PRELIMINARY VERSION

THE CHERN-HIRZEBRUCH-SERRE THEOREM FOR THE SIGNATURE MODULO EIGHT

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ABSTRACT. By the Chern-Hirzebruch-Serre Theorem, the signature of oriented Poincaré duality complexes is multiplicative in fibrations if the fundamental group of the basis acts trivially on the rational cohomology of the fibre. Our main theorem is the following new variant: If the fundamental group of the basis acts trivially on the middle-dimensional cohomology of the fibre with $\mathbb{Z}/2$-coefficients, then the signature mod 8 is multiplicative. The proof uses a cochain construction of the Pontrjagin square in the Serre spectral sequence, and the analysis of the associated mod 8 Arf invariant.

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1. Introduction and main theorem

Let $F \to M \to B$ be an oriented fibration of connected closed oriented Poincaré duality complexes. For short, we call this an **oriented PD fibration**. It is a long studied problem how the signature of the total space M is related to that of B and F. If the fundamental group $\pi_1(B)$ acts trivially on the rational cohomology of F then $\mathrm{sign}(M) = \mathrm{sign}(F)\,\mathrm{sign}(B)$ by the celebrated Chern-Hirzebruch-Serre Theorem [4]. Several decades later it was realized that besides the signature, there are other genera which are multiplicative under certain *additional* assumptions on the fibration. For example, if all spaces are manifolds and the fibres are spin, then all *elliptic* genera are multiplicative [18]. In this note we concentrate on the signature and on *weakening* the assumptions on the fibration. Our main result is as follows:

Theorem 1.1. *Let $F \to M \to B$ be a oriented PD fibration. If $\pi_1(B)$ acts trivially on the middle-dimensional cohomology $H^f(F^{2f};\mathbb{Z}/2)$, then*

$$\mathrm{sign}(M) \equiv \mathrm{sign}(F)\,\mathrm{sign}(B) \mod 8.$$

The signature of a fibration can only be nontrivial if the dimension of the total space M is a multiple of 4. The proof of the classical Chern-Hirzebruch-Serre Theorem which we recall in the next section actually shows that the signature vanishes if the fibres are odd dimensional (for any action of $\pi_1(B)$). So we henceforth assume

$$\dim F = 2f, \quad \dim B = 2b, \quad \dim M = 2m, \quad m \text{ even}.$$

There exist examples of fibrations with nontrivial action of $\pi_1(B)$ on $H^f(F;\mathbb{Q})$ where multiplicativity of the signature fails. For example, there are surface bundles over surfaces with $\mathrm{sign}(M) = 4$ [14], but clearly $\mathrm{sign}(B) = \mathrm{sign}(F) = 0$. An explicit

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example with this minimal signature defect was given only recently [5]. This shows that even working modulo 8 one needs some kind of an assumption on the fibration.

The paper is organized as follows: In the next section, we recall the Serre spectral sequence and the proof of the classical Chern-Hirzebruch-Serre Theorem, as this will be the basis for our generalization. In the third section, we recall the Pontrjagin square which will give us a quadratic refinement of the multiplicative structure in cohomology. By a theorem of Morita, on a closed manifold the Arf invariant of the Pontrjagin square is given by the signature mod 8. Thus, the proof of our main theorem consists essentially in a careful construction and analysis of the Pontrjagin squaring operation in the Serre spectral sequence by cochain methods. This is done in the forth section and leads to the proof of our main result using some algebraic properties of the $\mathbb{Z}/8$-valued Arf invariant which we collected in an appendix. In the last section we consider some examples of surface bundles over surfaces.

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2. The Serre spectral sequence and the classical Chern-Hirzebruch-Serre Theorem

In this section, we recall the construction of the Serre spectral sequence including its multiplicative properties, and give a short proof of the classical Chern-Hirzebruch-Serre Theorem in the slightly stronger form of [12], as this will be the basis for the proof of our main theorem.

Let R denote some commutative ring. A cochain complex (C^*, d) of R-modules with a decreasing filtration by subcomplexes

$$\mathcal{F}^0C^* \supset \mathcal{F}^1C^* \supset \dots$$

induces a spectral sequence [20] (chapter 5) (E_r^{**}, d_r) with

$$E_r^{p,q} := Z_r^{p,q}/(dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}),$$

$$Z_r^{p,q} := \{x \in \mathcal{F}^pC^{p+q} \mid dx \in \mathcal{F}^{p+r}C^{p+q+1}\}.$$

In order to construct the cohomology Serre spectral sequence with coefficients in R for any fibration $F \to M \to B$, we follow the simplicial approach in [11], chapter VI. We take the normalized singular cochain complex $C^* := C^*(M; R)$ of the total space M. The decreasing filtration $\mathcal{F}$ is induced from the increasing filtration of the total singular complex $S_\bullet(M)$ by taking the inverse images of the p-skeleta of $S_\bullet(B)$ as subcomplexes. Hence $\mathcal{F}^pC^n$ consists of those normalized n-cochains on M which vanish on all singular n-simplices in M that project to singular n-simplices in B which degenrate to $(p-1)$-simplices in B ($n \leq p$). In particular, $\mathcal{F}^0C^n = C^n$ and $\mathcal{F}^{n+1}C^n = 0$. This decreasing filtration on C^* induces a decreasing filtration on $H(C^*) = H^*(M; R)$ which we also call $\mathcal{F}$. Explicitely, $\mathcal{F}^pH^*(M; R)$ consists of the kernel of the restriction map to the inverse image of the p-skeleton and there is an isomorphism

$$E_\infty^{p,q} = \mathcal{F}^pH^{p+q}(M; R)/\mathcal{F}^{p+1}H^{p+q}(M; R).$$

Furthermore, the E_2-term is isomorphic to

$$E_2^{pq} = H^p(B; \mathcal{H}^q(F; R)),$$

where $\mathcal{H}^*(F;R)$ denotes cohomology with local coefficients over B. Then the spectral sequence above is the cohomology Serre spectral sequence

$$E_2^{pq} = H^p(B;\mathcal{H}^q(F;R)) \Longrightarrow H^{p+q}(M;R).$$

Clearly, this construction is natural in the category of fibrations: A commutative square

$$\begin{array}{ccccc} F & \rightarrow & M & \rightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ F' & \rightarrow & M' & \rightarrow & B' \end{array}$$

with the horizontal maps being fibrations induces a map of the associated Serre spectral sequences.

