Poincaré complex diagonals

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Abstract Let *M* be a Poincaré duality space of dimension $d \ge 4$. In this paper we describe a complete obstruction to realizing the diagonal map $M \to M \times M$ by a Poincaré embedding. The obstruction group depends only on the fundamental group and the parity of *d*.

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1 Introduction

A distinguishing feature of smooth manifolds is that they come equipped with a preferred bundle of tangent vectors. If M is a smooth manifold with tangent bundle τ_M , then the diagonal map

$$M \to M \times M$$
$$x \mapsto (x, x)$$

is a smooth embedding possessing a tubular neighborhood diffeomorphic to the total space of τ_M . Moreover, the set of tubular neighborhoods of the diagonal can be topologized in such a way that the resulting space is contractible.

In this note we shall consider a homotopy theoretic analog of the above in which M is now replaced by a space satisfying Poincaré duality. There are two parts question to consider: existence and uniqueness. Here we will be concerned with the existence problem.

Forty years ago, Spivak [18] proved that a Poincaré space possesses a "stable tangent bundle" in the sense of stable spherical fibration theory. By Atiyah duality [1], the Spivak tangent fibration is *stably* unique up to contractible choice. Furthermore, elementary obstruction theory shows that a stable spherical fibration over a *d*-dimensional Poincaré space lifts

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to an unstable (d-1)-spherical fibration. However, it is not unique: the number of such lifts is countably infinite when *d* is even and precisely two when *d* is odd.

Given a choice of unstable lift of the Spivak fibration to a (d-1)-spherical fibration $\tau: S(\tau) \to M$, one can then ask whether it appears as normal data for an embedding of the diagonal, where now by "embedding" we mean a *Poincaré embedding* in the sense the diagonal map extends to a homotopy equivalence

$$M \times M \simeq D(\tau) \cup_{S(\tau)} C$$
,

in which $D(\tau)$ is the mapping cylinder of τ and $(C, S(\tau))$ is a Poincaré pair. It may be the case that for some choice of τ there is such a Poincaré embedding. Our main result will be to provide a complete obstruction in dimensions at least four. It will turn out that the obstruction group depends only on the fundamental group of M and the parity of d.

To describe the obstruction group, let π be the fundamental group of M and let $\bar{\pi}$ be its set of conjugacy classes. We denote the conjugacy class of a group element x by \bar{x} . Let $\mathbb{Z}[\bar{\pi}]$ be the free abelian group with basis $\bar{\pi}$. Define an involution on $\mathbb{Z}[\bar{\pi}]$ by

$$\bar{x} \mapsto (-1)^d \bar{x}^{-1}, \qquad x \in \pi$$

and extending linearly. Let

$$Q_d(\pi) := \mathbb{Z}[\bar{\pi}]_{\mathbb{Z}_2}$$

be the resulting abelian group of coinvariants. When $\pi = e$ is the trivial group, $Q_d := Q_d(e)$ is infinite cyclic or cyclic of order two depending on whether d is even or odd. Let

$$\tilde{Q}_d(\pi) = \operatorname{coker}(Q_d \to Q_d(\pi))$$

be the cokernel of the evident homomorphism $Q_d \rightarrow Q_d(\pi)$.

Theorem A Associated with each connected Poincaré duality space M of dimension d and fundamental group π , there is an obstruction

$$\mu_M \in \tilde{Q}_d(\pi) \,,$$

which depends only on the homotopy type of M and which vanishes when the diagonal $M \rightarrow M \times M$ admits a Poincaré embedding.

If $d \ge 4$ and μ_M is trivial, then there is such a Poincaré embedding.

- *Remarks* (1) The group $\tilde{Q}_d(\pi)$ is trivial if and only if π is. Consequently, if M is simply connected, there is no obstruction and M diagonally Poincaré embeds, recovering [12, Corollary H].
- (2) Aside from manifolds, there are some special cases with non-trivial π which give trivial μ_M . For example, according to Byun [2], when *M* is obtained by gluing two compact smooth manifolds along their boundaries using a homotopy equivalence (i.e, a two-stage patch space) and the square-root closed condition is satisfied by the fundamental group of the boundary, then *M* admits a diagonal embedding.
- (3) I do not know if μ_M is always trivial, but I suspect it need not be. If we view uniqueness as a kind of relative existence, then some evidence against the global triviality of μ_M is provided by recent work of Longoni and Salvatore [15] which essentially exhibits two distinct concordance classes of diagonal Poincaré embeddings of the lens space L(7, 1).
- (4) Dupont [8,9], Sutherland [19] and Byun [3] showed that the Spivak tangent fibration always admits a preferred (d-1)-spherical lift. If $\mu_M = 0$, one also obtains a preferred lift. The relationship between the two lifts is still unclear to me.

To find the unquotiented group $Q_d(\pi)$, we were heavily influenced by the paper of Hatcher and Quinn [11], which gives a bordism theoretic approach to deciding, in the metastable range, when an immersion of smooth manifolds is regularly homotopic to an embedding. Their obstruction was defined geometrically by placing the immersion in self-transverse position, taking the resulting double point manifold and considering it as a bordism class. In the special case of the diagonal map of a smooth manifold M, the Hatcher–Quinn bordism group is isomorphic to $Q_d(\pi)$ (of course, since the diagonal of a smooth manifold is an embedding, its Hatcher-Quinn invariant is trivial).

In the Poincaré category we cannot use the Hatcher–Quinn approach because the requisite transversality theory is lacking. Nevertheless, there is a way to work *externally* to define a homotopy theoretic version of the Hatcher–Quinn invariant. A crude version of the idea is this: think of an immersion $P \rightarrow N$ as a "cycle." It is then Poincaré dual to a "cocycle" which in this case is a *stable* Thom-Pontryagin map $N \rightarrow P^{\nu}$, where P^{ν} is the Thom space of the stable normal bundle (=stable spherical fibration in the Poincaré setting). If the immersion were an embedding, this stable map would desuspend to an actual map. This suggests that the obstruction we seek should be the one which realizes when the map desuspends. It is well-known that the Hopf invariant provides such an obstruction, once an unstable representative of ν has been chosen. In this paper, we will use a π -equivariant version of the Hopf invariant currently being developed by Crabb and Ranicki to capture some additional information lost by the ordinary Hopf invariant.

In addition to the above, the other problem one needs to consider is the indeterminacy associated with choosing different lifts of ν to an unstable (n-p-1)-spherical fibration. In the case of the diagonal map, the indeterminacy vanishes when one takes the quotient of $Q_d(\pi)$ by the subgroup Q_d . This leads to the definition of μ_M .

Outline Section 2 consists of preliminary material. Most of the technical language and definitions can be found in this section. In Sect. 3 we define variants of the Thom-Pontryagin collapse of a Poincaré embedding. In Sect. 4 we consider the problem of compressing Poincaré embeddings into one dimension beyond the stable range. Section 5 develops the equivariant stable Hopf invariant and uses it to reinterpret the results of Sect. 4. In Sect. 6 we consider Poincaré embeddings of the diagonal for fixed choice of (d-1)-spherical lift of the stable tangent fibration. Section 7 explains how the set of equivalence classes of (d-1)-spherical lifts of the stable tangent fibration form a Q_d -torsor. In Sect. 8 we prove Theorem A.

2 Preliminaries

Spaces. The spaces of this paper are to be topologized with the compactly generated topology. A map $X \to Y$ of based spaces is a *weak equivalence* if it is a weak homotopy equivalence. A weak equivalence is written with the symbol " $\xrightarrow{\sim}$ ", and we use the notation $X \simeq Y$ to indicate that *X* and *Y* are equated by a chain of weak equivalences. We write [X, Y] for based homotopy classes of maps from *X* to *Y*.

