

#### PII: S0040-9383(98)00040-8

# TWISTED KNOT POLYNOMIALS: INVERSION, MUTATION AND CONCORDANCE

# PAUL KIRK and CHARLES LIVINGSTON

(Received 1 June 1996; in revised form 16 May 1998)

The twisted Alexander polynomial of a knot is applied in three areas of knot theory: invertibility of knots, mutation, and concordance. Three examples are used to illustrate the utility of this invariant. First, a simple proof that the knot  $8_{17}$  is non-invertible is given. It is then proved that  $8_{17}$  is not even concordant to its inverse. Finally, the twisted polynomial is shown to distinguish the concordance class of the pretzel knot P(-3,5,7,2) from that of its positive mutant, P(5, -3,7,2). This last example completes the solution to problem 1.53 of Kirby (1984, 1997) asking for a relation between mutation and concordance. © 1999 Elsevier Science Ltd. All rights reserved.

The twisted Alexander polynomial of a knot was first studied by Lin [15], and has since been generalized and studied by a number of authors, including [5, 13, 24]. In our own work [12], summarized below, we show that the polynomial is related to the twisted homology of the infinite cyclic cover of the knot. We also prove that the twisted polynomial provides a slicing obstruction related to Casson–Gordon invariants. In this paper we combine these previous results with several new observations to provide three applications of the twisted polynomial.

(1) Inversion: Distinguishing an oriented knot K in  $S^3$  from its inverse, that is K with its orientation reversed, is among the most subtle problems in classical knot theory. It was not until 1963 that Trotter [23] constructed an example of a non-invertible knot and it awaited the late 1970s before procedures appeared that could address the general problem effectively. (That the knot  $8_{17}$  is non-invertible was first proved by Kawauchi [6] and independently by Bonahon–Siebenmann in 1979 using geometric methods. It is the least crossing non-invertible knot, the only one with 8 crossings.) The problem has received renewed attention in recent years with the question of whether or not finite-type Vassiliev invariants or quantum invariants are ever sufficient to distinguishes a knot from its inverse, using the knot  $8_{17}$  as an example.

(2) Concordance to inverses: We next demonstrate that the twisted Alexander polynomial offers an obstruction to a far more delicate problem than invertibility; it can be used to prove that a knot is not even concordant to its inverse. The only previous results in this realm were achieved by the second author in [17] and generalized by Naik in [19]. Using the twisted polynomial we now have a simple tool to prove that a knot is not concordant to its inverse. In this light the example of  $8_{17}$  is particularly interesting and we use it as our primary example.

(3) *Mutation and concordance*. The operation of mutation is known to leave most invariants of a knot unchanged; references include [7, 11, 14, 22]. Hence, the question was asked [9, Problem 1.53] as to whether a knot and its mutant are always concordant.

However, since a knot and its inverse are mutants, the answer is no by the concordance results mentioned above. However, in the update to this question [10] it is asked whether a knot and its *positive mutant*, defined below, are always concordant. As a final example, we use the twisted polynomial to disprove this conjecture.

Together these examples provide deeper evidence for the rigidity that occurs in classical knot concordance. For instance, in higher dimensions any general construction on knots (such as reversing orientation) that preserves all abelian invariants does not change the concordance class, and it follows that there is a corresponding geometric procedure to build the concordance. In dimension three, however, these constructions do change the concordance class, and hence these geometric procedures cannot work in dimension four.

This work also serves to demonstrate the sensitivity of Casson–Gordon invariants in distinguishing closely related knots. For instance, distinguishing a knot from a positive mutant is among the more difficult problems in classical knot theory; distinguishing the knots up to concordance is, obviously, a more difficult problem. One merit of our approach is that it is algorithmic, easily applied (with the help of computer) to a wide range of examples.

This paper is organized as follows. In the first section we give a general summary of the twisted Alexander polynomial. A particular class of twisted polynomials arising from meta-cyclic representations of the knot group is also presented, as is its use in distinguishing  $8_{17}$  from its inverse. Section 2 summarizes the relationship between a class of twisted Alexander polynomials, Casson–Gordon invariants, and slicing obstructions. These results are used to prove that  $8_{17}$  is not concordant to its inverse. Here the calculations of Section 1 are essential. In the third section we prove that the pretzel knot P(-3,5,7,2) and its positive mutant, P(5, -3,7,2), are not concordant.

