Microbundles are Fibre Bundles*

By J. M. KISTER

Let $\mathcal{G}(n)$ be the space of all imbeddings of euclidean n-space E^n into itself provided with the compact-open topology. Let $\mathcal{H}(n)$ be the subspace of all onto homeomorphisms. Those elements in $\mathcal{F}(n)$ and $\mathcal{H}(n)$ which preserve the origin 0 will be denoted by $\mathcal{G}_0(n)$ and $\mathcal{H}_0(n)$ respectively. Briefly, the main result (Theorem 2)[1] of this paper is that every microbundle over a complex contains a fibre bundle (in the sense of [5], where fibre $= E^n$, group $= \mathcal{H}_0(n)$), and the fibre bundle is unique. This implies that every such microbundle is mb-isomorphic to a fibre bundle, and any two such fibre bundles are fb-isomorphic. The same result extends to microbundles over neighborhood retracts in E^n. In the special case of a topological manifold M and its tangent microbundle, a neighborhood U_x is selected for each point x in M so that U_x is an open cell and varies continuously with x.

The proof of Theorem 2 depends on extending homeomorphisms, and requires an examination of the non-closed subset $\mathcal{H}_0(n)$ in $\mathcal{G}_0(n)$. We show that $\mathcal{H}_0(n)$ is a weak kind of deformation retract of $\mathcal{G}_0(n)$. More precisely:

THEOREM 1. *There is a map* $F: \mathcal{G}_0(n)\times I \to \mathcal{G}_0(n)$, *for each* n, *such that*

(1) $F(g, 0) = g$ *for all* g *in* $\mathcal{G}_0(n)$

(2) $F(g, 1)$ *is in* $\mathcal{H}_0(n)$ *for all* g *in* $\mathcal{G}_0(n)$.

(3) $F(h, t)$ *is in* $\mathcal{H}_0(n)$ *for all* h *in* $\mathcal{H}_0(n)$, t *in* I.

For definitions and basic results about microboundles cf. [4]. In [2] an introduction and outline of this paper will be found.

The author wishes to express his gratitude to D.R. McMillan for several helpful conversations.

Definitions

The disk of radius r with center at 0 in E^n is denoted by D_r and, if K is a compact set in E^n containing 0, we define the *radius* of K to be $\max\{r \mid D_r \subset K\}$. Let d be the usual metric in E^n. If $g_1, g_2: K \to E^n$ are imbeddings of the compact set K, then we say g_1 and g_2 *are within* ε if for each x in K it is true that $d(g_1(x), g_2(x)) < \varepsilon$. If g is in $\mathcal{G}_0(n)$ and K is a compact set in E^n, $V(g, K, \varepsilon)$ denotes all elements h in $\mathcal{G}_0(n)$ such that $g \mid K$ and $h \mid K$ are within ε. The collection of all such $V(g, K, \varepsilon)$ is, of course, a basis for $\mathcal{G}_0(n)$.

Two compact sets in E^n, K_1 and K_2, are *ε-homeomorphic* if there is a

* Supported by a grant from the Institute for Advanced Study and by NSF grant G-24156.
[1] B. Mazur has obtained this result also.

homeomorphism h: $K_1 \to K_2$ within ε of the identity $1 : K_1 \to K_1$.

If $0 \leqq a < b < d$ and $a < c < d$ and t is in I, then we define $\theta_t(a, b, c, d)$ to be the homeomorpism of E^n onto itself, fixed on D_a and outside D_d as follows. Let L be a ray emanating from the origin and coordinatized by distance from the origin. Then θ_t is fixed on $[0, a]$ and on $[d, \infty)$, and it takes b onto $(1-t)b + tc$ and is linear on $[a, b]$ and $[b, d]$. We denote $\theta_1(a, b, c, d)$ by $\theta(a, b, c, d)$, and $\theta(0, b, c, d)$ by $\theta(b, c, d)$. Clearly $(t, a, b, c, d) \to \theta_t(a, b, c, d)$ is continuous, regarded as a function from a subset of E^5 into $\mathcal{H}_0(n)$.

When the dimension is unambigous $\mathcal{G}$, $\mathcal{H}_0$ etc. will be used for $\mathcal{G}(n)$, $\mathcal{H}_0(n)$ etc.