A non-empty (possibly unbased) space Z is 0-connected if it is path connected. It is r-connected with r > 0, if it is 0-connected and $[S^k, Z]$ has one element for $k \le r$. A map of non-empty (possibly unbased) spaces is r-connected if its homotopy fiber, taken with respect to all choices of basepoint, is (r-1)-connected.

A *stable map* from X to Y is a map $X \to QY$ where $Q = \Omega^{\infty} \Sigma^{\infty}$ is the stable homotopy functor. Alternatively, it is colimit under suspension of maps of the form $\Sigma^j X \to \Sigma^j Y$. Stable maps can be composed. We write $\{X, Y\}$ for the abelian group of stable homotopy classes

of maps from X to Y. A stable map is denoted with "+." We sometimes write $X \simeq_s Y$ to indicate that X and Y are related by a chain of stable weak equivalences.

If *G* is a group and *X* and *Y* are now based *G*-spaces, then an equivariant map $X \to Y$ is a weak equivalence if it is a weak homotopy equivalence of underlying topological spaces. We write $[X, Y]_G$ to indicate equivariant homotopy classes of equivariant based maps, and $\{X, Y\}_G$ for the equivariant stable homotopy classes. We note that the kind of equivariant homotopy we are considering here is the naive one in which the group is acting trivially on suspension coordinates, and all objects are weak equivalent to ones with free actions.

Poincaré spaces. A finitely dominated space *X* is a *Poincaré duality space* of dimension *d* if there exist a bundle of coefficients \mathcal{L} which is locally isomorphic to \mathbb{Z} , and a class $[X] \in H_d(X; \mathcal{L})$ such that the associated cap product homomorphism

$$\cap [X]: H^*(X; M) \to H_{d-*}(X; \mathcal{L} \otimes M)$$

is an isomorphism in all degrees. Here, M denotes any bundle of coefficients (cf. [14,21]). The pair $(\mathcal{L}, [X])$ is uniquely defined up to unique isomorphism; \mathcal{L} is called the *orientation bundle* and [X] a *fundamental class*.

Similarly, one has the definition of Poincaré space X with boundary ∂X (also called a *Poincaré pair* $(X, \partial X)$). Here, one assumes both X and ∂X are finitely dominated and there is a fundamental class $[X] \in H_d(X, \partial X; \mathcal{L})$ such that

$$\cap [X]: H^*(X; M) \to H_{d-*}(X; \partial X; \mathcal{L} \otimes M)$$

is an isomorphism. Additionally, if $[\partial X]$ is the image of [X] under the boundary homomorphism $H_d(X, \partial X; \mathcal{L}) \to H_{d-1}(\partial X; \mathcal{L}_{|\partial X})$, one also requires

$$(\mathcal{L}_{\mid \partial X}, [\partial X])$$

to equip ∂X with the structure of a Poincaré space.

Unless otherwise indicated, we usually work with *connected* Poincaré spaces. We say that a Poincaré space X is *closed* if it is without boundary.

Poincaré embeddings. Let

$$f: P \to N$$

be a map of connected Poincaré duality spaces, where

- *P* is closed, but *N* need not be;
- *P* has dimension *p*, *N* has dimension *n*;
- $p \leq n-1$.

A Poincaré embedding of f consists of an (n-p-1)-spherical fibration

$$\xi: S(\xi) \to P$$

and commutative diagram



such that

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• $D(\xi)$ is the mapping cylinder ξ (in particular, $D(\xi) \simeq P$), and f_{ξ} is the composite

$$D(\xi) \simeq P \stackrel{f}{\to} N$$
.

- The square in the diagram is a homotopy pushout.
- The composite $\partial N \to C \to N$ is the inclusion.
- The image of a fundamental class [N] under the composite

$$H_n(N, \partial N; \mathcal{L}) \to H_n(N, C; \mathcal{L}) \cong H_n(D(\xi), S(\xi); \mathcal{L}_{|D(\xi)})$$

gives $(D(\xi), S(\xi))$ the structure of a Poincaré space. Similarly, the image of a fundamental class under

$$H_n(N, \partial N; \mathcal{L}) \to H_n(N, D(\xi) \amalg \partial N; \mathcal{L}) \cong H_n(C, S(\xi) \amalg \partial N; \mathcal{L}_{|C})$$

equips $(C, D(\xi) \amalg \partial N)$ with the structure of a Poincaré space.

We warn the reader that to avoid notational clutter, we are not necessarily assuming in the above that $S(\xi) \amalg \partial N \to C$ is an inclusion, and we are slightly abusing notation when writing expressions such as $H_n(N, C; \mathcal{L})$ (the reader should substitute the appropriate mapping cylinder in such cases).

The spherical fibration ξ appearing above is called the *normal datum* of the Poincaré embedding. We say in this case that that we have a *codimension zero* Poincaré embedding of the map $f_{\xi}: D(\xi) \to N$.

We will define concordance only in the codimension zero setting. Note that a codimension zero Poincaré embedding is really just a certain kind of object

$$C \in \mathbf{Top}_{S(\xi) \sqcup \partial \to N}$$
,

where the latter is the category of spaces which factorize the given map $S(\xi) \amalg \partial \to N$. A morphism in this category is just a map of spaces which is compatible with the given factorizations. We say that such a morphism is a weak equivalence if it is a weak homotopy equivalence of underlying topological spaces. Given two Poincaré embeddings C, C', considered as objects in the above category, we say that they are *concordant* if there is a chain of weak equivalences connecting them.

Poincaré Immersions. Let $f: P^p \to N^n$ be as above, and suppose we are given an (n-p-1)-spherical fibration $\xi: S(\xi) \to P$. By analogy with Smale-Hirsch theory, a *Poincaré immersion* of (f, ξ) is a choice of (codimension zero) Poincaré embedding of the map

$$f_{\xi \oplus \epsilon^j} \colon D(\xi \oplus \epsilon^j) = D(\xi) \times D^j \xrightarrow{f_{\xi} \times \mathrm{id}_{D^j}} N \times D^j$$

for some $j \ge 0$. Here ϵ^j denotes the trivial (j-1)-spherical fibration over P and \oplus is the fiberwise join operation.

In [13], it was shown that (f, ξ) immerses if and only if there is a fiber homotopy equivalence

$$\nu_P \oplus \xi \simeq f^* \nu_N$$
,

where v_P and v_N are the Spivak fibrations of P and N, respectively.

3 The collapse

The classical collapse. Given (f, ξ) as above, and a Poincaré embedding



we get maps of pairs

$$(N, \partial N) \longrightarrow (N, C) \longleftarrow (D(\xi), S(\xi))$$

which yields collapse map

$$N/\partial N \longrightarrow N/C \xleftarrow{\sim} P^{\xi}$$

upon taking quotients, where $P^{\xi} = D(\xi)/S(\xi)$ is the *Thom space* of ξ .

This gives a well defined homotopy class

$$c_f \in [N/\partial N, P^{\xi}].$$

The equivariant version. Let

$$\tilde{N} \rightarrow N$$

be a universal cover with π its group of deck transformations. Then π is identified with the fundamental group of N and it acts freely on \tilde{N} . The orbit space \tilde{N}/π coincides with N.

For any map $X \to N$, we define

$$\tilde{X} := X \times_N \tilde{N}$$

to be the fiber product of X and \tilde{N} along N. Then \tilde{X} inherits a free π -action.

Returning again to our Poincaré embedding diagram, we pull all terms back along \tilde{N} to get a diagram of free π -spaces



where $\partial \tilde{N}$ is shorthand for $\partial N \times_N \tilde{N}$. Note that the square in this diagram is a homotopy pushout because it comes about by pulling back a homotopy pushout along \tilde{N} .