#### 1. TWISTED ALEXANDER POLYNOMIALS

Suppose that X is a finite complex and that we are given two homomorphisms,  $\varepsilon:\pi_1(X) \to Z$  and  $\rho:\pi_1(X) \to \operatorname{GL}(V)$ , where V is a finite-dimensional vector space over a field F. The homomorphism  $\varepsilon$  determines an infinite cyclic cover Y of X, and we consider the homology group  $H = H_1(Y, \{V\})$ , where the coefficients are twisted by the representation  $\rho$  restricted to  $\pi_1(Y)$ . The group of deck transformations act on H, and hence H is a finitely generated module over the PID,  $\Lambda = F[t, t^{-1}]$ . As a module  $H = \Lambda^n \bigoplus_i \Lambda / \langle p_i(t) \rangle$ , where the  $\{p_i(t)\}$  is a finite set of non-zero polynomials. The product of the  $p_i$  is denoted  $\Delta_{X,\varepsilon,\rho}(t)$ , and is called the twisted Alexander polynomial. It is well defined up to multiplication by  $at^n$  where  $a \in F$  and n is an integer. References for the twisted polynomial include [5, 12, 13, 15, 24], the relationship between the various descriptions is given in [12]. An algorithm for computing it, based on a presentation for the group and using the Fox Calculus, appears in [24].

# 1.1. Cyclic representations on cyclic covers of a knot

A particularly interesting class of examples arises in the following way. Let  $X_n$  be the *n*-fold cyclic cover of a knot complement,  $S^3 - K$ , where K is an oriented knot, and let  $\overline{X}_n$  be the associated branched cover. Suppose that we have a homomorphism  $\overline{\chi}: H_1(\overline{X}_n) \to Z_d$ . Via the inclusion this defines a representation  $\chi: H_1(X_n) \to Z_d$ . There is a one-dimensional representation of  $Z_d$  on  $Q(\xi_d)$  given by multiplication by  $\xi_d$ , where  $\xi_d$  is the primitive *d*-root

of unity  $e^{2\pi i/d}$ , and this composed with  $\chi$  defines a one dimensional representation  $\rho: \pi_1(X_n) \to \operatorname{GL}_1(Q(\xi_d)) = Q(\xi_d) - \{0\}$ . There is also a natural surjective representation  $\varepsilon: H_1(X_n) \to Z$  induced by the covering map,  $H_1(X_n) \to H_1(S^3 - K) = Z$ . (The projection is onto nZ, which is isomorphic to Z; note that the orientation of K determines the identification of  $H_1(S^3 - K)$  with Z.) If  $\overline{\chi}$  is multiplied by an invertible element  $\alpha$  in  $Z_d$ , the effect on the twisted polynomial is to apply the Galois automorphism  $\sigma_{\alpha}$  to its coefficients, which lie in  $Q(\xi_d)$ , where  $\sigma_{\alpha}(\xi_d) = \xi_d^{\alpha}$ . The associated twisted polynomial,  $\Delta_{X_n,\varepsilon,\rho}(t)$ , will be denoted simply  $\Delta_{K,\varepsilon,\rho}(t)$ .

One result that we need in this setting concerns connected sums. If the knot K is a connected sum then the spaces and characters described above all split in a natural way. The infinite cyclic cover used to define the twisted polynomial splits along a contractible space, the infinite cyclic cover of an annulus. Hence, a simple Mayer–Vietoris argument computes the polynomial in terms of those of the summands. The only slight complication arises because for non-trivial representations the  $H_0$  terms in the Mayer–Vietoris sequence may be zero. Noting this, one quickly attains the following result in the case that  $K = K_1 \# K_2$  with  $\rho$  restricting to  $\rho_i$  on the factors:

$$\Delta_{K,\varepsilon,\rho}(t) = \Delta_{K_1,\varepsilon,\rho_1}(t) \ \Delta_{K_2,\varepsilon,\rho_2}(t) \text{ or } \Delta_{K_1,\varepsilon,\rho_1}(t) \ \Delta_{K_2,\varepsilon,\rho_2}(t) \ (1-t)$$

with the second case occurring if and only if  $\rho_1$  and  $\rho_2$  are non-trivial.

