_k(X; V[\mathbf{Z}])$ is isomorphic to $F[\mathbf{Z}] \otimes H_k(X; V)$ and hence is free over $F[\mathbf{Z}]$ for each k. In particular, $\triangle_k = 1$ and $\tau = 1$.*
2. *Replacing $\varepsilon : \pi_1 X \to \mathbf{Z}$ by $\varepsilon' = m \cdot \varepsilon$ for some non-zero integer m replaces $\Delta_k(t)$ by $\Delta_k(t^m)$ and $\tau(t)$ by $\tau(t^m)$.*

Proof. If $\varepsilon : \pi_1 X \to \mathbf{Z}$ is trivial, then the infinite cyclic cover corresponding to ε is a union of components homeomorphic to X indexed by the integers, and the deck transformations correspond to translation in the integers. Thus

$$H_k(X; V[\mathbf{Z}]) = F[\mathbf{Z}] \otimes H_k(X; V)$$

and is therefore a free $F[\mathbf{Z}]$ module.

Replacing ε by $-\varepsilon$ changes the $F[\mathbf{Z}]$ module structure on $H_k(X; V[\mathbf{Z}])$ by replacing t by t^{-1}. Thus each factor $F[\mathbf{Z}]/p(t)$ in the decomposition of $H_k(X; V[\mathbf{Z}])$ into cyclic modules is replaced by $F[\mathbf{Z}]/p(t^{-1})$. It follows that $\Delta_k(t)$ is replaced by $\Delta_k(t^{-1})$ and so the same holds for $\tau(t)$.

Suppose ε is replaced by $\varepsilon' = m \cdot \varepsilon$ for some $m > 0$. Let $X'_\infty \to X$ denote the corresponding infinite cyclic cover. Then X'_∞ is a disjoint union of m copies of X_∞, and the covering group $\mathbf{Z} = \langle t \rangle$ acts by cyclically permuting the components so that t^m preserves components, and corresponds in each component to the generator of the deck transformations for X_∞. It follows that $\Delta_k(t)$ is replaced by $\Delta_k(t^m)$; similarly for $\tau(t)$. Q.E.D.

A useful tool for computing torsion is Theorem 3.2 of [16]. This theorem states that if $0 \to C' \to C \to C'' \to 0$ is a short exact sequence of compatibly based chain complexes, with given bases for their homology groups, then

$$\tau(C) = \tau(C')\tau(C'')\tau(\mathscr{H}) \tag{3.2}$$

where $\mathscr{H}$ denotes the long exact sequence associated to $0 \to C' \to C \to C'' \to 0$, viewed as a based, acyclic complex.

3.3. Examples and computations of the twisted polynomials

1. *The torus*: Consider the case of a torus with a representation $\rho: \mathbf{Z} \oplus \mathbf{Z} \to GL_n(F)$ and infinite cyclic cover defined by $\varepsilon: \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z}$. The chain complex $C_*(T; F^n[\mathbf{Z}])$ is non-zero in degrees 0, 1, and 2 only. Taking the natural cell structure on T with one 0-cell, two 1-cells, and one 2-cell one can identify $C_*(T; F^n[\mathbf{Z}])$ with

$$0 \to F[\mathbf{Z}]^n \overset{\partial_2}{\to} F[\mathbf{Z}]^{2n} \overset{\partial_1}{\to} F[\mathbf{Z}]^n \to 0.$$

Also,

$$\partial_2 = (Id - \rho(y)t^{\varepsilon(y)}, \quad \rho(x)t^{\varepsilon(x)} - Id) \quad \text{and } \partial_1 = \begin{pmatrix} \rho(x)t^{\varepsilon(x)} - Id \\ \rho(y)t^{\varepsilon(y)} - Id \end{pmatrix}$$

where x and y are the generators of $\pi_1 T$, represented by the 1-cells.

If the homomorphism ε is zero, then the homology is free over $F[\mathbf{Z}]$ by Proposition 3.6 and so $\Delta_k = 1$ for $k = 0, 1, 2$ and τ equals 1 according to our conventions. Suppose that ε is non-zero. Then the kernel of ∂_2 is zero so that $H_2(T; F^n[\mathbf{Z}]) = 0 = H_2(T; F^n(t))$. Moreover, $H_0(T; F^n(t)) = 0$ by Proposition 3.5 and since the Euler characteristic is zero the complex $C_*(X; F^n(t))$ is acyclic. Suppose that $\varepsilon(x)$ is non-zero by exchanging x and y if necessary. It follows from Theorem 4.1 in the next section that

$$\frac{\Delta_1}{\Delta_0} = \frac{\det(\rho(x)t^{\varepsilon(x)} - Id)}{\det(Id - \rho(x)t^{\varepsilon(x)})}.$$

Hence $\tau = \Delta_1/\Delta_0$ equals 1 and $\Delta_0 = \Delta_1$. Since $H_0(T; F^n[\mathbf{Z}])$ is the cokernel of ∂_1, Δ_0 equals the greatest common divisor of the $n \times n$ subdeterminants of the matrix representing ∂_1.

2. *Knots in* S^3: We next look at twisted polynomials for knots in S^3. The Wirtinger presentation corresponds to a cell structure for $X = S^3 - K$ with only 0, 1, and 2-cells if K is a knot in S^3. This has generators x_i, $i = 1, \cdots, n$ and $n - 1$ relations of the form $x_i x_j x_i^{-1} x_k^{-1}$.

Let $\rho: \pi_1 X \to GL_m(F)$ be a representation and let $\varepsilon: \pi_1 X \to \mathbf{Z}$ be the natural surjection, taking each x_i to t. Taking the matrix of Fox derivatives of the relations and tensoring with

$F^m[\mathbf{Z}]$ gives an $(n-1)\times n$ matrix, with entries that are elements of $M_{m\times m}(F[\mathbf{Z}])$. This matrix represents the differential

$$\partial_2 : C_2(X; F^m[\mathbf{Z}]) = F^m[\mathbf{Z}]^{n-1} \to C_1(X; F^m[\mathbf{Z}]) = (F^m[\mathbf{Z}])^n.$$

The row of this matrix corresponding to the relation $x_i x_j x_i^{-1} x_k^{-1}$ has $Id - \rho(x_j)t$ in the ith column, $\rho(x_i)t$ in the jth column, and $-Id$ in the kth column. The zero chains are $C_0(X; F^m[\mathbf{Z}]) = F^m[\mathbf{Z}]$ and ∂_1 is the column matrix with ith entry $\rho(x_i)t - Id$.

Dropping the last column of this matrix yields a $(n-1)\times(n-1)$ matrix with entries that are themselves $m\times m$ matrices. This may be viewed as an $(n-1)m\times(n-1)m$ matrix. It follows from Theorem 4.1 in the next section that $\tau = \Delta_1/\Delta_0$ is equal to the ratio of the determinant of this matrix and the determinant of $\rho(x_n)t - Id$. (Notice that $H_2(X; F^m[\mathbf{Z}]) = 0$.) References for the use of Fox derivatives include [4,5].

3. *Satellites*: We obtain a formula for the twisted polynomials of a satellite in a special case to illustrate computations. Consider $F = \mathbf{C}$ and ρ an *abelian* $U(m)$ representation. Since the Wirtinger generators x_i are conjugate, ρ takes the same value on each x_i; we may assume $\rho(x_i)$ is the diagonal matrix with entries $\xi_1, \dots, \xi_m$. Since ρ is completely reducible, the homology $H_i(X; \mathbf{C}^m[\mathbf{Z}])$ also splits into summands of the form $H_i(X; \mathbf{C}[\mathbf{Z}])$. Thus $\Delta_i(X, \rho)$ is the product of $\Delta_i(X, \rho_i)$ over the 1-dimensional representations $\rho_i(x_j) = \xi_i$.

From the description above and the formulation of the Alexander polynomial in terms of the Fox derivatives it is clear that $H_*(X; \mathbf{C}(t)_{\rho_i}) = 0$, $\Delta_0(X, \rho_i)(t) = 1 - \xi_i t$, and

$$\frac{\Delta_1(X, \rho_i)(t)}{\Delta_0(X, \rho_i)(t)} = \frac{A_K(\xi_i t)}{1 - \xi_i t}$$

where $A_k(t)$ is the Alexander polynomial of K. Hence $\Delta_1(X, \rho_i)(t) = A_K(\xi_i t)$. Returning to ρ we conclude that

$$\Delta_1(X, \rho)(t) = \prod_{i=1}^{m} A_K(\xi_i t) \quad \text{and} \quad \Delta_0(X, \rho)(t) = \prod_{i=1}^{m} 1 - \xi_i t. \tag{3.3}$$

Now suppose that L is a satellite of K with non-zero winding number n, and that $\rho : \pi_1(S^3 - L) \to U(m)$ is a representation whose restriction to $\pi_1(S^3 - K)$ is abelian, and such that $H_*(S^3 - L; \mathbf{C}^m(t)) = 0$. Let $L_U \subset S^3$ be the knot obtained by taking the corresponding satellite of the unknot. A theorem of Seifert [21] states that the (ordinary) Alexander polynomials satisfy

$$A_L(t) = A_K(t^n) A_{L_U}(t).$$

Write $S^3 - L = X \cup_T Y$ where X is the exterior of K, Y is the complement of a knot in a solid torus, and T is the separating torus. We have seen that $H_*(T; \mathbf{C}^m(t)) = 0$. Moreover, $H_*(X; \mathbf{C}^m(t)_\rho) = 0$. In fact, this follows from the remarks above since the representation is abelian on X.