Arguing just as in the classical case, we get a collapse map which is now π -equivariant. So we obtain an equivariant homotopy class

$$\tilde{c}_f \in [\tilde{N}/\partial \tilde{N}, \tilde{P}^{\xi}]_{\pi}$$

where $\tilde{P}^{\xi} := \tilde{D}(\xi)/S(\xi)$ is the *equivariant Thom space*. We call this construction the *equivariant collapse* of the Poincaré embedding.

If $\partial N = \emptyset$, then observe $\tilde{N}/\partial \tilde{N}$ is $\tilde{N}^+ =$ the union of \tilde{N} with a disjoint basepoint.

The equivariant stable collapse. Given $f: P \rightarrow N$, we consider the virtual spherical fibration

$$f^*\tau_N - \tau_P$$

over P. This formal difference is to be interpreted in the Grothendieck group of spherical fibrations over P.

It is well-known that every stable spherical fibration over a k-dimensional CW complex arises from an unstable (k - 1)-spherical fibration. It follows that If j is sufficiently large, the sum $\xi := f^* \tau_N - \tau_P \oplus \epsilon^j$ will admit an unstable spherical fibration representative (here, $j \ge 2p - n$ will suffice). It follows that when j is large, the composite

$$P \xrightarrow{f} N \subset N \times D^j$$

will admit a Poincaré embedding with normal datum ξ .

Then we get an equivariant collapse, which is an element of

$$\left[\Sigma^{j}\tilde{N}/\partial\tilde{N},\,P^{\xi}\right]_{\pi}$$

Letting j tend to infinity, we obtain an equivariant stable homotopy class

$$\tilde{c}_f \in \left\{ \tilde{N}/\partial \tilde{N}, \, \tilde{P}^{f^*\tau_N - \tau_P} \right\}_{\pi} \,,$$

where $\tilde{P}^{f^*\tau_N-\tau_P}$ is the *equivariant Thom spectrum* of the virtual spherical fibration $f^*\tau_N-\tau_P$.

We call \tilde{c}_f the *equivariant stable collapse*; it only depends on the homotopy class of $f: P \to N$.

4 Compression

If $Y \to N$ is a map, i.e., Y is a "space over" N, then we can form the *unreduced fiberwise* suspension

$$S_N Y := N \times 0 \cup_{Y \times 0} Y \times [0, 1] \cup_{Y \times 1} N \times 1$$

which is a space over $N \times D^1 \cong N \times [0, 1]$. Let

$$S_N^j Y$$

be its *j*-fold iterate; it is a space over $N \times D^{j}$.

Given a Poincaré embedding



we apply fiberwise suspension j-times to obtain a new Poincaré embedding

(Note that $S_N^j \partial N$ is $\partial (N \times D^j)$.) This is the *j*-fold decompression of the given Poincaré embedding.

We wish to find obstructions to reversing the above. That is, fix a Poincaré immersion of $f: P \rightarrow N$. Under what circumstances can one find a Poincaré embedding of $f: P \rightarrow N$ which decompresses to the given Poincaré immersion?

To answer this question, we take the equivariant stable collapse

$$\tilde{c}_f \in \left\{ \tilde{N}/\partial \tilde{N}, \, \tilde{P}^{f^*\tau_N - \tau_P} \right\}_{\pi}$$

and ask whether it comes from an unstable class.

More precisely, assume that 2p = n. Then by classical obstruction theory, there exists an unstable (p-1)-spherical fibration ξ over P which represents $f^*\tau_N - \tau_P$. Note that ξ is not necessarily unique.

Theorem 4.1 Assume n = 2p and $p \ge 4$. Choose a (p-1)-spherical fibration ξ representing $f^*\tau_N - \tau_P$. Then f admits a Poincaré embedding with normal datum ξ if and only if \tilde{c}_f destabilizes to an element of $\left[\tilde{N}/\partial \tilde{N}, \tilde{P}^{\xi}\right]_{\pi}$.

Proof This is a corollary of [12, Theorem B] and the proof is essentially the same as that of [12, Theorem E] (for the proof, see Sect. 4 of that paper). \Box

5 The equivariant stable Hopf invariant

Let Y be a based space. The quadratic construction on Y is the based space

$$D_2Y := (Y \wedge Y) \wedge_{\mathbb{Z}_2} E\mathbb{Z}_2^+,$$

which is the homotopy orbit space of \mathbb{Z}_2 -acting on the smash product $Y \wedge Y$ by permuting factors. Explicitly, let $E\mathbb{Z}_2$ denote the infinite sphere with antipodal action. Form the cartesian product $Y \times Y \times E\mathbb{Z}_2$ and collapse the subspace $(Y \vee Y) \times E\mathbb{Z}_2$ to a point. This results in a based \mathbb{Z}_2 -space. The associated orbit space is then D_2Y .

For based spaces X and Y, the stable Hopf invariant is a certain function

$$H: \{X, Y\} \to \{X, D_2Y\}$$

This function is a natural transformation in X, and hence, by the Yoneda lemma, it is determined by a map

$$QY \rightarrow QD_2Y$$

where Q is the stable homotopy functor. For a stable map $f: X \leftrightarrow Y$, the class H(f) is an obstruction to desuspending f to an unstable homotopy class. It is a complete obstruction in the metastable range.

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In this paper, we will need to generalize the stable Hopf invariant to the G-equivariant setting for a discrete group G. Let X and Y now be based G-spaces. Then we will describe a function

$$\tilde{H}: \{X, Y\}_G \to \{X, D_2Y\}_G,\$$

which gives an obstruction to equivariantly destabilizing a stable G-map.

The construction of \tilde{H} is basically the same as one of the well-known constructions of H; the details will appear in a forthcoming book of Crabb and Ranicki [6] (see [7] for a preliminary version). We will sketch their construction, which uses G. Segal's \mathbb{Z}_2 -equivariant stable homotopy functor.

Let \mathcal{U} be a complete universe of \mathbb{Z}_2 -representations. For example, we can take \mathcal{U} to be a countable direct sum of the regular representation. For $V \in \mathcal{U}$ we let S^V denote its one point compactification. Given Y as above, we set

$$Q_{\mathbb{Z}_2}(Y \wedge Y) := \operatorname{colim}_{V \in \mathcal{U}} \operatorname{map}\left(S^V, S^V \wedge Y \wedge Y\right),$$

where the function space appearing above is given the \mathbb{Z}_2 -action arising from conjugating maps. Then $Q_{\mathbb{Z}_2}(Y \wedge Y)$ is a $(G \times \mathbb{Z}_2)$ -space. Let

$$Q_{\mathbb{Z}_2}(Y \wedge Y)^{\mathbb{Z}_2}$$

denote the fixed point set of \mathbb{Z}_2 acting on $Q_{\mathbb{Z}_2}(Y \wedge Y)$. This is a *G*-space.

If Z is a based G-space, we let

$$\{Z, Y \wedge Y\}_G^{\mathbb{Z}_2}$$

denote the abelian group of *G*-equivariant homotopy classes of *G*-maps $Z \to Q_{\mathbb{Z}_2}(Y \wedge Y)^{\mathbb{Z}_2}$. Note that such a homotopy class is determined by specifying a $V \in \mathcal{U}$ and a $(G \times \mathbb{Z}_2)$ -equivariant map

$$S^V \wedge Z \to S^V \wedge Y \wedge Y$$

(however, not all homotopy classes need arise in this way). In particular, the reduced diagonal $\Delta_Y : S^0 \wedge Y \to S^0 \wedge Y \wedge Y$ is such a map.