*Example*:  $8_{17}$  is not invertible.

The knot  $J = 8_{17}$ , illustrated in Fig. 1, offers a simple example of the representations described above. Begin by fixing an orientation on J. Using standard methods (see for instance [20]) one can compute that  $H_1(\bar{X}_3) = Z_{13} \oplus Z_{13}$ . The group of deck transformations is a cyclic group of order 3 and acts on  $H_1(\bar{X}_3)$  via a map T. (Note: the orientation of J determines T; if the orientation of J is reversed, T is inverted.) As such,  $Z_{13} \oplus Z_{13}$  splits into eigenspaces of T,  $V_3 \oplus V_9$ . (In  $Z_{13}$  the elements 3 and 9 are the primitive third roots of unity.) There is a non-trivial character  $\bar{\chi}_3: H_1(\bar{X}_3) \to Z_{13}$  vanishing on  $V_3$ , and any such character is a multiple of  $\bar{\chi}_3$  by a non-zero element in  $Z_{13}$ . Similarly, let  $\bar{\chi}_9: H_1(\bar{X}_3) \to Z_{13}$  be a fixed non-trivial character on  $H_1(\bar{X}_3)$  vanishing on  $V_9$ . Using the algorithm of Wada [24] the associated twisted polynomial can be computed; for particular choices of  $\bar{\chi}_3$  and  $\bar{\chi}_9$ the results are listed as  $\Delta_{8_{17},\epsilon,\rho_3}(t)$  and  $\Delta_{8_{17},\epsilon,\rho_9}(t)$  in the table concluding the paper. (The fundamental group of  $X_3$  must be computed to apply Wada's algorithm; this can be done



Fig. 1.

using the Reidemeister–Schreier process [18].) The table also gives  $\Delta_{8_{17},\epsilon,\rho_0}(t)$ , where  $\rho_0$  is the trivial character.

The calculation of the previous paragraph can be repeated for the knot  $J^*$ , the knot  $8_{17}$  with its orientation reversed. The cover,  $X_3^*$ , is the identical space as  $X_3$  but the change in the orientation of J leads to the following changes in  $\varepsilon$  and  $\rho$ . Most simply,  $\varepsilon$  is replaced with  $-\varepsilon$ . Also, T is replaced with  $T^{-1}$  leading to a switch of eigenspaces. Hence,  $\bar{\chi}_3$  for  $J^*$  corresponds  $\alpha \bar{\chi}_9$  for J, where  $\alpha$  is some non-zero element of  $Z_{13}$ . It follows that  $\Delta_{8_{17}^*,\varepsilon,\rho_3}(t) = \sigma_{\alpha}(\Delta_{8_{17},\varepsilon,\rho_9}(t^{-1}))$ . (Similarly, for the mirror image of the knot  $8_{17},m(8_{17})$ , one has  $\Delta_{m(8_{17}),\varepsilon,\rho_3}(t) = \sigma_{\alpha}(\Delta_{8_{17},\varepsilon,\rho_9}(t^{-1}))$ ; the following argument then also shows that  $8_{17}$  is not positive amphicheiral.)

In order to understand the effect of applying a Galois automorphism to the coefficients of the polynomials given in the table, note that  $\{(\xi_{13})^i\}_{i=1,\dots,12}$  forms a basis of  $Q(\xi_{13})$  over Q and that the action of  $\sigma_{\alpha}$  is simply to permute these basis elements. In the table the coefficients are expressed in terms of this basis and it now becomes clear that  $\Delta_{8_{17},\epsilon,\rho_3}(t)$  and  $\sigma_{\alpha}(\Delta_{8_{17},\epsilon,\rho_3}(t^{-1}))$  are unequal for all  $\alpha$ .

#### 1.2. Comparison with Hartley's approach

The proof by Hartley [4] that  $8_{17}$  is not invertible can be described as follows. From the representations  $\chi_3$  and  $\chi_9$  one can construct representations of  $\pi_1(S^3 - J)$  onto a metacylic group  $Z_{13} > Z_3$  ( $Z_{13}$  is the normal subgroup in this semidirect product). To each such representation there is an irregular 13-fold covering space. Hartley uses the homology of the covers to distinguish J from J\*. These spaces that Hartley considers have 3-fold covers that are 13-fold covers of our spaces  $X_3$ . Our twisted polynomial, evaluated at t = 1, yields homology information about that cover.