The Mayer–Vietoris sequence for this decomposition shows that $H_*(Y; \mathbf{C}^m(t)_\rho) = 0$ and so

$$\tau(X; \rho)\tau(Y; \rho) = \tau(S^3 - L; \rho)\tau(T; \rho). \tag{3.4}$$

Using the fact that $\tau(T; \rho) = 1$, eq. (3.3), and Proposition 3.6, one concludes that

$$\tau(X; \rho)(t) = \prod_{i=1}^{m} A_K(\xi_i t^n)/(1 - \xi_i t^n). \tag{3.5}$$

We also have a decomposition $S^3 - L_U = X_U \cup_T Y$ where X_U is the exterior of the unknot, so $X_U = S^1 \times D^2$. Since ρ restricts to an abelian representation of X, it canonically defines a representation $\rho' : \pi_1(S^3 - L_U) \to U(m)$, which agrees with ρ on Y. Thus one obtains similarly

$$\tau(X_U; \rho')\tau(Y; \rho) = \tau(S^3 - L_U; \rho). \tag{3.6}$$

Since X_U is a homotopy circle and ρ' is abelian on $\pi_1(X_U)$, $\tau(X_U) = \Pi_i(1 - \xi_i t^n)^{-1}$.

Combining eqs. (3.4)–(3.6), substituting and cancelling one finally obtains

Theorem 3.7. *Let L be a satellite of a knot K with non-zero winding number n. Let $\rho : \pi_1(S^3 - L) \to U(m)$ be a representation whose restriction to $\pi_1(S^3 - K)$ is abelian, and let $\xi_1, \dots, \xi_m$ denote the eigenvalues of $\rho(\mu_K)$ where μ_K denotes the meridian of the companion K. Let L_U be the corresponding satellite of the unknot and $\rho' : \pi_1(S^3 - L_U) \to U(m)$ the corresponding representation. Then*

$$\tau(S^3 - L; \rho)(t) = \prod_{i=1}^{m} A_K(\xi_i t^n) \cdot \tau(S^3 - L_U; \rho')(t).$$

The fact that ρ restricted to an abelian representation on X was used to relate $\tau(S^3 - L)$ to $\tau(S^3 - L_U)$; the terms involving $\tau(Y)$ cancelled. In general one will not have this trick at one's disposal, and so a formula for the polynomial of a satellite will involve the term $\tau(Y)$.

4. *An example with $C_*(X; V(t))$ non-acyclic*: Our examples up to this point all have $C_*(X; V(t))$ acyclic if ε is non-zero. This is not always the case, and we now give examples of classical knot complements $S^3 - K$ and representations $\rho : \pi_1(S^3 - K) \to SO(3)$ such that $H_1(S^3 - K; V[\mathbf{Z}])$ has a free $F[\mathbf{Z}]$ summand. Here $\varepsilon : \pi_1(S^3 - K) \to \mathbf{Z}$ is the natural surjection. This is a significant point where the properties of the twisted invariants differ from the untwisted case.

Let $K \subset S^3$ be a winding number zero satellite knot with exterior X. Let $T \subset S^3 - K$ be the separating incompressible torus, dividing X into a knot complement X_1 and the complement of a nullhomologous knot in a solid torus X_2. Then the homomorphism ε restricts to the zero homomorphism on $\pi_1(X_2)$ and $\pi_1(T)$.

Suppose we are given a representation $\rho : \pi_1(X) \to SO(3)$ whose restriction to $\pi_1(X_1)$ is non-abelian, and which maps $\pi_1(T)$ into a maximal torus, a circle subgroup. One can find many such examples of representations of Whitehead doubles of knots in [12]. To be specific, the results in [12] show that the 0-twisted Whitehead double of the trefoil has infinitely many such representations.

Take $V = \mathbf{R}^3$ with the standard $SO(3)$ action. Then since the restriction $\rho : \pi_1(T) \to SO(3)$ maps into a maximal torus, it leaves a subspace of V fixed. Hence $H_0(T; V)$ is non-zero; it is 3- or 1-dimensional according to whether or not the restriction of ρ to $\pi_1(T)$ is trivial. On the other hand, $H_0(X_1; V) = 0$ since $\rho : \pi_1 X_1 \to SO(3)$ is non-abelian. By Proposition 3.6, then, $H_0(T; V[\mathbf{Z}])$ is a non-trivial free $F[\mathbf{Z}]$ module, and $H_0(X_1; V[\mathbf{Z}]) = 0$. The group $H_0(X_2; V[\mathbf{Z}])$ is torsion over $F[\mathbf{Z}]$ by Proposition 3.5.

Applying the Mayer–Vietoris sequence one sees that the image of $H_1(X; V[\mathbf{Z}])$ in $H_0(T; V[\mathbf{Z}])$ is a non-trivial free $F[\mathbf{Z}]$ module (of rank either 1 or 3). Therefore $H_1(X; V[\mathbf{Z}])$ has a non-trivial free $F[\mathbf{Z}]$ summand.

One can understand this situation geometrically as follows. The first *cohomology* of X_∞ with V coefficients can be defined in terms of the Zariski tangent space of the character "variety" $\mathrm{Hom}(\pi', SO(3))/\mathrm{conj}$. (Recall that $\pi' = \pi_1(X_\infty)$.) A similar argument shows that this first cohomology is also infinite dimensional over F. Thus the Zariski tangent space to the character variety is infinite dimensional. We now show that $\mathrm{Hom}(\pi', SO(3))/\mathrm{conj}$ is infinite dimensional. (We are using this terminology somewhat loosely since making the notion of an infinite dimensional variety and its tangent space precise is a delicate matter.)

Since the satellite has winding number zero, the separating incompressible torus has inverse image in X_∞ a disjoint image of infinitely many separating tori, say $\bigcup_{i=\infty}^{\infty} T_i$. The restriction of ρ to $\pi_1(T_i)$ lies in a circle subgroup of $SO(3)$ for each i and hence this restriction is invariant under the conjugation action of this circle subgroup. On the other hand, the restriction of ρ to each of the two path components complementary to T_i is non-abelian, and hence has trivial stabilizer. Thus one can "bend" the representation along T_i, i.e. conjugate the restriction to one of the complementary regions by an element in the S^1 stabilizer. This gives a 1-parameter family of deformations of $\rho : \pi' \to SO(3)$. The deformation corresponding to T_i is not equivalent to the deformation corresponding to T_j if $i \neq j$, and so there are infinitely many independent directions in which one can deform ρ. Hence $\mathrm{Hom}(\pi', SO(3))/\mathrm{conj}$ is infinite dimensional near ρ. (We thank E. Klassen for providing us with this interpretation.)

4. RELATIONSHIP TO PREVIOUS DEFINITIONS OF TWISTED ALEXANDER POLYNOMIALS

The notion of Alexander polynomials twisted by a representation has appeared in several papers [9, 13, 22]. In this section we discuss how the invariants of these articles are specializations of the Δ_i defined above. In particular, we prove that Wada's invariant [22] is equal to the quotient Δ_1/Δ_0.

We begin by recalling Wada's definition. (We will consider only the case of an infinite cyclic cover, although the case of any free abelian cover is considered in [22] and [9].) The zeroth and first homology of any connected complex can be computed from its group homology; in particular $H_i(X; V[\mathbf{Z}]) = H_i(\pi; V[\mathbf{Z}])$ for $i = 0$ and 1. The group cohomology can be computed using the Fox calculus; this is the starting point for Wada's construction.