A useful lemma

LEMMA. *Let g and h be in $\mathcal{G}_0(n)$ with $h(E^n) \subset g(E^n)$. Let a, b, c and d be real numbers satisfying $0 \leqq a < b$, $0 < c < d$, and such that $h(D_b) \subset g(D_c)$. Then there is an isotopy $\varphi_t(g, h; a, b, c, d) = \varphi_t(t \in I)$ of E^n onto itself satisfying*

(1) $\varphi_0 = 1$;

(2) $\varphi_1(h(D_b)) \supset g(D_c)$;

(3) *φ_t is fixed outside $g(D_d)$ and on $h(D_a)$.*

Furthermore $(g, h, a, b, c, d, t) \to \varphi_t$ is a continuous function from the appropriate subset of $\mathcal{G}_0 \times \mathcal{G}_0 \times E^5$ into $\mathcal{H}_0$.

PROOF. Let a' be the radius of $g^{-1}h(D_a)$; note that $a' < c$. Let b' be the radius of $g^{-1}h(D_b)$; note that $a' < b' \leqq c < d$.

We first shrink $h(D_a)$ inside $g(D_{a'})$ with a homeomorphism σ fixed outside $h(D_b)$. This can be done as follows. Let a'' be the radius of $h^{-1}g(D_{a'})$; note that $a'' \leqq a < b$. Define

$$\sigma = \begin{cases} h\theta(a, a'', b)h^{-1} & \text{on } h(D_b) \\ 1 & \text{elsewhere} . \end{cases}$$

Next we get an isotopy $\psi_t(t \in I)$ taking $g(D_{b'})$ onto $g(D_c)$, leaving $g(D_{a'})$, and the exterior of $g(D_d)$ fixed. Define

$$\psi_t = \begin{cases} g\theta_t(a', b', c, d)g^{-1} & \text{on g}(D_d) \\ 1 & \text{elsewhere} . \end{cases}$$

Finally define $\varphi_t = \sigma^{-1}\psi_t\sigma$. It is easy to verify that (1), (2) and (3) are satisfied. The continuity of φ_t depends on the following three propositions.

PROPOSITION 1. *Let g be in $\mathcal{G}_0$, and let r and ε be two positive numbers. Then there is a $\delta > 0$ so that, if g_1 is in $V(g, D_{r+\varepsilon}, \delta)$, then*

(1) *$g_1(D_{r+\varepsilon}) \supset g(D_r)$;*

(2) *$g_1^{-1}|g(D_r)$ and $g^{-1}|g(D_r)$ are within ε.*

PROOF. *Let*

$$\delta_1 = \min \{d(g(x), g(y)) \mid x \in D_r, y \notin \operatorname{int} D_{r+\varepsilon}\}$$

and

$$\delta_2 = \min \{d(g(x), g(y)) \mid x, y \in D_r, d(x, y) \geqq \varepsilon\} .$$

Let

$$\delta = \min \{\delta_1, \delta_2\} .$$

Suppose g_1 is in $V(g, D_{r+\varepsilon}, \delta)$. Then condition (1) is satisfied, for otherwise there is a z in Bd $g_1(D_{r+\varepsilon}) \cap g(D_r)$. Let $x = g^{-1}(z) \in D_r$, and $y = g_1^{-1}(z) \in \text{Bd } D_{r+\varepsilon}$. Then $\delta_1 \leqq d(g(x), g(y)) = d(g_1(y), g(y))$, contradicting the choice of g_1.

To see that condition (2) is satisfied, suppose not. Then there is a z in $g(D_r)$ such that, if $x = g^{-1}(z)$ and $y = g_1^{-1}(z)$, then $d(x, y) \geqq \varepsilon$ and x and y are in $D_{r+\varepsilon}$. It follows that $\delta_2 \leqq d(g(x), g(y)) = d(g_1(y), g(y))$, contradicting the choice of g_1.

PROPOSITION 2. *Let C be a compact set, $h : C \to E^n$ an imbedding, D a compact set in E^n containing $h(C)$ in its interior, and $g : D \to E^n$ another imbedding. For any $\varepsilon > 0$, there is a δ so that, if $g_1 : D \to E^n$, $h_1 : C \to E^n$ are imbeddings within δ of g and h respectively, then $g_1 h_1$ is defined and within ε of gh.*

PROOF. Since D contains $h(C)$ on its interior and $h(C)$ is compact, there is a $\delta_1 > 0$ such that the δ_1-nbd of $h(C)$ is contained in D. Let δ_2 be so small that $x, y \in D$ and $d(x, y) < \delta_2$ imply $d(g(x), g(y)) < \varepsilon/2$. Choose $\delta = \min(\delta_1, \delta_2, \varepsilon/2)$. Then if $g_1 : D \to E^n$ and $h_1 : C \to E^n$ are imbeddings within δ of g and h respectively, $g_1 h_1$ is defined, since $\delta \leqq \delta_1$. Let $z \in C$ and $x = h(z)$, $y = h_1(z)$. It follows that $d(x, y) < \delta \leqq \delta_2$, hence $d(g(x), g(y)) < \varepsilon/2$. Also $d(g(y), g_1(y)) < \delta \leqq \varepsilon/2$, hence $d(gh(z), g_1 h_1(z)) = d(g(x), g_1(y)) < \varepsilon$.