In the next section we will expand on relations to Casson–Gordon invariants. We note here that in [1] the point is made that to obtain obstructions to slicing it is not sufficient to use just a 3-fold cover; in [2] the behaviour of the  $3^d$ -fold covers as d goes to infinity is explored. Alternatively, one can examine the homology of the infinite cyclic cover; this is the approach of [2] and most other papers on Casson–Gordon invariants. The change from the infinite cyclic cover to the 3-fold cover is represented by setting t = 1 in the previous paragraph.

#### 2. CONCORDANCE TO INVERSES

In [12] it is shown that the twisted polynomial provides an obstruction to a knot being slice. Here is the statement of the theorem. The representations and notation are those of the previous section; that is, our representation acts on  $Q(\xi_d)$  and is induced by a surjective homomorphism  $\bar{\chi}: H_1(\bar{X}_n) \to Z_d$ .

THEOREM. If K is an oriented slice knot in  $S^3$ , p, q odd primes, so  $n = p^r$  and  $d = q^s$ , are odd, then there is a subgroup M of  $H_1(\bar{X}_n)$  satisfying  $order(M)^2 = order(H_1(\bar{X}_n))$  and so that for all  $Z_d$ -valued characters  $\bar{\chi}$  vanishing on M, the associated twisted polynomial of K factors as  $af(t) \overline{f(t^{-1})}(1-t)^s$ . Here  $a \in Q(\xi_d)$  and s = 1 if  $\bar{\chi}$  is non-zero and s = 0 if  $\bar{\chi}$  is trivial.

In fact, M is the kernel of the map on first homology induced by the inclusion of  $\overline{X}_n$  into the branched cover of the 4-ball over the slice disc, and in particular, is invariant under the automorphisms induced by the deck transformations.

Theorem 1 is proved in [12] where its connection to Casson–Gordon invariants is described, as we now summarize. Given a knot K and a character  $\bar{\chi}: H_1(\bar{X}_n) \to Z_d$ , as in the previous section, Casson and Gordon defined an invariant  $\tau(K, \bar{\chi})$  taking values in the Witt group  $W(Q(\xi_d)(t)) \otimes Z_{(2)}$ . Under the hypothesis of Theorem 1 they prove that  $\tau(K, \bar{\chi})$ vanishes for the appropriate characters. Under suitable restrictions (d odd) there is a discriminant invariant defined on  $W(Q(\xi_d)(t)) \otimes Z_{(2)}$  taking values in  $(Q(\xi_d)(t) - \{0\})/N$ , where N is generated by products of the form  $af(t) \overline{f(t^{-1})}$ . (See [3, 16] for further references on the discriminant of Casson–Gordon invariants.) The twisted polynomial represents this invariant. The proof of Theorem 1 in [12] is independent of this connection with Casson–Gordon invariants.

## 2.1. 8<sub>17</sub> is not concordant to its inverse

We now prove that  $8_{17}$  is not concordant to its inverse  $8_{17*}$ . We again abbreviate  $8_{17}$  by *J*. Structurally, the proof is much like that used in [17]. However, the detailed analysis of the Seifert pairing of a genus one knot in [17] would be difficult to repeat for the genus 3 knot  $8_{17}$ . It also appears difficult to compute Casson–Gordon signature invariants in the present case. This example illustrates the simplicity of the twisted polynomial approach.

In general, if K is concordant to its inverse, the knot  $K # -K^*$  is slice. (In this notation -K represents the inverse to K in the knot concordance group. Alternatively, -K is the mirror image of K with orientation reversed and so  $-K^* = m(K)$ .) We prove that for  $J = 8_{17}$  the above theorem applies to show that  $J # -J^*$  is not slice.