Suppose that $\pi = \pi_1 X$ has the presentation

$$\pi = \langle x_1, \dots, x_s | r_1 \dots r_t \rangle.$$

Let $\rho : \pi \to GL_n(F)$, and let $\varepsilon : \pi \to \mathbf{Z}$ be a non-trivial homomorphism. Write $\Lambda = F[\mathbf{Z}]$. Then $\varepsilon \otimes \rho$ defines a ring homomorphism $\mathbf{Z}[\pi] \to M_n(\Lambda)$, where $M_n(\Lambda)$ denotes the $n \times n$ matrices over Λ (the map takes $\gamma \in \pi$ to $t^{\varepsilon(\gamma)}\rho(\gamma)$). Let F_s denote the free group on the generators $x_1, \dots, x_s$, and denote by $\Phi : \mathbf{Z}[F_s] \to M_n(\Lambda)$ the composite of the surjection $\mathbf{Z}[F_s] \to \mathbf{Z}[\pi]$ induced by the presentation and the map $\mathbf{Z}[\pi] \to M_n(\Lambda)$ given by $\varepsilon \otimes \rho$.

Consider the tail end of the chain complex with homology $H_*(\pi; F^n[\mathbf{Z}])$,

$$\to (F[\mathbf{Z}] \otimes_F F^n) \otimes_{F[\pi]} F[\pi]^t \to (F[\mathbf{Z}] \otimes_F F^n) \otimes_{F[\pi]} F[\pi]^s \to (F[\mathbf{Z}] \otimes_F F^n) \otimes_{F[\pi]} F[\pi] \to 0.$$

Identifying $F[\mathbf{Z}] \otimes_F F^n$ with Λ^n gives the alternate description

$$\to (\Lambda^n)^t \xrightarrow{\partial_2} (\Lambda^n)^s \xrightarrow{\partial_1} \Lambda^n \to 0. \tag{4.1}$$

Then the Fox calculus implies that

$$\partial_2 = \left[\Phi\left(\frac{\partial r_i}{\partial x_j} \right) \right]_{tn \times sn}$$

and

$$\partial_1 = \begin{pmatrix} \Phi(x_1 - 1) \\ \cdots \\ \Phi(x_s - 1) \end{pmatrix}_{sn \times n}.$$

Lemma 1 of [22] asserts that for some index j the jth entry $\Phi(x_j - 1)$ of ∂_1 has non-zero determinant (as a map $\Lambda^n \to \Lambda^n$). Notice that this implies Proposition 3.5 above. Fixing such an index j, let $p_j : (\Lambda^n)^s \to (\Lambda^n)^{s-1}$ denote the obvious projection with kernel the jth copy of Λ^n. Wada defines the $Q_j \in \Lambda$ to be the the greatest common divisor of the determinants of the $n(s-1) \times n(s-1)$ submatrices of the matrix representing the linear map

$$\Lambda^{nt} = (\Lambda^n)^t \xrightarrow{\partial_2} (\Lambda^n)^s \xrightarrow{p_j} (\Lambda^n)^{s-1} = \Lambda^{n(s-1)}.$$

Finally Wada's invariant is defined to be

$$W = \frac{Q_j}{\det(\Phi(x_j - 1))}. \tag{4.2}$$

Wada proves that it is independent of the choice of index j provided $\det(\Phi(x_j - 1)) \neq 0$. (Notice that if the presentation has deficiency one, then there is only one $n(s-1) \times n(s-1)$ subdeterminant so one need not compute a gcd.)

The following theorem asserts that $\Delta_1 = W \cdot \Delta_0$. Since Δ_0 is often easy to compute, this gives a straightforward method of obtaining Δ_1. This is the method we use in the calculations in [10].

Theorem 4.1. *If $H_1(X; F^n[\mathbf{Z}])$ is torsion, then*

$$\Delta_1 = W \cdot \Delta_0.$$

Proof. Clearly Q_j is the order of the torsion of the cokernel of

$$(\Lambda^n)^t \xrightarrow{\partial_2} (\Lambda^n)^s \xrightarrow{p_j} (\Lambda^n)^{s-1} = \Lambda^{n(s-1)},$$

and $\det(\Phi(1 - x_j))$ is the order of the cokernel of the composite

$$\Lambda^n \hookrightarrow (\Lambda^n)^s \xrightarrow{\partial_1} \Lambda^n$$

where the first map is the inclusion of the jth coordinate.

We will need the following lemma.

Lemma 4.2. *Suppose that V, W, Y, and Z are free, finitely generated Λ modules such that W and Z have the same rank. Suppose that $a: V \to W$, $b: V \to Y$, $c: W \to Z$ and $d: Y \to Z$ are homomorphisms such that c is injective and so that*

$$0 \to V \xrightarrow{a \oplus b} W \oplus Y \xrightarrow{c+d} Z \to 0$$

is a complex. Then

$$\frac{\text{order}(\text{coker}\, b)}{\text{order}(\text{coker}\, c)} = \frac{\text{order}(H_1)}{\text{order}(H_0)} \in F(t)$$

where H_0 and H_1 are the zeroth and first homology modules of this complex.

Assuming this lemma, we complete the proof of Theorem 4.1. For simplicity of notation assume that $j = 1$. let $V = (\Lambda^n)^t/(\ker \partial_2), W = \Lambda^n, Y = (\Lambda^n)^{s-1}$ and $Z = \Lambda^n$. Take $a \oplus b : V \to W \oplus Y$ to be the map induced by ∂_2 and let $c + d$ be ∂_1. Thus the complex $0 \to V \to W \oplus Y \to Z \to 0$ is just the truncation of the complex (4.1). Hence the ith homology module of this complex equals $H_i(\pi; F^n[\mathbf{Z}]) = H_i(X; F^n[\mathbf{Z}])$ for $i = 0$ and 1, and

$$\frac{\mathrm{order}(\mathrm{coker}\, b)}{\mathrm{order}(\mathrm{coker}\, c)}$$

equals W. Thus Theorem 4.1 is proved.

Proof of Lemma 4.2. The sequence

$$0 \to \frac{\mathrm{image}\, c + d}{\mathrm{image} c} \to \frac{Z}{\mathrm{image}\, c} \to \frac{Z}{\mathrm{image}\, c + d} \to 0$$

is exact. This can be rewritten as

$$0 \to \frac{\mathrm{image}\, c + d}{\mathrm{image}\, c} \to \mathrm{coker}\, c \to H_0 \to 0. \tag{4.3}$$

Now define maps

$$\alpha : \frac{\ker c + d}{\mathrm{image}\, a \oplus b} \to \frac{Y}{\mathrm{image}\, b}$$

and

$$\beta : \frac{Y}{\mathrm{image}\, b} \to \frac{\mathrm{image}\, c + d}{\mathrm{image}\, c}$$

by the formulas $\alpha[w, y] = [y]$ and $\beta[y] = [d(y)]$ where the brackets denote cosets. Then a routine exercise shows that α and β are well-defined, α is injective, β is surjective, and image $\alpha = \ker \beta$. Since

$$\frac{\ker c + d}{\mathrm{image}\, a \oplus b} = H_1 \quad \text{and} \quad \frac{Y}{\mathrm{image}\, b} = \mathrm{coker}\, b,$$

these fit to give a short exact sequence

$$0 \to H_1 \xrightarrow{\alpha} \mathrm{coker}\, b \xrightarrow{\beta} \frac{\mathrm{image}\, c + d}{\mathrm{image}\, c} \to 0. \tag{4.4}$$

Splicing the two sequences (4.3) and (4.4) together gives an exact sequence

$$0 \to H_1 \to \mathrm{coker}\, b \to \mathrm{coker}\, c \to H_0 \to 0.$$

The lemma now follows from the fact that the product of the orders of the even terms in an exact sequence of torsion modules equals the product of the orders of the odd terms. Q.E.D.

We end this section with formulas relating other versions of the twisted Alexander polynomial to the Δ_i. In [9] Jiang and Wang define an invariant $A^\rho(M)$ for 3-manifolds M with representations $\varepsilon : \pi_1(M) \to \mathbf{Z}$ and $\rho : \pi_1(M) \to GL(V)$. The definition is similar to Wada's definition and one can show

$$A^\rho = \Delta_1 \cdot \mathrm{gcd}_j \left\{ \frac{\det(\Phi(x_j - 1))}{\Delta_0} \right\}.$$

In [13], Lin defines a twisted polynomial $A(K,\rho)$ for knots K in S^3 and $GL(n,\mathbf{C})$ representations using a free Seifert surface. Proposition 3.3 of [9] establishes the formula

$$A^{\rho}(S^3 - K) = A(K,\rho)/\det(I - t\rho(z))$$

where z is the meridian of K provided the Seifert surface represents non-trivially by ρ, otherwise $A^{\rho}(K) = A(K,\rho)$.

Thus we see that the earlier definitions of twisted Alexander polynomials are determined by Δ_1 and Δ_0.

5. SYMMETRIES OF THE TORSION

5.1. Duality and τ

The Alexander polynomial of a knot in S^3 is symmetric, i.e. $p(t) = \pm t^n p(t^{-1})$. Moreover, the Alexander polynomial of a slice knot satisfies $p(t) = \pm t^n f(t) f(t^{-1})$ for some polynomial $f(t)$. Proofs of these facts using the interpretation of the Alexander polynomial as a Reidemeister torsion were given by Milnor in [15]. His argument works in our case and we will describe the set-up in our context.