The multiplicative structure in the Serre spectral sequence can be constructed from the Alexander-Whitney cup product of cochains on M, which behaves multiplicative with respect to the filtration:

$$\cup : \mathcal{F}^p C^m \otimes \mathcal{F}^q C^n \longrightarrow \mathcal{F}^{p+q} C^{m+n}.$$

Hence, there is an induced product structure

$$\mu_r : E_r^{p,q} \otimes E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$

in the spectral sequence such that the differentials are graded derivations. Moreover, the product μ_∞ is induced from the product on $H^*(M;R)$, and the product μ_2 corresponds to the composition

$$H^p(B;\mathcal{H}^q(F;R)) \otimes H^{p'}(B;\mathcal{H}^{q'}(F;R)) \longrightarrow H^{p+p'}(B;\mathcal{H}^q(F;R)\otimes\mathcal{H}^q(F;R))$$
$$\longrightarrow H^{p+p'}(B;\mathcal{H}^{q+q'}(F;R))$$

induced by the cup product on $H^*(F;R)$ and by the cup product $H^*(B;S) \otimes H^*(B;S') \to H^*(B;S\otimes S')$ for $\pi_1(B)$-modules S and S'.

For an oriented PD fibration and coefficients in a field R, the product

$$\mu_r : \bigoplus_{p+p'=2b,q+q'=2f} (E_r^{p,q} \otimes E_r^{p',q'}) \longrightarrow E_r^{2b,2f} = R$$

is non-degenerate [4], with Poincaré duality interchanging the summands $E_r^{p,q} \otimes E_r^{p',q'}$ and $E_r^{p',q'} \otimes E_r^{p,q}$. In particular, the product

$$\mu_r : E_r^{b,f} \otimes E_r^{b,f} \longrightarrow R$$

is non-degenerate, too.

Now, the proof of the classical Chern-Hirzebruch-Serre Theorem uses the rational multiplicative structure and proceeds in the following three steps:

$$\begin{array}{rcll} \operatorname{sign}(M) & = & \operatorname{sign}(E_\infty^{b,f},\mu_\infty) & (1) \\ & = & \operatorname{sign}(E_2^{b,f},\mu_2) & (2) \\ & = & \operatorname{sign}(F)\operatorname{sign}(B) & (3). \end{array}$$

Step (1) holds by the sublagrangian lemma for the signature, because in the multiplicative descending filtration $H^*(M;\mathbb{Q}) = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \ldots$, the subspace $L := \mathcal{F}^{b+1}H^m(M;\mathbb{Q})$ is a sublagrangian in $H^m(M;\mathbb{Q})$ with orthogonal complement $L^\perp = \mathcal{F}^b H^m(M;\mathbb{Q})$. Step (2) follows by the sublagrangian lemma again, this time applied to $L_r := im(d_r) \subset E_r^{b,f}$ with $L_r^\perp = ker(d_r)$. For step (3) recall that by the assumption

$$E_2^{p,q} = H^p(B;\mathcal{H}^q(F;\mathbb{Q})) = H^b(B;\mathbb{Q}) \otimes H^f(F;\mathbb{Q})$$

with pairing given by $\mu_2 = \mu_B \otimes \mu_F$. Hence $\mathrm{sign}(E_2^{b,f}, \mu_2) = \mathrm{sign}(B)\,\mathrm{sign}(F)$ by the multiplicativity of the signature in tensor products of symmetric pairings. Indeed, this proves the following slightly stronger form of the Chern-Hirzebruch-Serre Theorem [12]:

Theorem 2.1. *Let $F \to M \to B$ be an oriented PD fibration. If the dimension of the fibre is odd, then $sign(M) = 0$. Otherwise,* $\mathrm{sign}(M) = \mathrm{sign}(E_2^{b,f}, \mu_2)$, *and if $\pi_1(B)$ acts trivially on $H^f(F;\mathbb{Q})$, then*

$$\mathrm{sign}(M) = \mathrm{sign}(E_2^{b,f}, \mu_2) = \mathrm{sign}(B)\,\mathrm{sign}(F).$$

It is not easy to find examples of oriented PD fibrations where multiplicativity of the signature fails. A famous class of examples is given by surface bundles over surfaces, which we will consider in the last section. Indeed, by the following theorem of W. D. Neumann, $\pi_1(B)$ has to be complicated in order to allow examples with failing multiplicativity. For any discrete group G, denote by $WU(G)$ the Witt group of finite dimensional hermitian representations of G over $\mathbb{C}$. Neumann defines a natural ring homomorphism $\psi_G : WU(G) \to \tilde{H}^{2*}(G;\mathbb{Q})$ and shows:

Theorem 2.2. [17] *If $\psi_{\pi_1(B)} = 0$, then the signature is multiplicative in any oriented PD fibration $F \to M \to B$.*

For example, Neumann proves that this is the case if $\pi_1(B)$ is finite, or free.