Now, given a stable G-map

$$f: X \leftrightarrow Y$$

we consider the difference

$$\delta(f) := (f \wedge f)\Delta_X - \Delta_Y f \in \{X, Y \wedge Y\}_G^{\mathbb{Z}_2}.$$

According to a formula of tom Dieck [20, Theorem 2] (cf. [5, Corollary A.3], [4], [16, p. 203]), there is a homotopically split fibration sequence of *G*-spaces

$$QD_2Y \xrightarrow{i} Q_{\mathbb{Z}_2}(Y \wedge Y)^{\mathbb{Z}_2} \xrightarrow{j} QY.$$

In particular, we have a (split) short exact sequence

$$0 \to \{X, D_2Y\}_G \xrightarrow{i_*} \{X, Y \land Y\}_G^{\mathbb{Z}_2} \xrightarrow{j_*} \{X, Y\}_G \to 0.$$

One observes that the composite

$$j_*\delta(f)$$

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is trivial. Consequently, there is a unique element

$$\tilde{H}(f) \in \{X, D_2Y\}_G$$

such that $i_*\tilde{H}(g) = \delta(g)$. We call $\tilde{H}(f)$ the π -equivariant stable Hopf invariant of f^{1} .

Notation We will usually use \tilde{H} when the group G is understood. When the group is ambiguous we resort to the notation H_G to refer to \tilde{H} . In particular, for the trivial group e, $H_e = H$ is the classical stable Hopf invariant.

Desuspension. Assume *X* has the equivariant weak homotopy type of a *G*-space obtained from a point by attaching free *G*-cells of dimension at most *k*. Assume *Y* is *r*-connected and *Y* has the equivariant homotopy type of a based *G*-CW complex, where the action of *G* is free away from the basepoint.

Proposition 5.1 Assume in addition $k \leq 3r + 1$. Then $\tilde{H}(f) = 0$ if and only if $f \in \{X, Y\}_G$ desuspends to an element of $[X, Y]_G$.

Proof The proof that the equivariant Hopf invariant is trivial for maps which desuspend follows immediately from the fact that $(f \land f) \circ \Delta_X = \Delta_Y \circ f$ for unstable *G*-maps $f: X \to Y$.

To prove the other direction, note that equivariant Hopf invariant in the universal case $\iota: QY \rightarrow Y$ (= adjoint to the identity map of QY) is represented by a *G*-map

$$\tilde{H}(\iota): QY \to QD_2Y$$

which coincides with the classical (unequivariant) stable Hopf invariant when the G-actions are forgotten.

Consequently, if $Y \rightarrow QY$ is the inclusion, we see that

$$Y \to QY \xrightarrow{\tilde{H}(l)} QD_2Y$$

is a homotopy fibre sequence of underlying topological spaces up through dimension 3r + 1 (cf. [5,17]; the above represents the classical EHP sequence in stable homotopy) having the additional property that each of the maps is *G*-equivariant. In particular the map from *Y* into the homotopy fiber of $\tilde{H}(\iota)$ is (3r + 1)-connected. Furthermore, observe that the naturality of \tilde{H} gives $\tilde{H}(f) = \tilde{H}(\iota) \circ \hat{f}$, where $\hat{f} : X \to QY$ is adjoint to f.

Consequently, if X is obtained from a point by attaching free G-cells of dimension $\leq k$, and $k \leq 3r + 1$, it follows from the above sequence that $\tilde{H}(f)$ is trivial if and only if f equivariantly desuspends.

Corollary 5.2 Assume $k \leq 3r + 1$ as above. Then the sequence of based sets

$$[X,Y]_G \longrightarrow \{X,Y\}_G \xrightarrow{\tilde{H}} \{X,D_2Y\}_G$$

is exact.

The composition formula. We will need to know how \tilde{H} behaves with respect to compositions. Suppose $f: X \leftrightarrow Y$ and $g: Y \leftrightarrow Z$ are stable *G*-maps.

Proposition 5.3

$$\tilde{H}(g \circ f) = \tilde{H}(g) \circ f + D_2(g) \circ \tilde{H}(f) \,.$$

¹ Crabb and Ranicki use the terminology geometric Hopf invariant for $\tilde{H}(f)$.

This will be proved in [6] (cf. [7]). The formula is well-known when G is the trivial group. When G is non-trivial one basically copies the proof for trivial G noticing that all constructions are equivariant.

Extension of groups. Let $K \to G$ be a group monomorphism, and suppose $f: X \to Y$ is an equivariant stable map of based *K*-spaces. We assume without loss in generality that *X* and *Y* are built up from a point by attaching free *K*-cells. Then we have a stable map of *G*-spaces

$$f_{K\uparrow G} = f \wedge \mathrm{id}_{G^+} \colon X \wedge_K (G^+) \nleftrightarrow Y \wedge_K (G^+)$$

given by *inducing* f along $K \to G$. One defines $X \wedge_K (G^+)$ by taking orbits with respect to the diagonal action of K on the smash product $X \wedge (G^+)$ (where K acts on G^+ by left multiplication).

The inclusion $Y \to Y \wedge_K (G^+)$ is *K*-equivariant and therefore induces a *K*-equivariant map of quadratic constructions

$$D_2Y \to D_2(Y \wedge_K (G^+))$$

which admits a preferred extension to a G-map

$$i_2: D_2Y \wedge_K (G^+) \to D_2(Y \wedge_K (G^+)).$$

The following lemma relates $H_G(f_{K\uparrow G})$ to $H_K(f)$. Its proof, which we omit, is a straightforward diagram chase.

Lemma 5.4

$$H_G(f_{K\uparrow G}) = i_2 \circ (H_K(f) \wedge_K \operatorname{id}_{G^+}).$$

Application to compression. We now apply 5.2 to the problem of compressing Poincaré embeddings. As in the previous section, we are given a map $f: P^p \to N^n$, with P and N Poincaré spaces, with P closed and n = 2p. Let π be the fundamental group of N.

Then we have the associated equivariant stable collapse

$$\tilde{c}_f \in \left\{\tilde{N}/\partial\tilde{N}, \,\tilde{P}^{f^*\tau_N-\tau_P}\right\}_{\pi}$$

Given a choice of unstable lift ξ of $f^*\tau_N - \tau_P$ to a (p-1)-spherical fibration, we obtain an explicit identification

$$\tilde{P}^{f^*\tau_N - \tau_P} := \tilde{P}^{\xi} . \tag{1}$$

which equates the equivariant Thom spectrum on the left with a representative equivariant Thom space on the right.

Using this identification, the equivariant stable Hopf invariant is realized as a function

$$\tilde{H}^{\xi} \colon \left\{ \tilde{N}/\partial \tilde{N}, \, \tilde{P}^{f^*\tau_N - \tau_P} \right\}_{\pi} \to \left\{ \tilde{N}/\partial \tilde{N}, \, D_2 \tilde{P}^{f^*\tau_N - \tau_P} \right\}_{\pi}$$

in which are now indicating ξ as a superscript the notation since it uses (1).

Proposition 5.5 Assume n = 2p and $p \ge 4$. Then $f_{\xi} \colon D(\xi) \to N$ admits a Poincaré embedding if and only if

$$\tilde{H}^{\xi}(\tilde{c}_f) \in \left\{ \tilde{N}/\partial \tilde{N}, D_2 \tilde{P}^{f^*\tau_N - \tau_P} \right\}_{\pi}$$

is trivial.

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Proof This is an immediate consequence 4.1 together with 5.2 (using k = n and r = n - p + 1).