Suppose that X is a compact PL manifold of dimension n with perhaps non-empty boundary. Let X' denote X with the dual cell structure. Let $^- : F \to F$ be an involution. Extend this involution to $F[\mathbf{Z}]$ and $F(t)$ by taking $\bar{t} = t^{-1}$.

Next let W be a right $F[\pi]$ module and $\{\ ,\ \} : V \times W \to F$ a non-degenerate inner product satisfying (2.4) and (2.5).

Define a right action of π on $F(t) \otimes_F W$ in the same way as for V, i.e. by

$$(p \otimes w) \cdot \gamma = (t^{\varepsilon(\gamma)} p) \otimes w \cdot \gamma.$$

This action then can be used to construct the chain complex

$$C_*(X', \partial X'; W(t)) = (F(t) \otimes W) \otimes_{F[\pi]} C_*(\tilde{X}', \partial \tilde{X}').$$

To make notation less cumbersome denote this chain complex by D_*, and denote the chain complex $(F(t) \otimes V) \otimes C_*(\tilde{X})$ by C_* for the rest of this section.

The following notation will be convenient. Given an $F(t)$ module M, let $\overline{\mathrm{Hom}}_{F(t)}(M, F(t))$ denote the set of homomorphisms from M to $F(t)$ with the $F(t)$ vector space structure given by $(r \cdot h)(m) = \bar{r} \cdot h(m)$ for $r \in F(t)$. Also let $\mathrm{AHom}_{F(t)}(M, F(t))$ denote the set of anti-linear homomorphisms from M to $F(t)$ (i.e. those h so that $h(rm) = \bar{r}h(m)$ for $r \in F(t)$), with the $F(t)$ structure $(r \cdot h)(m) = r \cdot h(m)$.

Now define an inner product

$$\langle\ ,\ \rangle : C_q \times D_{n-q} \to F(t) \tag{5.1}$$

by the formula

$$\langle g \otimes v \otimes z_1, f \otimes w \otimes z_2 \rangle = \sum_{\gamma \in \pi} (z_1 \cdot \gamma z_2) g \bar{f} t^{\varepsilon(\gamma)} \{v\gamma, w\}. \tag{5.2}$$

In this formula, $(z_1 \cdot \gamma z_2)$ denotes the algebraic intersection number (in $\mathbf{Z}$) of the simplex z_1 with the cell z_2.

This inner product is well defined and defines an $F(t)$-isomorphism

$$D_{n-q} \to \overline{\mathrm{Hom}}_{F(t)}(C_q, F(t))$$

via $d \mapsto (c \mapsto \langle c,d \rangle)$. This isomorphism takes the differential of D_* to the dual of the differential of C_* (up to sign), and hence induces the Poincaré duality isomorphism

$$H_{n-q}(D_*) \to H^q(\overline{\mathrm{Hom}_{F(t)}}(C_*, F(t))).$$

A choice of basis (over F) for V and lifts to $\tilde{X}$ of the simplices of X endows the chain complex C_* with a preferred $F(t)$-basis (of the form $1 \otimes v_i \otimes z_j$, where v_i is an element of the basis of V and z_j is a lift of a simplex of X). Then D_* has a natural dual basis (over $F(t)$) obtained by picking a basis for W dual to the basis for V via the inner product $\{\ ,\ \}$ and the dual cells $\tilde{X}'$ of the fixed lifts of the simplices of X. These bases are dual with respect to the inner product (5.2).

The universal coefficient theorem applied to the $F(t)$ chain complex C_* implies that evaluation induces an isomorphism

$$H^q(\overline{\mathrm{Hom}_{F(t)}}(C_*, F(t))) \cong \overline{\mathrm{Hom}_{F(t)}}(H_q(C_*), F(t))$$

and so the pairing of Eq. (5.1) induces a *non-singular* pairing on homology groups

$$H_q(X; V(t)) \times H_{n-q}(X, \partial X; W(t)) \to F(t). \tag{5.3}$$

Fix a basis for $H_q(X; V(t))$ and give $H_{n-q}(X, \partial X; W(t))$ the dual basis with respect to this pairing.

Thus we have specified bases for the chain complexes C_*, D_* and for their homology $H_*(C_*), H_*(D_*)$. The argument in [15, p. 142] shows that with respect to these bases, the Reidemeister torsions satisfy

$$\tau(D_*)\,\overline{\tau(C_*)}^{(-1)^n} = \pm 1. \tag{5.4}$$

Following [15] and also [16, Theorem 3.2] one obtains

Theorem 5.1. *Suppose that F is a subfield of $\mathbf{C}$ closed under conjugation and V be a unitary $F[\pi]$ module which is finite-dimensional over F. Let $^- : F(t) \to F(t)$ be the involution as described above. Suppose that X is a compact PL manifold with representation ρ: $\pi_1 X \to GL(V)$.*

1. *Suppose that $C_*(X; V(t))$ is acyclic. Then $C_*(X, \partial X; V(t))$ and $C_*(\partial X; V(t))$ are also acyclic, and their torsions satisfy*

$$\tau(\partial X) = \tau(X)\overline{\tau(X)}^{(-1)^n},$$

$$\tau(X, \partial X) = \overline{\tau(X)}^{(-1)^{n+1}}$$

and

$$\tau(X, \partial X)\tau(\partial X) = \tau(X).$$

2. *If $C_*(X; V(t))$ is not acyclic, but $C_*(\partial X; V(t))$ is acyclic, choose a basis for $H_q(X; V(t))$ for each q and give $H_{n-q}(X, \partial X; V(t))$ the dual basis. Then with respect to these bases*

$$\tau(\partial X) = \tau(X)\overline{\tau(X)}^{(-1)^n},$$

and

$$\tau(X, \partial X)\tau(\partial X)\tau(\mathscr{H}) = \tau(X)$$

where $\tau(\mathscr{H})$ denotes the torsion of the long exact sequence for the pair $(X, \partial X)$ with respect to the given homology bases. Equivalently, since we are assuming that

$C_*(\partial X; V(t))$ is acyclic, $\tau(\mathscr{H})$ is just the alternating product of the determinants of the inclusion maps $H_i(X; V(t)) \to H_i(X, \partial X; V(t))$.

The proofs of these assertions follow exactly as in Milnor's paper and will be omitted.

We will call representations V of π as in the statement of Theorem 5.1 *unitary representations over F*. This includes $U(n)$ representations as well as $O(n)$ representations. In our applications to knot slicing F is $\mathbf{Q}(\zeta)$ with ζ a non-trivial root of unity, $^{-}: F \to F$ is complex conjugation, and $V = F$; this is clearly unitary.

One immediately obtains the following corollaries from Theorem 5.1. Notice that these holds up to the indeterminacy of the Reidemeister torsion τ, such as in Corollary 3.2. The duality discussed in Section 7 can be used to give alternative proofs of these corollaries for the homology torsion.

Corollary 5.2. *Let X be an odd dimensional closed manifold. If $\rho: \pi_1 X \to GL(V)$ is a unitary representation and $C(X; V(t))$ is acyclic then τ is symmetric, i.e. $\bar{\tau} = \tau$, where $\bar{\tau}$ denotes the polynomial obtained by conjugating the coefficients and taking t to t^{-1}.*

Corollary 5.3. *Let X be an odd dimensional closed manifold. Suppose that $\rho: \pi_1 X \to GL(V)$ is a unitary representation and $C_*(X; V(t))$ is acyclic. Suppose further that X bounds a manifold W such that the representation ρ extends to $\pi_1 W$ and so that the corresponding complex $C_*(W; V(t))$ is also acyclic. Then $\tau = \tau(X)$ is a norm in $F(t)$, i.e.*

$$\tau = f\bar{f}$$

for some $f \in F(t)$.

In [11], Kitano proves Corollary 5.2 for $SO(n)$ representations of the complement of a knot in S^3. Although a knot complement is not closed, one obtains Kitano's result from Theorem 5.1 by using the calculation of Section 3 that $\tau(T) = 1$ if T is a torus.

It is easy to show (see [15] for the proof) that if X is obtained by zero-surgery on a knot in S^3 and V is the trivial 1-dimensional representation then the resulting torsion τ is equal to the quotient of the Alexander polynomial of the knot divided by $t - 1$. Therefore these two corollaries give the well-known results that the Alexander polynomial is symmetric and that the Alexander polynomial of a slice knot has the form $f(t)f(t^{-1})$ [20].

Corollary 5.3 suggests that the torsion can be used to obstruct slicing of knots. Theorem 6.2 gives a criterion for a knot to be slice in terms of the twisted Alexander polynomial Δ_1 for some family of representations, using Corollary 5.3.