For now denote the branched covers of  $S^3$  over J and  $-J^*$  by  $\overline{X}_3$  and  $\overline{Y}_3$ . The work of the previous section shows that  $G = H_1(\overline{X}_3 \# \overline{Y}_3) = (Z_{13})^2 \oplus (Z_{13})^2$ . With respect to the deck transformation T, G splits into a 3-eigenspace,  $\langle v_3, w_3 \rangle$  (that is, the span of  $\{v_3, w_3\}$ ) and a 9-eigenspace,  $\langle v_9, w_9 \rangle$ , with the  $v_i$  coming from the first summand, the  $w_i$  from the second.

A simple exercise shows that any T invariant summand of  $(Z_{13})^2 \oplus (Z_{13})^2$  will be spanned by eigenvectors. (In general, if a vector space is spanned by eigenvectors of a linear transformation T, then so is any invariant subspace—the minimal polynomial for T has distinct linear factors.) Hence, for the subgroup  $G_0$  in Corollary 1 there are three possibilities.

(1)  $G_0 = \langle v_3, w_3 \rangle$ , (2)  $G_0 = \langle v_9, w_9 \rangle$ , (3)  $G_0 = \langle \alpha v_3 + \beta w_3, \gamma v_9 + \delta w_9 \rangle$ .

In case (3) either  $\alpha$  or  $\beta$  is nonzero and either  $\gamma$  or  $\delta$  is nonzero in  $Z_{13}$ .

In each case we must determine the set of  $Z_{13}$ -valued characters on G vanishing on  $G_0$ . Letting the ordered set of characters  $\{\bar{\chi}_3, \bar{\chi}_9, \bar{\chi}'_3, \bar{\chi}'_9\}$  be the  $Z_{13}$ -dual basis to the ordered basis  $\{v_9, v_3, w_9, w_3\}$  of the  $Z_{13}$ -vector space G, these are easily seen to be the following:

(1)  $\tilde{G}_0 = \langle \bar{\chi}_3, \bar{\chi}'_3 \rangle$ , (2)  $\tilde{G}_0 = \langle \bar{\chi}_9, \bar{\chi}'_9 \rangle$ , (3)  $\tilde{G}_0 = \langle \beta \bar{\chi}_9 - \alpha \bar{\chi}'_9, \delta \bar{\chi}_3 - \gamma \bar{\chi}'_3 \rangle$ .

Cases (1) and (2) are most easily handled. In case (1) we consider the character  $\bar{\chi}_3$ . The associated twisted polynomial for this character is

$$\Delta_{8_{17},\epsilon,\rho_3}(t)\Delta_{-8_{17}^*,\epsilon,\rho_0}(t) = \Delta_{8_{17},\epsilon,\rho_3}(t)\Delta_{8_{17},\epsilon,\rho_0}(t).$$

(Here, we have used our product formula for connected sums.) From the table, which gives the irreducible factorizations of these polynomials we see that this product is not of the form  $af(t)\overline{f(t^{-1})}(1-t)$ , and hence case (1) is not possible. For case (2) the argument is the same, with  $\rho_9$  replacing  $\rho_3$ .

Case (3) is the most interesting. If  $\delta$  or  $\gamma$  is 0 (in  $Z_{13}$ ) we can proceed as in case (1). Otherwise, we find that the twisted polynomial associated to  $\delta\chi_3 - \gamma\tilde{\chi}'_3$  is the product  $(1-t)\sigma_\delta(\Delta_{8_{17},\epsilon,\rho_3}(t))\sigma_{-\gamma}(\Delta_{-8_{17}^*,\epsilon,\rho_3}(t))$ , where, for instance,  $\sigma_d$  is the Galois automorphism taking  $\xi_d$  to  $\xi_d^{\delta}$ . As in the previous section, where mirror images were discussed, this is equal to  $(1-t)\sigma_\delta(\Delta_{8_{17},\epsilon,\rho_3}(t))\sigma_{-\gamma}(\Delta_{8_{17},\epsilon,\rho_9}(t))$ . Again, using the factorizations given in the table along with our earlier observation concerning the action of the Galois group on polynomials, it is clear that this polynomial is not of the form  $af(t)\overline{f(t^{-1})}(1-t)$ . This concludes the proof.