6 Immersions of the diagonal

Suppose *M* is a closed Poincaré space of dimension $d \ge 4$ with fundamental group π and stable Spivak tangent bundle τ . We let $\tilde{M} \to M$ denote the universal cover. Then $\pi \times \pi$ is the fundamental group of $M \times M$ and $\tilde{M} \times \tilde{M} \to M \times M$ is its universal cover. Consider the $(\pi \times \pi)$ -equivariant stable collapse of the diagonal map $\Delta : M \to M \times M$:

$$\tilde{c}_{\Delta} \in \left\{ \tilde{M}^+ \wedge \tilde{M}^+, \hat{M}^{\tau} \right\}_{\pi \times \pi}$$

Here \hat{M}^{τ} is the $(\pi \times \pi)$ -equivariant Thom spectrum defined as follows: consider the regular $(\pi \times \pi)$ -covering space

$$\hat{M} \to M$$

defined by pulling back the universal cover of $M \times M$. Then \hat{M}^{τ} is the Thom spectrum of the pullback of τ to \hat{M} . Note that \hat{M} can also be considered as the effect of inducing \tilde{M} along the diagonal homomorphism $\pi \to \pi \times \pi$:

$$\hat{M} = \tilde{M} \times_{\pi} (\pi \times \pi)$$

and, likewise, $\hat{M}^{\tau} = \tilde{M}^{\tau} \wedge_{\pi} (\pi \times \pi)^+$, where \tilde{M}^{τ} is the Thom spectrum of the pullback of τ to \tilde{M} .

Now choose such a (d-1)-spherical lift $\xi : S(\xi) \to M$ of τ . Then we have the associated map

$$\Delta_{\xi} \colon D(\xi) \simeq M \xrightarrow{\Delta} M \times M.$$

Applying 5.5, we immediately get

Proposition 6.1 The map $\Delta_{\xi} : D(\xi) \to M \times M$ admits a Poincaré embedding if and only if

$$\tilde{H}^{\xi}(\tilde{c}_{\Delta}) \in \left\{ \tilde{M}^{+} \wedge \tilde{M}^{+}, D_{2}\hat{M}^{\tau} \right\}_{\pi \times \pi}$$

is trivial.

Our next step will be to identify the obstruction group appearing in 6.1. Let that $Q_d(\pi) = \mathbb{Z}[\bar{\pi}]_{\mathbb{Z}_2}$ be the abelian group defined in the introduction.

Theorem 6.2 There is a preferred isomorphism of abelian groups

$$\left\{\tilde{M}^+ \wedge \tilde{M}^+, D_2(\hat{M}^\tau)\right\}_{\pi \times \pi} \cong Q_d(\pi) \,.$$

The proof will make use of the following lemma.

Lemma 6.3 Let N be a connected Poincaré duality space of dimension n equipped with orientation sheaf \mathcal{L} and fundamental class [N]. Let E be a naive π -spectrum (or based stable π -space). Assume E is (n-1)-connected.

Then there is a preferred isomorphism of abelian groups

$$\left\{\tilde{N}^+, E\right\}_{\pi} \cong H_0(N; \mathcal{L} \otimes H_n(E)).$$

Proof Obstruction theory gives an isomorphism

$$\left\{\tilde{N}^+, E\right\}_{\pi} = H^n(N; H_n(E)),$$

where the right side is the cohomology in degree *n* of *N* with coefficients in the π -module $H_n(E)$. Poincaré duality identifies the right side with

$$H_0(N; \mathcal{L} \otimes H_n(E))$$
.

Proof of Theorem 6.2 The $(\pi \times \pi)$ -spectrum $D_2(\hat{M}^{\tau})$ is (2d - 1)-connected, so by 6.3, we have an isomorphism

$$\left\{\tilde{M}^{+}\wedge\tilde{M}^{+},D_{2}(\hat{M}^{\tau})\right\}_{\pi\times\pi}\cong H_{0}\left(M\times M;\left(\mathcal{L}\times\mathcal{L}\right)\otimes H_{2d}\left(D_{2}\left(\hat{M}^{\tau}\right)\right)\right),$$

where \mathcal{L} is the orientation bundle for M. Furthermore, we can rewrite the right side as the coinvariants of the action of \mathbb{Z}_2 on

$$H_0\left(M\times M; \left(\mathcal{L}\times\mathcal{L}\right)\otimes H_{2d}\left(\hat{M}^{\tau}\wedge\hat{M}^{\tau}\right)\right)$$

coming from the self map of $\hat{M}^{\tau} \wedge \hat{M}^{\tau}$ which switches factors.

Application of the Thom isomorphism gives an isomorphism of $(\pi \times \pi)$ -modules

$$(\mathcal{L} \times \mathcal{L}) \otimes H_{2d} \left(\hat{M}^{\tau} \wedge \hat{M}^{\tau} \right) \cong H_0 \left(\hat{M} \right) \otimes H_0 \left(\hat{M} \right).$$

With respect to this isomorphism, the \mathbb{Z}_2 -action on $H_0\left(\hat{M}\right) \otimes H_0\left(\hat{M}\right)$ switches factors and multiplies by $(-1)^d$. Consequently, we are reduced to computing

$$H_0\left(M \times M; H_0\left(\hat{M}\right) \otimes H_0\left(\hat{M}\right)\right)$$

with the given involution. The $(\pi \times \pi)$ -module $H_0(\hat{M})$ is $\mathbb{Z}[\pi^{ad}]$, where the latter is the free module on π given the adjoint action of $\pi \times \pi$. We therefore have an isomorphism of $(\pi \times \pi)$ -modules

$$H_0\left(\hat{M}
ight)\otimes H_0\left(\hat{M}
ight)\ \cong\ \mathbb{Z}\left[\pi^{\,\mathrm{ad}} imes\pi^{\,\mathrm{ad}}
ight]\,,$$

in which the involution on the left corresponds to the one on the right given by switching factors and multiplying by $(-1)^d$. Here $\pi^{ad} \times \pi^{ad}$ is given the diagonal $\pi \times \pi$ -action.

With respect to the above, the abelian group

$$H_0\left(M \times M; H_0\left(\hat{M}\right) \otimes H_0\left(\hat{M}\right)
ight)$$

is now identified with the coinvariants of $\pi \times \pi$ acting on $\mathbb{Z}[\pi^{ad} \times \pi^{ad}]$. But this group of coinvariants is just $\mathbb{Z}[\bar{\pi}]$: an isomorphism is given by mapping a conjugacy class \bar{x} to the pair (1, x), where $x \in \pi$ denotes any representative of \bar{x} . With respect to this isomorphism, the involution on $\mathbb{Z}[\bar{\pi}]$ is given by $\bar{x} \mapsto (-1)^d \bar{x}^{-1}$. This completes the proof of 6.2.

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7 Lifts of the tangent fibration

Let *M* be a closed, connected Poincaré space of dimension $d \ge 4$ with Spivak tangent fibration $\tau: M \to BG$. Here *BG* denotes the classifying space for stable spherical fibrations, and we let *BG_d* denote the classifying space for (d-1)-spherical fibrations. The (inclusion) map

$$i: BG_d \rightarrow BG$$

is *d*-connected. To simplify the exposition, we assume *i* is a fibration, by converting it to one.

It follows that τ is represented by an (d-1)-spherical fibration, i.e., there is a section $\xi: M \to BG_d$ of the fibration *i* along τ . The set of *lifts*

 \mathfrak{L}_{τ}

of τ is defined to be the homotopy classes of sections of *i* along τ . An element of \mathfrak{L}_{τ} is therefore represented by a (d-1)-spherical fibration ξ together with a choice of stable fiber homotopy equivalence $h: \xi \simeq_s \tau$.

The fiber of *i* at the basepoint is identified with G/G_d . It is (d - 1)-connected, and its *d*-homotopy group is canonically isomorphic to Q_d .