Remark. Proposition 2 shows that the semi-group $\mathcal{G}$ (or $\mathcal{G}_0$) whose multiplication consists of composition, is a topological semi-group, i.e., multiplication is continuous.

PROPOSITION 3. *Let g and h be in $\mathcal{G}_0$, and let a be a non-negative number such that $h(D_a) \subset g(E^n)$. Let $r =$ radius $g^{-1}h(D_a)$. Then $r = r(g, h, a)$ is continuous simultaneously in the variables g, h and a.*

PROOF. *Case* 1. $a > 0$. Let $T_{a_1} : E^n \to E^n$ be defined by $T_{a_1}(x) = (a_1/a)x$, for positive a_1. Clearly T_{a_1} varies continuously with a_1, hence Proposition 2 shows that, given any nbd N of h, there is a nbd M of h and a nbd P of a such that (h_1, a_1) in $M \times P$ implies that $h_1 T_{a_1}$ is in the nbd N of $h1 = h$. Using Propositions 1 and 2, we can conclude that, for any ε, there is a nbd L_1 of g, M_1 of h, and P_1 of a such that (g_1, h_1, a_1) in $L_1 \times M_1 \times P_1$ implies $g_1^{-1} h_1 T_{a_1} \mid D_a$ is defined and is within ε of $g^{-1}h \mid D_a$. This means $g^{-1}h(D_a)$ and $g_1^{-1}h_1(D_{a_1})$ are ε-homeomorphic, and it can easily be seen that $|r(g, h, a) - r(g_1, h_1, a_1)| < \varepsilon$.

Case 2. $a=0$. Then $r(g, h, a) = 0$ and, for any ε, there is a δ such that diameter $g^{-1}h(D_\delta) < \varepsilon$. As in *Case* 1, using Propositions 1 and 2, we can conclude that $g_1^{-1}h_1 \mid D_\delta$ varies continuously with g_1 and h_1; hence by restricting g_1 and h_1 to lie near g and h respectively, and for $a_1 \in [0, \delta]$, we have $r(g_1, h_1, a_1) < 2\varepsilon$. This finishes the proof of Proposition 3.

Going back to the proof of the Lemma we first show $\sigma = \sigma(g, h, a, b)$ is continuous. By applying Proposition 3 twice we see that a'' depends continuously on g, h, and a, hence $\theta(a, a'', b)$ depends continuously on g, h, a and b. Note that σ would be the same function if it were defined as $h\theta(a, a'', b)h^{-1}$ on the set $h(D_{b+2})$ and 1 elsewhere. Since $h(D_{b+1}) \subset \text{int } h(D_{b+2})$, there is a neighborhood N of h such that h_1 in N implies $h_1(D_{b+1}) \subset h(D_{b+2})$, hence if h_1 is in N, b_1 is in the interval $(0, b+1)$, and g_1 and a_1 satisfy the hypotheses of the Lemma, then $\sigma_1 = \sigma(g_1, h_1, a_1, b_1)$ can be defined as $h_1\theta(a_1, a_1'', b_1)h_1^{-1}$ on $h(D_{b+2})$ and 1 everywhere else, where $a_1'' = a''(a_1)$.

We may assume, using Proposition 1, that N has been chosen so that $h_1(D_{b+3}) \supset h(D_{b+2})$ for h_1 in N, hence $h_1^{-1} \mid h(D_{b+2})$ is defined. Proposition 1 also shows that $h_1^{-1} \mid h(D_{b+2})$ varies continuously with h_1. Using Proposition 2, we conclude that $\theta(a_1, a_1'', b_1)h_1^{-1} \mid h(D_{b+2})$ varies continuously with g_1, h_1, a_1 and b_1. Finally applying Proposition 2 again we see that $\sigma_1 \mid h(D_{b+2}) = h_1\theta(a_1, a_1'', b_1)h_1^{-1} \mid h(D_{b+2})$ varies continuously with g_1, h_1, a_1 and b_1, and hence $\sigma(g, h, a, b)$ is continuous.