3. The Pontrjagin square and a theorem of Morita

The Pontrjagin square is an unstable cohomology operation

$$\wp : H^n(X;\mathbb{Z}/2) \longrightarrow H^{2n}(X;\mathbb{Z}/4),$$

which is uniquely characterized by the following three properties [3]:

$$r\wp(x) = x^2, \qquad \wp(ry) = y^2, \qquad \wp'(x) = i(xSq^1x).$$

Here, r and i denote the coefficient homomorphisms associated to the short exact sequence $\mathbb{Z}/2 \xrightarrow{i} \mathbb{Z}/4 \xrightarrow{r} \mathbb{Z}/2$, $x \in H^n(X;\mathbb{Z}/2)$, $y \in H^n(X;\mathbb{Z}/4)$, and $\wp' := \sigma^{-1}\wp\sigma : H^n(X;\mathbb{Z}/2) \to H^{2n+1}(X;\mathbb{Z}/4)$ denotes the cohomology suspension of $\wp$ with $\sigma : H^*(X;A) \to H^{*+1}(SX;A)$ the cohomology suspension isomorphism. The operation $\wp'$ is called the Postnikov Square. It is also unstable, but linear, and its cohomology suspension vanishes, $\wp'' = 0$.

For sums and products, the following formulas hold:

Lemma 3.1. [3] [21]

$$\wp(x + x') = \wp(x) + \wp(x') + \begin{cases} i(x \cup x') & \text{for } |x| \text{ even} \\ 0 & \text{for } |x| \text{ odd} \end{cases},$$

$$\wp(xy) = \wp(x)\wp(y) + \wp'(x)\cdot Sq^{|y|-1}y + Sq^{|x|-1}x\cdot\wp'(y).$$

On cochain level, the Pontrjagin square is constructed as follows [16]:

$$\wp(x) := [y \cup_0 y + y \cup_1 dy \bmod 4].$$

Here, y denotes a singular integral cochain whose mod 2 reduction is a cocycle representing the cohomology class x, d denotes the singular coboundary operator and

$$\cup_i : C^m(X;R) \otimes C^n(X;R) \longrightarrow C^{m+n-i}(X;R)$$

denotes the cup-i product which can be defined for any commutative coefficient ring R. As this later will play a role we recall the construction.

Let $S := \mathbb{Z}[\pi]$ be the integral group ring of $\pi := \{1, T\} = \mathbb{Z}/2$ and

$$V := \{\dots \xrightarrow{\partial} Se_2 \xrightarrow{\partial} Se_1 \xrightarrow{\partial} Se_0 \xrightarrow{\epsilon} \mathbb{Z}\},$$
$$\partial(e_i) := (T + (-1)^i)e_{i-1}, \qquad \epsilon(e_0) := 1,$$

be the standard free resolution of the trivial S-module $\mathbb{Z}$ over S with basis $\{e_i\}$. This is the chain complex of the universal covering of $\mathbb{RP}^\infty$ with it's standard cell decomposition (one i-cell in each dimension i). Let $C^* := C^*(-;\mathbb{Z})$ be the normalized integral cochain functor on spaces or simplicial sets, and let π act on the tensor complex $C^*_{(2)} := C^* \otimes C^*$ via permutation. Considering V as a negatively graded cochain complex, we can form the cochain complex $V \otimes_\pi C^*_{(2)}$ over $\mathbb{Z}$ by dividing out the diagonal π-action. By the method of acyclic models, there exists a natural chain homomorphism

$$\phi : V \otimes_\pi C^*_{(2)} \longrightarrow C^*$$

which gives the diagonal $H^0(\phi) = \Delta^* : H^0(V \otimes_\pi C^*_{(2)}) = H^0(X \times X; \mathbb{Z}) \longrightarrow H^0(X;\mathbb{Z})$ on the 0th cohomology. This property fixes ϕ up to a natural chain homotopy. Now, the integral cup-i-products are defined by

$$x \cup_i y := \phi(e_i \otimes x \otimes y).$$

By definition, $\cup_0$ is the usual cup product. The following coboundary formula simply means that ϕ is a chain homomorphism:

$$d(x \cup_i y) = (-1)^i dx \cup_i y + (-1)^{i+m} x \cup_i dy + (-1)^{i+1} x \cup_{i-1} y + (-1)^{mn+1} y \cup_{i-1} x,$$

where $m := |x|$, $n := |y|$. As a famous application, the Steenrod squares are defined on cochain level by

$$sq^i : C^n(-;\mathbb{Z}/2) \longrightarrow C^{n+i}(-;\mathbb{Z}/2), \quad sq^i(x) := x \cup_{n-i} x + x \cup_{n-i+1} dx.$$

Now goying back to the definition of the Pontrjagin square, by the coboundary formula for the cup-1-product it follows straightforward that $y \cup_0 y + y \cup_1 dy$ mod 4 is a cocycle modulo 4 whose cohomology class depends only of that of x [16].

By the formula $\wp(x+y) = \wp(x) + \wp(y) + i(x \cup y)$ for $|x|$ even, we get for any connected closed oriented PD complex M of dimension $2m$ with m even a $\mathbb{Z}/4$-valued quadratic refinement

$$\wp_M : H^m(M;\mathbb{Z}/2) \to \mathbb{Z}/4, \quad \wp_M(x) := \langle \wp(x), [M] \rangle,$$

of the $\mathbb{Z}/2$-valued pairing on $H^m(M;\mathbb{Z}/2)$. For the convenience of the reader, we included some facts on $\mathbb{Z}/4$-valued quadratic forms and their $\mathbb{Z}/8$-valued Arf invariant in an appendix. The following theorem of Morita is crucial:

Theorem 3.2. [15] *For a connected closed oriented PD complex, it holds*

$$\mathrm{sign}(M) \equiv \mathrm{Arf}(\wp_M) \quad \bmod 8.$$

We also remark that $\mathrm{sign}(M) \equiv \wp_M(v_M)$ mod4, where $v_M \in H^m(M;\mathbb{Z}/2)$ denotes the middle dimensional Wu class of M.