By classical obstruction theory, there is a free and transitive action

$$H^d(M; \mathcal{L} \otimes \pi_d(G/G_d)) \times \mathfrak{L}_{\tau} \to \mathfrak{L}_{\tau}$$

where the cohomology group

$$H^d(M; \mathcal{L} \otimes \pi_d(G/G_d))$$
.

encodes the difference obstructions to lifts on the top dimension of M (recall that \mathcal{L} is the orientation sheaf). By Poincaré duality, we obtain an isomorphism of the above with $H_0(M; \pi_d(G/G_d)) = Q_d$. Summarizing,

Lemma 7.1 \mathfrak{L}_{τ} is a Q_d -torsor, i.e., there is a free and transitive action

 $Q_d \times \mathfrak{L}_{\tau} \to \mathfrak{L}_{\tau}$,

and therefore, if we choose a basepoint $\xi \in \mathfrak{L}_{\tau}$, we obtain an isomorphism of based sets $Q_d \cong \mathfrak{L}_{\tau}$.

A direct description of the action is as follows: a result of Wall [21, 2.4] shows that *M* possesses a top cell decomposition

$$M\simeq M_0\cup_\beta D^d$$
,

with M_0 a CW complex of dimension $\leq d-1$ if $d \geq 4$. For simplicity, fix $M = M_0 \cup D^n$. The set of lifts of τ along M_0 is trivial. Hence, a given lift ξ is completely described by its value on D^d . Now, suppose we are given representations of lifts $\xi : M \to BG_d$ and $\eta : M \to BG_d$ of τ . Then ξ and η are homotopic when restricted to M_0 . We can apply the homotopy lifting property to change η by a homotopy to a representative $\eta' : M \to BG_d$ whose restriction to M_0 identically coincides with τ . But η' and η represent the same lift. Consequently, we can assume without loss in generality that ξ and η identically coincide on M_0 .

Now form the (d-1)-spherical fibration over $S^d = D^d_- \cup D^d_+$ by taking η along the $D^d_$ and ξ along D^d_+ . This fibration comes equipped with a preferred stable trivialization, so it defines an element of $\pi_d(G/G_d) = Q_d$. Thus pairs of lifts "differ" by an element of Q_d . Note the special case when τ is trivial gives a preferred basepoint in \mathfrak{L}_{τ} , namely, the trivial lift. In this case, \mathfrak{L}_{τ} is just $[M, G/G_d]$, and the identification with Q_d is defined by mapping a homotopy class $\gamma : M \to G/G_d$ to $\gamma_*([M]) \in H_d(G/G_d) = Q_d$.

Remarks 7.2 The papers of Dupont [8,9] and Sutherland [19] concern explicit detection of the image of the function

$$\mathfrak{L}_{\tau} \to [M, BG_d].$$

When *d* is even, the Euler characteristic $\chi : [M, BG_d] \to \mathbb{Z}$ detects the image. When *d* is odd, the image has either one or two elements, and the invariant which detects them is subtler.

8 Proof of Theorem A

We will define a function

$$\phi\colon \mathfrak{L}_{\tau} \to \left\{ \tilde{M}^+ \wedge \tilde{M}^+, D_2 \hat{M}^{\tau} \right\}_{\pi \times \pi} \stackrel{\mathrm{Th}. \, 6.2}{\cong} Q_d(\pi) \, .$$

An element of \mathfrak{L}_{τ} is represented by a spherical fibration together with a stable fiber homotopy equivalence $h: \xi \simeq_s \tau$. Then h induces an identification $\hat{M}^{\xi} \simeq_s \hat{M}^{\tau}$, and the equivariant stable collapse \tilde{c}_{Δ} becomes a stable map

$$\tilde{M}^+ \wedge \tilde{M}^+ \leftrightarrow \hat{M}^{\xi}$$
.

Then

$$\phi(\xi) := \tilde{H}(\tilde{c}_{\Delta}) \in \left\{ \tilde{M}^+ \wedge \tilde{M}^+, D_2 \hat{M}^\tau \right\}_{\pi \times \pi}$$

Recall that \mathfrak{L}_{τ} comes equipped with a free and transitive action of Q_d . Recall that the inclusion of the trivial group into π induces an inclusion

$$i: Q_d \subset Q_d(\pi)$$

of subgroups.

Theorem 8.1 The function ϕ is one-to-one and has image equal to a coset of $Q_d \subset Q_d(\pi)$.

Remarks 8.2 In the category of smooth manifolds, one can give a short geometric proof of 8.1 using Whitney's method [22, Theorem 3] of locally modifying the double point set of a generic immersion by introducing an extra self-intersection. This method is not available in the Poincaré category. We will use homotopy theoretic methods to prove 8.1.

We first note how 8.1 implies Theorem A. By 5.5, the diagonal $\Delta: M \to M \times M$ will admit a Poincaré embedding if and only if the equation $\phi(\xi) = 0$ has a solution. By 8.1, $\phi(\mathfrak{L}_{\tau}) \subset Q_d(\pi)$ forms a coset of the subgroup $Q_d \subset Q_d(\pi)$. Consequently, solutions to the given equation exist if and only if the coset $\phi(\mathfrak{L}_{\tau})$ contains 0. Define the obstruction

$$\mu_M \in Q_d(\pi) \quad (= Q_d(\pi)/Q_d)$$

to be the equivalence class of the coset $\phi(\mathfrak{L}_{\tau})$. Then μ_M is the complete obstruction to finding a Poincaré embedding of the diagonal. The proof of Theorem A has now been reduced to establishing 8.1.

The proof of 8.1 will require some preparation. Suppose that $h \in Q_d$, $\xi, \eta \in \mathfrak{L}_{\tau}$ are elements satisfying $h \cdot \eta = \xi$. An unraveling of definitions allows use to think of *h* as a stable fiber homotopy equivalence

 $\eta \simeq_s \xi$

given by composing the pair of stable equivalences $\eta \simeq_s \tau \simeq_s \xi$. Then *h* induces a $(\pi \times \pi)$ -equivariant stable equivalence

$$\hat{h}: \hat{M}^{\xi} \leftrightarrow \hat{M}^{\eta}$$
.

The $(\pi \times \pi)$ -equivariant stable Hopf invariant \tilde{H} of \hat{h} is then an equivariant stable map

$$\tilde{H}(\hat{h}): \hat{M}^{\xi} \Rightarrow D_2 \hat{M}^{\eta},$$

which, in view of our identifications, we are entitled to regard is an equivariant stable map

$$\hat{M}^{\tau} \Rightarrow D_2 \hat{M}^{\tau}$$
.

Lemma 8.3 With respect to the above conventions,

$$\phi(\xi) - \phi(\eta) = \tilde{H}(\hat{h}) \circ \tilde{c}_{\Delta}.$$

Proof This is just a specific instance of the composition formula 5.3.

Lemma 8.4 The operation

$$h \mapsto \tilde{H}\left(\hat{h}\right) \circ \tilde{c}_{\Delta}$$

gives a homomorphism $\psi \colon Q_d \to Q_d(\pi)$.

Proof This follows again by the composition formula 5.3. If $g, h \in Q_d$, then we think of h as stably identifying ξ with η and g as stably identifying μ with ξ . Then

$$\tilde{H}\left(\hat{g}\circ\hat{h}\right)=\tilde{H}\left(\hat{g}\right)\circ\hat{h}+D_{2}\left(\hat{g}\right)\circ\tilde{H}\left(\hat{h}\right)$$

with respect to our identifications, this is the homomorphism property.

Lemma 8.5 The image of the homomorphism ψ is contained within the subgroup Q_d .

Proof A stable fiber homotopy equivalence $h: \xi \simeq_s \eta$ yields a stable π -map $\tilde{h}: \tilde{M}^{\xi} \to \tilde{M}^{\eta}$, which when induced along the diagonal homomorphism $\pi \to \pi \times \pi$, yields $\hat{h}: \hat{M}^{\xi} \to \hat{M}^{\eta}$. By 5.4, we have

$$\tilde{H}\left(\hat{h}\right) := H_{\pi \times \pi}\left(\hat{h}\right) = i_2 H_{\pi}\left(\tilde{h}\right)$$

which implies that $\psi(h)$ is in the image of the homomorphism

$$i_{2*} \colon \left\{ \tilde{M}^+ \wedge \tilde{M}^+, D_2 \tilde{M}^\tau \wedge_\pi (\pi \times \pi)^+ \right\}_{\pi \times \pi} \to \left\{ \tilde{M}^+ \wedge \tilde{M}^+, D_2 \hat{M}^\tau \right\}_{\pi \times \pi}$$

given by composing with i_2 . A calculation of the kind appearing in the proof of 6.2 shows that the domain of i_{2*} is isomorphic to Q_d , and that with respect to this identification, i_{2*} is the inclusion $Q_d \rightarrow Q_d(\pi)$.