The proof that $\psi_t = \psi(g, h, a, b, c, d, t)$ is continuous is virtually the same as that for σ. From Propositions 1 and 2, it is easy to see that $\mathcal{H}$ is a topological group, hence the product φ_t is continuous in σ and ψ_t, and therefore φ_t depends continuously on g, h, a, b, c, d and t. q.e.d.

Proof of Theorem 1

Before we give the proof of Theorem 1 we state and prove two more propositions.

PROPOSITION 4. *Let g be in $\mathcal{G}_0$, and r_i be the radius of $g(D_i)$ for each positive integer i. Then there is an element h in $\mathcal{G}_0$ such that $h(D_i) = D_{r_i}$, for each i, and h depends continuously on g.*

PROOF. Let L be any ray emanating from the origin in E^n. Coordinatize L by the distance from 0. We shall define h on L so that

$$h(L) = L \cap \left(\textstyle\bigcup_{i=1}^{\infty} D_{r_i}\right).$$

The segment $[0, 1]$ on L is mapped linearly onto $[0, r_1]$. More generally, $[i, i+1]$

is mapped linearly by h onto $[r_i, r_{i+1}]$, $i = 1, 2, \cdots$. It is easily seen that h is in $\mathcal{G}_0$.

To see that h is continuous as a function of g, we merely have to note that h depends only on the r_i, and that each r_i depends continuously on g according to Proposition 3.

PROPOSITION 5. *Let F: $\mathcal{G}_0 \times [0, 1) \to \mathcal{G}_0$ be continuous, and denote $F(g, t)$ by g_t. Suppose $g_t | D_n = g_{1-(1/2)^n} | D_n$ for all t in $[1 - (1/2)^n, 1)$, and $n = 1, 2, \cdots$. Then F can be extended to $\mathcal{G}_0 \times I$.*

PROOF. Define $F(g, 1)$ to be $\lim_{t\to 1} g_t = g_1$. Clearly g_1 is well-defined, continuous, and $1 - 1$, and by *invariance of domain*, g_1 is open, hence g_1 is in $\mathcal{G}_0$.

We verify continuity of F at $(g, 1)$. Let K be any compact set in E^n, $\varepsilon > 0$, and let $V(g_1, K, \varepsilon)$ be the neighborhood they determine in $\mathcal{G}_0$. Let n be large enough that K is contained in D_n. Then $g_{1-(1/2)^n}$ is in $V(g_1, K, \varepsilon)$, so by continuity of F at $(g, 1 - (1/2)^n)$, there is a neighborhood N of g such that

$$F\left(N \times 1 - \left(\frac{1}{2}\right)^n\right) \subset V(g_1, K, \varepsilon) .$$

It follows that

$$F\left(N \times \left[1 - \left(\frac{1}{2}\right)^n, 1\right]\right) \subset V(g_1, K, \varepsilon)$$

since $g'_t | D_n = g'_{1-(1/2)^n} | D_n$ for g' in N, t in $[1 - (1/2)^n, 1]$.

We return to the proof of Theorem 1. Let g in $\mathcal{G}_0$ be given. Use Proposition 4 to find $h = h(g)$. First we shall produce an isotopy $\alpha_t(t \in I) : E^n \to g(E^n)$ such that

(a) $\alpha_0 = h$;

(b) $\alpha_1(E^n) = g(E^n)$;

(c) $\alpha_t = \alpha(g, t)$ is continuous in g and t.

We do this in an infinite number of steps. To define $\alpha_t(t \in [0, 1/2])$ we use the Lemma for $a = 0$, $b = c = 1$, $d = 2$, and obtain $\varphi_t(t \in I)$. Define $\alpha_t = \varphi_{2t}h(t \in [0, 1/2])$. Then $\alpha_0 = h$, $\alpha_{1/2}(D_1) \supset g(D_1)$ and, by Proposition 4, the Lemma, and the remark after Proposition 2, $\alpha_t(t \in [0, 1/2])$ is continuous in g and t. Note that $\alpha_{1/2}(D_2) \subset g(D_2)$ by property (3) of the Lemma.