For a non-unitary representation the polynomial need not be symmetric. See [22] for an example. One can instead prove the following.

Corollary 5.4. *Let X be a closed n-manifold and let V be any right $F[\pi]$-module. Let $V^* = \mathrm{Hom}(V, F)$ be the dual module (so $(h \cdot \gamma)(v) = h(v \cdot \gamma^{-1})$). Suppose that $C_*(X; V(t))$ is acyclic. Then $C_*(X; V^*(t))$ is also acyclic and their torsions satisfy*

$$\tau(V^*)(t) = \tau(V)(t^{-1})^{(-1)^n}.$$

5.2. Determinants of inclusion maps

We finish this section with some algebraic observations concerning the chain complexes C_* and D_* and their homology which will be useful in the next section.

The pairing of Eq. (5.1) and the maps induced by inclusions determine a commutative diagram of $F(t)$ linear maps

$$\begin{array}{ccc} H_q(X; V(t)) & \rightarrow & \mathrm{AHom}_{F(t)}(H_{n-q}(X, \partial X; V(t)); F(t)) \\ \downarrow i_q & & \downarrow (-1)^{q(n-q)} \bar{i}^*_{n-q} \\ H_q(X, \partial X; V(t)) & \rightarrow & \overline{\mathrm{Hom}}_{F(t)}(H_{n-q}(X, V(t)), F(t)). \end{array} \tag{5.5}$$

In this diagram, the top horizontal map is given by $a \mapsto \langle a, \ \rangle$ and the bottom horizontal map is given by $c \mapsto \langle \ ,c \rangle$. The left vertical arrow is just the map induced by inclusion, and the right vertical arrow is the map which takes an anti-linear homomorphism h to the homomorphism

$$m \mapsto (-1)^{q(n-q)} \overline{h(i_{n-q}(m))}.$$

The bases of homology were chosen so that the two horizontal maps in the diagram (5.5) are expressed in these bases by the identity map. It follows that the vertical maps have the same determinants with respect to these bases.

LEMMA 5.5. *For each q fix a basis of $H_q(X; V(t))$ and give $H_{n-q}(X, \partial X; V(t))$ the dual basis with respect to the pairing of Eq.* (5.3). *Then with respect to these bases*

$$\det i_q = (-1)^{q(n-q)} \overline{\det} \, i_{n-q}$$

where $i_q : H_q(X; V(t)) \rightarrow H_q(X, \partial X; V(t))$ denotes the map induced by inclusion.

Moreover, if $n = 2k$, then with respect to these bases the determinant of i_k equals the determinant of the intersection form

$$H_k(X; V(t)) \times H_k(X; V(t)) \rightarrow H_k(X; V(t)) \times H_k(X, \partial X; V(t)) \rightarrow F(t)$$

where the first map is induced by inclusion and the second map is the pairing given in Eq. (5.3).

Proof. The first assertion follows easily from the commutativity of (5.5). For the second assertion, consider the diagram ($V(t)$ coefficients).

$$\begin{tikzcd} H_k(X) \times H_k(X) \arrow[r] \arrow[d, "1 \times i_k"'] & F(t) \\ H_k(X) \times H_k(X, \partial X) \arrow[ur] & \end{tikzcd}$$

The horizontal map is the intersection form and the diagonal map is the identity in the chosen bases. Since the vertical map is determined by i_k, the result follows. Q.E.D.

6. KNOT SLICING AND THE DETERMINANT OF THE CASSON–GORDON INVARIANTS

In this section we will show how to use Δ_1 and τ to detect non-sliceness of knots using Theorem 5.1 of the previous section. The method is to put τ into the context studied by Casson and Gordon in [2]. We will prove that τ is the determinant of the Casson–Gordon invariant. Our approach initially follows the Casson–Gordon argument [2]. However, the novelty of our point of view stems from the fact that no 4-manifold constructions are needed to define or compute the invariants.

6.1. The Casson–Gordon set-up

We begin by recalling the context in which the Casson–Gordon invariants are constructed.

Let $K \subset S^3$ be an oriented knot. Let X denote the p^r-fold cyclic cover of 0-framed surgery on K for some prime p. Then $\pi_1 X$ has a non-trivial homomorphism to $\mathbf{Z}$ given by the composite of the map induced by the covering projection to the complement of K and the natural surjection of $\pi_1(S^3 - K)$ to $\mathbf{Z}$ determined by the orientation of K. Let $\varepsilon : \pi_1 X \to \mathbf{Z}$ be the surjection obtained by restricting to the image of this composite.

We will take prime-power covers and prime-power characters. Thus let $m = p^r$ for some prime p and $d = q^s$ for some prime q.

Let B_m denote the m-fold cyclic branched cover of $S^3 - K$ and let E_m denote the m-fold cyclic cover of $S^3 - K$. The inclusion $i : E_m \subset X$ induces an isomorphism i_*: $H_1(E_m; \mathbf{Z}) \to H_1(X; \mathbf{Z})$ and the inclusion $j : E_m \subset B_m$ induces a surjection j^*: $H_1(E_m; \mathbf{Z}) \to H_1(B_m; \mathbf{Z})$. Thus $j_* \circ i_*^{-1}$ induces a surjection

$$H_1(X; \mathbf{Z}) \to H_1(B_m; \mathbf{Z}).$$

Let $\chi : H_1(B_m; \mathbf{Z}) \to \mathbf{Z}/d$ be some surjective homomorphism, where d is a prime power. Let $\zeta_d = e^{2\pi i/d}$. Then $\mathbf{Z}/d$ acts on $\mathbf{Q}(\zeta_d)$ by multiplication by ζ_d. Thus the homomorphism χ defines a representation

$$\rho_\chi : \pi_1 X \to GL_1(\mathbf{Q}(\zeta_d)) = \mathbf{Q}(\zeta_d)^\times$$

by precomposing χ with the Hurewitz map and the surjection $H_1(X; \mathbf{Z}) \to H_1(B_m; \mathbf{Z})$. Use ε and ρ_χ to define the chain complexes $C_*(X; \mathbf{Q}(\zeta_d)[\mathbf{Z}])$ and $C_*(X; \mathbf{Q}(\zeta_d)(t))$ as in the previous sections. So $F = \mathbf{Q}(\zeta_d)$ and $V = \mathbf{Q}(\zeta_d)$ is one dimensional.

6.2. Using τ and Δ_1 as slicing obstructions

Lemma 4 and its corollary of [2] shows that $H_*(X; \mathbf{Q}(\zeta_d)(t)) = 0$. Thus the Reidemeister torsion $\tau \in \mathbf{Q}(\zeta_d)(t)$ is defined up to $\{\pm \zeta_d^r t^s \mid r, s \in \mathbf{Z}\}$. The homology torsion is defined up to $\{rt^s \mid r \in \mathbf{Q}(\zeta_d), s \in \mathbf{Z}\}$. We remark that the proof of the lemma in [2] does not use any 4-dimensional constructions, in fact it holds for knots in any dimension. Moreover this same lemma implies that if K is a slice knot and the homomorphism χ extends to the m fold branched cover of the slice disc, then the chain complex $C_*(N; \mathbf{Q}(\zeta_d)(t))$ is acyclic, where N denotes the m-fold cover of the complement of the slice disc (so $\partial N = X$) endowed with the obvious extensions of ρ_χ and ε.

Applying Theorem 5.1 of the previous section to N and $\partial N = X$ we immediately obtain the following.

Proposition 6.1. *Let K, X, χ, ρ_χ, ε, $m = p^r$, $d = q^s$ be as above and suppose that K is slice. Suppose that the homomorphism χ extends over the m-fold branched cover of the complement of the slice disc. Then the torsion of X is a norm, i.e. there exists an element $f \in \mathbf{Q}(\zeta_d)(t)$ so that*

$$\tau(X, \rho_\chi) = f\bar{f}$$

modulo $\pm \zeta_d^r t^s$.

For a slice knot one can always find characters χ that extend over the branched cover of the slice disk, corresponding to metabolizers of the linking form on B_m (see [6]). Thus τ gives an obstruction to slicing knots. We reinterpret this proposition in a more usable form in the next theorem.

THEOREM 6.2. *If K is an oriented slice knot in S^3, p, q odd primes, so $m = p^r$ and $d = q^s$ are odd, then there is a subgroup M of $H_1(B_m; \mathbf{Z})$ satisfying* order $(M)^2$ = *order*$(H_1(B_m; \mathbf{Z}))$ *so that for all $\mathbf{Z}/d$-valued characters χ vanishing on M, the associated twisted polynomial $\Delta_1(E_m; \rho_\chi)$ (having coefficients in $\mathbf{Q}(\zeta_d)$) factors as $at^n f\bar{f}(t-1)^s$ where $a \in \mathbf{Q}(\zeta_d)$, $f \in \mathbf{Q}(\zeta_d)[\mathbf{Z}]$, $s = 1$ if χ is non-zero, and $s = 0$ if χ is the zero character.*

In fact, M is the kernel of the map on first homology induced by the inclusion of B_m into the branched cover of the 4-ball over the slice disc, and in particular is invariant under the automorphisms induced by the deck transformations.