Let

$$\epsilon \colon Q_d(\pi) \to Q_d$$

be the homomorphism induced by mapping a conjugacy class to the identity. Note that ϵ is a retraction to the inclusion $Q_d \subset Q_d(\pi)$.

The following lemma implies that ψ is one-to-one.

Lemma 8.6 The homomorphism

$$h \mapsto \epsilon \circ \tilde{H}\left(\hat{h}\right) \circ \tilde{c}_{\Delta}$$

is one-to-one.

Proof By 5.4, the homomorphism has an alternative description as

$$h \mapsto H(h^{\sharp}) \circ c_{\Delta} \in \left\{ M^{+} \wedge M^{+}, D_{2}M^{\tau} \right\} \cong Q_{d},$$

where $h^{\sharp}: M^{\xi} \to M^{\eta}$ is the stable map of Thom spaces associated with h,

$$H(h^{\sharp}): M^{\xi} \Rightarrow D_2 M^{\eta} \simeq D_2 M^{\tau}$$

its stable Hopf invariant and $c_{\Lambda}: M^+ \wedge M^+ \rightarrow M^{\xi}$ the stable collapse.

Our next step is to compute the homotopical dimension of the mapping cone of c_{Δ} . This will be done by giving an alternative description of c_{Δ} in terms of *S*-duality. Recall that c_{Δ} was defined using a Poincaré embedding of the decompressed diagonal $M \to M \times M \times D^k$. An equivalent description of c_{Δ} is to take the *umkehr map* of the stable diagonal map

$$\Delta^{-2\tau}: M^{-2\tau} \to M^{-\tau} \wedge M^{-\tau}$$

which is the thomification of the usual diagonal map of M with respect to the stable spherical fibration $-\tau \times -\tau$ (By "umkehr map," we mean the effect of taking maps into the sphere spectrum, i.e., the effect of taking the associated map of *S*-duals). By Atiyah duality, the *S*-dual of $M^{-\tau} \wedge M^{-\tau}$ is identified $M^+ \wedge M^+$, whereas the *S*-dual of $M^{-2\tau}$ is M^{τ} . The equivalence of these two descriptions is a consequence of Poincaré duality; for reasons of space we omit the proof.

Since the mapping cone of the diagonal map of M is has the homotopy type of a connected CW complex of dimension 2d, it follows that the mapping cone of $\Delta^{-2\tau}$ has the homotopy type of a (-2d)-connected CW spectrum of dimension 0. Then the *S*-dual of the mapping cone of $\Delta^{-2\tau}$ has the homotopy type of a CW spectrum of dimension 2d - 1. But the latter is identified with a single desuspension of the mapping cone of c_{Δ} . It follows that the homotopy cofiber of the stable map c_{Δ} has the homotopy type of a CW spectrum of dimension 2d.

Since $D_2 M^{\tau}$ is (2d-1)-connected, it follows that the homomorphism

$$c_{\Delta}^* \colon \left\{ M^{\tau}, D_2 M^{\tau} \right\} \to \left\{ M^+ \wedge M^+, D_2 M^{\tau} \right\}$$

given by composing with c_{Δ} is surjective. Again, straightforward calculation shows that the domain and codomain of this last homomorphism are both isomorphic to Q_d . But any surjection of Q_d onto itself is necessarily an isomorphism. It follows that c^*_{Δ} is an isomorphism.

Consequently, we are reduced to showing that

$$h \mapsto H(h^{\sharp}) \in \{M^{\tau}, D_2 M^{\tau}\} \cong Q_d$$

is one-to-one. This is proved below in 8.11.

Corollary 8.7 The homomorphism $\psi : Q_d \to Q_d(\pi)$ is one-to-one with image contained in Q_d .

In view of this corollary, we may to view ψ as a monomorphism $Q_d \rightarrow Q_d$.

Corollary 8.8 When d is odd, the monomorphism $\psi: Q_d \to Q_d$ is an isomorphism.

Proof Q_d is finite when d is odd. Since ϕ is one-to-one, it must also be onto.

Now assume that d is even. Then Q_d is infinite cyclic and we have a monomorphism 2: $Q_d \rightarrow \mathbb{Z}$ given by multiplication by 2.

Lemma 8.9 Assume d is even. Represent an element of Q_d by a stable fiber homotopy equivalence $h: \eta \simeq_s \xi$ of (d-1)-spherical fibrations over M. Then up to sign,

$$2\psi(h) = \chi(\xi) - \chi(\eta),$$

where χ is the Euler characteristic.

Proof When d is even, The homomorphism 2ψ can alternatively be described as

 $h \mapsto \operatorname{tr}(H(h^{\sharp})) \in \{M^{\tau}, M^{\tau} \wedge M^{\tau}\} \cong \mathbb{Z}$

where tr: $\{M^{\tau}, D_2 M^{\tau}\} \rightarrow \{M^{\tau}, M^{\tau} \land M^{\tau}\}$ is the transfer. By definition of the stable Hopf invariant, tr $(H(h^{\sharp}))$ coincides with the difference of reduced diagonal maps $\Delta_{\xi} \colon M^{\xi} \rightarrow M^{\xi} \land M^{\xi}$ and $\Delta_{\eta} \colon M^{\eta} \rightarrow M^{\eta} \land M^{\eta}$. But each diagonal represents the Euler characteristic of the fibration that it is subscripted by.

Corollary 8.10 $\psi: Q_d \to Q_d$ is an isomorphism when d is even.

Proof Choose a stable fiber homotopy equivalence $h: \xi \simeq_s \eta$ representing a generator of Q_d . We may assume ξ and η coincide on M_0 (= M with its top cell D^d removed). Then $\chi(\xi) - \chi(\eta) = \pm 2$, because one can describe the difference obstruction in this case as a map

$$S^d \to G/G_d$$

such that the composite $S^d \to G/G_d \to BG_d$ maps the upper hemisphere via ξ and its lower one via η . It is well-known that the Euler characteristic of this fibration over S^d is ± 2 , since the composite

$$Q_d = \pi_d(G/G_d) \longrightarrow \pi_d(BG_d) \xrightarrow{\chi} \mathbb{Z}$$

is multiplication by 2 when d is even (and trivial when d is odd).

Then Lemma 8.9 says that 2ψ takes value ± 2 . This implies that ψ takes value $\pm 1 \in Q_d$, so ψ is an isomorphism.

Recall that if $h: \xi \simeq_s \eta$ represents an element of Q_d , then $h^{\sharp}: M^{\xi} \simeq_s M^{\eta}$ denotes the associated stable equivalence of (unequivariant) Thom spaces.

The next result was used in the proof of 8.6.

Proposition 8.11 The homomorphism

$$Q_d \rightarrow \{M^{\tau}, D_2 M^{\tau}\},\$$

given by $h \mapsto H(h^{\sharp})$ is one-to-one.

Proof If $H(h^{\sharp})$ is trivial, then h^{\sharp} desuspends to an unstable weak equivalence of Thom spaces

$$M^{\xi} \simeq M^{\eta}$$
.