## 3. MUTATION AND CONCORDANCE

The mutant of a knot is formed by the following procedure. A ball intersecting the knot in two arcs is removed from  $S^3$  and replaced with a 180° rotation that freely permutes the four boundary points of the arcs. If K is oriented, then the mutant is naturally oriented so that the orientation of that part of the knot outside the ball is unchanged. It is well known that many knot invariants, including the classical Alexander polynomial, the Jones polynomial, SU(2)-quantum invariants, and hyperbolic invariants remain unchanged under mutation. (References include [14, 7, 11, 21, 22].)

In Kirby's revised problem list of 1984 [9] it was asked whether a knot is concordant to each of its mutants. However, since a knot and its inverse are easily seen to be mutants, the examples of [17] show that this is not necessarily the case. (The applicability of [17] was also observed in [4].) For a fixed ball intersecting a knot in two arcs, there is only one mutation that preserves the orientation of the arcs in the ball. In the updated problem list [10] it is asked by Kearton if a knot and this *positive* mutant are necessarily concordant. Here, we provide a counterexample.

The pretzel knot  $J_1 = P(-5, 3, 7, 2)$  is illustrated in Fig. 2. The numbers in the boxes represent half twists. A positive mutation converts this to the pretzel knot  $J_2 = P(3, -5, 7, 2)$ ; rotate that portion of the knot within the dotted oval by 180° about the center point



Fig. 2.

of the oval. To see that these are not concordant, we show that  $J_1 \# - J_2$  is not slice. The argument is much the same as in the previous section; we point out only where the details differ.

The 3-fold branched cover of  $S^3$  branched over either  $J_1$  or  $J_2$  has first homology  $Z_7 \oplus Z_7$ . Under the action of the group of deck transformations, the homology splits as the direct sum  $V_2 \oplus V_4$ , where  $V_2$  and  $V_4$  are, respectively, the two and four eigenspaces of the action. (The primitive cube roots of unity in  $Z_7$  are 2 and 4.) Hence, we can compute the twisted polynomials associated to the representations that vanish on each of these eigenspaces. (Since each knot is invertible, in this case it does not matter which eigenspace is used, the polynomials are the same.) These polynomials are given in the table.

The argument of the previous section now applies to show that if  $J_1 \# - J_2$  were slice, either a product of the two twisted polynomials, or the product of one of the twisted polynomials and the untwisted polynomial would be a norm. This is easily seen not to be the case, since the polynomials in the table are all irreducible in  $Q(\zeta)[t]$ . (Again there is the technical condition that one must in fact consider Galois conjugates of these polynomials, but they are written so that the action of the Galois group is obvious.)

### 4. TABLE OF POLYNOMIALS

The twisted (and untwisted) polynomials for the knots  $8_{17}$ , P(-3, 5, 7, 2), and P(5, -3, 7, 2) are given below. The polynomials as written (where we, as indicated divided out by (1 - t) when possible) are irreducible in  $Q(\xi)[t]$ , where  $\xi = \xi_{13}$  for  $8_{17}$  and  $\xi = \xi_7$  for the pretzel knots, as described in the text. The program Maple was used in calculating the polynomials and their factorizations.

$$\begin{split} \Delta_{8_{17},\epsilon,\rho_0}(t) &= 1 - t - 34t^2 - 34t^2 - 101t^3 - 34t^4 - t^5 + t^6. \\ \Delta_{8_{17},\epsilon,\rho_3}(t)/(1-t) \\ &= 1 \\ &+ t(2\xi + 2\xi^2 + 2\xi^3 + 4\xi^4 + 2\xi^5 + 2\xi^6 + \xi^7 + \xi^8 + 2\xi^9 + 4\xi^{10} + \xi^{11} + 4\xi^{12}) \\ &+ t^2(-15\xi - 10\xi^2 - 15\xi^3 - 15\xi^4 - 10\xi^5 - 10\xi^6 - 10\xi^7 - 10\xi^8 - 15\xi^9 - 15\xi^{10} \\ &- 10\xi^{11} - 15\xi^{12}) \\ &+ t^3(4\xi + \xi^2 + 4\xi^3 + 2\xi^4 + \xi^5 + \xi^6 + 2\xi^7 + 2\xi^8 + 4\xi^9 + 2\xi^{10} + 2\xi^{11} + 2\xi^{12}) \\ &+ t^4 \\ \Delta_{8_{17},\epsilon,\rho_3}(t)/(1-t) \\ &= 1 \end{split}$$