Proof. The condition on the order of M is Lemma 3 of [2]. Elementary considerations show that if χ vanishes on M, then χ extends to the branched cover of the slice disc, provided that one extends the range of χ to $\mathbf{Z}/d^k \supset \mathbf{Z}/d$ for some positive integer k. Let X denote the m-fold branched cover of 0-surgery on K.

LEMMA 6.3. *With the notation as above,*

$$\Delta_1(E_m; \rho_\chi) = \begin{cases} \tau(X; \rho_\chi)(t-1)^{-1} & \text{if} \quad \chi \text{ is non-trivial,} \\ \tau(X; \rho_\chi) & \text{if} \quad \chi \text{ is trivial.} \end{cases}$$

Proof. Consider the solid torus $Z = S^1 \times D^2 \subset X$ with complement E_m. Since ε is non-trivial on $\pi_1(Z)$, $H_0(Z; \mathbf{Q}(\zeta_d)(t)) = 0$ by Proposition 3.5 and since Z is a homotopy circle $C_*(Z; \mathbf{Q}(\zeta_d)(t))$ is acyclic. Moreover, its torsion is easily computed to be $t-1$, this follows from the fact that χ is a character on the branched cover and so must send the generator of $\pi_1(Z)$ to zero. Theorem 5.1 implies that $C_*(Z, \partial Z; \mathbf{Q}(\zeta_d)(t))$ is acyclic with torsion $\overline{t-1} = t^{-1} - 1$. Excision implies that $C_*(X, E_m; \mathbf{Q}(\zeta_d)(t))$ is also acyclic. Since $C_*(X; \mathbf{Q}(\zeta_d)(t))$ is acyclic, $C_*(E_m; \mathbf{Q}(\zeta_d)(t))$ is also acyclic and

$$\tau(X, E_m)\tau(E) = \tau(X).$$

Switch now to $\mathbf{Q}(\zeta_d)[\mathbf{Z}]$ coefficients and homology torsion. Then $\tau(X, E_m) = \tau(Z, \partial Z) = t^{-1} - 1 = 1 - t$. Therefore

$$\frac{\Delta_1(E_m)}{\Delta_0(E_m)\Delta_2(E_m)} = \tau(E_m) = \tau(X)(1-t)^{-1}.$$

Since E_m collapses to a 2-complex, $H_2(E_m; \mathbf{Q}(\zeta_d)[\mathbf{Z}])$ is free, but it must be zero since $H_2(E_m; \mathbf{Q}(\zeta_d)(t)) = 0$. Thus $\Delta_2(E_m) = 0$ and so

$$\Delta_1(E_m) = \tau(X)(1-t)^{-1}\Delta_0(E_m).$$

From its definition $\Delta_0(E_m)$ is the greatest common divisor of the linear polynomials $\zeta_d^{\chi(\gamma)} t^{\varepsilon(\gamma)} - 1$ where γ ranges over $\pi_1(E_m)$. Since the meridian μ satisfies $\chi(\mu) = 0$ and $\varepsilon(\mu) = 1$,

$$\Delta_0(E_m) = \begin{cases} 0 & \text{if} \quad \chi \text{ is non-trivial,} \\ 1-t & \text{if} \quad \chi \text{ is trivial.} \end{cases}$$

The lemma follows. Q.E.D.

Continuing with the proof of Theorem 6.2, note that Proposition 6.1 implies that $\tau(X; \rho_\chi) \in \mathbf{Q}(\zeta_d)(t)$ factors as $f\bar{f}$ for some f in the (larger) field $\mathbf{Q}(\zeta_{d^k})(t)$. Since $\Delta_1(E_m; \rho_\chi)$ is a Laurent polynomial, it is not hard to see that there exists a polynomial $g \in \mathbf{Q}(\zeta_{d^k})[t]$ so that

$$\Delta_1(E_m; \rho_\chi) = at^n\, g\bar{g}(t-1)^s.$$

Here $a \in \mathbf{Q}(\zeta_d)$ and $n \in \mathbf{Z}$; these arise from the indeterminacy of the twisted polynomials. Also, $s = 0$ or 1 according to whether χ is trivial or not. (Note that $(1-t)^{-1} \cdot (1-t)\overline{(1-t)} = (1-t)$.)

Theorem 6.2 now follows from the next lemma.

LEMMA 6.4. *Suppose that* $f \in \mathbf{Q}(\zeta_d)[t^{\pm 1}]$ *and* f *factors as* $g\bar{g}$ *for some* $g \in \mathbf{Q}(\zeta_{d^k})[t^{\pm 1}]$. *Then if* d *is odd,* f *also factors as* $h\bar{h}$ *for some* $h \in \mathbf{Q}(\zeta_d)[t^{\pm 1}]$.

Proof. Let $G = Gal(\mathbf{Q}(\zeta_{d^k}), \mathbf{Q}(\zeta_d))$ and recall that the order of G is d^{k-1}, which is odd by assumption. (This follows from the fact that $\mathbf{Q}(\zeta_{d^k})$ is a degree $d^{k-1}(d-1)$ extension of $\mathbf{Q}$.) Taking the product of the conjugates of $f = g\bar{g}$ over all elements of G yields $f^{d^{k-1}} = h\bar{h}$ where $h \in \mathbf{Q}(\zeta_d)[t, t^{-1}]$.

Now, let $q \in \mathbf{Q}(\zeta_d)[t, t^{-1}]$ be an irreducible factor of f. If q and $\bar{q}$ are associates (i.e. $\bar{q} = aq$ for some $a \in \mathbf{Q}(\zeta_d)$), then we see that q has even exponent in $f^{d^{k-1}}$, and hence even exponent in f also since d is odd. On the other hand, if q and $\bar{q}$ are not associates, then it is clear that q and $\bar{q}$ have equal exponents in $f^{d^{k-1}}$ and hence in f as well.

Consequently, all irreducible factor of f appear either in conjugate pairs with equal exponents or are self-conjugate elements with even exponent.

The lemma follows, and so Theorem 6.2 is proven. Q.E.D.

See our article [10] for applications of Theorem 6.2 to concordance and invertability questions for knots in S^3.

6.3. The determinant of the Casson–Gordon invariant equals τ

An oriented knot K and a character $\chi : \pi_1 X \to \mathbf{Z}/d$ as above define a homomorphism $\varepsilon \times \chi : \pi_1 X \to \mathbf{Z} \times \mathbf{Z}/d$. Since the 3-dimensional bordism group over $\mathbf{Z} \times \mathbf{Z}/d$ is finite, in fact d-torsion, some multiple, say nX of X bounds a 4-manifold M over $\mathbf{Z} \times \mathbf{Z}/d$. Thus one obtains a homomorphism $\pi_1 M \to \mathbf{Q}(\zeta)(t)^\times$ with $\zeta = \zeta_d = e^{2\pi i/d}$. The twisted intersection form of M with coefficients in $\mathbf{Q}(\zeta)(t)$ is non-singular since nX is acyclic over $\mathbf{Q}(\zeta)(t)$, and hence defines a class $I(M; \mathbf{Q}(\zeta)(t))$ in the Witt group $W(\mathbf{Q}(\zeta)(t))$ of non-singular Hermitian forms over $\mathbf{Q}(\zeta)(t)$.

The (untwisted) intersection form $I(M)$ of M defines a class in the Witt group of $\mathbf{Z}$. To be more precise, $I(M)$ is a singular form in general and so one must take the quotient of this form by its kernel to obtain a non-singular form. Denote $I(M)/\ker$ by $\tilde{I}(M)$. Then the Casson–Gordon invariant is defined to be

$$CG(K, m, \chi) = (I(M; \mathbf{Q}(\zeta)(t)) - \tilde{I}(M)) \otimes \frac{1}{n} \in W(\mathbf{Q}(\zeta)(t)) \otimes \mathbf{Q}. \tag{6.1}$$

(Include $\tilde{I}(M)$ in $W(\mathbf{Q}(\zeta)(t))$ via the homomorphism on Witt groups induced by the canonical map $\mathbf{Z} \to \mathbf{Q}(\zeta)(t)$.) Note that we may as well assume n divides d. When d is odd this implies that the Casson–Gordon invariant lies in $W(\mathbf{Q}(\zeta_d)(t)) \otimes \mathbf{Z}_{(2)}$.