In the above definition of the Pontrjagin square, some choice of an integral lifting cochain y is involved. Indeed, it is possible to make this choice universally, which will simplify the proof of our main theorem. To this end, we recall some facts on cochain operations from [8]. By definition, a cochain operation is a natural transformation of normalized cochain functors on the category of simplicial sets (and thus also on topological spaces via the singular functor $S_\bullet(-)$):

$$\theta : C^m(-;R) \longrightarrow C^{m'}(-;R').$$

For example, the differential d in the cochain complex $d : C^m(-;R) \to C^{m+1}(-;R)$ can be viewed as a cochain operation. There are also 2-variable cochain operations, for example the integral cup-i-products

$$\cup_i : C^m(-;\mathbb{Z}) \times C^n(-;\mathbb{Z}) \longrightarrow C^{m+n-i}(-;\mathbb{Z}).$$

Suppose that θ maps cocycles to cocycles and that the cohomology class of $\theta(z)$ depends only on that of the cocycle z. In [8] the first author proves that this holds if and only if θ satisfies the condition

$$d\theta d = 0,$$

which then gives a cohomology operation

$$[\theta] : H^m(-;R) \longrightarrow H^{m'}(-;R'), \quad [z] \mapsto [\theta(z)].$$

Every (primary) cohomology operation can be represented in this way. Moreover, the condition $d\theta d = 0$ is equivalent to the existence of a cochain operation θ' with $\theta d = d\theta'$ [8]. The cochain operations θ' represents the cohomology suspension of the cohomology operation associated to θ.

In order to define a cochain operation $\theta : C^n(-;\mathbb{Z}/2) \longrightarrow C^{2n}(-;\mathbb{Z}/4)$ inducing the Pontragin square, $[\theta] = \wp$, we have to be a little careful as

$$z \mapsto \phi(z) := z \cup_0 z + z \cup_1 dz$$

can be regarded as a cochain operation from R- to R-coefficients (with $R = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/4, ...$) but not from $\mathbb{Z}/2$- to $\mathbb{Z}/4$-coefficients. Thus we need a universal lift from $\mathbb{Z}/2$-cochains to $\mathbb{Z}/4$-cochains. Denote by $K_\bullet(R,n)$ and $L_\bullet(R,n+1)$ the standard models of simplicial minimal Eilenberg-MacLane spaces, respectively their path spaces [13]. These spaces represent the normalized cocycle and cochain functors for simplicial sets $X_\bullet$:

$$Z^n(X_\bullet;R) = map_\Delta(X_\bullet, K_\bullet(R,n)), \quad C^n(X_\bullet;R) = map_\Delta(X_\bullet, L_\bullet(R,n+1)),$$

where map_Δ denotes the set of simplicial maps and the natural isomorphisms are given by pulling back the universal cocycle and cochain, repectively. By the Yoneda lemma, natural transformations between these functors (i.e., cochain operations) are given by simplicial maps between the representing spaces. Hence, for two abelian groups R and R' we have

$$Trans(C^n(-;R), C^n(-;R')) = map_\Delta(L_\bullet(R,n+1), L_\bullet(R',n+1)) =$$
$$= C^n(L_\bullet(R,n+1);R') = \{s : R \to R' \mid s(0) = 0\}$$

because $L_n(R,n+1) = R$ with $0 \in R$ being the only degenerate n-simplex. In particular, the nonlinear splitting function $s : \mathbb{Z}/2 \to \mathbb{Z}/4$ with $s(0) := 0$ and $s(1) := 1$ of the short exact coefficient sequence $\mathbb{Z}/2 \xrightarrow{i} \mathbb{Z}/4 \xrightarrow{r} \mathbb{Z}/2$ corresponds to a cochain operation which we also denote by s:

$$s : C^n(-;\mathbb{Z}/2) \longrightarrow C^n(-;\mathbb{Z}/4).$$

This operation satisfies $rs = \mathrm{id}$ and $s(x+y) = s(x)+s(y)+i(x{\cdot}y)$ where $x{\cdot}y$ denotes the point-wise product of the cochains x and y (regarding them as functions which assign a number to any simplex). By the quadratic property of s, it does not commute with d. Indeed, from $r(ds - sd) = 0$ we obtain $ds - sd = i\beta$ with a uniquely defined cochain operation $\beta : C^n(-;\mathbb{Z}/2) \to C^{n+1}(-;\mathbb{Z}/2)$. (In fact, it is not difficult to see that β is given by the simplicial formulae $\beta(x) = \sum_j d^{2j+1}x + \sum_{j<k} d^j x \cdot d^k x$.) Furthermore, $id\beta = i\beta d = dsd$, hence $d\beta = \beta d$ and $d\beta d = 0$. By the construction above it is straightforward to see that the induced cohomology operation of β is just given by the Bockstein Sq^1. By the first author's results in [8], [9], there are exactly two cochain operations representing the Bockstein, namely sq^1 and $sq^1 + d$. In particular, $\beta(x)$ coincides with one of them.

As s gives a universal lift from $\mathbb{Z}/2$-cochains to $\mathbb{Z}/4$-cochains, we get a cochain operation

$$\theta := \phi s : C^n(-;\mathbb{Z}/2) \longrightarrow C^{2n}(-;\mathbb{Z}/4)$$
$$\theta(x) = s(x) \cup_0 s(x) + s(x) \cup_1 ds(x)$$

which induces the Pontrjagin square by definition. The condition $d\theta d = 0$ yields existence of θ' with $\theta d = d\theta'$, where θ' represents the Postnikov square $\wp'$. Indeed, a straightforward computation shows $\theta'(x) = s(x) \cup_0 ds(x) + i(dx \cup_1 \beta(x))$. Here and in the following, we use the fact that $iy \cup_k iz = 0$ and $iy \cup_k z = i(y \cup_k rz)$ relating $\cup_k$-products for $\mathbb{Z}/2$- and $\mathbb{Z}/4$-coefficients, which follows from the corresponding facts for the coefficients.

We will need the following result refining the sum formulae of the Pontrjagin square in the next section.