$$\begin{aligned} &+ t(6\xi + 5\xi^2 + 6\xi^3 + 6\xi^4 + 5\xi^5 + 5\xi^6 + 5\xi^7 + 5\xi^8 + 6\xi^9 + 6\xi^{10} + 5\xi^{11} + 6\xi^{12}) \\ &+ t^2(-13\xi - 12\xi^2 - 13\xi^3 - 13\xi^4 - 12\xi^5 - 12\xi^6 - 12\xi^7 - 12\xi^8 - 13\xi^9 - 13\xi^{10} \\ &- 12\xi^{11} - 13\xi^{12}) \\ &+ t^3(6\xi + 5\xi^2 + 6\xi^3 + 6\xi^4 + 5\xi^5 + 5\xi^6 + 5\xi^7 + 5\xi^8 + 6\xi^9 + 6\xi^{10} + 5\xi^{11} + 6\xi^{12}) \\ &+ t^4 \end{aligned}$$

$$\begin{split} & \Delta p_{(-5,3,7,2),\kappa,p_s}(l) \\ &= 1 - 3t + 6t^2 - 7t^3 + 24t^4 - 18t^5 + 22t^6 - t^7 + 22t^8 - 18t^9 + 24t^{10} - 7t^{11} \\ &+ 6t^{12} - 3t^{13} + t^{14}. \end{split}$$

$$\begin{aligned} & \Delta p_{(-5,3,7,2),\kappa,p_s}(l)/(1-t) \\ &= 1 \\ &+ t(2\xi^3 + 2\xi^5 + 2\xi^6) \\ &+ t^2(-5\xi^5 - 5\xi^2 - \xi^3 - 5\xi^4 - \xi^5 - \xi^6) \\ &+ t^3(3\xi + 3\xi^2 + 4\xi^3 + 3\xi^4 + 4\xi^5 + 4\xi^6) \\ &+ t^4(4\xi + 4\xi^2 + 3\xi^3 + 4\xi^4 + 3\xi^5 + 3\xi^6) \\ &+ t^6(-7\xi - 7\xi^2 - 7\xi^3 - 7\xi^4 - 7\xi^5 - 7\xi^6) \\ &+ t^6(-7\xi - 7\xi^2 - 7\xi^3 - 7\xi^4 - 7\xi^5 - 7\xi^6) \\ &+ t^8(3\xi + 3\xi^2 + 4\xi^3 + 3\xi^4 + 4\xi^5 + 4\xi^6) \\ &+ t^8(3\xi + 3\xi^2 + 4\xi^3 + 3\xi^4 + 4\xi^5 + 4\xi^6) \\ &+ t^8(3\xi + 3\xi^2 + 4\xi^3 + 3\xi^4 + 4\xi^5 + 3\xi^6) \\ &+ t^{10}(-\xi - \xi^2 - 5\xi^3 - \xi^4 - 5\xi^5 - 5\xi^6) \\ &+ t^{10}(-\xi - \xi^2 - 5\xi^3 - \xi^4 - 5\xi^5 - 5\xi^6) \\ &+ t^{11}(2\xi + 2\xi^2 + 2\xi^4) \\ &+ t^{12} \end{aligned}$$

$$\begin{aligned} & \Delta p_{(3, -5, 7, 2), \kappa, p_s}(l)/(1-t) \\ &= 1 \\ &+ t(\xi + \xi^2 - \xi^3 + \xi^4 + \xi^5 + \xi^6) \\ &+ t^6(\xi + \xi^2 - \xi^3 + \xi^4 - \xi^5 - \xi^6) \\ &+ t^6(\xi + \xi^2 - \xi^3 + \xi^4 - \xi^5 - \xi^6) \\ &+ t^6(\xi + \xi^2 - \xi^3 + \xi^4 - \xi^5 - \xi^6) \\ &+ t^6(12\xi + 12\xi^2 + 10\xi^3 + 12\xi^4 + 10\xi^5 + 10\xi^6) \\ &+ t^6(10\xi + 10\xi^2 + 12\xi^3 + 10\xi^4 + 12\xi^5 + 12\xi^6) \\ &+ t^8(1-\xi - \xi^2 + \xi^3 - \xi^4 + \xi^5 + \xi^6) \\ &+ t^9(3\xi + 3\xi^2 - 4\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^9(13\xi + 3\xi^2 + 4\xi^3 + 3\xi^4 + 4\xi^5 + 4\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi^5 - 3\xi^6) \\ &+ t^{10}(-3\xi - 3\xi^2 - 3\xi^3 - 3\xi^4 - 3\xi$$