Let $N \subset \mathbf{Q}(\zeta)(t)^\times$ denote the (multiplicative) subgroup generated by norms $f\bar{f}$, ζ, t^n, and ± 1. The determinant induces a homomorphism

$$\det : W(\mathbf{Q}(\zeta)(t)) \to \mathbf{Q}(\zeta)(t)^\times / N \tag{6.2}$$

since the determinant of a metabolic form is a norm up to sign.

If c is an odd integer and $I \in W(\mathbf{Q}(\zeta)(t))$, $\det(cI) = \det(I)$ modulo N since $\det(2I) = \det(I)\det(I)$ is a norm because I is Hermitian. Therefore, the determinant of Eq. (6.2) extends to

$$\det : W(\mathbf{Q}(\zeta)(t)) \otimes \mathbf{Z}_{(2)} \to \mathbf{Q}(\zeta)(t)^{\times}/N$$

by the formula

$$\det\left(\frac{p}{q} I\right) = \det(I)^p. \tag{6.3}$$

Thus, when d is odd, we use Eq. (6.3) to define the determinant of the Casson–Gordon invariant, so

$$\det(CG(K, M, \chi)) = \det(I(M; \mathbf{Q}(\zeta)(t)))/\det(\tilde{I}(M)). \tag{6.4}$$

THEOREM 6.5. *Suppose that p and q are odd primes,* $m = p^r, d = q^s$. *Then*

$$\det(CG(K,m,\chi)) = \tau(X, \rho_\chi) \ \ \text{modulo} N$$

where X *is the m-fold cyclic cover of* 0-*surgery on* K, *and* $\rho_\chi : \pi_1 X \to \mathbf{Q}(\zeta_d)$ *is the homomorphism given by composing* χ *with* $\mathbf{Z}/d \to \mathbf{Q}(\zeta_d)$.

Proof. Take M the 4-manifold bounded by n copies of X as above, where n divides d. Note that n is odd since d is. Since M is a homotopy 3-complex, $H_4(M; \mathbf{Q}(\zeta)(t)) = 0$. Also $H_0(M; \mathbf{Q}(\zeta)(t)) = 0$ by Proposition 3.5 above since ε is non-trivial.

Fix a basis for $H_q(M; \mathbf{Q}(\zeta)(t))$ for each q and give $H_{4-q}(M, nX; \mathbf{Q}(\zeta)(t))$ the dual basis with respect to the intersection form, as in Section 5. For ease of notation, let $I_1 = I(M; \mathbf{Q}(\zeta)(t))$ and $I_2 = \tilde{I}(M)$ in $W(\mathbf{Q}(\zeta)(t))$. Equation (6.4) says that the determinant of $CG(K, m\chi)$ equals $\det(I_1)/\det(I_2)$.

Consider the long exact sequence for the pair (M, nX). Since $H_*(X; \mathbf{Q}(\zeta)(t)) = 0$ by the Lemma 4 of 2 (see above), this sequence reduces to isomorphisms

$$i_q : H_q(M; \mathbf{Q}(\zeta)(t)) \to H_q(M, nX; \mathbf{Q}(\zeta)(t)),\ q = 1, 2, 3.$$

Let $\tau(\mathscr{H})$ denote the torsion of the long exact sequence with respect to the given homology bases; this is equal to

$$\frac{\det(i_2)}{\det(i_1)\det(i_3)}.$$

Lemma 5.5 implies that $\det(i_1) = -\overline{\det(i_3)}$ and $\det(i_2) = \det(I_1)$. Therefore $\tau(\mathscr{H})$ equals $\det(I_1)$ modulo N.

Theorem 5.1 states that

$$\tau(M, nX)\tau(nX)\tau(\mathscr{H}) = \tau(M)$$

and that with respect to these choices of bases,

$$\tau(M, nX) = \overline{\tau(M)}^{\,-1}.$$

Thus

$$\tau(nX) = \tau(M)\overline{\tau(M)}\tau(\mathscr{H}).$$

Hence $\tau(nX)$ equals $\tau(\mathscr{H})$ modulo N. Clearly $\tau(nX) = \tau(X)^n$. Moreover $\tau(X) = \overline{\tau(X)}$ by Theorem 5.1 and since n is odd $\tau(X)^n$ equals $\tau(X)$ modulo N. Therefore $\tau(X)$ is equal to $\tau(\mathscr{H})$

modulo N. Combining this with the previous paragraph we see that $\tau(X)$ equals $\det(I_1)$ modulo N.

Since we assumed p is odd and $m = p^r$, the homology of B_m is a direct double [19] and thus has order a square, say $|H_1(B_m;\mathbf{Z})| = b^2$. An easy argument shows that if M is any 4-manifold and $\tilde{I}(M)$ denotes the intersection form of M made non-singular by dividing by the kernel, then the determinant of $\tilde{I}(M)$ equals the order of the torsion of the first homology of ∂M times the square of an integer. Since $H_1(X) = H_1(B_m)\oplus\mathbf{Z}$ and $\partial M = nX$, $\det(I_2) = (b^2\cdot k^2)^n$ for some integer k. Thus

$$\begin{aligned}\det(CG(K,\chi)) &= \det\left(\frac{1}{n}I_1\right)/\det\left(\frac{1}{n}I_2\right)\\ &= \det(I_1)/\det(I_2)\\ &= \tau(X)\cdot(b^2k^2)^{-n}\\ &= \tau(X).\end{aligned}$$

Q.E.D.

Remark. By the same argument one can reach the weaker conclusion that $\tau(X)$ equals $\det(CG(K,\chi))\cdot|H_1(B_m)|$ modulo N if p is even. It is not true in general that $|H_1(B_m)|$ is a square if B_m is a 2^r-fold branched cover of a knot, but if K is algebraically slice then $|H_1(B_m)|$ is a square and so Theorem 6.5 remains valid for $p = 2^k$-fold covers of algebraically slice knots.

The references [14] and [7] are relevant here. In [14] Litherland uses the determinant of appropriate Casson–Gordon invariants to show that a family of genus two algebraically slice knots are not slice. In [7] Gilmer and Livingston use a similar approach to deal with families of genus one knots. For genus one knots the determinant takes value in $\mathbf{Q}(\zeta)$, and showing that it is not a norm is more subtle than for polynomials in $\mathbf{Q}(\zeta)(t)$. These example illustrate the value of working with Reidemeister torsion; since the homology torsion is only defined modulo $\mathbf{Q}(\zeta)$ it could not be applied in the genus one case.

7. MILNOR DUALITY AND SIGNATURE INVARIANTS

7.1. Milnor duality

In this section we extend Milnor's duality theorem for infinite cyclic covers to the twisted case and derive a few interesting consequences. There is no need to restrict to representations pulled up from the base. Thus we take $M_\infty \to M$ to be some infinite cyclic cover of a compact manifold M and $\rho:\pi'\to GL(V)$ a representation (recall π' denotes $\pi_1 M_\infty$), giving V a right $F[\pi']$ module structure. We form the associated chain complex

$$C_*(M_\infty;V) = V\otimes_{F[\pi']}C_*(\tilde{M})$$

and denote its homology by $H_q(M_\infty;V)$.

We will need to work with cohomology. As usual assume the field F is equipped with an involution $r\mapsto\bar{r}$, let W be right $F[\pi']$ module and suppose that there is a non-degenerate pairing $\{\ ,\ \}:V\times W\to F$ satisfying (2.4) and (2.5) for $\gamma\in\pi'$.

Then the cochain complexes

$$\mathrm{Hom}_F(V\otimes_{F[\pi']}C_*(\tilde{M}),F)$$

and

$$\mathrm{Hom}_{F[\pi']}(C_*(\tilde{M}),W)$$

are anti-isomorphic, i.e. isomorphic as F-chain complexes provided one of the two is given the conjugate F-vector space structure $(r \cdot h)(x) = \bar{r}h(x)$ for $r \in F$. We denote (as is standard) the second cochain complex by $C^*(M_\infty; W)$ and its homology by $H^*(M_\infty; W)$, and let $\overline{H}^*(M_\infty; W)$ denote the same homology group with the conjugate vector space structure.

The universal coefficient theorem implies that

$$H^q(\mathrm{Hom}_F(C_*(M_\infty; V), F)) = \mathrm{Hom}_F(H_q(M_\infty, V), F)$$

and so

$$\overline{H}^q(M_\infty; W) = \mathrm{Hom}_F(H_q(M_\infty, V), F).$$

The previous constructions can also be applied to a pair $(M_\infty, \partial M_\infty)$. The inner product $\{\ ,\ \}$ defines a cup product

$$H^q(M_\infty, \partial M_\infty; V) \times H^p(M_\infty; W) \to H^{q+p}(M_\infty, \partial M_\infty; F) \tag{7.1}$$

which is Hermitian, i.e. linear in the first factor and anti-linear in the second. Here $H^*(M_\infty, \partial M_\infty; F)$ denotes the cohomology with *untwisted* coefficients.