Lemma 3.3. *For $x, y \in C^m(M;\mathbb{Z}/2)$ with m even,*

$$\begin{aligned} \theta(x+y) &= \theta(x) + \theta(y) + i(x \cup_0 y) \\ &-d(s(x) \cup_1 s(y)) + s(y) \cup_1 s(dx) - s(dx) \cup_1 s(y) \\ &+i\big(d(y \cup_2 \beta(x)) + dy \cup_2 \beta(x) + y \cup_2 \beta(dx) \\ &+d((x+y) \cup_1 x \cdot y) + x \cdot y \cup_1 (dx + dy) + (dx+dy) \cup_1 x \cdot y\big). \end{aligned}$$

Proof: First we remark that $u \cup_0 v - v \cup_0 u = d(u \cup_1 v) + du \cup_1 v + u \cup_1 v$, hence

$$\phi(u+v) = \phi(u) + \phi(v) + 2(u \cup_0 v) - d(u \cup_1 v) + v \cup_1 du - du \cup_1 v$$

for $u, v \in C^m(M;\mathbb{Z}/4)$. Because of $\phi(ix) = 0$ for $x \in C^m(M;\mathbb{Z}/2)$, we get

$$\phi(u + ix) = \phi(u) + i\big(d(ru \cup_1 v) + v \cup_1 dru + dru \cup_1 v\big).$$

The claim follows from $\theta(x+y) = \phi(sx + sy + ix \cdot y)$, $ds(z) = s(dz) + i\beta(z)$ and $z \cup_1 \beta(z') + \beta(z') \cup_1 z = d(z \cup_2 \beta(z')) + dz \cup_2 \beta(z') + z \cup_2 \beta(dz')$ for any $z, z' \in C^m(M;\mathbb{Z}/2)$. □

The reason for transforming the deviation $\theta(x+y) - \theta(x) - \theta(y) - i(x \cup_0 y)$ to this form will become clear in the next section. The point is that the terms are either a coboundary or contain some factor dx or dy (a factor $ds(x)$ would not be enough).

4. Proof of the main theorem

The crucial part of the proof consists in constructing Pontrjagin squaring operations in the E_r-terms of the Serre spectral sequence of an oriented PD fibration, and finding its relations to the filtration on $H^*(M)$, to the differentials d_r and to the E_2-term. Using our cochain operation θ we will obtain $\mathbb{Z}/4$-valued quadratic refinements

$$\wp_{(r)} : E_r^{b,f} \longrightarrow \mathbb{Z}/4$$

of the pairings μ_r with coefficients $\mathbb{Z}/2$. Then, the proof of our main result 1.1 will be finished in three steps (4.5, 4.6 and 4.7), again:

$$\begin{array}{rcll} \mathrm{Arf}(\wp_M) & = & \mathrm{Arf}(\wp_\infty) & (1) \\ & = & \mathrm{Arf}(\wp_2) & (2) \\ & = & \mathrm{Arf}(\wp_F)\,\mathrm{Arf}(\wp_B) & (3) \text{ for trivial operation.} \end{array}$$

Any cochain operation preserves the Serre filtration by naturality. The filtration properties of the $\cup_i$-products are stronger:

Lemma 4.1. [10] *(p.84) For a fibration $F \to E \to B$ and $x \in \mathcal{F}^pC^m$, $y \in \mathcal{F}^qC^n$, $x \cup_i y \in C^{m+n}$ has filtration*

$$\max(p+q-i, \langle \frac{p+q}{2} \rangle),$$

where $\langle a \rangle :=$ least integer $\geq a$.

The following obvious construction of spectral operations for the Serre spectral sequence is implicit in [10], chapter 1-7.

Lemma 4.2. *Let $\theta : C^n(-;R) \to C^{n'}(-;R')$ be some cochain operation, and assume that there are numbers p, q, r and p', q', r' with $p+q = n$ and $p'+q' = n'$, such that the following holds: For any fibration $F \to M \to B$ with associated Serre spectral sequences $E_*^{**}(R)$ and $E_*^{**}(R')$ (with R- and R'-coefficients), and any cochain $x \in Z_r^{pq}(R) \subset C^n(M;R)$, the cochain $\theta(x) \in C^{n'}(M;R')$ is an element in $Z_{r'}^{p'q'}(R')$, and the class $[\theta(x)] \in E_{r'}^{p'q'}(R')$ depends only on the class $[x] \in E_r^{pq}(R)$. Then θ induces a spectral operation*

$$[\theta] : E_r^{pq}(R) \longrightarrow E_{r'}^{p'q'}(R')$$

which behaves natural on the category of fibrations.

As an application, it follows by straightforward computation that the cochain operation sq^i which gives the Steenrod square $Sq^i = [sq^i]$ in $\mathbb{Z}/2$-cohomology, induces spectral operations in the Serre spectral sequence with $\mathbb{Z}/2$-coefficients [10]

$$\begin{array}{llll} Sq^i: & E_r^{pq} & \longrightarrow & E_r^{p,q+i} & \text{for } 0 \leq i \leq q, \\ Sq^i: & E_r^{pq} & \longrightarrow & E_{r+min(i-q,r-2)}^{p+i-q,2q} & \text{for } q \leq i \leq p+q. \end{array}$$

This was proved independently by Araki [1] and Vazques [19].