Next we define, $\alpha_t(t \in [1/2, 3/4])$ by again using the Lemma, this time for "h" $= \alpha_{1/2}$, $a = 1$, $b = c = 2$, $d = 3$, and we obtain $\varphi_t(t \in I)$. Now define $\alpha_t = \varphi_{4t-2}\alpha_{1/2}(t \in [1/2, 3/4])$. Then α_t is an extension of that obtained in the first step, $\alpha_{3/4}(D_2) \supset g(D_2)$, and since $\alpha_{1/2}$ depends continuously on g, we can conclude as before that $\alpha_t(t \in [1/2, 3/4])$ is continuous in g and t. Note that $\alpha_{3/4}(D_3) \subset g(D_3)$, and that $\alpha_t | D_1 = \alpha_{1/2} | D_1$ for t in $[1/2, 3/4]$, by property (3) of the Lemma.

We continue in this manner defining for each integer n, $\alpha_t(t \in [1 - (1/2)^n,$

$1-(1/2)^{n+1}]\big)$ such that $\alpha_{1-(1/2)^n}(D_n) \supset g(D_n)$ and $\alpha_t \mid D_n = \alpha_{1-(1/2)^n} \mid D_n$ for t in $[1-(1/2)^n, 1-(1/2)^{n+1}]$.

Proposition 5 allows us to define α_1 so that $\alpha_t(t \in I)$ depends continuously on g and t, and $\alpha_1(E^n) = g(E^n)$.

In the second stage, we produce an isotopy $\beta_t(t \in I)$: $E^n \to E^n$ such that

(a) $\beta_0 = h$,

(b) $\beta_1 = 1$,

(c) $\beta_t = \beta(g, t)$ is continuous in g and t.

This we do again in an infinite number of steps, first obtaining $\beta_t\big(t \in [0, 1/2]\big)$ as follows. We have $h(D_1) = D_{r_1}$ where $r_1 =$ radius of $g(D_1)$, since h was constructed so as to take round disks onto round disks. We shall preserve this property throughout the isotopy $\beta_t(t \in I)$. Let L be any ray emanating from the origin in E^n and coordinatized by distance from the origin. For t in I, let φ_t take the interval $[0, r_1]$ in L linearly onto $[0, (1-t)r_1 + t]$ and translate $[r_1, \infty)$ to $[(1-t)r_1 + t, \infty)$. This defines φ_t in $\mathcal{H}_0$ for each t in I. Now let $\beta_t = \varphi_{2t}h\big(t \in [0, 1/2]\big)$. Then $\beta_0 = h$ and $\beta_{1/2} \mid D_1 = 1$, and since r_1 and h depend continuously on g, then φ_{2t} and hence β_t are continuous in g and t.

Let s_2 be such that $\beta_{1/2}(D_2) = D_{s_2}$, and define $\beta_t\big(t \in [1/2, 3/4]\big)$ as follows. Let L be any ray as before, and let $\varphi_t(t \in I)$ take $[1, s_2]$ in L linearly onto $[1, (1-t)s_2 + 2t]$, translate $[s_2, \infty)$ onto $[(1-t)s_2 + 2t, \infty)$, and leave $[0, 1]$ fixed. Define $\beta_t = \varphi_{4t-2}\beta_{1/2}\big(t \in [1/2, 3/4]\big)$. Then this extends $\beta_t\big(t \in [0, t]\big)$, $\beta_{3/4} \mid D_2 = 1$, and β_t depends continuously on g and t.

Continuing in this manner, as in the first stage, we obtain an isotopy $\beta_t(t \in I)$ which depends continuously on g and t.

Now define

$$F(g, t) = \begin{cases} \alpha_{1-2t}\alpha_1^{-1}g & \text{for } t \text{ in } [0, 1/2] \\ \beta_{2t-1}\alpha_1^{-1}g & \text{for } t \text{ in } [1/2, 1]\,. \end{cases}$$

It is easy to check that F satisfies (1) and (2). An immediate consequence of Proposition 4 is that h is onto if g is. Each φ_t that occurs in a step of the construction of α_t and β_t is onto, hence α_t and β_t, and finally $F(g, t)$ is onto if g is, so property (3) holds. Continuity of F follows from that of α_t and β_t and from Propositions 1 and 2.

Admissible bundles

A microbundle $\mathbf{x}$: $B \xrightarrow{i} E \xrightarrow{j} B$, having fibre dimension n, *admits a bundle* providing there is an open set E_1 in E containing the 0-section $i(B)$ such that $j \mid E_1 : E_1 \to B$ is a fibre bundle with fibre E^n and structural group $\mathcal{H}_0$. The fibre bundle in this case will be called an *admissible bundle* for $\mathbf{x}$.

Let X_n be the statement that every microbundle over a locally-finite n-dimensional complex admits a bundle. Let U_n be the statement that any two admissible bundles for the same microbundle over a locally-finite n-dimensional complex are isomorphic. An isomorphism in this case is a homeomorphism between the total spaces which preserves fibres and is the identity on the 0-section.