The argument in [17, Section 4] generalizes step-by-step to the twisted case and gives the following theorem.

THEOREM 7.1. *Let M be an oriented compact n-dimensional manifold with infinite cyclic cover M_∞. Let V,W,$\{\ ,\ \}$ be as above. If $H_*(M_\infty; V)$ and $H_*(M_\infty; F)$ are finitely generated over F, then $H^{n-1}(M_\infty, \partial M_\infty; F) \cong F$, $H^{q-1}(M_\infty; V) \cong H_{n-q}(M_\infty, \partial M_\infty; V)$, and the cup product*

$$H^{q-1}(M_\infty; V) \times H^{n-q}(M_\infty, \partial M_\infty; W) \to H^{n-1}(M_\infty, \partial M_\infty; F) = F$$

of eq. (7.1) *is a non-degenerate Hermitian pairing.*

In general, the hypotheses of Theorem 7.1 will not hold, even for the complement of knots in S^3, as the example in Section 3 shows. However, in [18] Neumann shows how to remedy this problem to obtain an intersection form in the following way. Suppose that M has dimension $2k+1$ and suppose for convenience that M is closed. Let $N \subset M_\infty$ be a codimension 1 separating submanifold obtained by taking one of the path components of $p^{-1}(S)$, where $p: M_\infty \to M$ is the infinite cyclic cover and S is a closed, connected submanifold representing the Poincaré dual to the class $[\varepsilon] \in H^1(M; \mathbf{Z})$.

Then the composite

$$H^k(M_\infty; V) \times H^k(M_\infty; V) \xrightarrow{\cup} H^{2k}(M_\infty; F) \to H^{2k}(N; F) = F$$

defines a hermitian form. This form may be degenerate, but Lemma 2.1 of [18] shows that the non-degenerate form induced by modding out the kernel of this form is finite dimensional, and the covering transformation induces an isometry of this reduced form.

7.2. Signature invariants

Theorem 7.1 can be used just as in [17] to construct signature invariants. Assume that M is closed and has odd dimension $2k+1$. Restrict to a unitary representation V of $\pi_1(M)$ in the sense of Section 5 with respect to the involution to $F[\mathbf{Z}]$ by sending t to t^{-1}. Notice that we are assuming here that ρ is defined on $\pi_1(M)$, not just $\pi_1(M_\infty)$ so that the homology $H_*(M_\infty; V)$ and cohomology $H^*(M_\infty; V)$ are $F[\mathbf{Z}]$ modules. If M_∞ and ρ satisfy the hypotheses of Theorem 7.1, then the non-degenerate cup product of the theorem can be combined with the covering action to form a pairing

$$\langle\ ,\ \rangle : H^k(M_\infty; V) \times H^k(M_\infty; V) \to F$$

by the formula

$$\langle x, y\rangle = (t^*x)\cup y - x\cup(t^*y) \tag{7.2}$$

where $t^* : H^k(M_\infty; V) \to H^k(M_\infty; V)$ is the map induced by the generator of the group of deck transformations. This pairing satisfies the symmetries

$$\langle x, y\rangle = (-1)^{k+1}\overline{\langle y,x\rangle} \text{ and } \langle px, y\rangle = \langle x, \bar{p}y\rangle \text{ for } p\in F[\mathbf{Z}].$$

If the hypotheses of Theorem 7.1 are not satisfied, one can nevertheless repeat the definition using the non-degenerate form constructed by Neumann described above.

Since $(t^*x)\cup(t^*y) = x\cup y$, it follows that $\langle x, y\rangle = 0$ for all y if and only if $(t+1)(t-1)x = 0$. Theorem 2.1 implies that $H_k(M_\infty; V) = H_k(M; V[\mathbf{Z}])$ and Theorem 7.1 then implies that the order of the torsion of $H^k(M_\infty; V)$ equals $\Delta_k(M, V_\rho, \varepsilon)$. Thus the pairing of eq. (7.2) is non-degenerate if and only if 1 and -1 are not roots of $\Delta_k(M, V_\rho)$.

Restrict now to M a 3-manifold. The case of most interest for us is if M is a cyclic cover of 0-surgery on a knot is S^3. The pairing of (7.2) is Hermitian (and is non-degenerate if $\Delta_1(M,\rho)$ is not divisible by $t-1$ or $t+1$). We can thus define the *twisted signature of* (M,ρ,ε), $\sigma(M;\rho,\varepsilon)$ to be the signature of this pairing.

As in [17] this signature decomposes into a sum of signatures in the following way. Decompose the $F[\mathbf{Z}]$ module $H^1(M_\infty; V)$ into its $p(t)$ primary summands. Since $\langle p\cdot x, y\rangle = \langle x,\bar{p}\cdot y\rangle$ for $p\in F[\mathbf{Z}]$, the p primary summand is orthogonal to the q primary summand unless $\langle p\rangle = \langle\bar{q}\rangle$. Thus the only contribution to the signature $\sigma(M;\rho,\varepsilon)$ comes from those primary summands for which $\langle p\rangle = \langle\bar{p}\rangle$. Denote by $\sigma_p(M;\rho,\varepsilon)$ the signature of $\langle\ ,\ \rangle$ restricted to the p primary summand of $H^1(M_\infty; V)$ for the self-conjugate irreducible polynomials p. Hence $\sigma(M;\rho,\varepsilon)$ equals the sum of $\sigma_p(M;\rho,\varepsilon)$ over those irreducible p dividing $\Delta_1(M,V_\rho,\varepsilon)$.

So for example, if ρ is a $U(n,\mathbf{C})$ representation, The irreducible polynomials over $\mathbf{C}$ are linear, and the self-conjugate ones are precisely those $t-a$ with a on the unit circle.

The usual argument shows that if M bounds a 4-manifold W over which ρ and ε extend and $H_*(W_\infty; F)$ and $H_*(W_\infty; V)$ are finite-dimensional over F then the signatures $\sigma_p(M;\rho,\varepsilon)$ vanish for $p\neq t-1$ or $t+1$. (If $p = t-1$ or $t+1$ then the restriction of $\langle\ ,\ \rangle$ to the p-primary summand is degenerate and so one cannot conclude its signature is zero if M bounds.)

An important case when these constructions can be carried out is the context described in Section 6. Thus M is the $m = p^r$-fold branched cover of 0-surgery on a knot K and $\chi : \pi_1 M \to \mathbf{Z}/d$ is a non-trivial homomorphism. Rather than mapping to $\mathbf{Q}(\zeta_d)^\times$, it suffices to consider the composite with the inclusion $\mathbf{Q}(\zeta_d)^\times \subset \mathbf{C}^\times$. Hence one obtains a representation $\rho : \pi_1 M \to \mathbf{C}^\times$ and twisted homology $H_*(M; \mathbf{C}[\mathbf{Z}]_\rho)$. Since $H_*(M; \mathbf{C}[\mathbf{Z}]_\rho) = H_*(M; \mathbf{Q}(\zeta)[\mathbf{Z}]_\rho)\otimes_{\mathbf{Q}(\zeta)}\mathbf{C}$, it follows from Lemma 4 of [2] (as in Section 6) that $H_*(M; \mathbf{C}[\mathbf{Z}]_\rho) = H_*(M_\infty; \mathbf{C}_\rho)$ is finite dimensional over $\mathbf{C}$. That $H_*(M_\infty; \mathbf{C})$ (untwisted coefficients) is finite dimensional over $\mathbf{C}$ is well-known.

Thus the hypotheses of Theorem 7.1 apply in this context and so one can define the twisted signatures $\sigma(M;\rho,\varepsilon)$ and $\sigma_{t-a}(M;\rho,\varepsilon)$ for a a unit complex number different from ± 1. These signatures will vanish if K is a slice knot.

In [2], Casson–Gordon define signature invariants $\sigma_a(CG(K,\chi))$ for each unit complex number a. The σ_a are homomorphisms $W(\mathbf{Q}(\zeta)(t))\otimes\mathbf{Q} \to \mathbf{Q}$ taking a matrix $A(t)$ with coefficients in $F(t)$ representing an element in $W(\mathbf{Q}(\zeta)(t))$ to the signature of $A(a)$ if $A(a)$ has finite entries, and to the average of the two one-sided limits $\lim_{\beta \to a^{\pm}} \operatorname{sign}(A(\beta))$ if $A(a)$ is not finite.

The relationship between the twisted Milnor signatures $\sigma_{t-a}(M; \rho, \varepsilon)$ and the Casson–Gordon signatures $\sigma_a(CG(K, \chi))$ is expressed by the main theorem of [18].

To conclude, we note that this duality offers alternative proofs of the symmetry of the twisted polynomial and the slice condition on torsion, Corollaries 5.2 and 5.3.

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