As the Pontrjagin square is a refinement of the highest Steenrod square, a natural guess would be that θ induces a spectral operation $E_r^{p,q}(\mathbb{Z}/2) \longrightarrow E_{r'}^{2p,2q}(\mathbb{Z}/4)$. Unfortunately, this is *not* true:

Lemma 4.3. *If $x \in \mathcal{F}^pC^n$, then $\theta(x) \in \mathcal{F}^{2p-1}C^{2n}$.*

Proof: For $x \in \mathcal{F}^pC^n(M;\mathbb{Z}/2)$, we also have $s(x) \in \mathcal{F}^pC^n(M;\mathbb{Z}/4)$ by the injectivity of s, and $ds(x) \in \mathcal{F}^pC^{n+1}(M;\mathbb{Z}/4)$ by naturality. Hence $s(x) \cup_0 s(x)$ has filtration $2p$, but $s(x) \cup_1 ds(x)$ has filtration $2p-1$, only. □

The problem is that for $x \in \mathcal{F}^pC^n$ and $dx \in \mathcal{F}^{p+r}C^{n+1}$, the filtration of $ds(x)$ can be smaller than $p+r$ (in fact, by naturality it has only to be $\geq p$). This is in contrast to $s(dx)$ which has filtration $p+r$. Nevertheless, in our case of an oriented PD fibration, this causes no harm if we evaluate on the fundamental class of the total space:

Theorem 4.4. *Let $F \to E \to B$ be an oriented PD fibration and E_r^{**} the associated Serre spectral sequences for cohomology with $\mathbb{Z}/2$-coefficients. Then there are Pontrjagin squaring operations*

$$\wp_r : E_r^{b,f} \longrightarrow \mathbb{Z}/4, \quad \wp_r([x]) := \langle \theta(x), [M] \rangle$$

with $x \in Z_r^{b,f} \subset C^m(M;\mathbb{Z}/2)$, which are quadratic refinements of the products μ_r.

Proof: For $x \in Z_r^{b,f}$, i.e., $x \in \mathcal{F}^b C^m$ and $dx \in \mathcal{F}^{b+r}C^{m+1}$, we have $\theta(x) \in \mathcal{F}^{2b-1}C^{2m}(M;\mathbb{Z}/4)$ which can be evaluated on the $\mathbb{Z}/4$-fundamental class $[M]$ of M. We have to prove that $\wp_r$ is well-defined on $E_r^{b,f}$. If $x = dy \in dZ_{r-1}^{b-r+1,f+r-2}$, then by $\theta(dy) = d\theta'(y)$ and by Stokes' formulae $\wp_r(x) = \langle d\theta'(y), [M] \rangle = 0$. If $x \in Z_{r-1}^{b+1,f-1}$, then by lemma 4.3 $\theta(x) \in \mathcal{F}^{2b+1}C^{2n}$, which vanishes as evaluation on M factors over $\mathcal{F}^{2b+1}H^{2n}(M;\mathbb{Z}/4) = 0$. By lemma 3.3, we get for $x, y \in Z_r^{b,f}$ that $\theta(x+y) = \theta(x) + \theta(y) + i(x \cup_0 y) +$ boundary terms + terms of filtration larger than $2b$ (here, for $r = 2$ one has to use the fact that β raises the filtration by 1 as it represents the Bockstein). These additional terms vanish after evaluation on M, showing that $x \mapsto \langle \theta(x), [M] \rangle$ is a quadratic refinement of the cup product on cochain level. Moreover, it follows that $\wp_r$ is well-defined because for $x \in Z_r^{b,f}$ and $y \in dZ_{r-1}^{b-r+1,f+r-2}$ or $y \in Z_{r-1}^{b+1,f-1}$, we have $\langle x \cup_0 y, [M] \rangle = 0$ by Stokes' formulae and filtration, respectively. □

For the next two lemmas, we use the quadratic sublagrangian lemma which we prove in the appendix: If $q : V \to \mathbb{Z}/4$ is a nondegenerate quadratic form on a finite vector space V over $\mathbb{Z}/2$ and $L \subset V$ is a quadratic sublagrangian, i.e. $q(L) = 0$, then q induces a nondegenerate quadratic form $q' : V' \to \mathbb{Z}/4$ on $V' := L/L^\perp$ and $\mathrm{Arf}(q) = \mathrm{Arf}(q')$.

Lemma 4.5. *Let $F \to E \to B$ be an oriented PD fibration, then* $\mathrm{Arf}(\wp_M) = \mathrm{Arf}(\wp_\infty)$.

Proof: We already know that $L := \mathcal{F}^{b+1}H^m(M;\mathbb{Z}/2)$ is a multiplicative sublagrangian with $L^\perp = \mathcal{F}^b H^m$. If $x \in \mathcal{F}^{b+1}C^m(M;\mathbb{Z}/2)$, then $\theta(x) \in \mathcal{F}^{2b+1}C^{2m}$. Hence, $\wp_M([x]) = 0$, i.e. L is also a quadratic sublagrangian and we are done by the quadratic sublagranian lemma. □

Lemma 4.6. *Let $F \to E \to B$ be an oriented PD fibration, then* $\mathrm{Arf}(\wp_r) = \mathrm{Arf}(\wp_{r+1})$.

Proof: We already know that $L := im(d_r) \subset E_r^{b,f}$ is a multiplicative sublagrangian with $L^\perp = ker(d_r)$. We note that d_r is given by d on $C^*(M;\mathbb{Z}/2)$, thus for $[x] = [dy] \in im(d_r)$, we obtain $[\theta(x)] = [d\theta'(y)] \in im(d_r)$, too. Hence, $\wp_M([x]) = 0$ by Stokes' formulae, i.e. L is also a quadratic sublagrangian and we are done by the quadratic sublagranian lemma, again. □

The last step consists of the identification of $\wp_2$ and finishs the proof of our main theorem 1.1:

Theorem 4.7. *Let $F \to E \to B$ be an oriented PD fibration such that $\pi_1(B)$ acts trivially on $H^f(F^{2f};\mathbb{Z}/2)$, then* $\mathrm{Arf}(\wp_2) = \mathrm{Arf}(\wp_F)\,\mathrm{Arf}(\wp_B)$.