THEOREM 2. *X_n and U_n are true for all n.*

PROOF. The proof will be by induction on n. X_0 and U_0 follow immediately from the fact that microbundles over a 0-dimensional set are all trivial.

Next we show X_{n-1} and U_{n-1} imply X_n. Let $\mathbf{x}$ be a microbundle over a locally-finite n-complex K with diagram: $K \xrightarrow{i} E \xrightarrow{j} K$. For each n-simplex σ in K, we find an admissible (and trivial) bundle ξ_σ for $\mathbf{x} \mid \sigma$. Thus we have a homeomorphism h_σ: $\sigma \times E^n \to E(\xi_\sigma)$, where $E(\xi_\sigma)$ is the total space of ξ_σ, such that $jh_\sigma(p, q) = p$ and $h_\sigma(p, 0) = i(p)$, for all p in σ and q in E^n. Let D be an open set in E containing $i(K)$ such that $j^{-1}(\sigma) \cap D$ is contained in $E(\xi_\sigma)$. Let K^{n-1} denote the $(n-1)$-skeleton of K, and $\mathbf{y}$ the microbundle: $K^{n-1} \xrightarrow{i'} j^{-1}(K^{n-1}) \cap D \xrightarrow{j'} K^{n-1}$, where i' and j' are the restrictions of i and j. By X_{n-1}, $\mathbf{y}$ admits a bundle η. Let σ be any n-simplex in K. By the choice of D, for each point p in $\partial\sigma$, the η-fibre over p is contained in the ξ_σ-fibre over p. Then $\eta \mid \partial\sigma$ and $\xi_\sigma \mid \partial\sigma$ are both admissible bundles for $\mathbf{x} \mid \partial\sigma$, and since the second is trivial, by U_{n-1} it follows that $\eta \mid \partial\sigma$ is trivial also. Hence, we have a homeomorphism h_η: $\partial\sigma \times E^n \to E(\eta \mid \partial\sigma)$ such that $jh_\eta(p, q) = p$ and $h_\eta(p, 0) = i(p)$, for all p in $\partial\sigma$ and q in E^n.

For each p in $\partial\sigma$, define g^p: $E^n \to E^n$ by $h_\sigma^{-1}h_\eta(p, q) = (p, g^p(y))$. Of course, g^p is just the imbedding of the η-fibre over p in the ξ_σ-fibre over p relative to the coordinates given by h_η and h_σ, hence g^p is in $\mathcal{G}_0(n)$. It is easy to check that $p \to g^p$ is continuous. Let σ_1 be a smaller concentric n-simplex contained in σ. Identify points in $\sigma - \operatorname{int} \sigma_1$ with $\partial\sigma \times I$, with p in $\partial\sigma$ identified with $(p, 0)$. Let F be the map guaranteed by Theorem 1 and, for each point (p, t) in $\sigma - \operatorname{int} \sigma_1$, denote $F(g^p, t)$ by g_t^p. Finally let $E_1 = E(\eta) \cup \{h_\sigma((p, t), g_t^p(q)) \mid (p, t)$ in $\sigma - \operatorname{int} \sigma_1$, q in $E^n\} \cup E(\xi_\sigma \mid \sigma_1)$. We claim that $j \mid E_1$: $E_1 \to K^{n-1} \cup \sigma$ is an admissible bundle for $\mathbf{x} \mid K^{n-1} \cup \sigma$.

We verify local triviality over σ. Let f: $(\sigma - \operatorname{int} \sigma_1) \times E^n \to E_1$ be given by $f((p, t), q) = h_\sigma((p, t), g_t^p(q))$. Define e^p in $\mathcal{H}_0(n)$ for each $(p, 1)$ in $\partial\sigma_1$ by $e^p(q) = \pi_2 f^{-1} h_\sigma((p, 1), q)$, where π_2: $\sigma \times E^n \to E^n$ is projection onto the second factor. Now define e: $\sigma \times E^n \to j^{-1}(\sigma) \cap E_1$, an onto homeomorphism, by:

$$e \mid \sigma_1 \times E^n = h_\sigma \mid \sigma_1 \times E^n$$

and

$$e((p, t), q) = f((p, t), e^p(q))$$

for (p, t) in $\sigma - \operatorname{int} \sigma_1$.

To verify that e is well-defined, we let $(p, 1)$ be any point in $\partial\sigma_1$. Then $f^{-1}h_\sigma(p, 1), q) = ((p, 1), e^p(q))$, by definition of e^p, hence $h_\sigma(p,1),q) = f((p, 1), e^p(q))$. This proves local triviality over $\operatorname{int} \sigma$.