#### REFERENCES

- Casson, A. and Gordon, C. McA., On slice knots in dimension three. *Proceedings of the Symposium in Pure Mathematics*, 1978, 30, 39–53.
- 2. Casson, A. and Gordon, C. McA., Cobordism of classical knots. Preprint, Orsay. (Reprinted in *A la Recherche de la Topologie Perdue*, ed. A. Marin and L. Guillou, Progress in Mathematics 62, Birkhauser, 1986.)
- 3. Gilmer, P. and Livingston, C., Discriminants of Casson–Gordon invariants. *Mathematical Proceedings of the Cambridge Philosophical Society*, 1992, **112**, 127–139.
- 4. Hartley, R., Identifying non-invertible knots. Topology, 1983, 22, 137-145.
- 5. Jiang, B. and Wang, S., Twisted topological invariants associated with representations. *Topics in Knot Theory*. Kluwer Academic Publishers, Netherlands, 1993, pp. 211–227.
- 6. Kawauchi, A., The invertibility problem for amphicheiral excellent knots. *Proceedings of the Japanese Academy Series A*, (10) 1979, **55**, 399–402.
- 7. Kearton, C., Mutation of knots, Proceedings of the American Mathematical Society, 1989, 105, 206–208.
- Kirby, R., Problems in low dimensional manifold theory. Algebraic and Geometric Topology. Proceedings of the Symposium Pure Mathematics, 1978, 32, 273–312.
- 9. Kirby, R., 4-Manifold problems. *Four-Manifold Theory*, Contemporary Mathematics vol. 35, eds. C. Gordon and R. Kirby. AMS, Providence, RI, 1984.
- Kirby, R., Problems in low dimensional topology. Proceeding of University of Georgia Mathematics Conference, AMS/IP Studies in Advanced Mathematics, ed. Willian H. Kazez, 1997.
- Kirk, P., Mutations of homology spheres and Casson's invariant. Mathematical Proceedings of the Cambridge Philosophical Society, 1989, 105, 313–318.
- 12. Kirk, P. and Livingston, C., Twisted Alexander invariants, Reidemeister Torsion, and Casson-Gordon invariants, *Topology*, **38**, 635-661.
- 13. Kitano, T., Twisted Alexander polynomial and Reidemeister torsion. Pacific Journal of Mathematics, to appear.
- Lickorish, W. B. R., Distinct 3-manifolds with all SU(2)<sub>q</sub> invariants the same. Proceedings of the American Mathematical Society, 1993, 117, 285–292.
- 15. Lin, X.-S., Representations of knot groups and twisted Alexander polynomials, Preprint, 1990.
- 16. Litherland, R., A formula for the Casson-Gordon invariants of a knot. Preprint.
- 17. Livingston, C., Knots which are not concordant to their reverses. *Quarterly Journal of Mathematics Oxford*, 1983, **34**(2), 323–328.
- 18. Magnus, M., Karass, A. and Solitar, D., Combinatorial Group Theory. Interscience Publishers, New York, 1966.
- 19. Naik, S., Casson–Gordon invariants and concordance to reverses, *Journal of Knot Theory and its Ramifications*, to appear.
- 20. Rolfsen, D., Knots and Links. Publish or Perish, Inc., Berkeley, 1976.
- 21. Rong, Y. W., Mutation and witten invariants. Topology, 1994, 33, 499-507.
- 22. Ruberman, D., Mutation and volumes of knots in S<sup>3</sup>, Inventory Mathematics, 1987, 90, 189–215.
- 23. Trotter, H., Non-invertible knots exist. Topology, 1963, 2, 275-280.
- 24. Wada, M., Twisted Alexander polynomial for finitely presentable groups. Topology, 1994, 33, 241-256.

Department of Mathematics Indiana University Bloomington, IN 47405 U.S.A.