Proof: We recall that E_2^{**} is a subquotient of $C^*(M;\mathbb{Z}/2)$. We have $E_2^{b,0} = H^b(B;\mathbb{Z}/2)$ and, by assumption, $E_2^{0,f} = H^f(F;\mathbb{Z}/2)$, with isomorphisms induced by projection to the basis and inclusion of the fibre, respectively. By assumption, the product μ_2 (cup product) gives the isomorphism $H^b(B;\mathbb{Z}/2)\otimes H^f(F;\mathbb{Z}/2) = E_2^{b,f}$. Thus a basis of $E_2^{b,f}$ is given by the products $\mu_2(p^*[x_i],[z_j]) = [p^*x_i \cup z_j]$ with a basis $[x_i]$ of $H^b(B;\mathbb{Z}/2)$ and cochains z_j restricting to a basis $[y_j] = [i^*z_j]$ of $H^f(F;\mathbb{Z}/2)$. Now it is essential, that the product formula 3.1 of Wu also holds on the level of cochains ([21] p.157). By naturality and the product formula for the Pontrjagin square, we obtain

$$\wp_2([p^*x_i \cup z_j]) = \langle p^*\theta(x_i)\cup\theta(z_j),[M]\rangle,$$

because the summands containing the Postnikov square $\wp'$ vanish by dimensional reasons. As $p^*\theta(x_i)\in E_2^{2b,0}$ can pair non-trivially with elements in $E_2^{0,2f}$ only, the right hand side is equal to

$$\langle\theta(x_i),[B]\rangle\langle\theta(y_j),[F]\rangle.$$

Thus $\wp_2 = \wp_B\otimes\wp_F$ on pure tensors, which for f even (thus also b even) is a product of quadratic forms. Then the claim follows from the product formulae of the Arf invariant (see the appendix). For f (and b) odd, we have to be careful as $\wp_F$ and $\wp_B$ are linear in this case (see 3.1). Moreover, the intersection pairing of a connected closed oriented $(4n+2)$-dimensional PD complex is even, because $x^2 = Sq^{2n+1}x = Sq^1Sq^{2n}x = v_1Sq^{2n}x = 0$ by the vanishing of the first Wu class v_1. Hence $\wp_B$, $\wp_F$ and $\wp_2$ take values in $\mathbb{Z}/2\subset\mathbb{Z}/4$ and their generalized Arf invariants coincide with the usual $\mathbb{Z}/2$-valued Arf invariants. Now, it is enough to consider the tensor product of two hyperbolic summands with basis $x,y\in H^b(B;\mathbb{Z}/2)$ and $x',y'\in H^f(F;\mathbb{Z}/2)$, respectively. In the tensor product, a hyperbolic basis is given by $x_1 := x\otimes x'$ $y_1 := y\otimes y'$, and $x_2 := x\otimes y'$, $y_2 := y\otimes x'$. Thus $\mathrm{Arf}(\wp_2) = \wp_2(x_1)\wp_2(y_1) + \wp_2(x_2)\wp_2(y_2) = \wp_B(x)\wp_F(x')\wp_B(y)\wp_F(y') + \wp_B(x)\wp_F(y')\wp_B(y)\wp_F(x') = 0$. □

5. Surface bundles over surfaces (Not completed)

For a nice class of examples, we briefly recall some known facts about fibre bundles with F and B being closed oriented surfaces. Let g be the genus of the fibre. Then such bundles are given by a classifying map

$$\pi_1(B)\xrightarrow{c}\Gamma_g,$$

where Γ_g is the mapping class group of the fibre. It was shown in [14] that for $g\geq 3$ the signature of the total space M^4 is given by

$$\mathrm{sign}(M) = 4\cdot\deg(c),$$

where the degree $\deg(c)$ is defined by picking a generator in $H^2(\Gamma_g)\cong\mathbb{Z}$ and evaluating its pullback under c on the fundamental class in $H_2(\pi_1(B)) = H_2(B)$. This implies in particular, that all multiples of 4 occur as $\mathrm{sign}(M)$ because elements in the second homology of any space can be realized by maps of surfaces.

Now recall that we are interested in fibre bundles where $\pi_1(B)$ acts trivially on $H^1(F;\mathbb{Z}/p)$ for some prime p. This just means that the classifying map lies in the kernel K of the epimorphism

$$\Gamma_g\longrightarrow \mathrm{Sp}(2g,\mathbb{Z}/p) =: G$$

given by its action on the mod p intersection form of the surface. Hence the possible signatures are given by a factor of 4 times the index of the image k of $H_2(K)$ in

$H_2(\Gamma_g)$. Since the Schur multiplier $H_2(G)$ of our finite group G vanishes, the integer k is the order of the cyclic group

$$H_1(G; H_1(K))/d_2(H_3(G))$$

as follows from the Serre spectral sequence for our group extension

$$1 \longrightarrow K \longrightarrow \Gamma_g \longrightarrow G \longrightarrow 1$$

If our result is best possible, we would expect that $k = 2$ for $p = 2$ and $k = 1$ for $p > 2$.

The following hopefully useful information is taken from [6]. First note that by Minkowski's theorem the group K acts freely on Teichmueller space $\mathcal{T}_g$ for $p > 2$. In particular, K is torsionfree in this case and has a $(6g-6)$-dimensional classifying space $\mathcal{T}_g/K$. Note that G still acts on $\mathcal{T}_g/K$, the moduli space of level p curves, by permuting the level p structures (i.e. a basis of $H^1(F;\mathbb{Z}/p)$). This action has non-free orbits, corresponding to complex structures on F with larger symmetry group, namely the isotropy group of the orbit.

Moreover, $H^2(\Gamma_g)$ is generated by the first Chern class of the determinant bundle (of the Dolbeaux operator), an orbifold bundle over moduli space (or a Γ_g-equivariant bundle over $\mathcal{T}_g$). By the above remark, this determinant bundle becomes an honest complex line bundle L over $\mathcal{T}_g/K$. It is by construction G-equivariant and indivisible (under tensor product) as G-equivariant line bundle. The signatures in question are calculated by checking the precise divisibility of this line bundle, after forgetting the G-equivariance.