To verify local triviality on $\partial\sigma$, let $(p, 0)$ be any point in $\partial\sigma$. Let N_1 be a neighborhood of $(p, 0)$ in K^{n-1} such that $\eta | N_1$ is trivial. Then we have a homeomorphism h_1: $N_1 \times E^n \to j^{-1}(N_1) \cap E(\eta)$ such that $jh_1(q, r) = q$ and $h_1(q, 0) = i(q)$. Define e^q in $\mathcal{H}_0(n)$ by $e^q(r) = \pi_2 f^{-1}h_1(q, r)$. Let $N_2 = \{(q, t) | t < 1, q \text{ in } N_1 \cap \partial\sigma\}$ and $N = N_1 \cup N_2$. Then N is a neighborhood of $(p, 0)$ in $K^{n-1} \cup \sigma$. Define e: $N \times E^n \to j^{-1}(N) \cap E_1$ by:

$$e | N_1 \times E^n = h_1$$

and

$$e((q, t), r) = f((q, t), e^q(r))$$

for (q, t) in N_2.

As before, e is seen to be a well-defined onto homeomorphism, and this completes our demonstration of the local triviality of $j | E_1$: $E_1 \to K^{n-1} \cup \sigma$. Thus we have extended η to an admissible bundle over $K^{n-1} \cup \sigma$ and, by repeating this process on each n-simplex σ, we get an admissible bundle for $\mathbf{x}$.

Finally we show X_n implies U_n, and the proof for Theorem 2 will be finished. Let $\sigma_1, \sigma_2, \cdots, \sigma_\alpha, \cdots$ $(\alpha < \alpha_0)$ be a well-ordering of those simplexes in the n-complex K which are not faces of some higher dimensional simplex in K. Let ξ_1 and ξ_2 be two admissible bundles for $\mathbf{x}$, a microbundle over K, with diagram $K \xrightarrow{i} E \xrightarrow{j} K$. By X_n there is no loss in generality in assuming $E(\xi_1)$ is contained in $E(\xi_2)$. Let f_0: $E(\xi_1) \to E(\xi_2)$ be the inclusion. Let $N(\sigma_\alpha)$ be the closed star neighborhood of σ_α in the second barycentric subdivision. Let $K_\alpha = \bigcup_{\beta \leq \alpha} \sigma_\beta$, a subcomplex. Suppose for each $\beta < \alpha$ we have defined f_β: $E(\xi_1) \to E(\xi_2)$, an imbedding taking fibres into fibres, and f_β is the identity on $i(K)$. Suppose further that $f_\beta | K_\beta$ is an isomorphism from $\xi_1 | K_\beta$ onto $\xi_2 | K_\beta$ and that, for each point p in $E(\xi_1) - j^{-1}(N(\sigma_\beta))$, there is a $\gamma < \beta$ and a neighborhood N of p such that $f_\beta | N = f_{\beta'} | N$ for $\gamma \leqq \beta' \leqq \beta$. We construct f_α, satisfying these properties.

Let g_α: $E(\xi_1) \to E(\xi_2)$ be $f_{\alpha-1}$ if $\alpha - 1$ exists. Otherwise $g_\alpha = \operatorname{limit}_{\beta\to\alpha} f_\beta$, which exists because of the last induction property and since each point in K lies in only finitely-many $N(\sigma_\beta)$'s. Then $g_\alpha(E(\xi_1))$ is the total space of a bundle η_α over K in a natural way, with the projection map j restricted. Since $N(\sigma_\alpha)$ is contractible, $\eta_\alpha | N(\sigma_\alpha)$ and $\xi_2 | N(\sigma_\alpha)$ are both trivial. Let c_α: $N(\sigma_\alpha) \times E^n \to E(\eta_\alpha | N(\sigma_\alpha))$ and d_α: $N(\sigma_\alpha) \times E_n \to E(\xi_2 | N(\sigma_\alpha))$ be isomorphisms, so for example,