We need to show that $h: \xi \simeq_s \eta$ desuspends to an unstable fiber homotopy equivalence. Let BF_d be the classifying space of *d*-dimensional spherical fibrations equipped with section. Since the map

$$BG_d \rightarrow BF_d$$

given by $\xi \mapsto \xi \oplus \epsilon$ is (2d-3)-connected (see e.g., [10]), and $d \ge 4$, the map $[M, BG_d] \rightarrow [M, BF_d]$ is a bijection. It will therefore be enough to show that

$$\xi \oplus \epsilon, \eta \oplus \epsilon \colon M \to BF_d$$

are homotopic. Let S^{ξ} and S^{η} denote the total spaces of these fibrations. Note that M^{ξ} is just S^{ξ} with *M* collapsed to a point.

For a space *B* equipped with sectioned fibrations $E_1 \rightarrow B$ and $E_2 \rightarrow B$, let $[E_1, E_2]_B$ be the fiberwise homotopy classes of maps $E_1 \rightarrow E_2$ which cover the identity of *B*, and preserve the preferred sections. Let $\{E_1, E_2\}_B$ denote the associated stable fiberwise homotopy classes.

Consider the commutative diagram

where the horizontal arrows are given by sending a fibration map its induced map of Thom spaces, and the vertical ones are given by stabilization. We also have an identification

$$[M^{\xi}, M^{\eta}] \cong [S^{\xi}, M^{\eta} \times M]_M,$$

so the horizontal arrows of the diagram can also be described as arising from the *d*-connected map $S^{\eta} \rightarrow M^{\eta} \times M$ of fibrations over *M* with section by taking homotopy classes of maps from S^{ξ} . The right vertical arrow of (2) sits in a short exact sequence of based sets whose third term is $\{M^{\xi}, D_2 M^{\eta}\}$, and the map onto this term is surjective for dimensional reasons (cf. 5.2). Similarly, the left vertical arrow of (2) sits in a short exact sequence whose third term is

$$\left\{S^{\xi}, D_2^{\bullet}S^{\eta}\right\}_M$$

where $D_2^{\bullet}S^{\eta}$ is the quadratic construction applied fiberwise to S^{η} , and the map onto this term is again surjective for dimensional reasons.

The induced function

$$\left\{S^{\xi}, D_{2}^{\bullet}S^{\eta}\right\}_{M} \to \left\{M^{\xi}, D_{2}M^{\eta}\right\} \cong Q_{d}$$

$$\tag{3}$$

arises from a map $D_2^{\bullet}S^{\eta} \to D_2^{\bullet}(M^{\eta} \times M)$ by applying fiberwise stable maps out of S^{ξ} to the domain and codomain. The function (3) is therefore a homomorphism. It is straightforward to check that the map inducing it is 2*d*-connected. Since S^{ξ} has the dimension of a cell complex with cells in dimensions at most 2*d*, it follows that (3) is surjective.

Assertion. There is an isomorphism

$$Q_d \cong \left\{ S^{\xi}, D_2^{\bullet} S^{\eta} \right\}_M$$
.

Here is a proof: first consider the abelian group

$$\left\{S^{\xi}, S^{2\eta}\right\}_M$$

where $S^{2\eta}$ denotes the total space of $\eta \oplus \eta \oplus \epsilon$. This has a fiberwise involution whose homotopy orbits gives $D_2^{\bullet}S^{\eta}$. Since ξ and η are stable fiber homotopy equivalent, we can identify $\{S^{\xi}, S^{2\eta}\}_M$ with

$$\left\{S^{\eta}, S^{2\eta}\right\}_{M} \cong \left\{S^{0} \times M, S^{\eta}\right\}_{M},$$

where this last isomorphism comes from taking fiberwise smash product with S^{η} . The abelian group $\{S^0 \times M, S^{\eta}\}_M$ is the fiberwise homotopy classes of stable sections of S^{η} . It is also known as the stable cohomotopy of M^+ twisted by the spherical fibration τ (compare [5, p. 5]). By Atiyah duality [1], it is isomorphic to the zeroth stable homotopy group of M^+ , i.e.,

$$\pi_0^{\mathrm{st}}(M^+) \cong \mathbb{Z}$$
.

Taking into account how the involution acts, we see that $\{S^{\xi}, D_2^{\bullet}S^{\eta}\}_M$ is isomorphic to the coinvariants of a certain involution on \mathbb{Z} . It is not difficult to check that this involution is given by $(-1)^d$. This completes the proof of the assertion.

The proposition is now completed as follows: since the map (3) is identified with a surjective map of Q_d , it is an isomorphism. Consequently, the diagram (2) is a cartesian square of abelian groups. The element *h* lives in the lower left corner of the square, and by hypothesis, its image in the lower right corner lifts to the upper right corner. Consequently, *h* lifts to unstable fiber homotopy equivalence $\xi \oplus \epsilon \simeq \xi \oplus \epsilon$ of sectioned fibrations.

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References

- 1. Atiyah, M.F.: Thom complexes. Proc. Lond. Math. Soc. 11, 291-310 (1961)
- 2. Byun, Y.: Poincaré embedding of the diagonal. Trans. Amer. Math. Soc. 350, 3537-3553 (1998)
- 3. Byun, Y.: Tangent fibration of a Poincar complex. J. Lond. Math. Soc. 59, 1101–1116 (1998)
- 4. Carlsson, G.: A survey of equivariant stable homotopy theory. Topology **31**, 1–27 (1992)
- Crabb, M.C.: Z/2-Homotopy Theory. London Mathematical Society Lecture Note Series, vol. 44, Cambridge University Press, London (1980)
- 6. Crabb, M.C., Ranicki, A.: (in preparation)
- Crabb, M.C., Ranicki, A.: Quadratic structures in surgery theory. http://www.maths.ed.ac.uk/~aar/slides/ wallbeam.pdf
- 8. Dupont, J.L.: On homotopy invariance of the tangent bundle I. Math. Scand. 26, 5–13 (1970)
- 9. Dupont, J.L.: On homotopy invariance of the tangent bundle II. Math. Scand. 26, 200-220 (1970)
- 10. Haefliger, A.: Differential embeddings of S^n in S^{n+q} for q > 2. Ann. Math. 83, 402–436 (1966)
- Hatcher, A., Quinn, F.: Bordism invariants of intersections of submanifolds. Trans. Am. Math. Soc. 200, 327–344 (1974)
- Klein, J.R.: Embedding, compression and fiberwise homotopy theory. Alg. Geom. Topol. 2, 311–336 (2002)
- 13. Klein, J.R.: Poincaré immersions. Forum Math. 11, 717–734 (1999)
- Klein, J.R.: Poincaré duality spaces Surveys on Surgery Theory, vol. 1. Ann. Math. Stud. 145, 135–165 (2000)

- 15. Longoni, R., Salvatore, P.: Configuration spaces are not homotopy invariant. Topology 44, 375–380 (2005)
- May, J.P. et al.: Equivariant Homotopy and Cohomology Theory. CBMS Regional Conference Series in Mathematics. vol. 91, American Mathematical Society (1996)
- Milgram, R.J.: Unstable homotopy from the stable point of view. vol. 368, Lecture Notes in Mathematics. Springer, Heidelberg (1974)
- 18. Spivak, M.: Spaces satisfying Poincaré duality. Topology 6, 77–101 (1967)
- 19. Sutherland, W.A.: The Browder–Dupont invariant. Proc. Lond. Math. Soc. 33, 94–112 (1976)
- 20. tom Dieck, T.: Orbittypen und äquivariante Homologie. II. Arch. Math. (Basel) 26, 650-662 (1975)
- 21. Wall, C.T.C.: Poincaré complexes. I. Ann. Math. 86, 213–245 (1967)
- 22. Whitney, H.: The self-intersections of a smooth *n*-manifold in 2*n*-space. Ann. Math. 45, 220–246 (1944)