Appendix A. The sublagrangian lemma for the generalized Arf invariant

We recall here some facts about the usual and generalized Arf invariant. The latter was introduced in topology by Brown [2] in his work on generalizations of the Kervaire invariant. A survey on its algebraic properties can be found in [2]. See also the appendix of the author's thesis [7], from where we have taken the proof of the sublagrangian lemma.

Let V be a finite dimensional vector space over $\mathbb{Z}/2$ and $q : V \to \mathbb{Z}/2$ be a nondegenerate quadratic form, i.e. $\mu(x, y) := q(x+y) - q(x) - q(y)$ is bilinear in x, y and nondegenerate. Up to isomorphism, the form q is classified by the dimension of V and its Arf invariant $\text{Arf}(q) \in \mathbb{Z}/2$, which can be defined as the 'democratic invariant' as follows:

$$\text{Arf}(q) := \begin{cases} 1 & \text{iff the majority of vectors } x \in V \text{ has } q(x) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Because we have $\mu(x, x) = q(2x) - 2q(x) = 0$, the associated symmetric bilinear form μ of q is always even. In order to deal also with quadratic forms on V with associated odd bilinear form μ, one has to allow that q takes values in $\mathbb{Z}/4$. Thus a generalized nondegenerate quadratic form is a map $q : V \to \mathbb{Z}/4$ such that $\mu : V \times V \to \mathbb{Z}/4$ is again bilinear (and thus takes values in $\mathbb{Z}/2 \subset \mathbb{Z}/4$) and nondegenerate. Up to isomorphism, the form q is classified by dimension, type, and the generalized Arf invariant

$$\text{Arf}(q) \in \mathbb{Z}/8$$

defined as in the first section (see [7]).

We have the following properties: For two forms $q : V \to \mathbb{Z}/4$ and $q' : V' \to \mathbb{Z}/4$, we have

$$\operatorname{Arf}(q \oplus q') = \operatorname{Arf}(q) + \operatorname{Arf}(q'),$$
$$\operatorname{Arf}(q \otimes q') = \operatorname{Arf}(q) \cdot \operatorname{Arf}(q'),$$

where $(q \oplus q')(x, x') := q(x) + q'(x')$ and $(q \otimes q')(x \otimes x') := q(x) \cdot q'(x')$. If q takes values in $\mathbb{Z}/2 \subset \mathbb{Z}/4$, then Arf coincides with the usual Arf invariant via the embedding $\mathbb{Z}/2 \subset \mathbb{Z}/8$.

Furthermore, the following result holds for forms which are reductions of integral quadratic forms [2]: Let $\mu : F \times F \to \mathbb{Z}$ be a symmetric unimodular bilinear form on a finitely generated free abelian group F. Let $V := F \otimes \mathbb{Z}/2$ and define $q : V \to \mathbb{Z}/4$ by $q(x + 2F) := \mu(x, x) \bmod 4$. Then q is well-defined and (V, q) is a generalized nondegenerate quadratic form with

$$\operatorname{Arf}(q) \equiv \operatorname{sign}(\mu) \bmod 8.$$

The following sublagrangian lemma is taken from the first author's thesis [7]:

Lemma A.1. *Let $q : V \to \mathbb{Z}/4$ be a generalized nondegenerate quadratic form which vanishes on a sub-Lagrangian $V_- \subset V$. Define $V_0 := V_-^\perp / V_-$ and $q_0 : V_0 \to \mathbb{Z}/4$ by $q_0(v + V_-) := q(v)$. Then q_0 is a generalized nondegenerate quadratic form on V_0 and*

$$\operatorname{Arf}(q) = \operatorname{Arf}(q_0).$$

Proof: For $v_- \in V_-$ and $v \in V_-^\perp$ we get $q(v + v_-) = q(v)$, thus q_0 is well-defined. Clearly, μ is nondegenerate on V_0, thus also q_0. Let $V_+ := V/V_-^\perp$, then $V \cong V_- \oplus V_0 \oplus V_+$ and μ decomposes as

	V_-	V_0	V_+
V_-	0	0	$\mu_\pm$
V_0	0	μ_0	$*$
V_+	$\mu_\pm$	$*$	$*$

where the restriction $\mu_\pm : V_- \times V_+ \to \mathbb{Z}/2 \subset \mathbb{Z}/4$ is also nondegenerate. With $q(v_-) = 0$, this gives $q(v_- + v_0 + v_+) = q(v_0) + q(v_+) + \mu(v_-, v_+) + \mu(v_0, v_+)$, showing that

$$\sum_{x \in V} i^{q(x)} = \sum_{v_- \in V_-, v_0 \in V_0, v_+ \in V_+} i^{q(v_- + v_0 + v_+)}$$
$$= \sum_{v_0 \in V_0} i^{q(v_0)} \left(\sum_{v_+ \in V_+} i^{q(v_+) + \mu(v_0, v_+)} \left(\sum_{v_- \in V_-} i^{\mu(v_-, v_+)} \right) \right).$$

But $\sum_{v_- \in V_-} i^{\mu(v_-, v_+)}$ is 0 for fixed $v_+ \neq 0$, because half of the elements v_- of V_- are mapped to $i^{\mu(v_-, v_+)} = +1$, and the other half is mapped to -1. This follows from the nondegeneracy of $\mu_\pm$. For $v_+ = 0$, one has $\sum_{v_- \in V_-} i^{\mu(v_-, v_+)} = |V_-|$ (number of elements). So

$$\sum_{v \in V} i^{q(v)} = \sum_{v_0 \in V_0} i^{q(v_0)} \left(i^{q(0) + 0} \cdot |V_-| \right) = \left(\sum_{v_0 \in V_0} i^{q(v_0)} \right) \cdot |V_-|,$$

which gives $\operatorname{Arf}(q) = \operatorname{Arf}(q_0)$. □

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