$jc_\alpha(p, q) = p$ and $c_\alpha(p, 0) = i(p)$. Let h^p in $\mathcal{G}_0(n)$, for each p in $N(\sigma_\alpha)$, be defined by $d_\alpha^{-1}c_\alpha(p, q) = (p, h^p(q))$. As before $p \to h^p$ is continuous. Let $t\colon K \to I$ be a map such that $t(K - N(\sigma_\alpha)) = 0$ and $t(\sigma_\alpha) = 1$. If F is the function guaranteed in Theorem 1, let $h_t^p = F(h^p, t(p))$. Define $h\colon E(\eta_\alpha) \to E(\xi_2)$ by

$$h(r) = d_\alpha\big(j(r), h_t^{j(r)}\pi_2 c_\alpha^{-1}(r)\big) \qquad \text{for } j(r) \text{ in } N(\sigma_\alpha)\ ,$$

and h is the identity elsewhere. To see that h is continuous, suppose $j(r)$ is in $N(\sigma_\alpha) \cap \mathrm{Cl}\,(K - N(\sigma_\alpha))$. Then $t(j(r)) = 0$ and $h_t^{j(r)} = h^{j(r)}$; hence

$$\begin{aligned} h(r) &= d_\alpha\big(j(r), h^{j(r)}\pi_2 c_\alpha^{-1}(r)\big) \\ &= d_\alpha\big(j(r), \pi_2 d_\alpha^{-1} c_\alpha\big(j(r), \pi_2 c_\alpha^{-1}(r)\big)\big) \\ &= d_\alpha\big(j(r), \pi_2 d_\alpha^{-1}(r)\big) \\ &= r\ . \end{aligned}$$

Note that if $t = 1$, then h_t^p is onto, hence h takes the η_α-fibres over σ_α onto the ξ_2-fibres over σ_α. Furthermore if the η_α-fibre over p coincides with the ξ_2-fibre over p, then h^p is onto, as is h_t^p by property (3) of Theorem 1, hence the image under h of the η_α-fibre coincides with the ξ_2-fibre. Finally, define $f_\alpha = hg_\alpha$. It is easy to see that f_α satisfies the induction properties.

The isomorphism from ξ_1 onto ξ_2 is defined to be $\text{limit}_{\alpha\to\alpha_0} f_\alpha$. This finishes the proof of Theorem 2.

COROLLARY 1. *If B is a neighborhood retract in E^n (for example, any separable metric topological manifold) then any microbundle over B admits a unique bundle.*

PROOF. Let V be an open set in E^n containing B and $\rho\colon V \to B$, a retraction. Then if $\mathbf{x}$ is a microbundle over B, $\rho^*(\mathbf{x})$ may be regarded as an extension of $\mathbf{x}$ to all of V. But V can be triangulated, and Theorem 2 applied to give both the existence and uniqueness.

Denote by $\mathcal{H}_0^+(n)$ those elements in $\mathcal{H}_0(n)$ which preserve orientation.

COROLLARY 2. *For large enough n, the canonical homomorphism $\pi_7(\mathrm{SO}(n) \to \pi_7(\mathcal{H}_0^+(n))$ is not an isomorphism.*

PROOF. It is shown in [4] that the homomorphism $k_0S^8 \to k_{top}S^8$ is not an isomorphism. It is well known that each vector n-bundle over S^8 determines an element in $\pi_7(\mathrm{SO}(n))$. By Theorem 2 each microbundle over S^8 having fibre dimension n determines an element in $\pi_7(\mathcal{H}_0^+(n))$. Corollary 2 follows from the fact that only isomorphic bundles (vector bundles) determine the same element in $\pi_7(\mathcal{H}_0^+)(\pi_7(\mathrm{SO}(n)))$, and trivial bundles determine the identity element (cf. [5, p. 97]).

On the other hand it is a consequence of [1] and [3] that, for $n \leqq 3$, the homomorphisms $\pi_1(\mathrm{SO}(n)) \to \pi_i(\mathcal{H}_0^+(n))$, $i = 1, 2, 3, \cdots$ are isomorphisms, hence any microbundle over a sphere having fibre dimension $\leqq 3$ can be represented by a vector bundle.

INSTITUTE FOR ADVANCED STUDY AND
UNIVERSITY OF MICHIGAN

REFERENCES

1. J. CERF, *Groupes d'automorphismes et groupes de diffeomorphismes*, Bull. Soc. Math. France, 87 (1959), 319-329.
2. J. M. KISTER, *Microbundles are fibre bundles*, Bull. Amer. Math. Soc., 69 (1963), 854-857.
3. H. KNESER, *Die deformationssätze der einfach zusammenhängenden Flächen*, Math. Z., 25 (1926), 362-372.
4. J. MILNOR, Microbundles, Part I, Princeton University, 1963 (mimeographed).
5. N. STEENROD, The Topology of Fibre Bundles, Princeton University Press, 1951.

(Received August 27, 1963)